Dual Mean Minkowski Measures and the Grünbaum Conjecture for Affine Diameters

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Abstract

For a convex body $\mathcal{C}$ in a Euclidean vector space $\mathcal{X}$ of dimension $n\,(\geq 2)$, we define two sub-arithmetic monotonic sequences $\{\sigma_{\mathcal{C},k}\}_{k\geq 1}$ and $\{\sigma^o_{\mathcal{C},k}\}_{k\geq 1}$ of functions on the interior of $\mathcal{C}$. The k-th members are "mean Minkowski measures in dimension k" which are pointwise dual: $\sigma^o_{\mathcal{C},k}(O) = \sigma_{\mathcal{C}^O,k}(O)$, where $O \in \operatorname{int}\mathcal{C}$, and $\mathcal{C}^O$ is the dual (polar) of $\mathcal{C}$ with respect to O. They are measures of (anti-)symmetry of $\mathcal{C}$ in the following sense:

$$1 \leq \sigma_{\mathcal{C},k}(O), \sigma^o_{\mathcal{C},k}(O) \leq \frac{k+1}{2}.$$

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The lower bound is attained iff $\mathcal{C}$ has a k-dimensional simplicial slice or simplicial projection. The upper bound is attained iff $\mathcal{C}$ is symmetric with respect to O. In 1953 Klee showed that the lower bound $\mathfrak{m}^*_{\mathcal{C}} > n-1$ on the Minkowski measure of $\mathcal{C}$ implies that there are $n+1$ affine diagonals meeting at a critical point $O^* \in \mathcal{C}$. In 1963 Grünbaum conjectured the existence of such a point in the interior of any convex body (without any conditions). While this conjecture remains open (and difficult), as a byproduct of our study of the dual mean Minkowski measures, we show that

$$\frac{n}{\mathfrak{m}^*_{\mathcal{C}}+1} \leq \sigma^o_{\mathcal{C},n-1}(O^*)$$

always holds, and for sharp inequality Grünbaum's conjecture is valid.

1 Preliminaries and Statment of Results

Let $\mathcal{X}$ be an n-dimensional Euclidean vector space ($n \geq 2$) with scalar product $\langle \cdot, \cdot \rangle$ and distance function d. We consider a *convex body* $\mathcal{C} \subset \mathcal{X}$, a *compact* convex set in $\mathcal{X}$ with *non-empty interior*. Let $\partial\mathcal{C}$ denote the boundary of $\mathcal{C}$. Given an interior point $O \in \operatorname{int}\mathcal{C}$ we consider all the *chords* of $\mathcal{C}$ passing through O. For $C \in \partial\mathcal{C}$, let $\lambda_{\mathcal{C}}(C, O)$ denote the ratio into which O divides the chord of $\mathcal{C}$ starting at C, passing through O, and ending up at the *opposite* $C^o \in \partial\mathcal{C}$ of C (with respect to O). This defines the *distortion function* $\lambda_{\mathcal{C}} : \partial\mathcal{C} \times \operatorname{int}\mathcal{C} \to \mathbb{R}$:

$$\lambda_{\mathcal{C}}(C, O) = \frac{d(C, O)}{d(C^o, O)}, \quad C \in \partial\mathcal{C},\ O \in \operatorname{int}\mathcal{C}.$$

For the involution of $\partial\mathcal{C}$ given by $C \mapsto C^o$ (with $(C^o)^o = C$), we have $\lambda_{\mathcal{C}}(C^o, O) = 1/\lambda_{\mathcal{C}}(C, O)$, $C \in \partial\mathcal{C}$.
The (maximum) *Minkowski ratio* of $\mathcal{C}$ at O is defined as

$$\mathfrak{m}_{\mathcal{C}}(O) = \sup_{C \in \partial\mathcal{C}} \lambda_{\mathcal{C}}(C, O) \geq 1.$$

(Due to compactness of $\mathcal{C}$ and continuity of the distortion function $\lambda_{\mathcal{C}}$ [17, Lemma 1], the supremum is attained. This is also the case for all infima and suprema that we encounter in this paper.)

Let $\delta\mathcal{C}$ denote the (compact) space of all hyperplanes supporting $\mathcal{C}$. (Associating to each $\mathcal{H} \in \delta\mathcal{C}$ the unit normal that points inward $\mathcal{C}$, say, gives rise to a topological equivalence of $\delta\mathcal{C}$ and the unit sphere $\mathcal{S} \subset \mathcal{X}$.) For $\mathcal{H} \in \delta\mathcal{C}$, we define the ratio

$\rho_{\mathcal{C}}(\mathcal{H}, O) = d(\mathcal{H}, O)/d(\mathcal{H}^o, O)$, where $\mathcal{H}^o \in \delta\mathcal{C}$ is the (unique) parallel opposite of $\mathcal{H}$ such that $\mathcal{C}$ is between $\mathcal{H}$ and $\mathcal{H}^o$. This gives rise to the function $\rho_{\mathcal{C}} : \delta\mathcal{C} \times \operatorname{int}\mathcal{C} \to \mathbb{R}$. For the involution of $\delta\mathcal{C}$ given by $\mathcal{H} \mapsto \mathcal{H}^o$, $\mathcal{H} \in \delta\mathcal{C}$, we have $\rho_{\mathcal{C}}(\mathcal{H}^o, O) = 1/\rho_{\mathcal{C}}(\mathcal{H}, O)$, $\mathcal{H} \in \delta\mathcal{C}$.
It is well-known that

$$\mathfrak{m}_{\mathcal{C}}(O) = \sup_{C \in \partial\mathcal{C}} \lambda_{\mathcal{C}}(C, O) = \sup_{\mathcal{H} \in \delta\mathcal{C}} \rho_{\mathcal{C}}(\mathcal{H}, O), \quad O \in \operatorname{int}\mathcal{C}. \tag{1}$$

(See [5]. It is customary to define $\rho_{\mathcal{C}}(\mathcal{H}, O)$ for a hyperplane $\mathcal{H}$ *containing* O as the ratio ≥ 1 that $\mathcal{H}$ divides the distance between the two supporting hyperplanes $\mathcal{H}', \mathcal{H}'' \in \delta\mathcal{C}$ that are parallel to $\mathcal{H}$. In our study we need more control of the choice of the supporting hyperplane, henceforth we altered this definition accordingly. Since we are taking suprema these two definitions are equivalent.)

A technically more convenient reformulation of this second concept is as follows. Let $\mathrm{aff} = \mathrm{aff}(\mathcal{X})$ denote the $(n+1)$-dimensional vector space of affine functionals $f : \mathcal{X} \to \mathbb{R}$. We call $f \in \mathrm{aff}$ *normalized* for $\mathcal{C}$ if $f(\mathcal{C}) = [0, 1]$, that is, the zero-sets $\mathcal{H} = \{X \,|\, f(X) = 0\}$ and $\mathcal{H}^o = \{X \,|\, 1 - f(X) = 0\}$ are two parallel hyperplanes supporting and enclosing $\mathcal{C}$. We let $\mathrm{aff}_{\mathcal{C}} \subset \mathrm{aff}$ denote the (compact) subspace of affine functionals normalized for $\mathcal{C}$. (Associating to each $f \in \mathrm{aff}_{\mathcal{C}}$ the single zero-set $\mathcal{H}$ as above gives rise to a topological equivalence of $\mathrm{aff}_{\mathcal{C}}$ and $\delta\mathcal{C}$. Indeed, any $\mathcal{H} \in \delta\mathcal{C}$ and its opposite $\mathcal{H}^o$ uniquely define a normalized affine functional with the respective zero-sets as above.) Note that $\mathrm{aff}_{\mathcal{C}}$ has the obvious involution given by $f \mapsto 1 - f$, $f \in \mathrm{aff}_{\mathcal{C}}$.
Using the notations above, (1) gives

$$\inf_{f \in \mathrm{aff}_{\mathcal{C}}} f(O) = \inf_{f \in \mathrm{aff}_{\mathcal{C}}} (1 - f(O)) = \frac{1}{\sup_{\mathcal{H} \in \delta\mathcal{C}} \rho_{\mathcal{C}}(\mathcal{H}, O) + 1} = \frac{1}{\mathfrak{m}_{\mathcal{C}}(O) + 1}, \; O \in \operatorname{int}\mathcal{C}. \tag{2}$$

The two aspects of the Minkowski ratio above can be interpreted in terms of duality between the convex body $\mathcal{C}$ and its *dual* (also called *polar*) $\mathcal{C}^O$ with respect to the given interior point $O \in \operatorname{int}\mathcal{C}$. (For the definition of the dual and its properties, see the next section. Note that when dealing with duality we will frequently use the *bipolar theorem* $(\mathcal{C}^O)^O = \mathcal{C}$ without explicit mention; [4, Chapter 1.9] or [13, Theorem 1.6.1].)
First, as a technical tool, we will introduce and study the "musical equivalencies"

$$\flat = \flat_{\mathcal{C},O} : \partial\mathcal{C} \to \mathrm{aff}_{\mathcal{C}^O} \quad \text{and} \quad \sharp = \sharp_{\mathcal{C},O} : \mathrm{aff}_{\mathcal{C}} \to \partial\mathcal{C}^O.$$

(For simplicity, we will suppress the subscripts whenever no confusion arises. In Riemannian geometry the introduction of a Riemannian metric on a manifold gives rise to "musical isomorphisms" between the tangent bundle and its dual. Due to the descriptive nature of this concept and analogy we took the liberty to borrow this term for our setting.) The musical equivalencies satisfy

$$(C^o)^\flat = 1 - C^\flat \quad \text{and} \quad (f^\sharp)^o = (1-f)^\sharp, \quad C \in \partial\mathcal{C}, \ f \in \operatorname{aff}\mathcal{C}. \tag{3}$$

In addition, as the name suggests, they are inverses of each other:

$$\sharp_{\mathcal{C}^O,O} \circ \flat_{\mathcal{C},O} = \mathrm{id}_{\partial\mathcal{C}} \quad \text{and} \quad \flat_{\mathcal{C}^O,O} \circ \sharp_{\mathcal{C},O} = \mathrm{id}_{\operatorname{aff}_\mathcal{C}}. \tag{4}$$

