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Linearized pseudo-Einstein equations on the Heisenberg group

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ABSTRACT

We study the pseudo-Einstein equation $R_{1\bar{1}} = 0$ on the Heisenberg group $\mathbb{H}_1 = \mathbb{C}\times\mathbb{R}$. We consider first order perturbations $\theta_\epsilon = \theta_0 + \epsilon\,\theta$ and linearize the pseudo-Einstein equation about θ_0 (the canonical Tanaka–Webster flat contact form on $\mathbb{H}_1$ thought of as a strictly pseudoconvex CR manifold). If $\theta = e^{2u}\theta_0$ the linearized pseudo-Einstein equation is $\Delta_b u - 4|Lu|^2 = 0$ where Δ_b is the sublaplacian of $(\mathbb{H}_1, \theta_0)$ and $\bar{L}$ is the Lewy operator. We solve the linearized pseudo-Einstein equation on a bounded domain $\Omega \subset \mathbb{H}_1$ by applying subelliptic theory i.e. existence and regularity results for weak subelliptic harmonic maps. We determine a solution u to the linearized pseudo-Einstein equation, possessing Heisenberg spherical symmetry, and such that $u(x) \to -\infty$ as $|x| \to +\infty$.

1. Pseudohermitian Ricci curvature

Pseudohermitian geometry (a term coined by S.M. Webster in 1977) was simultaneously discovered by S.M. Webster and N. Tanaka, cf. [1]. Objects in the pseudohermitian category are *pseudohermitian manifolds*, that is CR manifolds $(M, T_{1,0}(M))$ endowed with a globally defined pseudohermitian structure i.e. a nowhere zero C^∞ section $\theta : M \to H(M)^\perp$ in the conormal bundle associated to the Levi, or maximally complex, distribution $H(M)$ of $(M, T_{1,0}(M))$. Cf. § 2 for a review of the basic notions in CR and pseudohermitian geometry (following [1]). Under an assumption of nondegeneracy (of the Levi form of $T_{1,0}(M)$) for each pseudohermitian structure θ on M there is a unique linear connection $\nabla = \nabla^\theta$ [the *Tanaka–Webster connection* of (M, θ)] parallelizing the Levi distribution $H(M)$, the complex structure along $H(M)$, and the Levi form, and having pure torsion (cf. e.g. [1]). If R^∇ is the curvature tensor field of the Tanaka–Webster connection ∇ the *pseudohermitian Ricci* tensor is (cf. [1])

$$R_{\mu\bar{\nu}} = \text{Trace}\left\{V \in T(M)\otimes\mathbb{C} \mapsto R^\nabla(V, T_{\bar{\nu}})T_\mu\right\}.$$

A nondegenerate pseudohermitian manifold is *pseudo-Einstein* (cf. [1]) if the pseudohermitian Ricci curvature is proportional to the Levi form

$$R_{\mu\bar{\nu}} = (R/n)\,g_{\mu\bar{\nu}} \tag{1}$$

where $R = g^{\mu\bar{\nu}}R_{\mu\bar{\nu}}$ is the *pseudohermitian scalar curvature* (cf. [1]). Equation (1) is the *pseudo-Einstein equation* on M. Any odd dimensional sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ is a strictly pseudoconvex CR manifold [with the CR structure induced on S^{2n+1}

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by the complex structure of $\mathbb{C}^{n+1}$] and the pseudohermitian structure $\theta = \iota^* \left\{ \frac{i}{2} \left(\overline{\partial} - \partial \right) |z|^2 \right\}$ is pseudo-Einstein. Any pseudohermitian structure on a nondegenerate 3-dimensional CR manifold is automatically pseudo-Einstein, eventually of nonzero pseudohermitian scalar curvature. Let $\mathbb{H}_n$ be the Heisenberg group i.e. the Lie group $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t) \cdot (w, s) = (z + w, \; t + s + 2\,\mathrm{Im}\,(z \cdot \overline{w})), \quad (z, t), \, (w, s) \in \mathbb{H}_n . \tag{2}$$

