Special Topic I: Quadratic, Cubic, and Quartic Equations

†[1] ♣ Beyond the pure existence of complex roots of polynomials of degree n, guaranteed by the FTA, the next question to ask concerns their constructibility in terms of radicals such as the quadratic formula for $n = 2$. Although it was known around 300 B.C. by the Greeks[2], it was not until François Viète (1540-1603) that the quadratic formula was cast in its present form due to its creator's insistence of systematic use of letters to represent constants and variables. For later purposes, it is instructive to derive the quadratic formula in a somewhat nontraditional way. In what follows it will be convenient to normalize our polynomials by dividing them with the leading coefficient. We will call a polynomial with leading coefficient one *monic.* To emphasize that from now on we are dealing with complex variables, we replace the real variable x by the complex variable z. In particular, we take our quadratic polynomial in the form

$$P(z) = z^2 + pz + q = (z - z_1)(z - z_2).$$

Here we already factored P into root factors. The complex roots z_1, z_2 are to be determined. We will allow the coefficients p and q to be complex numbers. Multiplying out, we obtain

$$z_1 + z_2 = -p \quad \text{and} \quad z_1 z_2 = q.$$

These formulas are symmetric with respect to the interchange $z_1 \leftrightarrow z_2$. Since the *discriminant*

$$\delta = (z_1 - z_2)^2$$

is also symmetric, it is resonable to expect that we can express δ in terms of the coefficients p and q. This is indeed the case since

$$\delta = (z_1 + z_2)^2 - 4z_1 z_2 = p^2 - 4q.$$

[1]This symbol indicates that the material in this section is more technical than the average text.

[2]The fact that extraction of square roots can be used to solve quadratic equations had already been recognized by the Sumerians. Furthermore, an ancient Babylonian tablet (c. 1600 B.C.) states problems that reduce to the solution of quadratic equations.

The two values $\pm\sqrt{\delta} = \pm(z_1 - z_2)$ can be combined with the expression for the sum of the roots above. We obtain

$$2z_1, 2z_2 = -p \pm \sqrt{\delta} = -p \pm \sqrt{p^2 - 4q}.$$

Substituting $p = b/a$ and $q = c/a$, this is the quadratic formula for the polynomial $az^2 + bz + c$.

To find the roots of a cubic polynomial we will employ a method due to Lagrange. This is somewhat tedious but less ad hoc than the more traditional approach. The further advantage of presenting the Lagrange method now is that it will reappear in a more subtle setting for the solution of quintics. The first step is the same for all methods. We reduce the general monic cubic equation

$$z^3 + az^2 + bz + c = 0, \;\; a, b, c \in \mathbf{C},$$

to the special cubic equation

$$P(z) = z^3 + pz + q = 0$$

by means of the substitution $z \mapsto z - a/3$. (Once again, a generalization of this seemingly innocent reduction (termed Tschirnhaus transformation) will gain primary importance in solving quintic equations. We could also have performed this reduction for quadratic equations, but that would have been the same as completing the square, a standard way to derive the quadratic formula.)

Assume now that z_1, z_2, z_3 are the roots of P:

$$z^3 + pz + q = (z - z_1)(z - z_2)(z - z_3).$$

Multiplying out, we have

$$z_1 + z_2 + z_3 = 0, \;\; z_1z_2 + z_2z_3 + z_3z_1 = p, \;\; z_1z_2z_3 = -q.$$

Remark.

As an application, we show that z_1, z_2, z_3 are the vertices of an equilateral triangle iff

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

First, we notice that this equation remains unchanged when z_1, z_2, z_3 are simultaneously subjected to the substitution $z \mapsto z - d$, where d is any complex number. (Geometrically, this corresponds to translation of the triangle with translation vector

$-d$.) Choosing $d = (z_1 + z_2 + z_3)/3$ (the centroid of the triangle), and adjusting the notations, we may thus assume that $z_1 + z_2 + z_3 = 0$ holds. Since

$$(z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1) = 0$$

the stated criterion splits into two equations

$$z_1 + z_2 + z_3 = 0 \quad \text{and} \quad z_1 z_2 + z_2 z_3 + z_3 z_1 = 0.$$

Now consider the monic cubic polynomial with roots z_1, z_2, z_3. Expanding, we have

$$(z - z_1)(z - z_2)(z - z_3) = z^3 + az^2 + bz + c,$$

Our conditions translate into $a = b = 0$. Equivalently, z_1, z_2, z_3 are the solutions of the equation $z^3 + c = 0$. Since these are the three cubic roots of $-c$, they are equally spaced on the circle $|z| = \sqrt[3]{|c|}$. The claim follows. (For another solution to this problem, we could have used the factorization

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = (z_1 + z_2\omega + z_3\omega^2)(z_1 + z_2\omega^2 + z_3\omega),$$

where $\omega = z(2\pi/3)$ is the primitive third root of unity, but we preferred a less ad hoc approach.)