These formulas (applied to the dual pair $\mathcal{C}$ and $\mathcal{C}^O$) imply that the musical equivalencies are actually homeomorphisms of the respective spaces.
The following formulas show that the two aspects of Minkowski ratios are dual constructions applied to $\mathcal{C}$ and its dual $\mathcal{C}^O$:

$$C^\flat(O) = \frac{1}{\lambda_\mathcal{C}(C,O)+1}, \quad C \in \partial\mathcal{C}, \ O \in \operatorname{int}\mathcal{C}, \tag{5}$$

and

$$f(O) = \frac{1}{\lambda_{\mathcal{C}^O}(f^\sharp,O)+1}, \quad f \in \operatorname{aff}_\mathcal{C}, \ O \in \operatorname{int}\mathcal{C}. \tag{6}$$

Taking the infima on the respective sets in (5)-(6) and using (2), we obtain

$$\inf_{C\in\partial\mathcal{C}} C^\flat(O) = \frac{1}{\mathfrak{m}_\mathcal{C}(O)+1} = \inf_{f\in\operatorname{aff}_\mathcal{C}} f(O) = \frac{1}{\mathfrak{m}_{\mathcal{C}^O}(O)+1}, \quad O \in \operatorname{int}\mathcal{C}.$$

This gives

$$\mathfrak{m}_\mathcal{C}(O) = \mathfrak{m}_{\mathcal{C}^O}(O), \quad O \in \operatorname{int}\mathcal{C}. \tag{7}$$

The *Minkowski measure* of $\mathcal{C}$ is defined as

$$\mathfrak{m}^*_\mathcal{C} = \inf_{O\in\operatorname{int}\mathcal{C}} \mathfrak{m}(O).$$

The set of interior points where this infimum is attained is called the *critical set*

$$\mathcal{C}^* = \{O^* \in \operatorname{int}\mathcal{C} \,|\, \mathfrak{m}_\mathcal{C}(O^*) = \mathfrak{m}^*_\mathcal{C}\}. \tag{8}$$

The critical set $\mathcal{C}^* \subset \mathcal{C}$ is compact and convex, and we have Klee's inequality

$$(1 \leq)\ \mathfrak{m}^*_\mathcal{C} + \dim\mathcal{C}^* \leq n$$

improving the classical Minkowski-Radon inequality (in which the dimension of the critical set is absent). (See [9].) Clearly, $\mathfrak{m}^*_\mathcal{C} = 1$ iff $\mathcal{C}$ is symmetric with respect to

then unique regular point. It is also straightforward to show that the upper bound is attained for simplices. Conversely, Minkowski and Radon also proved that $\mathfrak{m}^*_{\mathcal{C}} = n$ implies that $\mathcal{C}$ is a simplex.
For $O^* \in \mathcal{C}^*$ critical, by (7), we have

$$\mathfrak{m}^*_{\mathcal{C}} = \mathfrak{m}_{\mathcal{C}}(O^*) = \mathfrak{m}_{\mathcal{C}^{O^*}}(O^*) \geq \mathfrak{m}^*_{\mathcal{C}^{O^*}}.$$

Whether equality holds, that is $O^* \in \mathcal{C}^*$ is also a critical point of the dual $\mathcal{C}^{O^*}$, seems to be a difficult problem in general.

Recall that a chord $[C, C^o]$ of $\mathcal{C}$ is an *affine diameter* if there are parallel supporting hyperplanes $\mathcal{H}$ and $\mathcal{H}^o$ of $\mathcal{C}$ at the endpoints of the chord, that is $C \in \mathcal{H}$ and $C^o \in \mathcal{H}^o$. (For a general survey on affine diameters and related problems, see [14, 15].) As discussed above, we describe these hyperplanes as the zero-sets of a normalized affine functional $f \in \mathrm{aff}_{\mathcal{C}}$, that is we have $\mathcal{H} = \{X \in \mathcal{X} \,|\, f(X) = 0\}$ and $\mathcal{H}^o = \{X \in \mathcal{X} \,|\, 1 - f(X) = 0\}$. Under the musical equivalencies, affine diameters of $\mathcal{C}$ correspond to affine diameters of $\mathcal{C}^O$ in the sense that if $[C, C^o]$ is an affine diameter of $\mathcal{C}$ with parallel supporting hyperplanes given by $f \in \mathrm{aff}_{\mathcal{C}}$ then $[f^\sharp, (f^\sharp)^o] = [f^\sharp, (1-f)^\sharp]$ is an affine diameter of $\mathcal{C}^O$ with parallel supporting hyperplanes given by $C^\flat \in \mathrm{aff}_{\mathcal{C}^O}$. (For the proof, see Section 2.)

We now introduce the sequence $\{\sigma_{\mathcal{C},k}\}_{k\geq 1}$ of *mean Minkowski measures* of $\mathcal{C}$. (We give here a concise summary; for details, see [16, 17].) The *k-th measure* $\sigma_{\mathcal{C},k} : \mathcal{C} \to \mathbb{R}$, $k \geq 1$, is a function on the interior of $\mathcal{C}$ defined as follows. First, a *(point) k-configuration of* $\mathcal{C}$ with respect to O is a *multi-set* $\{C_0, \dots, C_k\} \subset \partial\mathcal{C}$ (with repetition allowed) such that the convex hull $[C_0, \dots, C_k]$ contains O. (We use square brackets to indicate convex hull rather than "conv.") With this we define

$$\sigma_{\mathcal{C},k}(O) = \inf_{\{C_0,\dots,C_k\} \in \mathfrak{C}_{\mathcal{C},k}(O)} \sum_{i=0}^{k} \frac{1}{\lambda_{\mathcal{C}}(C_i, O) + 1}, \quad O \in \mathrm{int}\, \mathcal{C}, \tag{9}$$

where $\mathfrak{C}_{\mathcal{C},k}(O)$ denotes the set of all k-configurations of $\mathcal{C}$ (with respect to O).
Algebraically, $\sigma_{\mathcal{C},k}$ is a "k-average" of the rescaled distortion, and, as we will see below, geometrically $\sigma_{\mathcal{C},k}(O)$ measures how far the k-dimensional slices of $\mathcal{C}$ across O are from a k-simplex.
A k-configuration $\{C_0, \dots, C_k\} \in \mathfrak{C}_{\mathcal{C},k}(O)$ at which the infimum in (9) is attained is called *minimizing*, or simply *minimal.* Since $\mathfrak{C}_{\mathcal{C},k}(O)$ inherits a compact topology from that of $\partial\mathcal{C}$ and the distortion is continuous, minimal configurations always exist. (As examples show, they are by no means unique.)
For $k = 1$, a 1-configuration of O is an opposite pair of points $\{C_0, C_1\} \subset \partial\mathcal{C}$, $C_1 = C_0^o$. Since $\lambda_{\mathcal{C}}(C_0^o, O) = 1/\lambda_{\mathcal{C}}(C_0, O)$, we have $\sigma_{\mathcal{C},1}(O) = 1$, $O \in \mathrm{int}\, \mathcal{C}$.

Since a (minimal) k-configuration can always be extended to a $(k+l)$-configuration by adding l copies of a boundary point at which the distortion $\lambda_{\mathcal{C}}(\cdot, O)$ attains its maximum $\mathfrak{m}_{\mathcal{C}}(O)$, we have *sub-arithmeticity*:

$$\sigma_{\mathcal{C},k+l}(O) \leq \sigma_{\mathcal{C},k}(O) + \frac{l}{\mathfrak{m}_{\mathcal{C}}(O)+1}, \quad O \in \operatorname{int}\mathcal{C},\ k, l \geq 1. \tag{10}$$

By *Carathéodory's theorem*, for $k > n$, a k-configuration always contains an n-configuration. In addition, any subconfiguration of a minimal configuration is minimal, and, at the complementary configuration points, the distortion $\lambda_{\mathcal{C}}(\cdot, O)$ attains its maximum $\mathfrak{m}_{\mathcal{C}}(O)$. We see that the sequence $\{\sigma_{\mathcal{C},k}(O)\}_{k\geq 1}$ is *arithmetic with difference* $1/(\mathfrak{m}_{\mathcal{C}}(O)+1)$ *from the n-th term onwards.*

For $1 \leq k \leq n$, we have

$$\sigma_{\mathcal{C},k}(O) = \inf_{O\in\mathcal{E}\subset\mathcal{X},\, \dim\mathcal{E}=k} \sigma_{\mathcal{C}\cap\mathcal{E},k}(O), \quad O \in \operatorname{int}\mathcal{C}, \tag{11}$$

where the infimum is over affine subspaces $\mathcal{E} \subset \mathcal{X}$ of dimension k which contain O. This holds because the affine span of any k-configuration $\{C_0, \ldots, C_k\} \in \mathfrak{C}_{\mathcal{C},k}(O)$ is contained in an affine subspace $\mathcal{E}(\ni O)$ of dimension k; therefore the infimum in (9) can first be taken for configurations that are contained in a specific $\mathcal{E}$, yielding $\sigma_{\mathcal{C}\cap\mathcal{E},k}(O)$, and then for all k-dimensional affine subspaces $\mathcal{E}$ (which contain O) as in (11).

The mean Minkowski measures are *measures of of symmetry* (or asymmetry for some authors) in the following sense:

$$1 \leq \sigma_{\mathcal{C},k}(O) \leq \frac{k+1}{2}, \quad O \in \operatorname{int}\mathcal{C}. \tag{12}$$

(For measures of symmetry in general, see the seminal work of Grünbaum [5].) Assuming $k \geq 2$, the upper bound is attained iff $\mathcal{C}$ is symmetric with respect to O. For the lower bound, if, for some $k \geq 1$, $\sigma_{\mathcal{C},k}(O) = 1$ at $O \in \operatorname{int}\mathcal{C}$ then $k \leq n$, and $\mathcal{C}$ has a k-dimensional simplicial intersection across O, that is there exists a k-dimensional affine subspace $\mathcal{E} \subset \mathcal{X}$ such that $\mathcal{C} \cap \mathcal{E}$ is a k -simplex (and consequently $\sigma_{\mathcal{C},k} = 1$ identically on $\mathcal{C} \cap \mathcal{E}$).