$\mathbb{H}_n$ is a strictly pseudoconvex CR manifold with the CR structure obtained by identifying $\mathbb{H}_n$ to the boundary of the Siegel domain $\Omega_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \mathrm{Im}(w) > |z|^2\}$. The canonical pseudohermitian structure $\theta_0 = dt + i \left(z^\mu \, d\overline{z}_\mu - \overline{z}_\mu \, dz^\mu \right)$ is Tanaka–Webster flat and in particular a solution to

$$R_{\mu\overline{\nu}} = 0. \tag{3}$$

Solving for θ in (3) is a nontrivial problem (similar to solving the Einstein equations for empty space in general relativity theory) even in the CR dimension $n = 1$ case. It is our purpose in the present paper to consider first order perturbations

$$\theta_\epsilon = \theta_0 + \epsilon\, \theta \tag{4}$$

($\epsilon \ll 1$) and linearize Eq. (3) about the canonical pseudohermitian structure θ_0 on $\mathbb{H}_1$. There is a unique $u \in C^\infty(\mathbb{H}_1, \mathbb{R})$ such that $\theta = e^{2u}\theta_0$. Substitution from (4) into (3) [followed by dropping terms of order $O(\epsilon^2)$ and higher] leads to

$$\Delta_b u - 4|Lu|^2 = 0 \tag{5}$$

where Δ_b is the sublaplacian (cf. [2]) of $(\mathbb{H}_1, \theta_0)$ and $\overline{L}$ is the Lewy operator (cf. [1]). Using techniques of subelliptic theory (cf. e.g. [3]) we solve the Dirichlet problem for the linearized pseudo-Einstein (5) on a bounded domain $\Omega \subset \mathbb{H}_1$. The situation we consider bears a strong analogy to the classical work by A. Einstein (cf. [4]) where the gravitational field equations for empty space are linearized about the (flat) Minkowski metric. The following table lists objects in pseudohermitian geometry on the 3-dimensional Heisenberg group $\mathbb{H}_1$ and their analogs in general relativity theory.

$\mathbb{H}_1$	$\mathbb{R}^4$
$\theta_0 = dt + i\,(z\,d\overline{z} - \overline{z}\,dz)$ canonical contact form on $\mathbb{H}_1$	$g_0 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2$ Minkowski metric on $\mathbb{R}^4$
∇^{θ_ϵ} Tanaka–Webster connection of $\theta_\epsilon = \theta_0 + \epsilon\,\theta$	∇^{g_ϵ} Levi-Civita connection of $g_\epsilon = g_0 + \epsilon\,h$
Geodesics equations for ∇^{θ_ϵ}	Geodesic motion equations of a particle in the gravitational field g_ϵ
$R_{1\overline{1}} = 0$ pseudo-Einstein equations	$\mathrm{Ric}(g) = 0$ gravitational field equations for empty space
Δ_b sublaplacian of $(\mathbb{H}_1, \theta_0)$	Δ_0 Laplacian of $\mathbb{R}^3$
$\Delta_b u - 4|Lu|^2 = 0$ Linearized pseudo-Einstein equation	$\Delta_0 u = 0$ Linearized gravitational field equation
$\epsilon\,e^{2u}$ with $\epsilon \ll 1$ and $u(x) \to -\infty$ for $|x| \to +\infty$	weak field $\epsilon\,h$ decaying to zero at space infinity

At the present stage of development of subelliptic theory (cf. e.g. [5] and [6]) a treatment of Dirichlet problem on external domains, with conditions at infinity, is missing within the mathematical literature. Therefore, to pursue the analogy in the table above (and produce pseudohermitian analogs to gravitational fields decaying to zero far from the gravitating body) we derive an explicit solution u to the linearized pseudo-Einstein equations (5) such that $u(x) \to -\infty$ as $|x| \to +\infty$. Here $|x| = \left(|z|^4 + t^2\right)^{1/4}$ is the Heisenberg norm of $x = (z, t) \in \mathbb{H}_1$. The solution turns out to be singular at $x = 0$ yet locally integrable on $\mathbb{H}_1$ and then a weak solution to (5) in an appropriate manner (cf. our § 6).

