Solutions Manual

for

Glimpses of Algebra and Geometry

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This is a text file in Latex, the dvi and postscript formats are in solutions.dvi and solutions.ps.

In this manual we solve the hardest problems posed in the Glimpses. We focus on those problems that have been left out from the Hints for Selected Problems, and whose solutions require novel ideas.

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Section 1.

1.(b) Write the number in the form $7k$, $7k \pm 1$,$7k \pm 2$, or $7k \pm 3$, take the cubes of these, and consider the remainders modulo 7.

2. By Problem 1(b), we need to show that $7a - 1$ is not a square. This follows as in 1(b) by squaring the numbers $7k$, $7k \pm 1$,$7k \pm 2$, and $7k \pm 3$, and studying the remainders modulo 7.

Section 2.

2. Write $\cos 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}$ and repeat, with appropriate modifications, the proof of irrationality of e as in the remark following Theorem 1.

4. Using the geometric series formula, a Riemann sum can be worked out as

$$\sum_{k=1}^{m} (\alpha^k)^n (\alpha^k - \alpha^{k-1}) = \frac{\alpha - 1}{\alpha} \sum_{k=1}^{m} \alpha^{(n+1)k}$$
$$= (\alpha - 1)\alpha^n \frac{\alpha^{m(n+1)} - 1}{\alpha^{(n+1)} - 1} = \frac{a^{n+1} - 1}{\alpha^n + \alpha^{n-1} + \ldots + \alpha + 1}.$$

For $m \to \infty$, we have $\alpha = \sqrt[m]{a} \to 1$ so that the Riemann sum converges to

$$\frac{a^{n+1}-1}{n+1}.$$

8. The pyramidal staircase can be cut into a large square pyramid of volume $\frac{n^3}{3}$, and two sets of n triangular prisms of heights $1, 2, \ldots, n$, common base area $\frac{1}{2}$, and total volume $\frac{1+2+\ldots+n}{2}$. Notice that the two sets of prisms overlap in n small square pyramids with total volume $\frac{n}{3}$, and this has to be subtracted form the volume. The volume of the pyramidal staircase is thus $\frac{n^3}{3} + (1+2+\ldots+n) - \frac{n}{3}$. Now use Problem 5.

10. Write $a = \sum_{n=0}^{N} c_n 2^n$ with $c_n \in \{0,1\}$. Notice that $c_n = 1$ iff a_n is odd.

Section 3.

3. For a, b, c consecutive, we have $a + c = 2b$. Substituting this into the Pythagorean equation, we obtain $4a = 3b$. Since a, b are relatively prime $a = 3$ and $b = 4$ follow.

8. Let (x_0, y_0), $y_0 \neq 0$, be a rational point on the Bachet's curve given by $y^2 = x^3 - a$. Implicit differentiation gives $2yy' = 3x^2$ so that the slope m of the tangent line at (x_0, y_0) is $m = \frac{3x_0^2}{2y_0}$. We need to couple the equation of the tangent line $y - y_0 = m(x - x_0)$ with Bachet's equation $y^2 = x^3 - a$ to obtain the coordinates of the intersection point. Since $y_0^2 = x_0^3 - a$, we write Bachet's equation as $y^2 - y_0^2 = x^3 - x_0^3$. After factoring, we have $(y - y_0)(y + y_0) = (x - x_0)(x^2 + xx_0 + x_0^2)$. Substituting $y - y_0 = m(x - x_0)$ and canceling the factor $x - x_0$, we obtain

$$m(y + y_0) = x^2 + xx_0 + x_0^2.$$

Substituting again $y - y_0 = m(x - x_0)$, we finally arrive at

$$m(2y_0 + m(x - x_0)) = x^2 + xx_0 + x_0^2.$$

This is a quadratic equation in x that has x_0 as a root since $m = \frac{3x_0^2}{2y_0}$. Factoring, we obtain

$$(x - x_0)\left(x - \frac{x_0^4 + 8ax_0}{4y_0^2}\right) = 0.$$

Bachet's formula follows.

13. $78 = 2 \cdot 3 \cdot 13 = \frac{ab}{2}$ and $a^2 + b^2 = c^2$. In particular, a, b are relatively prime since $6^2 + 26^2 = 712$ is not a square. Setting $a = 2st$, $b = t^2 - s^2$, and $c = t^2 + s^2$, we obtain $2 \cdot 3 \cdot 13 = st(t-s)(t+s)$ with $t > s$ relatively prime and of different parity. This is impossible.

Section 4.