The functions $\sigma_{\mathcal{C},k} : \operatorname{int}\mathcal{C} \to \mathbb{R}$, $k \geq 1$, are continuous on $\operatorname{int}\mathcal{C}$ and extend continuously to $\partial\mathcal{C}$ as

$$\lim_{d(O,\partial\mathcal{C})\to 0} \sigma_{\mathcal{C},k}(O) = 1. \tag{13}$$

The limiting behavior in (13) follows from sub-arithmeticity in (10) ($k = 1$ and $l = k-1$ and $\sigma_{\mathcal{C},1}(O) = 1$), and the lower estimate in (12). (For a different proof, see Theorem D/(b) in [16].

The sequence $\{\sigma_{\mathcal{C},k}(O)\}_{k\geq 1}$ is *super-additive*:

$$\sigma_{\mathcal{C},k+l}(O) - \sigma_{\mathcal{C},k}(O) \geq \sigma_{\mathcal{C},l}(O) - \sigma_{\mathcal{C},1}(O), \quad O \in \operatorname{int}\mathcal{C},\ k,l \geq 1. \tag{14}$$

In particular ($l = 1$), the sequence $\{\sigma_{\mathcal{C},k}(O)\}_{k\geq 1}$ is *monotonic*: $\sigma_{\mathcal{C},k}(O) \leq \sigma_{\mathcal{C},k+1}(O)$, $k \geq 1$.

Finally, note the obvious lower bound

$$\frac{k+1}{\mathfrak{m}_{\mathcal{C}}(O)+1} \leq \sigma_{\mathcal{C},k}(O), \quad O \in \operatorname{int}\mathcal{C},\ k \geq 1. \tag{15}$$

The main new technical tool of the present paper is the "dual construction." Let $k \geq 1$. First, a *dual (or supporting) k-configuration of O* is a multi-set $\{f_0, \dots, f_n\} \subset \operatorname{aff}_{\mathcal{C}}$ (repetition allowed) such that the intersection

$$\bigcap_{i=0}^{k} \{X \in \mathcal{X} \mid f_i(X) \leq 0\} = \emptyset. \tag{16}$$

With this, the *k-th dual mean Minkowski measure* $\sigma^o_{\mathcal{C},k} : \operatorname{int}\mathcal{C} \to \mathbb{R}$ is defined as

$$\sigma^o_{\mathcal{C},k}(O) = \inf_{\{f_0,\dots,f_k\}\in\mathfrak{C}^o_{\mathcal{C},k}(O)} \sum_{i=0}^{k} f_i(O), \quad O \in \operatorname{int}\mathcal{C}, \tag{17}$$

where $\mathfrak{C}^o_{\mathcal{C},k}(O)$ denotes the set of all dual k-configurations of O.
A dual k-configuration $\{f_0, \dots, f_k\} \in \mathfrak{C}^o_{\mathcal{C},k}(O)$ at which the infimum in (17) is attained is called *minimizing* or *minimal* for short. Since $\mathfrak{C}^o_{\mathcal{C},k}(O)$ inherits a compact topology from that of $\delta\mathcal{C}$ and the sum in (17) is continuous with respect to $(f_0, \dots, f_k) \in (\operatorname{aff}_{\mathcal{C}})^{k+1}$, minimal configurations always exist.
For $k = 1$, a dual 1-configuration of any $O \in \operatorname{int}\mathcal{C}$ is an opposite pair of affine functionals $\{f_0, f_1\} \subset \operatorname{aff}_{\mathcal{C}}$, $f_1 = 1 - f_0$, and we have $\sigma^o_{\mathcal{C},1} = 1$ identically on $\operatorname{int}\mathcal{C}$.

Note, by (2), the obvious lower bound

$$\frac{k+1}{\mathfrak{m}_{\mathcal{C}}(O)+1} \leq \sigma^o_{\mathcal{C},k}(O), \quad O \in \operatorname{int}\mathcal{C},\ k \geq 1. \tag{18}$$

The first and most obvious property of the dual mean Minkowski measures is that, being infima of affine functions, $\sigma^o_{\mathcal{C},k} : \operatorname{int}\mathcal{C} \to \mathbb{R}$, $k \geq 1$, are automatically *concave functions*. This is in striking contrast with the mean Minkowski measures $\sigma_{\mathcal{C},k} : \operatorname{int}\mathcal{C} \to \mathbb{R}$, $k \geq 1$ which, albeit concave in dimension $n = 2$ (Theorem E in [17]), for $n \geq 3$, they, in general, fail to satisfy any concavity properties. The following example illustrates this point.

Example 1 Let $\mathcal{C}$ be an n-cube, $n \geq 3$. Then the function $\sigma_{\mathcal{C},n-1}$ is identically 1 on the complement of the (open) cross-polytope $\mathcal{C}_0$ inscribed in $\mathcal{C}$ (since the vertex figures provide $n-1$ dimensional simplicial intersections), but in the interior of $\mathcal{C}_0$ we have $\sigma_{\mathcal{C},n-1} > 1$. Thus, $\sigma_{\mathcal{C},n-1}$ is not concave. A somewhat more involved argument shows that $\sigma_{\mathcal{C},n}$ is also non-concave. (For a much more general result, see [18, Theorem D].) As a byproduct, we see that, for the n-cube $\mathcal{C}$, $n \geq 3$, $\sigma_{\mathcal{C},n}$ and $\sigma^o_{\mathcal{C},n}$ are different functions.

The following pointwise duality is the cornerstone of our study:

Theorem 1 *Let $\mathcal{C} \subset \mathcal{X}$ be a convex body, and $O \in \operatorname{int} \mathcal{C}$. For $k \geq 1$, we have*

$$\sigma^o_{\mathcal{C},k}(O) = \sigma_{\mathcal{C}^O,k}(O), \tag{19}$$

where C^O is the dual of $\mathcal{C}$ with respect to O.

Remark It is important to note that on the right-hand side of (19) the mean Minkowski measure has a double dependency on the point O; not only in the argument but also in forming the dual $\mathcal{C}^O$. For this reason duality can only be used pointwise.

The crux of the proof of Theorem 1 (Section 3) is the equivalence

$$\{f_0, \dots, f_k\} \in \mathfrak{C}^o_{\mathcal{C},k}(O) \quad \Leftrightarrow \quad \{f_0^\sharp, \dots, f_k^\sharp\} \in \mathfrak{C}_{\mathcal{C}^O,k}(O). \tag{20}$$

As a byproduct of the proof, it will also follow that, under this equivalence, minimal configurations correspond to each other.

Pointwise duality allows the properties of the mean Minkowski measures to carry over to the dual. Replacing $\mathcal{C}$ with $\mathcal{C}^O$ in (10) and using (7) and (19), we have *sub-arithmeticity*:

$$\sigma^o_{\mathcal{C},k+l}(O) \leq \sigma^o_{\mathcal{C},k}(O) + \frac{l}{\mathfrak{m}_{\mathcal{C}}(O)+1}, \quad O \in \operatorname{int} \mathcal{C},\ k, l \geq 1. \tag{21}$$

In addition, the sequence $\{\sigma^o_{\mathcal{C},k}(O)\}_{k \geq 1}$ is arithmetic with difference $1/(\mathfrak{m}_{\mathcal{C}}(O)+1)$ from the n-th term onwards.

Remark It is worthwile noting that, the direct proof of arithmeticity (without the use of duality) beyond the dimension is an application of (the contrapositive of)

Helly's theorem (instead of Carathéodory's): For $k > n$, any dual k-configuration (characterized by (16)) contains an n-configuration.

To state the dual version of (11), for $1 \leq k \leq n$, we denote by $\mathfrak{P}_k = \mathfrak{P}_{\mathcal{X},k}$ the space of all orthogonal projections $\Pi : \mathcal{X} \to \mathcal{X}$ onto k-dimensional affine subspaces $\Pi(\mathcal{X}) = \mathcal{E} \subset \mathcal{X}$. We then have

$$\sigma^o_{\mathcal{C},k}(O) = \inf_{\Pi \in \mathfrak{P}_k} \sigma^o_{\Pi(\mathcal{C}),k}(\Pi(O)), \quad O \in \operatorname{int} \mathcal{C}. \tag{22}$$

(In the infimum $\Pi(O)$ can be replaced by O if we require $O \in \Pi(\mathcal{X}) = \mathcal{E}$.)

By duality, the bounds in (12) stay the same for the dual mean Minkowski measures. To characterize the convex bodies for which the lower bound is attained is somewhat more complex (to be expounded in Section 3). We summarize these concisely in the following:

Theorem 2 *Let $\mathcal{C} \subset \mathcal{X}$ be a convex body. For $k \geq 1$, we have*

$$1 \leq \sigma^o_{\mathcal{C},k}(O) \leq \frac{k+1}{2}, \quad O \in \operatorname{int} \mathcal{C}. \tag{23}$$

Assuming $k \geq 2$, the upper bound in (23) is attained iff $\mathcal{C}$ is symmetric with respect to O. If, for some $k \geq 1$, $\sigma^o_{\mathcal{C},k}(O) = 1$ at $O \in \operatorname{int} \mathcal{C}$ then $\sigma^o_{\mathcal{C},k} = 1$ identically on $\operatorname{int} \mathcal{C}$; we have $k \leq n$, and $\mathcal{C}$ has an orthogonal projection to a k-simplex.

The functions $\sigma^o_{\mathcal{C},k} : \operatorname{int} \mathcal{C} \to \mathbb{R}$, $k \geq 1$, are continuous on $\operatorname{int} \mathcal{C}$. As in the non-dual case, the lower bound in (23) along with sub-arithmeticity ($k = 1$ and $l = k - 1$ in (21) with $\sigma^o_{\mathcal{C},1} = 1$), we have continuity up to the boundary via

$$\lim_{d(O,\partial\mathcal{C}) \to 0} \sigma^o_{\mathcal{C},k}(O) = 1. \tag{24}$$

Example 2 Let $\mathcal{C}$ be a tetrahedron ($n = 3$) truncated near all four vertices (by vertex figures, say). Then $\sigma_{\mathcal{C},2} = 1$ identically as $\mathcal{C}$ has triangular intersections through any of its interior points. On the other hand, $\sigma^o_{\mathcal{C},2} > 1$ everywhere since $\mathcal{C}$ has no triangular projection. We see once again that, in general, the functions $\sigma_{\mathcal{C},k}$ and its dual $\sigma^o_{\mathcal{C},k}$ are different.