To build a 3-dimensional pseudohermitian analog to general relativity theory one may require $R_{1\overline{1}} = 0$ to describe gravity in free space. A finding in this paper is that, while $R_{1\overline{1}} = 0$ would be *a priori* a postulate, its linearized counterpart (5) may be shown to be part of the Euler–Lagrange system $\delta\, S_D(\theta, u) = 0$ for gravity coupled with a sigma model

$$S_D\,(\theta, u) = \frac{1}{2} \int_D \left[-\frac{R}{2} + \frac{1}{\lambda^2}\, F^{\mu\nu} u_{|\mu} u_{|\nu} e^{4u} \right] d\,\mathrm{vol}(F_\theta)$$

on the total space $\mathfrak{M}$ (with $D \subset\subset \mathfrak{M}$) of the canonical circle bundle $S^1 \to \mathfrak{M} \to \mathbb{H}_1$ (cf. [1]), thus relating the tentative 3-dimensional gravity theory based on pseudohermitian geometry to a well established physical theory (cf. [7]).

Base maps $\phi : \mathbb{H}_1 \to N$ associated to S^1 invariant wave maps $\Phi : \mathfrak{M} \to N$ [from the Fefferman space–time $(\mathfrak{M}, F_{\theta_0})$ into a Riemannian manifold N] where previously shown (cf. [8]) to be subelliptic harmonic (in the sense of [3]). Consequently, a finding in this paper is that mathematical analysis of existence and regularity of weak solutions to (5) may be performed by recognizing (5) as the subelliptic harmonic maps equation for maps from the pseudohermitian manifold $(\mathbb{H}_1, \theta_0)$ into the Riemannian manifold $N = (\mathbb{R}, \; e^{4t}\, dt \otimes dt)$.

2. Lewy operator, canonical contact structure

The *Lewy operator* is the first order differential operator

$$\overline{L} = \partial/\partial\overline{z} - iz\,\partial/\partial t$$

discovered by H. Lewy (cf. e.g. [1]) in connection with the boundary behavior of holomorphic functions on the Siegel domain $\Omega_2 = \{(z,w) \in \mathbb{C}^2 : \mathrm{Im}(w) > |z|^2\}$. Indeed if $F \in \mathcal{O}(\Omega_2) \cap C^1(\overline{\Omega}_2)$ is a holomorphic function C^1 up to the boundary then its boundary values $f = F|_{\partial\Omega}$ obey to $\overline{L}(f \circ H) = 0$ where $H : \mathbb{H}_1 \to \partial\Omega_2$ is the C^∞ diffeomorphism $H(z,t) = (z,\ t + i|z|^2)$.

In a bundle theoretic recast (cf. [9]) $\mathbb{H}_1$ carries the natural left invariant *CR structure* $T_{1,0}(\mathbb{H}_1) = \mathbb{C}L$ spanned by L. The boundary of the Siegel domain inherits a CR structure $T_{1,0}(\partial\Omega_2) = [T(\partial\Omega_2) \otimes \mathbb{C}] \cap T^{1,0}(\mathbb{C}^2)$ as a real hypersurface in $\mathbb{C}^2$, so that H is a *CR isomorphism* i.e. a C^∞ diffeomorphism and $(d_xH)T_{1,0}(\mathbb{H}_1)_x = T_{1,0}(\partial\Omega_2)_{H(x)}$ for any $x \in \mathbb{H}_1$. Here $T^{1,0}(\mathbb{C}^2)$ is the holomorphic tangent bundle over $\mathbb{C}^2$ i.e. the span of $\{\partial/\partial z,\ \partial/\partial w\}$. The equations