5. Let P and Q be polynomials of degrees m and n with rational coefficients and leading coefficient one such that $P(c_1) = Q(c_2) = 0$. Let P and Q have roots $c_1 = \alpha_1, \ldots, \alpha_m$ and $c_2 = \beta_1, \ldots, \beta_n$. Consider the polynomial $R(x) = \prod_{i=1}^m \prod_{j=1}^n (x - (\alpha_i + \beta_j))$. The coefficients of R are symmetric in α_i and β_j, and therefore, by the fundamental theorem of symmetric polynomials, they can be expressed as polynomials (with integral coefficients) in the coefficients of P and Q. In particular, the coefficients of R are rational. Since $R(c_1 + c_2) = 0$, $c_1 + c_2$ is algebraic.

6. By the addition formulas

$$\begin{aligned} \cos((n+1)\alpha) &= \cos(n\alpha)\cos(\alpha) - \sin(n\alpha)\sin(\alpha) \\ \sin((n+2)\alpha) &= \sin((n+1)\alpha)\cos(\alpha) + \cos((n+1)\alpha)\sin(\alpha) \end{aligned}$$

For $\alpha = \cos^{-1}(x)$, these rewrite as

$$\begin{aligned} T_{n+1}(x) &= xT_n(x) - (1-x^2)U_{n-1}(x) \\ U_{n+1}(x) &= xU_n(x) + T_{n+1}(x). \end{aligned}$$

Since $T_1(x) = x$ and $U_0(x) = 1$, these recurrence relations imply that T_n and U_n are polynomials. Now $\cos(\pi/n)$ is algebraic since it is a root of the polynomial $T_n + 1$.

12. The existence and uniqueness of x_0 follow from the properties of the exponential function. Let m be rational. If x_0 is rational then so is $e^{x_0} = mx_0 + 1$, and this contradicts to the corollary to Theorem 1 in Section 2. Transcendentality of x_0 for algebraic m follows from Lindemann's theorem.

Section 5.

2. The equation of the circle is

$$\left(x-\frac{a-c}{2a}\right)^2+\left(y+\frac{b}{2a}\right)^2=\frac{(a-c)^2+b^2}{4a^2}.$$

Setting $x=1$ and solving for y, we obtain the quadratic formula for r_1 and r_2. A generalization of this construction to cubic polynomials would mean that the roots of a cubic polynomial with constructible coefficients would be constructible. This is false (cf. Problem 3 on page 19).

Section 6.

4. The two lines are parallel iff $\frac{z_1-z_2}{z_3-z_4}$ is real. Thus, we may assume that the two lines intersect. The condition $\Re(\frac{z_1-z_2}{z_3-z_4})=0$ is invariant under translation (that amounts to add a constant to each variable) so that we may assume that the lines intersect at the origin. Then z_1, z_2 and z_3, z_4 are real constant multiples of each other, and the condition reduces to $\Re(z_1/z_3)=0$. Now write z_1 and z_3 in polar form and consider the argument of the ratio z_1/z_3.

5. As in Problem 4, the equation is invariant under adding a constant to each variable z_1, z_2, z_3 so that we may assume that the centroid of the triangle is at the origin, $z_1+z_2+z_3=0$. Squaring this, we see that the condition is equivalent to $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1=0$. On the other hand, $z_1z_2+z_2z_3+z_3z_1=z_1(z_2+z_3)+z_2z_3=-z_1^2+z_2z_3$. Thus, $z_1^2=z_2z_3$. Similarly, $z_2^2=z_1z_3$ and $z_3^2=z_1z_2$. In particular, $z_1^3=z_2^3=z_3^3$ and hence $|z_1|=|z_2|=|z_3|$. Now write z_1, z_2, and z_3 in polar form.

7. The condition remains invariant when we translate, rotate, and scale each triangle individually. (For example, translation corresponds to the row operation of adding a constant multiple of the first row to the row the triangle is represented by.) We can thus reduce the problem to the case when $z_1=w_1=0$ and $z_2=w_2=1$. The condition now reduces to $z_3=w_3$.

8. $2\cdot 3=(1+\sqrt{5}i)(1-\sqrt{5}i)$.

Section 7.

4. We have

$$|z-w|^2 = (z-w)(\bar{z}-\bar{w}) = |z|^2 - z\bar{w} - \bar{z}w + |w|^2$$

and

$$|1-z\bar{w}|^2 = (1-z\bar{w})(1-\bar{z}w) = 1 - z\bar{w} - \bar{z}w + |z|^2|w|^2.$$

The stated inequality is thus equivalent to $0 < (1-|z|^2)(1-|w|^2)$.

Section 8.