Next, again by duality, we note super-additivity

$$\sigma^o_{\mathcal{C},k+l}(O) - \sigma^o_{\mathcal{C},k}(O) \geq \sigma^o_{\mathcal{C},l}(O) - \sigma^o_{\mathcal{C},1}(O), \quad O \in \operatorname{int} \mathcal{C}, \ k, l \geq 1,$$

and, as a consequence, monotonicity: $\sigma^o_{\mathcal{C},k}(O) \leq \sigma^o_{\mathcal{C},k+1}(O)$, $k \geq 1$.

Most of the properties of the dual mean Minkowski measures discussed above are consequences of the pointwise duality asserted by Theorem 2. They have, however, additional and more refined properties showing that, as measures, they are better adapted convex bodies than their non-dual counterparts. Our next result asserts the striking fact that the n-th dual mean Minkowkski measure can be *explicitly* calculated at the critical points of a convex body.

Theorem 3 *Let $\mathcal{C} \subset \mathcal{X}$ be a convex body and $\mathcal{C}^* \subset \mathcal{C}$ its critical set. For any critical point $O^* \in \mathcal{C}^*$, we have*

$$\sigma^o_{\mathcal{C},n}(O^*) = \frac{n+1}{\mathfrak{m}^*_{\mathcal{C}}+1}. \tag{25}$$

The proof of Theorem 3 (Section 3) relies heavily on Klee's delicate analysis of the critical set and the proof of his improved Minkowski-Radon inequality.

Remark It is natural to ask if (25) holds for the n-th (non-dual) mean Minkowski measure $\sigma_{\mathcal{C},n}$. While this remains unsolved, it seems to depend on whether a critical point $O^* \in \mathcal{C}^*$ is also a critical point for the dual $(\mathcal{C})^{O^*}$ or not. For the class of *convex bodies of constant width* the answer is affirmative as follows. (For a general reference on convex bodies of constant width, see [2].) For a convex body $\mathcal{C}$ of constant width d, the critical set $\mathcal{C}^*$ is a singleton, and the unique critical point O^* is the common center of the circumcircle $\mathcal{S}_{R_{\mathcal{C}}}(O^*)$ and the incircle $\mathcal{S}_{r_{\mathcal{C}}}(O^*)$ with circumradius $R_{\mathcal{C}}$ and inradius $r_{\mathcal{C}}$. The latter can be expressed in terms of the Minkowski measure as

$$R_{\mathcal{C}} = \frac{\mathfrak{m}^*_{\mathcal{C}}}{\mathfrak{m}^*_{\mathcal{C}}+1} d \quad \text{and} \quad r_{\mathcal{C}} = \frac{1}{\mathfrak{m}^*_{\mathcal{C}}+1} d.$$

In particular, we have $R_{\mathcal{C}} + r_{\mathcal{C}} = d$ and

$$\mathfrak{m}^*_{\mathcal{C}} = \frac{R_{\mathcal{C}}}{r_{\mathcal{C}}}.$$

(For these results, see [8], and (for some) also [1, 63] and [4, Theorem 53 and its Corollary, p. 125].) Another classical fact is that $O^* \in [\partial\mathcal{C} \cap \mathcal{S}_{R_{\mathcal{C}}}(O^*)]$, so that, by Carathéodory's theorem, O^* is in the convex hull of at most $n+1$ boundary points of $\mathcal{C}$ on the circumcircle $\mathcal{S}_{R_{\mathcal{C}}}(O^*)$. It follows that the circumcircle contains an n-configuration of O^*. Thus, for a convex body $\mathcal{C}$ of constant width, equality holds in (25) for the (non-dual) mean Minkowski measure $\sigma_{\mathcal{C},n}$.

For $k = n$, an n-configuration $\{C_0, \dots, C_n\} \in \mathfrak{C}_{\mathcal{C},n}(O)$, $O \in \operatorname{int}\mathcal{C}$, is called *simplicial* if $[C_0, \dots, C_n]$ is an n-simplex with O is in its interior. We let $\Delta_{\mathcal{C}}(O) \subset \mathfrak{C}_{\mathcal{C},n}(O)$ denote the (non-compact) space of all simplicial configurations. (The concept of simplicial k-configurations, $1 \leq k \leq n$, can be defined analogously using relative interiors, but we will not need this here.) In (9) the infimum can be restricted to $\Delta_{\mathcal{C}}(O)$, but a minimizing sequence of simplicial configurations may not (sub)converge. If degeneracy at the infima does not occur, that is all minimal n-configurations are simplicial then we call $O \in \operatorname{int} \mathcal{C}$ a *regular point*. The set of regular points is denoted by $\mathcal{R}_{\mathcal{C}} \subset \operatorname{int} \mathcal{C}$.

We now turn to the dual construction (Section 4). A dual n-configuration $\{f_0, \ldots, f_n\} \in \mathfrak{C}^o_{\mathcal{C},n}(O)$ is called *simplicial* if the intersection

$$\bigcap_{i=0}^{n} \{X \in \mathcal{X} \mid f_i(X) \geq 0\}$$

is an n-simplex. Using musical equivalences, this is equivalent to $\{f_0^\sharp, \ldots, f_n^\sharp\} \in \mathfrak{C}_{\mathcal{C}^o,n}(O)$ being simplicial. We let $\Delta^o_{\mathcal{C}}(O) \subset \mathfrak{C}^o_{\mathcal{C},n}$ denote the space of all simplicial dual configurations. As before, in (17) the infimum can be restricted to $\Delta^o_{\mathcal{C}}(O)$, but a minimizing sequence of simplicial dual configurations may not (sub)converge. If all minimal dual n-configurations are simplicial then we call $O \in \operatorname{int} \mathcal{C}$ a *dual regular point*. The set of dual regular points is denoted by $\mathcal{R}^o_{\mathcal{C}} \subset \operatorname{int} \mathcal{C}$.
The concept of regularity meshes well with duality, and Theorem 2 gives

$$O \in \mathcal{R}^o_{\mathcal{C}} \quad \Leftrightarrow \quad O \in \mathcal{R}_{\mathcal{C}^o}, \quad O \in \operatorname{int} \mathcal{C}. \tag{26}$$

The significance of these concepts lie in the fact that at any regular or dual regular points $n+1$ affine diameters meet.
This is closely related to *Grünbaum's Conjecture*: Any convex body $\mathcal{C}$ has an interior point O at which $n+1$ affine diameters meet. (See [5, 6.4.3, p. 254].)
A study of subconvergence of minimizing sequences then gives the following consequence of Theorem 3:

Theorem 4 *Let $O^* \in \mathcal{C}^* \subset \mathcal{C}$ be as in Theorem 3. Then we have*

$$\frac{n}{\mathfrak{m}^*_{\mathcal{C}} + 1} \leq \sigma^o_{\mathcal{C},n-1}(O^*). \tag{27}$$

If strict inequality holds then $O^* \in \mathcal{R}^o_{\mathcal{C}}$ and the Grünbaum conjecture is valid for $\mathcal{C}$: There are $n+1$ affine diameters that meet at O^*.

Remark 1 Klee in [9] proved Grünbaum's conjecture under the condition $\mathfrak{m}_{\mathcal{C}}(O^*) > n-1$. This is much more restrictive than (27) since $\sigma_{\mathcal{C},n-1}(O^*) \geq 1$ automatically holds.

Remark 2 The geometric interpretation of the right-hand side in (27) follows from (22): $\sigma^o_{\mathcal{C},n-1}(O^*)$ is the infimum of $\sigma^o_{\Pi(\mathcal{C}),n-1}(\Pi(O^*))$ for all projections $\Pi \in \mathfrak{P}_{\mathcal{C},n-1}$ of $\mathcal{C}$ to hyperplanes in $\mathcal{X}$.

Remark 3 Equality holds in (27) if $\mathcal{C}$ is symmetric (necessarily with center O^*). In this case the Grünbaum conjecture obviously holds.

Remark 4 Let $\mathcal{C}$ be a convex body of constant width. By the remark after Theorem 3, Theorem 4 holds for the (non-dual) mean Minkowski measure. Whether the respective inequality is strict or not depends on the (unique) critical point $O^* \in \mathcal{C}^*$ being regular or not. This, in turn, depends on whether O^* is in the convex hull of boundary points of $\mathcal{C}$ contained in a (proper) great sub-sphere of the circumsphere $\mathcal{S}_{R_\mathcal{C}}(O^*)$. Note that the construction of raising the dimension for convex bodies of constant width shows that non-regular points can well occur; see [10, Theorem 6].