Returning to the main line, once again we consider the discriminant

$$\delta = \prod_{1 \le j < l \le 3} (z_j - z_l)^2 = (z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2.$$

A mildly unpleasant calculation gives

$$\begin{aligned} \delta &= -4(z_1 z_2 + z_2 z_3 + z_3 z_1)^3 - 27(z_1 z_2 z_3)^2 \\ &= -4p^3 - 27q^2 = -108 \left(\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2 \right). \end{aligned}$$

Let $\omega = z(2\pi/3) = (-1 + \sqrt{3})/2$ be the primitive third root of unity, and consider the so-called *Lagrange substitutions*

$$\xi_l = \frac{1}{3} \sum_{j=1}^{3} \omega^{(j-1)l} z_j, \quad l = 0, 1, 2.$$

Expanding the sums and using $\omega^3 = 1$, we obtain

$$\begin{aligned} 3\xi_0 &= z_1 + z_2 + z_3, \\ 3\xi_1 &= z_1 + \omega z_2 + \omega^2 z_3, \\ 3\xi_2 &= z_1 + \omega^2 z_2 + \omega z_3. \end{aligned}$$

Since our cubic is reduced, ξ_0 vanishes. Moreover, ξ_1^3 and ξ_2^3 remain unchanged when the roots z_1, z_2, z_3 are subjected to even (in our case cyclic) permutations. Thus, it is reasonable to expect that ξ_1^3 and ξ_2^3 can be expressed as polynomials in p, q and $\sqrt{\delta}$ since the latter three seem to be more 'elementary' and possess the same symmetries. Once again a somewhat tedious calculation gives

$$\begin{aligned}
\xi_1^3 &= -\frac{q}{2} + \frac{\sqrt{3}}{18} i\sqrt{\delta} = -\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}, \\
\xi_2^3 &= -\frac{q}{2} - \frac{\sqrt{3}}{18} i\sqrt{\delta} = -\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}.
\end{aligned}$$

The second formula follows from the first because

$$\begin{aligned}
9\xi_1\xi_2 &= z_1^2 + z_2^2 + z_3^2 + (\omega + \omega^2)(z_1 z_2 + z_2 z_3 + z_3 z_1) \\
&= (z_1 + z_2 + z_3)^2 - 3(z_1 z_2 + z_2 z_3 + z_3 z_1) = -3p.
\end{aligned}$$

Here we used Proposition 3 to the effect that $1 + \omega + \omega^2 = 0$. Summarizing, we find

$$\begin{aligned}
\xi_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}}, \\
\xi_2 &= \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}},
\end{aligned}$$

where the cubic roots are chosen such that $\xi_1\xi_2 = -p/3$ is satisfied. Finally, the Lagrange substitutions can be solved uniquely for z_1, z_2, z_3 in terms of ξ_1 and ξ_2 since the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3(\omega^2 - \omega) = -3\sqrt{3}i \neq 0.$$

The solution is

$$z_j = \sum_{l=1}^{2} \omega^{-(j-1)l} \xi_l, \quad j = 1, 2, 3.$$

Indeed, substituting we have

$$\begin{aligned}
\sum_{j=1}^{3} \omega^{(j-1)l} z_j &= \sum_{j=1}^{3} \left(\sum_{k=1}^{2} \omega^{(j-1)l} \omega^{-(j-1)k} \right) \xi_k \\
&= \sum_{k=1}^{2} \left(\sum_{j=1}^{3} \omega^{(j-1)(l-k)} \right) \xi_k = 3\xi_l
\end{aligned}$$

since the sum in the last brackets is 3 for $k = l$ and zero for $k \neq l$. Putting everything together, we finally obtain the roots of the reduced cubic:

$$\begin{aligned} z_1 &= \xi_1 + \xi_2, \\ z_2 &= \omega^2 \xi_1 + \omega \xi_2, \\ z_3 &= \omega \xi_1 + \omega^2 \xi_2, \end{aligned}$$

where ξ_1 and ξ_2 are given in terms of p and q above.

Remark.

Recall that in the formulas for ξ_1 and ξ_2 we have to choose the values of the cubic roots such that the constraint $\xi_1 \xi_2 = -p/3$ holds. With a fixed choice of cubic roots, z_1, z_2, z_3 give the solutions to our cubic equation. Notice however that ξ_l, $\omega \xi_l$, and $\omega^2 \xi_l$, are the 3 cubic roots of ξ_l^3, $l = 1, 2$, so that our three solutions can (and usually are) written more concisely as

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}}$$

with the understanding that we choose the cubic roots appropriately.