4.(a) If P is an odd degree polynomial with real coefficients then $\lim_{x\to\pm\infty} P(x) = \pm\,\mathrm{sgn}\,(c_n)\,\infty$, where $\mathrm{sgn}\,(c_n)$ is the sign of the leading coefficient c_n of P. By continuity, P must have a real zero. (b) The polynomial Q_k, $k \in \mathbf{Z}$, has coefficients that are, up to sign, the elementary symmetric polynomials in the variables $\alpha_i + \alpha_j + k\alpha_i\alpha_j$, $1 \leq i < j \leq n$. But an elementary symmetric polynomial in these variables is symmetric in α_i, $i = 1, \ldots, n$, and hence it can be written as a real polynomial in the coefficients of P. Thus the coefficients of Q_k are real. The degree of Q_k is $\frac{n(n-1)}{2} = 2^{m-1}a(2^m a - 1)$, $m \geq 1$. Since $a(2^m a - 1)$ is odd, the induction hypothesis applies. Thus, for each k, there are indices $1 \leq i < j \leq n$ such that the root $\alpha_i + \alpha_j + k\alpha_i\alpha_j$ is a complex number. Since there are finitely many roots α_i but infinitely many choices of k, for some $1 \leq i < j \leq n$, $\alpha_i + \alpha_j$ and $\alpha_i\alpha_j$ are both complex numbers. The polynomial $z^2 - (\alpha_i + \alpha_j)z + \alpha_i\alpha_j = (z-\alpha_i)(z-\alpha_j)$ has complex coefficients, so that, by the complex form of the quadratic formula, α_i and α_j are complex. (c) If P is a complex polynomial then $P\bar{P}$ has real values, in particular, its coefficients must be real. By the above, $P\bar{P}$ has a complex root. Since $|P|^2 = P\bar{P}$, it is also a complex root of P.

Section 9.

1. The regular hexagon is obtained by slicing the cube with a plane through the origin and normal vector $(1,1,1)$. The plane extensions of the three sides of the cube meeting at $(1,1,1)$ cut an equilateral triangle out of this plane, and the plane extensions of the other three sides of the cube meeting at $(-1,-1,-1)$ further truncate this triangle to a regular hexagon.

3.(a) The vertices of P_{2n} are those of P_n plus the midpoints of the circular arcs over the sides of P_n. Thus, half of a side of P_n and a side of P_{2n} meeting at a vertex of P_n are two sides of a right triangle whose third side is $1-\sqrt{1-(s_n/2)^2}$. We thus have

$$s_{2n}^2 = \left(\frac{s_n}{2}\right)^2 + \left(1-\sqrt{1-\left(\frac{s_n}{2}\right)^2}\right)^2.$$

This gives

$$s_{2n} = \sqrt{2-\sqrt{4-s_n^2}}.$$

In particular, since $s_4 = \sqrt{2}$, we have

$$s_{2^n} = \sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}},$$

with $n-1$ nested square roots. (b) Let A_n denote the area of P_n. Since half of the sides of P_n serve as heights of the $2n$ isosceles triangles that make up P_{2n}, we have $A_{2n} = \frac{ns_n}{2}$. In particular, we have

$$A_{2^n} = 2^{n-1}s_{2^n} = 2^{n-1}\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}},$$

with $n-1$ nested square roots.

Section 10.

1. Since G is finite it cannot contain translations. The argument on pp. 75-76 shows that if R_1 and R_2 are two rotations about different centers and different rotation angles then their compositions $R_1 \circ R_2$ and $R_2 \circ R_1$ are also rotations with different centers but the same rotation angle. Then $R_2^{-1} \circ R_1^{-1} \circ R_2 \circ R_1$ is a translation.

2. Type 5.

4. Type 2.

5. We may assume that $\omega = z(2\pi/n)$, $n \geq 3$. The rotation $R_{2\pi/n}$ leaves $\mathbf{Z}[\omega]$ invariant since it is multiplication by ω. If $L = \mathbf{Z}[\omega]$ is a lattice, then $R_{2\pi/n}$ is in the point group $\bar{G}$. By the crystallographic restriction, $n = 3, 4$ or 6.

Section 11.

2. From the geometry of the stereographic projection it follows that, given a circle S on the extended plane, $h_N^{-1}(S)$ is a great circle on S^2 iff $S \cap S^1$ contains an antipodal pair of points. The rest follows from the remark on pp. 102-103.