Example 3 Let $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1,\, y \geq 0\}$ be the unit half-disk. A simple computation shows that $\mathfrak{m}_\mathcal{C}$ attains its minimum at the (unique) critical point $O^* = (0, \sqrt{2} - 1)$. (See also [6].) We thus have $\mathfrak{m}^*_\mathcal{C} = \sqrt{2}$, and, by Theorem 3, $\sigma^o_{\mathcal{C},2}(O^*) = 3/(\sqrt{2} + 1)$. Since $\sigma^o_{\mathcal{C},1} = 1$, in (27) strict inequality holds, in particular, $O^* \in \mathcal{R}^o_\mathcal{C}$. (Note that the centroid $g(\mathcal{C}) = (0, 4/3\pi)$ of $\mathcal{C}$ is different form O^*.) We claim that $\mathcal{R}^o_\mathcal{C} = \operatorname{int}\Delta$, where $\Delta = [C_0, C_-, C_+]$ is the triangle with vertices $C_0 = (0, 1)$ and $C_\pm = (\pm 1, 0)$. Given $O = (a, b) \in \operatorname{int}\mathcal{C}$ there may be at most three affine diameters passing through (a, b), those that also pass through C_0, C_-, and C_+. This immediately gives $\mathcal{R}^o_\mathcal{C} \subset \operatorname{int}\Delta$. For equality, let $O = (a, b) \in \operatorname{int}\Delta$ with $a \geq 0$ (by symmetry). Define $f_0 \in \operatorname{aff}_\mathcal{C}$ by $f_0(x, y) = y$ (with zero-set the first axis), and let $f_\pm \in \operatorname{aff}_\mathcal{C}$ have its zero-set the tangent line to the unit circle at the opposite $C^o_\pm$ with respect to O. A simple comparison of ratios shows that $f_-(O) + f_+(O) < 1$ and $f_0(O) + f_+(O) < 1$. On the other hand, we have $1/(\mathfrak{m}_\mathcal{C}(O) + 1) = \min(f_0(O), f_-(O))$, and we obtain

$$f_0(O) + f_-(O) + f_+(O) < 1 + \frac{1}{\mathfrak{m}_\mathcal{C}(O) + 1}.$$

Since $\{f_0, f_-, f_+\} \in \mathfrak{C}^o_\mathcal{C}(O)$, a dual 2-configuration, we see that O is a dual regular point. The claim follows.

A simple consideration of the affine coordinates associated to a simplex shows that the interior of a simplex consists of dual regular points only. (See Section 3.) In the other extreme it is natural to expect that the interior of a symmetric convex body does not have any dual regular points. This is indeed the case asserted by the following:

Theorem 5 *In a symmetric body $\mathcal{C}$ there are no dual regular points.*

Remark The same holds for (non-dual) regular points; see [18, Theorem A]. This, however, does not imply Theorem 5 due to the fact that the duality in Theorem 2 is only pointwise.

Example 4 Let $\mathcal{C} = \Delta \times I \subset \mathbb{R}^3$ be a prism, where $\Delta \subset \mathbb{R}^2$ is a triangle and $I \subset \mathbb{R}$ is a closed interval. Then there are no dual regular points in the interior of $\mathcal{C}$. This shows that the converse of Theorem 5 is not true. In addition, since $\mathfrak{m}^*_\Delta = 2$, we have $\mathfrak{m}^*_\mathcal{C} = 2$, and $\sigma_{\mathcal{C},2} = 1$ identically (since $\mathcal{C}$ has the triangular projection Δ). We see that equality holds in (27). On the other hand, through any interior points of $\mathcal{C}$ there are 4 affine diameters so that Grünbaum's conjecture holds for $\mathcal{C}$. This shows that in trying to remove the condition in (27) one needs to consider non-symmetric convex bodies with no dual regular points. (As it was pointed out by Hammer and Sobczyk in [7], $\mathcal{C}$ is a convex body with 1-dimensional critical set $\mathcal{C}^*$. In addition, for $\mathcal{C}$ equality holds in Klee's inequality showing that it is sharp.)

2 Duality via the Musical Equivalencies

We define the dual of a convex body $\mathcal{C} \subset \mathcal{X}$ with respect to an interior point $O \in \operatorname{int} \mathcal{C}$ as follows.
First, let $\mathcal{C}_0 \subset \mathcal{X}$ be a convex body with $0 \in \mathcal{C}_0$, the origin in $\mathcal{X}$. We define the dual of $\mathcal{C}_0$ with respect to 0 as

$$\mathcal{C}_0^0 = \{X \in \mathcal{X} \mid \sup_{C \in \mathcal{C}_0} \langle C, X \rangle \leq 1\}. \tag{28}$$

Clearly, $0 \in \operatorname{int} \mathcal{C}_0$, and by the bipolar theorem, we have $(\mathcal{C}_0^0)^0 = \mathcal{C}_0$.
The general case ($O \in \operatorname{int} \mathcal{C}$) is reduced to this by employing translations $T_V : \mathcal{X} \to \mathcal{X}$, $V \in \mathcal{X}$, where $T_V(X) = X + V$, $X \in \mathcal{X}$.
We first let $\mathcal{C}_0 = (T_O)^{-1}(\mathcal{C})$ (so that the point $O \in \operatorname{int} \mathcal{C}$ is moved to the origin $0 \in \operatorname{int} \mathcal{C}_0$), and then define

$$\mathcal{C}^O = T_O(\mathcal{C}_0^0), \quad \mathcal{C}_0 = (T_O)^{-1}(\mathcal{C}). \tag{29}$$

Clearly, $O \in \operatorname{int} \mathcal{C}^O$, and, by the above, we also have $(\mathcal{C}^O)^O = \mathcal{C}$.

The translations $T_V : \mathcal{X} \to \mathcal{X}$, $V \in \mathcal{X}$, act on the space of affine functionals $\mathrm{aff} = \mathrm{aff}(\mathcal{X})$ by $T_V^o : \mathrm{aff} \to \mathrm{aff}$, $V \in \mathcal{X}$, defined by ${T_V}^o(f) = f \circ T_V^{-1}$, $f \in \mathrm{aff}$. Using the notations above, for $O \in \operatorname{int} \mathcal{C}$, the linear map T_O^o restricts to $T_O^o : \mathrm{aff}_{\mathcal{C}_0} \to \mathrm{aff}_\mathcal{C}$, $\mathcal{C}_0 = T_O^{-1}(\mathcal{C})$, between the normalized affine functionals of the respective convex bodies. (Indeed, for $f_0 \in \mathrm{aff}\, \mathcal{C}_0$, we have $f_0(\mathcal{C}_0) = T_O^o(f_0)(\mathcal{C}) = [0, 1]$.) Since, by (29), $\mathcal{C}_0^0 = T_O^{-1}(\mathcal{C}^O)$, we also have the restriction $T_O^o : \mathrm{aff}_{\mathcal{C}_0^0} \to \mathrm{aff}_{\mathcal{C}^O}$.

In this spirit, the definition of the musical equivalencies

$$\flat_{\mathcal{C},O} : \partial\mathcal{C} \to \mathrm{aff}_{\mathcal{C}^O} \quad \text{and} \quad \sharp_{\mathcal{C},O} : \mathrm{aff}_{\mathcal{C}} \to \partial\mathcal{C}^O$$

can be reduced to

$$\flat_{\mathcal{C}_0,0} : \partial\mathcal{C}_0 \to \mathrm{aff}_{\mathcal{C}_0^0} \quad \text{and} \quad \sharp_{\mathcal{C}_0,0} : \mathrm{aff}_{\mathcal{C}_0} \to \partial\mathcal{C}_0^0$$

by the formulas

$$\flat_{\mathcal{C},O} = T_O^o \circ \flat_{\mathcal{C}_0,0} \circ T_O^{-1} \quad \text{and} \quad \sharp_{\mathcal{C},O} = T_O \circ \sharp_{\mathcal{C}_0,0} \circ (T_O^{-1})^o. \tag{30}$$

It remains to define the musical equivalencies for $\mathcal{C}_0$ with respect to $0 \in \mathrm{int}\,\mathcal{C}_0$ satisfying (3)-(6). For simplicity, we now suppress the subscript 0 and set $\mathcal{C} = \mathcal{C}_0$ with $0 \in \mathrm{int}\,\mathcal{C}$.
For $C \in \partial\mathcal{C}$, we let $C^\flat : \mathcal{X} \to \mathbb{R}$ be the affine functional given by

$$C^\flat(X) = \frac{1}{\lambda_{\mathcal{C}}(C,0)+1}\left(1 - \langle C, X\rangle\right), \quad X \in \mathcal{X}. \tag{31}$$

Evaluating this at the origin 0, (5) immediately follows.
The opposite of $C \in \partial\mathcal{C}$ (with respect to the origin 0) is $C^o = -C/\lambda_{\mathcal{C}}(C,0)$. Replacing C by C^o in (31), a simple computation gives the first formula in (3). Now a quick look at the definition of the dual $\mathcal{C}^0$ in (28) shows that $C^\flat$ is normalized for $\mathcal{C}^0$. We conclude that the musical map $\flat : \partial\mathcal{C} \to \mathrm{aff}_{\mathcal{C}^0}$ is well-defined.
For $f \in \mathrm{aff}_{\mathcal{C}}$, we write $f(X) = \langle A, X\rangle + a$, $A \in \mathcal{X}$ and $a \in (0,1)$ (since f is normalized). We then define

$$f^\sharp = -\frac{A}{a}. \tag{32}$$

Since f is normalized, (28) shows that this point is on the boundary of the dual $\mathcal{C}^0$. Once again, we obtain that the musical map $\sharp : \mathrm{aff}_{\mathcal{C}} \to \partial\mathcal{C}^0$ is well-defined.
Using (28) and (32) with $1-f$ in place of f, we obtain

$$(1-f)^\sharp = \frac{A}{1-a} = (f^\sharp)^o,$$

and the second formula in (3) follows. Since $-A/a$ and $A/(1-a)$ are opposites in $\mathcal{C}^0$, as a byproduct, we obtain (6).

Finally, it remains to show that the musical equivalencies are inverses of each other, that is (4) holds. Indeed, combining (31) and (32), we obviously have $(C^\flat)^\sharp = C$, $C \in \partial\mathcal{C}$, and the first relation in (4) follows. For the second, letting $f(X) = \langle A, X\rangle + a$

as above and using (6), we have $(f^\sharp)^\flat(X) = a(1 + \langle A, X\rangle/a) = f(X)$, $X \in \mathcal{X}$. The second relation in (4) also follows.