$$\overline{L}(f) = 0 \tag{6}$$

are the *tangential Cauchy–Riemann equations*. A C^1 solution to (6) is a *CR function*. For instance $\psi(z,t) = |z|^2 - it$ is a CR function (C^∞ on $\mathbb{H}_1$). Let $\mathrm{Aut}_{\mathrm{CR}}(\mathbb{H}_1)$ consist of all CR isomorphisms of $\mathbb{H}_1$ into itself. *CR invariants* are $\mathrm{Aut}_{\mathrm{CR}}(\mathbb{H}_1)$-invariant geometric objects on $\mathbb{H}_1$. As shown by S.M. Webster (cf. [1]) CR invariants may be computed in terms of pseudohermitian invariants. Indeed we may consider the real 1-form $\theta_0 \in \Omega^1(\mathbb{H}_1)$ given by

$$\theta_0 = dt + i\,(z\,d\overline{z} - \overline{z}\,dz)$$

so that $\theta_0 \wedge d\theta_0$ is a volume from on $\mathbb{H}_1$ i.e. θ_0 is a *contact form*. Also θ_0 is a *pseudohermitian structure* on $\mathbb{H}_1$ i.e. a nowhere vanishing C^∞ section in the conormal bundle

$$H(\mathbb{H}_1)^\perp_x = \left\{\omega \in T^*_x(\mathbb{H}_1) : \mathrm{Ker}(\omega) \supset H(\mathbb{H}_1)_x\right\}, \quad x \in \mathbb{H}_1,$$

where $H(\mathbb{H}_1) = \mathrm{Re}\left\{T_{1,0}(\mathbb{H}_1) \oplus T_{0,1}(\mathbb{H}_1)\right\}$ (the *Levi distribution*).

As $H(\mathbb{H}_1)^\perp \to \mathbb{H}_1$ is a real line bundle, any other pseudohermitian structure $\theta \in C^\infty(H(\mathbb{H}_1)^\perp)$ is given by $\theta = \lambda\,\theta_0$ for some C^∞ function $\lambda : \mathbb{H}_1 \to \mathbb{R} \setminus \{0\}$. The *Levi form* is

$$G_\theta(X,Y) = (d\theta)(X,JY), \quad X,Y \in H(\mathbb{H}_1),$$

where J is the complex structure along $H(\mathbb{H}_1)$ i.e. $J(Z+\overline{Z}) = i(Z-\overline{Z})$ for any $Z \in T_{1,0}(\mathbb{H}_1)$. Then $G_\theta = \lambda\,G_{\theta_0}$ and $G_{\theta_0}(L,\overline{L}) = 1$. In particular G_{θ_0} is positive definite i.e. the CR structure $T_{1,0}(\mathbb{H}_1)$ is *strictly pseudoconvex*. The *Reeb vector* of $(\mathbb{H}_1,\theta)$ is the globally defined, nowhere zero, tangent vector field $\xi \in \mathfrak{X}(\mathbb{H}_1)$ transverse to the Levi distribution $H(\mathbb{H}_1)$, uniquely determined by

$$\theta(\xi) = 1, \xi \rfloor d\theta = 0.$$

The Reeb vector of $(\mathbb{H}_1,\theta_0)$ is $\xi_0 = \partial/\partial t$. Let g_θ be the *Webster metric* i.e. the Riemannian metric on $\mathbb{H}_1$ determined by

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,\xi) = 0, \quad g_\theta(\xi,\xi) = 1,$$

for any $X,Y \in H(\mathbb{H}_1)$. The *Tanaka–Webster connection* of $(\mathbb{H}_1,\theta)$ is the unique linear connection $\nabla = \nabla^\theta$ on $\mathbb{H}_1$ such that (i) $H(\mathbb{H}_1)$ is parallel with respect to ∇ i.e. $X \in H(\mathbb{H}_1) \Longrightarrow \nabla_V X \in H(\mathbb{H}_1)$ for any $V \in \mathfrak{X}(\mathbb{H}_1)$, (ii) $\nabla J = 0$, $\nabla g_\theta = 0$, (iii) the torsion T_∇ of ∇ is *pure* i.e.