3. Let C_1, C_2 be great circles on S^2 meeting in an angle α at $p \in S^2$. If p is the South Pole then angle preservance of h_N is clear since α is the dihedral angle between the vertical planes that contain C_1 and C_2, and the h_N-images of C_1 and C_2 are the intersections of these planes with $\hat{\mathbf{C}}$. In general, let R be a reflection to a great circle C that carries p into the South Pole. Then $R(C_1)$ and $R(C_2)$ meet in angle α at the South Pole, and, by the previous argument, $h_N(R(C_1))$ and $h_N(R(C_2))$ also meet in angle α. Now write these circles as $(h_N \circ R \circ h_N^{-1})(h_N(C_1))$ and $(h_N \circ R \circ h_N^{-1})(h_N(C_2))$ and use the fact that the reflection $h_N \circ R \circ h_N^{-1}$ to the circle $h_N(C)$ is angle preserving. (This can be seen directly from the remark on pp. 102-103; see also the argument on pp. 112-113.)

Section 12.

7. Setting $z = |z|z(\theta)$, the image of a circle $|z| = r < 1$ is an ellipse with semi-major axis $r + 1/r$ and semi-minor axis $r - 1/r$. The image of a half-line emanating from the origin is a hyperbola. The ellipses and the hyperbolas are confocal.

8.(a) If the two circles intersect then apply a Möbius transformation that sends one of the intersection points to ∞. It thus remains to consider two disjoint nonconcentric circles C_1 and C_2. Applying a suitable isometry, we may assume that the line through their centers is the real axis. Show that there is a (unique) circle C that intersects both circles C_1 and C_2, and the real axis at right angles (see Problem 2 of Section 13). Choose a linear fractional tranformation with real coefficients that maps C to a vertical line. Since the real axis is preserved, the images of C_1 and C_2 intersect the real axis and the vertical line at right angles. Hence they must be concentric.

Section 13.

1. Problem 4 of Section 7 implies that the given linear fractional trans-

formations are self maps of the unit disk D^2. Theorem 9 asserts that the group of linear fractional transformations of H^2 onto itself is 3 dimensional (with $a, b, c, d \in \mathbf{R}$ being the parameters subject to the constraint $ad - bc = 1$). Since D^2 and H^2 are equivalent through a linear fractional transformation (Problem 5 of Section 12), it follows that the group of linear fractional transformations that leave D^2 invariant is also 3 dimensional. On the other hand, the linear fractional transformations given in the problem form a group and they depend on 3 parameters, $\Re(w)$, $\Im(w)$ and θ. Thus, these give all the linear fractional transformations preserving D^2.

2. Applying a suitable isometry if necessary, l_1 can be assumed to be vertical with ideal point $c \in \mathbf{R}$. Then l_2 is a semi-circle. Draw a Euclidean line from c tangent to l_2. The hyperbolic line perpendicular to both l_1 and l_2 is represented by the semi-circle with center at c that contains the point of tangency.

4. We may assume that the triangle has vertices i, ti, $t > 1$, and $z(\theta)$, $0 < \theta < \frac{\pi}{2}$. Let α and β be the angles at $z(\theta)$ and at ti. (a) From the hyperbolic distance formula, we obtain

$$\cosh a = \frac{t^2+1}{2t}, \;\; \cosh b = \frac{1}{\sin\theta}, \;\; \cosh c = \frac{t^2+1}{2t\sin\theta}.$$

The Pythagorean theorem follows. (b) Let $r > 0$ be the radius and $c \in \mathbf{R}$ the center of the semi-circle that represents the hyperbolic line through ti and $z(\theta)$. Let δ be the angle at c between by the radial segment connecting c and $z(\theta)$, and the real axis. We have

$$\sin\beta = \frac{t}{r}, \;\; r\cos\delta = r\cos\beta + \cos\theta, \;\; r\sin\delta = \sin\theta.$$

The identity $\cos^2\delta + \sin^2\delta = 1$ gives

$$r^2 = r^2\cos^2\beta + 2r\cos\beta\cos\theta + 1$$

so that we have

$$t^2 = r^2\sin^2\beta = 2r\cos\beta\cos\theta + 1.$$

Using this and $\alpha = \theta - \delta$, we compute

$$\begin{aligned}\cos\alpha &= \cos(\theta-\delta) = \cos\theta\cos\beta + \frac{1}{r}\\ &= \frac{t}{r}\Big(\frac{t^2-1}{2t} + \frac{1}{t}\Big)\\ &= \sin\beta\cosh a.\end{aligned}$$

Swithching the roles of a, b and α, β, we also have

$$\cos\beta = \sin\alpha\cosh b.$$

Now the identity

$$\sinh a \tan\beta = \tanh b$$

follows by eliminating α.