As a final preparatory step, as stated in the previous section, we need to work out the dual of an affine diameter. Let $[C, C^o] \subset \mathcal{C}$ be an affine diameter with parallel supporting hyperplanes $\mathcal{H}, \mathcal{H}^o \in \delta\mathcal{C}$ at both ends, that is $C \in \mathcal{H}$ and $C^o \in \mathcal{H}^o$. As above, we let $f \in \mathrm{aff}_{\mathcal{C}}$ be the normalized affine function with zero sets $\mathcal{H} = \{X \,|\, f(X) = 0\}$ and $\mathcal{H}^o = \{X \,|\, 1 - f(X) = 0\}$. We have $f(C) = 0$ and $f(C^o) = 1$. Letting $0 = O$ and $f(X) = \langle A, X\rangle + a$, $X \in \mathcal{X}$, as above, we have

$$C^\flat(f^\sharp) = \frac{1}{\lambda_{\mathcal{C}}(C,0)+1}\left(1 - \left\langle C, -\frac{A}{a}\right\rangle\right) = \frac{1}{a(\lambda_{\mathcal{C}}(C,0)+1)} f(C) = 0,$$

and

$$\begin{aligned} C^\flat((f^\sharp)^o) &= \frac{1}{\lambda_{\mathcal{C}}(C,0)+1}\left(1 - \left\langle C, \frac{A}{1-a}\right\rangle\right) \\ &= \frac{1}{(1-a)(\lambda_{\mathcal{C}}(C,0)+1)}\left(1 - a - \langle C, A\rangle\right) = 1, \end{aligned}$$

since

$$f(C^o) = \langle A, C^o\rangle + a = -\frac{1}{\lambda_{\mathcal{C}}(C,0)}\langle A, C\rangle + a = 1.$$

We see that $[f^\sharp, (f^\sharp)^o]$ is an affine diameter of the dual $\mathcal{C}^0$ with parallel supporting hyperplanes $C^\flat, (C^o)^\flat \in \delta\mathcal{C}^0$ at the endpoints.
We conclude that the dual of an affine diameter configuration is also an affine diameter configuration.

3 Proofs of Theorems 1-3

Proof of Theorem 1. We will show that $\sigma_{\mathcal{C},k}(O) = \sigma^o_{\mathcal{C}^O,k}(O)$. Since $(\mathcal{C}^O)^O = \mathcal{C}$, this will imply the theorem.
We first claim that, for any $\{C_0, \dots, C_k\} \subset \partial\mathcal{C}$, we have

$$O \in [C_0, \dots, C_k] \quad \Leftrightarrow \quad \bigcap_{i=0}^{k} \{X \in \mathcal{X} \,|\, C_i^\flat(X) \leq 0\} = \emptyset, \tag{33}$$

where $\flat = \flat_{\mathcal{C},O} : \partial\mathcal{C} \to \mathrm{aff}_{\mathcal{C}^O}$ is the musical equivalence.
Without loss of generality, we may set $O = 0 \in \mathrm{int}\,\mathcal{C}$, the origin.

First, assume that $0 \in [C_0, \ldots, C_k]$, that is we have $\sum_{i=0}^k \lambda_i C_i = 0$ with $\sum_{i=0}^k \lambda_i = 1$, $\lambda_i \in [0,1]$, $i = 0, \ldots, k$. Assume that there exists $X \in \mathcal{X}$ such that $C_i^\flat(X) \leq 0$, $i = 0, \ldots, k$. By (31), this means that $\langle C_i, X \rangle \geq 1$, $i = 0, \ldots, k$. Summing up, we obtain

$$\sum_{i=0}^k \lambda_i \langle C_i, X \rangle = \left\langle \sum_{i=0}^k \lambda_i C_i, X \right\rangle = 0 \geq \sum_{i=0}^k \lambda_i = 1,$$

a contradiction.

Conversely, assume that $0 \notin [C_0, \ldots, C_k]$ so that 0 and the convex hull $[C_0, \ldots, C_k]$ can be (strictly) separated by a hyperplane $\mathcal{H} \subset \mathcal{X}$. A unit normal $N \in \mathcal{X}$ of $\mathcal{H}$ then satisfies $\langle C_i, N \rangle > 0$, $i = 0, \ldots, k$. For $t > 0$ large enough, we then have $\langle C_i, tN \rangle \geq 1$, $i = 0, \ldots, k$. Thus, tN belongs to the intersection $\bigcap_{i=0}^k \{X \in \mathcal{X} \,|\, C_i^\flat(X) \leq 0\}$. The converse follows.

The claim just proved can be reformulated as

$$\{C_0, \ldots, C_k\} \in \mathfrak{C}_{\mathcal{C},k}(O) \quad \Leftrightarrow \quad \{C_0^\flat, \ldots, C_k^\flat\} \in \mathfrak{C}^o_{\mathcal{C}^O,k}(O).$$

Using (5), we now calculate

$$\begin{aligned}
\sigma_{\mathcal{C},k}(0) &= \inf_{\{C_0,\ldots,C_k\} \in \mathfrak{C}_{\mathcal{C},k}(0)} \sum_{i=0}^k \frac{1}{\lambda_{\mathcal{C}}(C_i, 0) + 1} \\
&= \inf_{\{C_0^\flat,\ldots,C_k^\flat\} \in \mathfrak{C}^o_{\mathcal{C}^O,k}(0)} \sum_{i=0}^k C_i^\flat(0) \\
&= \inf_{\{f_0,\ldots,f_k\} \in \mathfrak{C}^o_{\mathcal{C}^O,k}(0)} \sum_{i=0}^k f_i(0) = \sigma^o_{\mathcal{C}^O,k}(0)
\end{aligned}$$

Theorem 1 follows.

Remark Dually, for $\{f_0, \ldots, f_k\} \subset \mathrm{aff}_{\mathcal{C}}$, we also have

$$\bigcap_{i=0}^k \{X \in \mathcal{X} \,|\, f_i(X) \leq 0\} = \emptyset \quad \Leftrightarrow \quad O \in [f_0^\sharp, \ldots, f_k^\sharp].$$

This is the same as the equivalency asserted in (20). As a byproduct of the computation above we also see that under the musical equivalencies minimal configurations correspond to each other.

We now turn to the proof of (22). Given a dual k-configuration $\{f_0, \ldots, f_k\} \in \mathfrak{C}^o_{\mathcal{C},k}(O)$, let $\mathcal{E} \subset \mathcal{X}$ be a k-dimensional affine subspace containing the duals $f_0^\sharp, \ldots, f_k^\sharp \in$

$\mathcal{X}$ (and, by (20), also O). We have $\{f_0|_{\mathcal{E}}, \ldots, f_k|_{\mathcal{E}}\} \in \mathfrak{C}^o_{\Pi(\mathcal{C}),k}(O)$, where $\Pi \in \mathfrak{P}_k$ is the orthogonal projection of $\mathcal{X}$ to $\mathcal{E}$. The affine functionals f_i, $i = 0, \ldots, k$, are constant along (the fibres of) Π, and we also have $\sum_{i=0}^k f_i(O) = \sum_{i=0}^k (f_i|_{\mathcal{E}})(O)$. We conclude that the infimum for $\sigma^o_{\mathcal{C},k}(O)$ in (22) can first be taken for dual k-configurations in $\mathfrak{C}^o_{\Pi(\mathcal{C}),k}(\Pi(O))$ for a given $\Pi \in \mathfrak{P}_k$, thus yielding $\sigma^o_{\Pi(\mathcal{C}),k}(\Pi(O))$, and finally followed by the infimum for all $\Pi \in \mathfrak{P}_k$. The claim follows.

Proof of Theorem 2. As noted previously, the bounds in (23) follow by duality via Theorem 1.

We now consider the upper bound in (23). Let $k \geq 2$, and assume that $\sigma^o_{\mathcal{C},k}(O) = (k+1)/2$. Dualizing, again by Theorem 1, we have $\sigma_{\mathcal{C}^O,k}(O) = (k+1)/2$. Hence, $\mathcal{C}^O$ is symmetric with respect to O. Since duality (with respect to the center) preserves symmetry, we obtain that $\mathcal{C} = (\mathcal{C}^O)^O$ is symmetric with respect to O.

It remains to consider the lower bound in (23). Assume that, for some $k \geq 1$, we have $\sigma^o_{\mathcal{C},k}(O) = 1$ at an interior point $O \in \operatorname{int} \mathcal{C}$. Since $\sigma^o_{\mathcal{C},k}$ is a concave function on $\operatorname{int} \mathcal{C}$ and, by (24), it assumes the value 1 on the boundary, we see that $\sigma^o_{\mathcal{C},k} = 1$ identically on $\mathcal{C}$.

If $k > n$ then, by arithmeticity and (23), we have

$$1 = \sigma^o_{\mathcal{C},k}(O) = \sigma^o_{\mathcal{C},n}(O) + \frac{k-n}{\mathfrak{m}_{\mathcal{C}}(O)+1} \geq 1 + \frac{k-n}{\mathfrak{m}_{\mathcal{C}}(O)+1} > 1.$$

This is a contradiction. Thus $k \leq n$. (Alternatively, again by duality, $\sigma^o_{\mathcal{C},k}(O) = \sigma_{\mathcal{C}^O,k}(O) = 1$ so that $k \leq n$.)

For the last statement, let the infimum in (22) be attained at an orthogonal projection $\Pi \in \mathfrak{P}_k$ (onto a k-dimensional affine subspace), so that we have $\sigma^o_{\Pi(\mathcal{C}),k}(\Pi(O)) = 1$. As before, $\sigma^o_{\Pi(\mathcal{C}),k} = 1$ identically on $\Pi(\mathcal{C})$. Let O^* be a critical point of $\Pi(\mathcal{C})$. By the obvious lower bound in (18) applied to the k-dimensional convex body $\Pi(\mathcal{C})$ (and O^*), we have

$$\frac{k+1}{\mathfrak{m}^*_{\Pi(\mathcal{C})}+1} \leq \sigma^o_{\Pi(\mathcal{C}),k}(O^*) = 1.$$

This gives $k \leq \mathfrak{m}_{\Pi(\mathcal{C})}(O^*)$. By the Minkowski-Radon inequality, $\mathfrak{m}^*_{\Pi(\mathcal{C})} \leq k$, so that equality holds and $\Pi(\mathcal{C})$ is a k-simplex. Theorem 2 follows.