As in the cubic case, the solution of the general quartic equation

$$z^4 + az^3 + bz^2 + cz + d = 0, \quad a, b, c, d \in \mathbf{C},$$

can be reduced to the special case

$$P(z) = z^4 + pz^2 + qz + r = 0,$$

by means of the substitution $z \mapsto z - a/4$. Assuming that z_1, z_2, z_3, z_4 are the roots of this reduced quartic, expanding the root factors, we obtain the following equations

$$\begin{gathered} z_1 + z_2 + z_3 + z_4 = 0, \quad z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 = p \\ z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = -q, \quad z_1 z_2 z_3 z_4 = r. \end{gathered}$$

To follow our earlier path we would need to work out the discriminant

$$\delta = \prod_{1 \leq j < k \leq 4} (z_j - z_k)^2$$

in terms of p, q, r, but this seems a rather unpleasant task. Instead, based on the analogy with $\sqrt{\delta}$, we look for polynomials in z_1, z_2, z_3, z_4 that possess only partial symmetries when these variables are permuted. We set

$$\begin{aligned} z_1^* &= (z_1+z_2)(z_3+z_4), \\ z_2^* &= (z_1+z_3)(z_2+z_4), \\ z_3^* &= (z_1+z_4)(z_2+z_3). \end{aligned}$$

Each of these new variables is symmetric only with respect to specific permutations of z_1, z_2, z_3, z_4. The elementary symmetric polynomials $z_1^* + z_2^* + z_3^*$, $z_1^* z_2^* + z_2^* z_3^* + z_3^* z_1^*$, and $z_1^* z_2^* z_3^*$ produced from z_1^*, z_2^*, z_3^*, however, are symmetric with respect to all permutations, and thereby they should be expressible in terms of the coefficients p, q, r! For example, expanding, we obtain

$$z_1^* + z_2^* + z_3^* = 2(z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4) = 2p.$$

Similarly, we have

$$z_1^*z_2^* + z_2^*z_3^* + z_3^*z_1^* = p^2 - 4r, \text{ and } z_1^*z_2^*z_3^* = -q^2.$$

We conclude that z_1^*, z_2^*, z_3^* are the three roots of the cubic

$$P^*(z) = z^3 - 2pz^2 + (p^2 - 4r)z + q^2.$$

But to find the roots of a cubic is exactly what we just accomplished! Thus, using our earlier reduction and formulas, z_1^*, z_2^*, z_3^* can be explicitly expressed in terms of p, q, r. Finally, to pass from the roots of P^* to those of our original quartic P is now easy. Since P is reduced, we have $z_1 + z_2 + z_3 + z_4 = 0$ and the formulas defining z_1^*, z_2^*, z_3^* can be resolved yielding

$$\begin{aligned} z_1 + z_2 &= \sqrt{-z_1^*}, & z_3 + z_4 &= -\sqrt{-z_1^*} \\ z_1 + z_3 &= \sqrt{-z_2^*}, & z_2 + z_4 &= -\sqrt{-z_2^*} \\ z_1 + z_4 &= \sqrt{-z_3^*}, & z_2 + z_3 &= -\sqrt{-z_3^*}. \end{aligned}$$

Once again, we have to fix the ambiguity inherent in the choice of the square roots. As computations confirm, we need

$$\sqrt{-z_1^*}\sqrt{-z_2^*}\sqrt{-z_3^*} = -q,$$

to be satisfied. Finally, solving the system above for z_1, z_2, z_3, z_4, we obtain

$$\begin{aligned} 2z_1 &= \sqrt{-z_1^*} + \sqrt{-z_2^*} + \sqrt{-z_3^*} \\ 2z_2 &= \sqrt{-z_1^*} - \sqrt{-z_2^*} - \sqrt{-z_3^*} \\ 2z_3 &= -\sqrt{-z_1^*} + \sqrt{-z_2^*} - \sqrt{-z_3^*} \\ 2z_4 &= -\sqrt{-z_1^*} - \sqrt{-z_2^*} + \sqrt{-z_3^*}. \end{aligned}$$

One final remark. Since we have

$$\begin{aligned} z_1^* - z_2^* &= -(z_1 - z_4)(z_2 - z_3), \\ z_1^* - z_3^* &= -(z_1 - z_3)(z_2 - z_4), \\ z_2^* - z_3^* &= -(z_1 - z_2)(z_3 - z_4), \end{aligned}$$

the discriminant of P^* is the same as the discriminant of P! From the explicit formula of the discriminant for cubics, we obtain that the discriminant of P is

$$\delta = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$

The cubic P^* that helped to solve the quartic equation is called the *resolvent cubic.*