Section 14.

1. (a) A parabolic isometry g is the composition of two reflections in hyperbolic lines that meet at a common endpoint on the boundary of H^2. Conjugating g with an isometry that carries this endpoint to ∞, the two hyperbolic lines become vertical Euclidean lines, and the conjugated g has the form $z \mapsto z + a$, $a \in \mathbf{R}$. Now conjugate this with $z \mapsto \frac{1}{a}z$ to obtain $z \mapsto z + 1$. (b) This is given on pp. 127-128. (c) Two nonintersecting hyperbolic lines with no common endpoints on the boundary of H^2 can be mapped to concentric Euclidean semi-circles by an isometry of H^2 (see Problem 2 of Section 13). Now a horizontal translation brings this configuration to concentric circles centered at the origin. The corresponding composition of reflections is of the form $z \mapsto kz$, $k \neq 0$.

2. Parabolic isometries have a unique fixed point on $\mathbf{R} \cup \{\infty\}$. Elliptic isometries have a unique fixed point in H^2. Hyperbolic isometries have two fixed points on $\mathbf{R} \cup \{\infty\}$.

3. Since a linear fractional transformation determines the corresponding matrix in $SL(2, \mathbf{C})$ up to sign, trace^2 is well defined. Given an isometry g, the condition

$$g(z) = \frac{az+b}{cz+d} = z, \quad a, b, c, d \in \mathbf{R},$$

for z to be a fixed point of g amounts to solve a quadratic equation with discriminant $(a-d)^2 - 4bc = (a+d)^2 - 4 = \text{trace}^2(g) - 4$. If $\text{trace}^2(g) = 4$ then g has a unique fixed point on $\mathbf{R} \cup \{\infty\}$. Conjugating g by a suitable isometry, we may assume that this fixed point is ∞. This means that $c = 0$ and $b \neq 0$ (by unicity of the fixed point). As in Problem 1, we obtain that g is parabolic. If $0 \leq \text{trace}^2(g) < 4$ then there are two fixed points of g and they are conjugate complex numbers. Thus there is a unique fixed point in H^2. The argument on pp. 127-128 shows that g is elliptic. Finally, if $\text{trace}^2(g) > 4$ then g has two fixed points on $\mathbf{R} \cup \{\infty\}$ that may be assumed to be 0 and ∞. We thus have $b = c = 0$, and g is hyperbolic.

4. This follows form Problems 2-3.

6. If $z_0 \in \mathbf{R} \cup \{\infty\}$ is the unique fixed point of the parabolic g_1 then, by the stated commutativity, $g_2(z_0)$ is left fixed by g_1 and we must have $g_2(z_0) = z_0$. Thus g_2 is either parabolic or hyperbolic. If $w_0 \in \mathbf{R} \cup \{\infty\}$ were another fixed point of g_2 then $g_1(w_0)$ would also be left fixed by g_2. This is a contradiction since $g_1(w_0) \neq w_0, z_0$.

8. This follows from Example 8 on pp. 142-143 by minor modifications.

Section 15.

1. Using the Euler formula for complex exponents, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

The complex extension of cosine is therefore defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Since the exponential function is periodic with period $2\pi i$, the cosine function is periodic with period 2π. We now claim that it is one-to-one on any vertical strip $k\pi < \Re(z) < (k+1)\pi$, $k \in \mathbf{Z}$, and each strip is mapped onto the whole complex plane with cuts $(-\infty, -1)$ and $(1, \infty)$ along the real axis. Indeed, using the trigonometric identity

$$\cos z_1 - \cos z_2 = -2\sin\left(\frac{z_1 - z_2}{2}\right)\sin\left(\frac{z_1 + z_2}{2}\right),$$

we see that $\cos z_1 = \cos z_2$ iff $z_1 \pm z_2 = 2m\pi$, for some $m \in \mathbf{Z}$. If z_1 and z_2 are in a vertical strip then $z_1 = z_2$ follows. For surjectivity, notice that the equation

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = w$$

is quadratic in e^{iz} so that

$$z = -i\log(w \pm \sqrt{w^2 - 1}) = \pm i\log(w + \sqrt{w^2 - 1}).$$

(The arguments $w \pm \sqrt{w^2 - 1}$ are reciprocals of each other.)

Section 17.

8. By the construction of the dodecahedron on pp. 186-187, every vertex of the dodecahedron is the vertex of at most two cubes. The 5 inscribed cubes have the total of 40 vertices. Since the dodecahedron has 20 vertices, it follows that every vertex of the dodecahedron is the vertex of exactly two cubes. These two cubes have a common diagonal. There are exactly 20/2=10 diagonals. 10 is also the number of different pairs of cubes. Thus the statement is true.