Example 5 An n-simplex $\Delta = [C_0, \ldots, C_n]$ with vertices $C_0, \ldots, C_n \in \mathcal{X}$ possesses a unique minimal dual n-configuration for any interior point, the *affine coordinate system* $\{f_0, \ldots, f_n\} \subset \operatorname{aff}_\Delta$ associated to Δ. (For $i = 0, \ldots, n$, $f_i \in \operatorname{aff}_\Delta$ is the normalized affine functional that vanishes on the i-th face $[C_0, \ldots, \widehat{C}_i, \ldots, C_n]$ (opposite to the vertex C_i), and $f_i(C_i) = 1$.) For $O \in \operatorname{int} \Delta$ with $O = \sum_{i=0}^n \lambda_i C_i$, $\sum_{i=0}^n \lambda_i = 1$, $\lambda_i \in (0,1)$, we have $f_i(O) = \lambda_i$, $i = 0, \ldots, n$. Since (16) obviously holds, we have

$\sigma^o_{\Delta,n}(O) \leq \sum_{i=0}^n f_i(O) = \sum_{i=0}^n \lambda_i = 1$. By (23), equality must hold. We see that $\{f_0, \dots, f_n\} \in \mathfrak{C}^o_{\Delta,n}(O)$ is the (unique) minimal dual n-configuration for all $O \in \operatorname{int}\Delta$. As a byproduct, we see that all interior points of an n-simplex are dual regular, that is $\mathcal{R}^o_\Delta = \operatorname{int}\Delta$. (Note that the same holds for (non-dual) regular points.)

Remark The previous example can be used to show directly that if $\sigma^o_{\mathcal{C},n}(O) = 1$ then $\mathcal{C}$ is an n-simplex. This gives an alternative proof of the last part of Theorem 2 (for $\Pi(\mathcal{C})$ instead of $\mathcal{C}$) without the recourse of the Minkowski-Radon theorem.
Assume $\sigma^o_{\mathcal{C},n}(O) = 1$ for some $O \in \operatorname{int}\mathcal{C}$. First, any minimal dual n-configuration of O must be simplicial. Indeed, otherwise a minimal dual n-configuration would contain a proper subconfiguration, and we would have arithmeticity: $1 = \sigma^o_{\mathcal{C},n} = \sigma^o_{\mathcal{C},n-1} + 1/(\mathfrak{m}_\mathcal{C}(O)+1) > 1$, a contradiction. Second, let $\{f_0, \dots, f_n\} \in \Delta^o_\mathcal{C}(O)$ be a minimal simplicial dual configuration. The corresponding n-simplex $\Delta = \bigcap_{i=0}^n \{X \in \mathcal{X} \,|\, f_i(X) \geq 0\}$ contains $\mathcal{C}$. For each $i = 0, \dots, n$, let $\tilde{f}_i \in \operatorname{aff}_\Delta$ be the normalized affine functional such that the zero-sets $\{X \in \mathcal{X} \,|\, f_i(X) = 0\} = \{X \in \mathcal{X} \,|\, \tilde{f}_i(X) = 0\}$. Now, assume that $\mathcal{C}$ is *not* a simplex. Then $f_i(O) < \tilde{f}_i(O)$ for some $i = 0, \dots, n$. We then have $1 = \sigma^o_{\mathcal{C},n}(O) = \sum_{i=0}^n f_i(O) < \sum_{i=0}^n \tilde{f}_i(O) = \sigma_{\Delta,n}(O) = 1$, where the last two equalities follow from Example 5. This is a contradiction, so that $\mathcal{C}$ must be an n-simplex.

Proof of Theorem 3. We first introduce some notation. We define

$$\mathcal{M}(O) = \{C \in \partial\mathcal{C} \,|\, \lambda_\mathcal{C}(C, O) = \mathfrak{m}_\mathcal{C}(O)\}, \quad O \in \operatorname{int}\mathcal{C},$$

where $\mathfrak{m}_\mathcal{C} : \operatorname{int}\mathcal{C} \to \mathbb{R}$ is the maximal Minkowski ratio. Clearly, $\mathcal{M}(O) \subset \partial\mathcal{C}$ is compact, and for every $C \in \mathcal{M}(O)$, the chord $[C, C^o]$ of $\mathcal{C}$ is an affine diameter. (This is an elementary fact; also noted in [9, 3.2].)

We now turn to the proof in which we will use several results of Klee in [9]. Let $\mathcal{N}(O^*) = \mathcal{M}(O^*)^o \subset \partial\mathcal{C}$ be the opposite set of $\mathcal{M}(O^*) \subset \partial\mathcal{C}$ with respect to O^*. Denote by $\mathfrak{G}$ the family of closed half-spaces that intersect $\mathcal{N}(O^*)$ but disjoint from $\operatorname{int}\mathcal{C}$. Clearly, for each $\mathcal{G} \in \mathfrak{G}$, the boundary $\mathcal{H} = \partial\mathcal{G}$ is a hyperplane supporting $\mathcal{C}$ at a point in $\mathcal{N}(O^*)$. Conversely, for any hyperplane $\mathcal{H}$ supporting $\mathcal{C}$ at a point in $\mathcal{N}(O^*)$, the closed half-space $\mathcal{G}$ with boundary $\mathcal{H}$ and disjoint from $\mathcal{C}$ belongs to $\mathfrak{G}$. In a technical lemma, Klee in [9, 3.1] proved

$$\bigcap \mathfrak{G} = \bigcap_{\mathcal{G} \in \mathfrak{G}} \mathcal{G} = \emptyset.$$

Taking interiors, the family

$$\mathfrak{I} = \operatorname{int}\mathfrak{G} = \{\operatorname{int}\mathcal{G}, \,|\, \mathcal{G} \in \mathfrak{G}\}$$

of open half-spaces is in Klee's terminology 0-closed. This means that, for any sequence $\{\mathcal{I}_k\}_{k\geq 1} \subset \mathfrak{G}$ which is Kuratowski convergent to a limit $\mathcal{I}$, we have $\operatorname{int}\mathcal{I} \in \mathfrak{G}$. (Note that, by definition, any Kuratowski limit is a closed set.)
We now need Klee's extension of Helly's theorem for 0-closed families: If any $n+1$ members of an 0-closed family has non-empty intersection then the interior of the intersection of all members of the family is non-empty (see [9, 3.2]).
We apply this to our family $\mathfrak{I}$ of open half-spaces above. Since $\bigcap \mathfrak{I} = \emptyset$ (as $\bigcap \mathfrak{G} = \emptyset$) we see that that there are $n+1$ open half-spaces $\mathcal{I}_0, \dots, \mathcal{I}_n \in \mathfrak{I}$ such that $\bigcap_{i=0}^n \mathcal{I}_k = \emptyset$.
Let $i = 0, \dots, n$. We select $C_i \in \mathcal{M}(O^*)$ such that the opposite $C_i^o \in \bar{\mathcal{I}}_i$ (with respect to O^*). Then $[C_i, C_i^o]$ is an affine diameter with $\lambda_{\mathcal{C}}(C_i, O^*) = \mathfrak{m}_{\mathcal{C}}(O^*) = \mathfrak{m}_{\mathcal{C}}^*$. We let $f_i \in \operatorname{aff}_{\mathcal{C}}$ be the (unique) normalized affine functional with zero-set $\partial\mathcal{I}_i$. Since $C_i^o \in \partial\mathcal{I}_i$, we have $f_i(C_i^o) = 0$ and hence $f_i(C_i) = 1$. We calculate

$$f_i(O^*) = \frac{d(C_i^o, O^*)}{d(C_i^o, C_i)} = \frac{1}{d(C_i, O^*)/d(C_i^o, O^*) + 1} = \frac{1}{\lambda_{\mathcal{C}}(C_i, O^*) + 1} = \frac{1}{\mathfrak{m}_{\mathcal{C}}^* + 1}.$$

Summing up, we obtain

$$\sigma_{\mathcal{C},n}^o(O^*) \leq \sum_{i=0}^n f_i(O^*) = \frac{n+1}{\mathfrak{m}_{\mathcal{C}}^* + 1}.$$

On the other hand, by (18), the right-hand side is an obvious lower bound for $\sigma_{\mathcal{C},n}^o(O^*)$. Theorem 3 follows.

4 Regular Points and the Grünbaum Conjecture

Let $\mathcal{C} \subset \mathcal{X}$ be a convex body. Recall that $O \in \operatorname{int}\mathcal{C}$ is a regular point if all minimal n-configurations in $\mathfrak{C}_{\mathcal{C},n}(O)$ are simplicial, that is they belong to $\Delta_{\mathcal{C}}(O)$. Since minimal simplicial configurations do not contain any proper (necessarily minimal) subconfigurations, this condition can be conveniently reformulated in terms of the mean Minkowski measures: $O \in \operatorname{int}\mathcal{C}$ is regular iff in (10) strict sub-arithmeticity holds:

$$\sigma_{\mathcal{C},n}(O) < \sigma_{\mathcal{C},n-1}(O) + \frac{1}{\mathfrak{m}_{\mathcal{C}}(O) + 1}. \tag{34}$$

(For more details, see [16].) Since the mean Minkowski measures are continuous, we see that the set of all regurar points $\mathcal{R}_{\mathcal{C}} \subset \operatorname{int}\mathcal{C}$ is open.

Let $O \in \mathcal{R}_{\mathcal{C}}$ be a regular point, and $\{C_0, \dots, C_n\} \in \Delta_{\mathcal{C}}(O)$ a minimal simplicial configuration. Since O is in the interior of the n-simplex $[C_0, \dots, C_n]$, by (9), for

each $i = 0, \ldots, n$, the distortion $\lambda_{\mathcal{C}}(\cdot, O)$ attains a *local maximum* at C_i. It is well-known that at local maxima of the distortion the corresponding chord (through O) is an *affine diameter*. (See, for example [6] or [17].) We conclude that, for each $i = 0, \ldots, n$, the chord $[C_i, C_i^o]$ is an affine diameter. Thus, at any regular point $O \in \mathcal{R}_{\mathcal{C}}$, $n+1$ affine diameters meet.
In 1963 Grünbaum conjectured that any convex body has a common point of $n+1$ affine diameters. We see that if $\mathcal{R}_{\mathcal{C}} \neq \emptyset$ then we have an affirmative answer to Grünbaum's conjecture: At any regular point $n+1$ affine diameters meet.