$$\tau \circ J + J \circ \tau = 0,$$

$$T_\nabla(Z,W) = 0, \quad T_\nabla(Z,\overline{W}) = 2i\,G_\theta(Z,\overline{W})\,\xi,$$

for any $Z,W \in T_{1,0}(\mathbb{H}_1)$. Here $\tau(V) = T_\nabla(\xi,V)$ (the *pseudohermitian torsion*). Cf. [1], p. 25. Let R^∇ be the curvature tensor field of ∇ and let us set

$$R_A{}^D{}_{BC}T_D = R^\nabla(T_B,T_C)T_A, \quad A,B,C,\ldots \in \{1,\ \overline{1},\ 0\},$$

$$T_1 = L, \quad T_{\overline{1}} = \overline{L}, \quad T_0 = \xi,$$

$$R_{1\overline{1}} = R_1{}^1{}_{1\overline{1}}, \quad \rho = g^{1\overline{1}}\,R_{1\overline{1}},$$

$$g_{1\overline{1}} = G_\theta(L,\ \overline{L}), \quad g^{1\overline{1}} = 1/g_{1\overline{1}}\,.$$

Here $g_{1\overline{1}}$, $R_{1\overline{1}}$ and ρ are respectively the *Levi invariant*, the *pseudohermitian Ricci tensor*, and the *pseudohermitian scalar curvature* of $(\mathbb{H}_1,\theta)$.

3. Linearized pseudo-Einstein equation

Let θ be a positively oriented contact form on $\mathbb{H}_1$ and let us set

$$\theta_\epsilon = \theta_0 + \epsilon\, \theta, \quad \epsilon \ll 1. \tag{7}$$

If $g_{1\bar{1}}(\epsilon)$ is the Levi invariant of $(\mathbb{H}_1, \theta_\epsilon)$ then $g_{1\bar{1}}(\epsilon) = 1 + \epsilon\, g_{1\bar{1}}$ and [by dropping terms of order $O(\epsilon^2)$]

$$g^{1\bar{1}}(\epsilon) \approx 1 - \epsilon\, g_{1\bar{1}} = 1 - \epsilon\, e^{2u},$$

where $u \in C^\infty(\mathbb{H}_1, \mathbb{R})$ is given by $\theta = e^{2u}\,\theta_0$. Let ∇^ϵ be the Tanaka–Webster connection of $(\mathbb{H}_1, \theta_\epsilon)$ and $\Gamma^A_{BC}(\epsilon)$ its coefficients with respect to the frame $\{L,\ \bar{L},\ \xi_\epsilon\}$ where $\xi_\epsilon \in \mathfrak{X}(\mathbb{H}_1)$ is the Reeb vector corresponding to θ_ϵ. As

$$\theta_\epsilon = e^{2u_\epsilon}\,\theta_0, \quad u_\epsilon = \log\sqrt{1 + \epsilon\, e^{2u}},$$

the Reeb vectors ξ_ϵ and ξ are related by

$$\xi_\epsilon = e^{-2u_\epsilon}\left\{\xi_0 + i\,L(u_\epsilon)\bar{L} - i\,\bar{L}(u_\epsilon)\,L\right\}.$$

or

$$\xi_\epsilon = \left(1 - \epsilon\, e^{2u}\right)\, \xi_0 + i\,\epsilon\, e^{2u}\left\{L(u)\bar{L} - \bar{L}(u)\,L\right\} + O(\epsilon^2). \tag{8}$$

In particular the following commutation formulas hold

$$\left[L, \bar{L}\right] = -2i\,\xi_0, \tag{9}$$

$$[L, \xi_\epsilon] = -i\epsilon\, e^{2u}\left\{2|Lu|^2 + L\bar{L}u\right\}L + i\epsilon\, e^{2u}\left\{2L(u)^2 + L^2u\right\}\bar{L}. \tag{10}$$

By a result in [1]

$$\Gamma^1_{11}(\epsilon) = g^{1\bar{1}}(\epsilon)\left\{L\left(g_{1\bar{1}}(\epsilon)\right) - g_{\theta_\epsilon}\left(L, \left[L, \bar{L}\right]\right)\right\}, \tag{11}$$

$$\Gamma^1_{\bar{1}1}(\epsilon) = g^{1\bar{1}}(\epsilon)\, g_{\theta_\epsilon}\left(\left[\bar{L}, L\right], \bar{L}\right), \tag{12}$$

$$\Gamma^1_{01}(\epsilon) = g^{1\bar{1}}(\epsilon)\, g_{\theta_\epsilon}\left([\xi_\epsilon, L], \bar{L}\right). \tag{13}$$