10. (a) The extensions of the sides of a spherical triangle give 3 great circles that divide the sphere into 8 regions; the original triangle whose area we denote by A and its opposite triangle, 3 spherical triangles with a common side to the original triangle whose areas we denote by X, Y and Z, and their opposites. Let α, β, and γ denote the spherical angles of the original triangle opposite to the sides that are common with the other 3 triangles with areas X, Y, and Z. Notice that $A+X$ is the area of a spherical wedge with spherical angle α. We thus have $A + X = 2\alpha$. Similarly, $A + Y = 2\beta$ and $A + Z = 2\gamma$. Adding, we obtain $3A + X + Y + Z = 2(\alpha + \beta + \gamma)$. On the other hand, $2(A + X + Y + Z) = 4\pi$, the area of the entire sphere. Subtracting, the statement follows.

17.(a) In Figure 17.28 the triangles $\triangle p_0p_3p_4$ and $\triangle p_1p_3p_4$ are similar (by the definition of the golden section). Thus all three angles at p_3 are congruent and thereby equal to $\frac{\pi}{5}$. It follows that the segment connecting p_0 and p_4 is the side of a decagon inscribed in a unit circle.

18. The vertices of the octahedron are the midpoints of the faces of

the cube circumscribed around the reciprocal pair of tetrahedra. The vertex figure at a vertex v of the octahedron is parallel to the face of the cube whose midpoint is v. Homothety of the two cubes follows. Since the edges of the reciprocal cube bisect those of the octahedron perpendicularly, the ratio of magnification is $\frac{1}{2}$.

19. The 20 vertices of the 5 tetrahedra are vertices of a regular polyhedron with symmetry group $\mathcal{A}_5$. Thus, it must be a dodecahedron. At a fixed vertex v of the colored icosahedron all 5 colors are represented. Consider the 5 triangular faces, one for each tetrahedron, that are extensions of the 5 icosahedral faces meeting at v. The 5 vertices closest to v in each of these triangular faces form a regular pentagon since the order 5 rotation with axis through v permutes these vertices cyclically. The pentagon is perpendicular to this axis and hence parallel to the vertex figure of the icosahedron at v. Homothety of the two dodecahedra follows.

22. Inscribe a Euclidean dodecahedron in D^3 with vertices $v_1, \ldots, v_{20} \in S^2$. By the hyperbolic geometry of D^3, for $0 < t < 1$, there is a unique hyperbolic dodecahedron in D^3 with vertices $tv_1, \ldots, tv_{20}$. Let $\delta(t)$ be the common dihedral angle of this hyperbolic dodecahedron. δ is a continuous decreasing function on $(0, 1)$. We have $\lim_{t\to 1} \delta(t) < \pi/2$, and the limit $\lim_{t\to 0} \delta(t)$ is the dihedral angle $> \frac{\pi}{2}$ of the Euclidean dodecahedron. Actually, this dihedral angle is $\delta = \pi - 2\psi$, where

$$\tan\psi = \frac{\tau}{\sqrt{\tau^2+1}}$$

(see Problem 20). By continuity, there is a value $t_0 \in (0, 1)$ for which $\delta(t_0) = \frac{\pi}{2}$. The fact that hyperbolic dodecahedra with this dihedral angle tesselate D^3 follows since at each vertex the three faces are mutually prependicular.

Section 19.

The solutions in this section were written by Joseph Gerver.

12. If we assume that no more than three countries touch at one point (so that the corresponding graph is a triangulation) then a necessary and sufficient condition is that each country have an even number of neighbors.

13. Four colors are needed for a checker board, but no finite number will suffice for every map.

14. Six.

15. Take a complete graph on 27 vertices, and color each edge red or green. Then we can find a subgraph which is either (i) a triangle with all edges colored green, or (ii) a complete graph on eight vertices with all edges colored red.

16. Use a stereographic projection to go between maps on the sphere and maps on the plane. One point (the point at infinity) will be missing from the sphere, but that does not affect the number of colors, because countries that only touch at one point are not required to have different colors.

17. Establish customs stations at every frontier between neighboring countries, and build roads from each customs station to the capitals of both countries. If two roads should cross, rebuild them by changing the connections so that they do not cross. The capitals are the vertices and the roads are the edges of a planar graph. (The roads should not cross any frontiers except at the customs stations. This can be done provided every country is pathwise connected.)