We now consider the dual scenario. Recall that a dual n-configuration $\{f_0, \ldots, f_n\} \in \mathfrak{C}^o_{\mathcal{C},n}(O)$ is called *simplicial* if $\{f_0^\sharp, \ldots, f_n^\sharp\} \in \mathfrak{C}_{\mathcal{C}^O,n}(O)$ is simplicial, where $\sharp = \sharp_{\mathcal{C},O} : \mathrm{aff}_{\mathcal{C}} \to \partial\mathcal{C}^O$ is the musical equivalence. As noted previously, geometrically, a dual n-configuration $\{f_0, \ldots, f_n\} \in \mathfrak{C}^o_{\mathcal{C},n}(O)$ is simplicial if and only if $\bigcap_{i=0}^n \{X \in \mathcal{X} \,|\, f_i(X) \geq 0\}$ is an n-simplex with O in its interior. The set of dual simplicial configurations is denoted by $\Delta^o_{\mathcal{C}}(O)$. By (20), for $\{f_0, \ldots, f_n\} \subset \mathrm{aff}_{\mathcal{C}}$, we have

$$\{f_0, \ldots, f_n\} \in \Delta^o_{\mathcal{C}}(O) \quad \Leftrightarrow \quad \{f_0^\sharp, \ldots, f_n^\sharp\} \in \Delta_{\mathcal{C}^O}(O).$$

Recall that an interior point $O \in \mathrm{int}\,\mathcal{C}$ is called *dual regular* if any minimal dual n-configuration in $\mathfrak{C}^o_{\mathcal{C},n}(O)$ is simplicial. The set of all dual regular points is denoted by $\mathcal{R}^o_{\mathcal{C}} \subset \mathrm{int}\,\mathcal{C}$. As in the dual case, $O \in \mathcal{R}^o_{\mathcal{C}}$ if and only if

$$\sigma^o_{\mathcal{C},n}(O) < \sigma^o_{\mathcal{C},n-1}(O) + \frac{1}{\mathfrak{m}_{\mathcal{C}}(O) + 1}, \tag{35}$$

in particular, $\mathcal{R}^o_{\mathcal{C}} \subset \mathrm{int}\,\mathcal{C}$ is open.
Now, comparing (34) and (35), Theorem 1 along with (7) give (26).

Let $O \in \mathcal{R}^o_{\mathcal{C}}$ be a dual regular point and $\{f_0, \ldots, f_n\} \in \Delta^o_{\mathcal{C}}(O)$ a minimal simplicial configuration. We have $O \in \mathcal{R}_{\mathcal{C}^O}$, and, by Theorem 1, $\{f_0^\sharp, \ldots, f_n^\sharp\} \in \Delta_{\mathcal{C}^O}(O)$ is a minimal simplicial configuration. By the discussion above, for each $i = 0, \ldots, n$, the chord $[f_i^\sharp, (f_i^\sharp)^o]$ is an affine diameter of $\mathcal{C}^O$. Let $\mathcal{K}_i$ and $\mathcal{K}_i^o$ be parallel hyperplanes at the endpoints of $f_i^\sharp$ and $(f_i^\sharp)^o$. Finally, let $g_i \in \mathrm{aff}_{\mathcal{C}^O}$ be the normalized affine functional with zero-sets $\mathcal{K}_i = \{X \in \mathcal{X} \,|\, g_i(X) = 0\}$ and $\mathcal{K}_i^o = \{X \in \mathcal{X} \,|\, 1 - g_i(X) = 0\}$. By the discussion at the end of Section 2, for each $i = 0, \ldots, n$, the chord $[g_i^\sharp, (g_i^\sharp)^o]$ is an affine diameter of $\mathcal{C} = (\mathcal{C}^O)^O$, and the parallel supporting hyperplanes at the endpoints are given by the respective zero-sets of the original affine functional $f_i = (f_i^\sharp)^\flat$. Letting $C_i = g_i^\sharp \in \partial\mathcal{C}$, we see that the zero-sets $\mathcal{H}_i = \{X \in \mathcal{X} \,|\, f_i(X) = 0\}$ and $\mathcal{H}_i^o = \{X \in \mathcal{X} \,|\, 1 - f_i(X) = 0\}$ are parallel supporting hyperplanes of $\mathcal{C}$ with affine diameters $[C_i, C_i^o] \subset \mathcal{C}$, $i = 0, \ldots, n$.
We claim that the affine diameters $[C_i, C_i^o]$, $i = 0, \ldots, n$, are distinct. Assume that $[C_i, C_i^o] = [C_j, C_j^o]$ for some $i \neq j$, $i, j = 0, \ldots, n$. (This means that this common

affine diameter has two pairs of parallel supporting hyperplanes, $\mathcal{H}_i$, $\mathcal{H}_i^o$ and $\mathcal{H}_j$, $\mathcal{H}_j^o$.) Since $C_i = C_j$ or $C_i = C_j^o$, in the dual, we have $g_i = g_j$ or $g_i = 1 - g_j$. In particular, the affine diameters $[f_i^\sharp, (f_i^\sharp)^o]$ and $[f_j^\sharp, (f_j^\sharp)^o]$ of $\mathcal{C}^O$ share a single pair of parallel supporting hyperplanes, $\mathcal{K}_i = \mathcal{K}_j$, $\mathcal{K}_i^o = \mathcal{K}_j^o$, or $\mathcal{K}_i = \mathcal{K}_j^o$, $\mathcal{K}_i^o = \mathcal{K}_j$. On the other hand, in a minimal simplicial configuration of a regular point (such as $\{f_0^\sharp, \dots, f_n^\sharp\} \in \Delta_{\mathcal{C}^O}(O)$ with $O \in \mathcal{R}_{\mathcal{C}^O}$) two affine diameters cannot share the same parallel supporting hyperplanes since otherwise we can slide one in the respective hyperplanes (along a line segment) to the other to obtain another minimal configuration with multiple points or a pair of antipodal points. These contradict to regularity.
We conclude that if $O \in \mathcal{R}_{\mathcal{C}}^o$ then $n+1$ affine diameters meet at O.

Proof of Theorem 4. Let $O^* \in \mathcal{C}^*$ be a critical point of $\mathcal{C}$. Sub-arithmeticity in (21) gives

$$\sigma_{\mathcal{C},n}^o(O^*) \leq \sigma_{\mathcal{C},n-1}^o(O^*) + \frac{1}{\mathfrak{m}_{\mathcal{C}}^* + 1}.$$

The equality in (25) of Theorem 3 reduces this to (27), and the first statement of Theorem 4 follows. Strict inequality holds iff $O^* \in \mathcal{R}_{\mathcal{C}}^o$, a dual regular point. By the discusssion above, this implies the existence of $n+1$ affine diameters across O^*. The second statement of Theorem 4 follows.

Proof of Theorem 5. Let $\mathcal{C}$ be a symmetric convex body with center O_0. Assume that $O \in \operatorname{int}\mathcal{C}$ is a dual regular point. Since the center O_0 is obviously not dual regular, we may assume that $O \neq O_0$. Let $\{f_0, \dots, f_n\} \in \mathfrak{C}_{\mathcal{C},n}^o(O)$ be a minimal configuration. Since $O \in \mathcal{R}_{\mathcal{C}}^o$, this configuration is simplicial. Fix $i = 0, \dots, n$, and, for simplicity, suppress the subscript and set $f = f_i \in \mathrm{aff}_{\mathcal{C}}$. By the discussion before the proof of Theorem 4, $\mathcal{C}$ has an affine diameter $[C, C^o] \subset \mathcal{C}$ with supporting hyperplanes $\mathcal{H} = \{X \in \mathcal{X} \,|\, f(X) = 0\}$ and $\mathcal{H}^o = \{X \in \mathcal{X} \,|\, 1 - f(X) = 0\}$ such that $C \in \mathcal{H}$ and $C^o \in \mathcal{H}^o$. (Here the opposite is with respect to O.)
Let $A \in \partial\mathcal{C}$ be the point at which the ray $\mathfrak{r}$ emanating from O_0 and passing through O meets the boundary of $\mathcal{C}$. We claim that $[A, A^o]$ is an affine diameter of $\mathcal{C}$, and, beyond A, this ray $\mathfrak{r}$ enters into the half-space $\{X \in \mathcal{X} \,|\, f(X) \leq 0\}$. Since $\mathfrak{r}$ is independent of $i = 0, \dots, n$, this means that the intersection in (16) is non-empty; a contradiction.
If C is on $\mathfrak{r}$ then $A = C$ and we are done. Thus we may assume that the points C, O, and O_0 are not collinear.
Let $C_0^o \in \partial\mathcal{C} \cap \mathcal{H}^o$ be the opposite of C with respect to the center O_0. By symmetry, we have $[C^o, C_0^o] \subset \partial\mathcal{C} \cap \mathcal{H}^o$.
Let $A_1 \in \partial\mathcal{C}$ be the opposite of C_0^o with respect to O. Moving along the line segment $[C^o, C_0^o]$ and taking the opposites (with respect to O), we see that $A_1 \in \mathcal{H}$ since $f(O)$ is a local minimum in $\mathrm{aff}_{\mathcal{C}}$. Since $\mathcal{H}$ supports $\mathcal{C}$, we have $[A_1, C] \subset \partial\mathcal{C} \cap \mathcal{H}$. We

now define A_k, $k \geq 1$, inductively as follows. Assume that $A_k \in \partial\mathcal{C}$ is constructed with $[A_k, C] \subset \partial\mathcal{C} \cap \mathcal{H}$. We take the opposite of A_k with respect to O_0 followed by the opposite with respect to O. This gives the point A_{k+1}. As before, we have $[A_{k+1}, C] \subset \partial\mathcal{C} \cap \mathcal{H}$. The sequence $\{A_k\}_{k\geq 1}$ is actually collinear and converges to $A \in \partial\mathcal{C}$ which then must be on $\mathcal{H}$. (In fact, an elementary argument shows that the sequence $\{d(A_k, A)\}_{k\geq 1}$ is geometric.) By construction, the chord $[A, A^o]$ is an affine diameter, where A^o is the opposite of A with respect to O. After A the ray $\mathfrak{r}$ enters the open half-space $\{X \in \mathcal{X} \mid f(X) < 0\}$. The claim follows and the proof of Theorem 5 is complete.

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