20. There cannot be a vertex of degree 3; else we could remove that vertex, color the remaining graph with four colors, replace the vertex, and color it differently from its three neighbors. If there is a vertex of degree 4, and we use four colors to color every other vertex, then the four neighbors of the vertex of degree 4 must require four different colors, say red, green, yellow, and blue; otherwise the missing color could be used to color the vertex of degree 4. We can now use the same argument as in the proof of the five color theorem, looking at red-green paths or yellow-blue paths, and swapping either red and green or yellow and blue for part of the graph.

21. The red-blue path might cross the red-green path in Figure 19.6.

Section 21.

5. This follows by adjusting the argument on pp. 262-263 to the quaternionic case. The action of $S^3 \subset \mathbf{H}$ on $\mathbf{H}^2$ is given by $g : (p, q) \mapsto$

$(p \cdot g^{-1}, q \cdot g^{-1})$, $g \in S^3$.

Section 22.

2. The binary octahedral group $\mathbf{O}^*$ contains $\mathbf{T}^*$ as an index 2 subgroup. Multiplication by $\frac{1+i}{\sqrt{2}}$ has the effect of adding $\frac{\pi}{4}$ to the coordinates of the points on the middle Clifford torus corresponding to the elements of $\mathbf{T}^*$ (see Figure 22.3).

3. A rotation generating the orbit of the 5 tetrahedra circumscribed around the colored icosahedron has angle $\frac{2\pi}{5}$ and its axis lies in the coordinate plane orthogonal to the first axis and the slope of the axis is the golden section τ (see Figure 17.29). This is because at the common vertex $(0, 1, \tau)$ of the icosahedron and a golden rectangle all 5 colors are represented. This axis goes through

$$\left(0, \frac{1}{\sqrt{\tau^2+1}}, \frac{\tau}{\sqrt{\tau^2+1}}\right).$$

Using the notations in Problem 6 of Section 21, we have $\theta = \frac{2\pi}{5}$ and

$$\begin{aligned} a &= \cos\left(\frac{\pi}{5}\right) = \frac{\tau}{2}, \\ b &= \frac{\tau}{\sqrt{\tau^2+1}} \sin\left(\frac{\pi}{5}\right) = \frac{1}{2}, \\ c &= \frac{1}{\sqrt{\tau^2+1}} \sin\left(\frac{\pi}{5}\right) = \frac{1}{2\tau}, \\ d &= 0. \end{aligned}$$

Hence the pair of quaternions that corresponds to this rotation is

$$\pm\frac{1}{2}\left(\tau + i + \frac{j}{\tau}\right).$$

4. Throughout this problem we use the notations and the results in Problem 5 of Section 11. In particular, we write the elements of the tetrahedral, octahedral and icosahedral groups as linear fractional transformations. We choose the regular tetrahedron such that its vertices are alternate vertices of the cube, and the cube is inscribed in S^2 such that its faces are orthogonal to the coordinate axes. We also

agree that the positive octant contains a vertex of the tetrahedron. This vertex must be

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

The other 3 vertices are

$$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

The 3 half-turns around the coordinate axes plus the identity give the dihedral group D_2, a subgroup of the tetrahedral group:

$$z \mapsto \pm z, \ \pm\frac{1}{z}.$$

Working out the 8 linear fractional transformations corresponding to the order 3 rotations, we obtain

$$z \mapsto \pm i\frac{z+1}{z-1}, \pm i\frac{z-1}{z+1}, \pm\frac{z+i}{z-i}, \pm\frac{z-i}{z+i}.$$

Putting these together, we arrive at the 12 elements of the tetrahedral group:

$$z \mapsto \pm z, \pm\frac{1}{z}, \pm i\frac{z+1}{z-1}, \pm i\frac{z-1}{z+1}, \pm\frac{z+i}{z-i}, \pm\frac{z-i}{z+i}.$$

We choose the octahedron such that its vertices are the six intersections of the coordinate axes with S^2. This octahedron is homothetic to the intersection of the tetrahedron above and its reciprocal. The octahedral group is generated by the tetrahedral group and a symmetry of the octahedron that interchanges the tetrahedron with its reciprocal. An example to the latter is the qurter-turn around the first axis. This quarter-turn corresponds to the linear fractional transformation $z \mapsto iz$. Thus the 24 elements of the octahedral group are as follows:

$$\begin{aligned} z \mapsto\ & i^k z, \frac{i^k}{z}, i^k\frac{z+1}{z-1}, i^k\frac{z-1}{z+1}, i^k\frac{z+i}{z-i}, i^k\frac{z-i}{z+i}, \\ & k = 0, 1, 2, 3. \end{aligned}$$

Finally, we work out the icosahedral group. We inscribe the icosahedron in S^2 such that the North and South Poles become vertices. The icosahedron can now be considered as being made up of a northern and southern pentagonal pyramid separated by a pentagonal antiprism. The order 5 rotations with axis through the poles are symmetries of the icosahedron and they correspond to the linear fractional transformations

$$z \mapsto \omega^j z, \quad j = 0, \ldots, 4, \tag{1}$$

where $\omega = e^{i\frac{2\pi}{5}}$ is the fifth root of unity. We still have the freedom to rotate the icosahedron around this vertical axis for a convenient position. We fine tune the position of the icosahedron by agreeing that the second coordinate axis must go through the midpoint of one of the cross edges of the pentagonal antiprism. The half-turn U around this axis thus becomes a symmetry of the icosahedron and it corresponds to the linear fractional transformation

$$U: \quad z \mapsto -\frac{1}{z}.$$

The rotations listed above do not generate the entire symmetry group of the icosahedron because they both leave the equator invariant. We choose for another generator the half-turn V whose axis is orthogonal to the axis of U and goes through the midpoint of an edge in the base of the upper pentagonal pyramid. Since U and V are (commuting) half-turns with orthogonal rotation axes, their composition $W = UV$ is also a half-turn whose rotation axis is orthogonal to those of U and V. With the identity, U, V, and W form a dihedral subgroup D_2 of the icosahedral group. Using the explicit form of the vertices of the icosahedron projected to $\hat{\mathbf{C}}$ in p. 208, the axis of the rotation V goes through the point

$$\left(\frac{\tau}{\sqrt{\tau^2+1}}, 0, \frac{1}{\sqrt{\tau^2+1}}.\right)$$

With this, the linear fractional transformation corresponding to the half-turn V finally writes as

$$V: \quad z \mapsto \frac{(\omega^2 - \omega^3)z + (\omega - \omega^4)}{(\omega - \omega^4)z - (\omega^2 - \omega^3)}.$$

Composing this with U, we obtain the linear fractional transformation corresponding to $W = UV$:

$$W: \quad z \mapsto \frac{-(\omega - \omega^4)z + (\omega^2 - \omega^3)}{(\omega^2 - \omega^3)z + (\omega - \omega^4)}.$$

Making all possible combinations of these linear fractional transformations, we finally arrive at the 60 elements of the icosahedral group:

$$\begin{aligned} z &\mapsto \omega^j z,\ -\frac{1}{\omega^j z}, \\ & \omega^j \frac{-(\omega-\omega^4)\omega^k z + (\omega^2-\omega^3)}{(\omega^2-\omega^3)\omega^k z + (\omega-\omega^4)}, \\ & \omega^j \frac{(\omega^2-\omega^3)\omega^k z + (\omega-\omega^4)}{(\omega-\omega^4)\omega^k z - (\omega^2-\omega^3)}, \\ & \quad j,k = 0,\ldots,4. \end{aligned}$$

Section 23.

1. A vertex figure of the 4-dimensional cube is a regular 3-dimensional polyhedron with four vertices since exactly four edges meet at a vertex. It is thus a regular tetrahedron. The 3-dimensional hyperplane through the origin with normal vector $(1,1,1,1)$ gives an octahedral slice of the cube; the four 3-dimensional cubic faces that meet at $(1,1,1,1)$ intersect this hyperplane in a regular tetrahedron while the other four cubic faces meeting at $(-1,-1,-1,-1)$ further truncate this tetrahedron to obtain the octahedron. For an analogy, consider Problem 1 in Section 9.

2.(a) Take a good look at Figure 23.2. The Schläfli symbol $\{4,3,3\}$ means that the 3-dimensional faces of the 4-dimensional cube are ordinary 3-dimensional cubes with Schläfli symbol $\{4,3\}$, and each edge is surrounded with 3 of these. (b) Let V_n, E_n, and F_n denote the number of vertices, edges, and faces of the n-dimensional cube. We have $V_n = 2^n$, $E_{n+1} = 2E_n + V_n$, $E_1 = 1$, and $F_{n+1} = 2F_n + E_n$, $F_1 = 0$. These recurrences can be solved easily and they give $E_n = n2^{n-1}$ and $F_n = \frac{n(n-1)}{2}2^{n-2}$.

3.(a) A 2-dimensional projection of the 4-dimensional tetrahedron is the pentagram star (top of Figure 19.4). (b) Each edge is surrounded by 3 ordinary tetrahedral faces; thus the Schläfli symbol is $\{3,3,3\}$