As $g_{\theta_\epsilon}(\xi_\epsilon, L) = 0$ and $g_{\theta_\epsilon}(\xi_\epsilon, \xi_\epsilon) = 1$, formula (8) implies

$$g_{\theta_\epsilon}(\xi_0, L) = -i\epsilon\, e^{2u}\, L(u).$$

Consequently [by (9)–(13)]

$$\Gamma^1_{11}(\epsilon) = 4\epsilon\, e^{2u}\, L(u), \tag{14}$$

$$\Gamma^1_{\bar{1}1}(\epsilon) = -2\epsilon\, e^{2u}\, \bar{L}(u), \tag{15}$$

$$\Gamma^1_{01}(\epsilon) = i\epsilon\, e^{2u}\, \left\{2|Lu|^2 + L\bar{L}u\right\}. \tag{16}$$

In particular $\Gamma^A_{BC} = O(\epsilon)$. Let R^{∇^ϵ} be the curvature tensor field of ∇^ϵ and let $R_1{}^1{}_{1\bar{1}}(\epsilon)$ be its components with respect to $\{L,\ \bar{L},\ \xi_\epsilon\}$ so that

$$R_{1\bar{1}}(\epsilon) = R_1{}^1{}_{1\bar{1}}(\epsilon) = L\left(\Gamma^1_{\bar{1}1}(\epsilon)\right) - \bar{L}\left(\Gamma^1_{11}(\epsilon)\right) + 2i\,\Gamma^1_{01}(\epsilon). \tag{17}$$

Substitution from (14)–(16) leads to

$$R_{1\bar{1}}(\epsilon) = 4\epsilon\, e^{2u}\left(\Delta_b u - 4|Lu|^2\right). \tag{18}$$

Here Δ_b is the *sublaplacian* of $(\mathbb{H}_1, \theta_0)$ i.e. the formally self-adjoint second order differential operator given by

$$\Delta_b u = -\mathrm{div}\left(\nabla^H u\right), \quad u \in C^2(\mathbb{H}_1).$$

Cf. e.g. Definition 2.1 in [1], p. 111. If $\Psi = \theta_0 \wedge d\theta_0$ then let $\mathrm{div}(V)$ be given by $\mathcal{L}_V\Psi = \mathrm{div}(V)\,\Psi$ for any $V \in \mathfrak{X}(\mathbb{H}_1)$, where $\mathcal{L}_V$ is the Lie derivative. With respect to the frame $\{E_a : a \in \{1, 2\}\}$ defined by $E_1 = (1/\sqrt{2})X$ and $E_2 = (1/\sqrt{2})\,Y$

$$\Delta_b u = -\sum_{a=1}^{2}\left\{E_a(E_a u) - (\nabla^0_{E_a}E_a)u\right\} = -\frac{1}{2}\left\{X^2u + Y^2u\right\}$$

hence Δ_b is degenerate elliptic (in the sense of J.M. Bony, [10]). Also $L = \frac{1}{2}(X - iY)$ yields

$$\Delta_b u = -\left(L\bar{L} + \bar{L}L\right)u, \quad u \in C^2(\mathbb{H}_1). \tag{19}$$

In particular (by a result in [11], cf. also Lemma 2.1 in [1], p. 114) Δ_b is *subelliptic* of order 1/2 i.e. for any point $x \in \mathbb{H}_1$ there is an open neighborhood $U \subset \mathbb{H}_1$ of x such that

$$\|u\|^2_{\frac{1}{2}} \leq C\left((\Delta_b u, u)_{L^2} + \|u\|^2_{L^2} \right)$$

for any $u \in C^\infty_0(U)$, where $\|u\|_r = \left(\int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^r \left|\hat{u}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}$ is the Sobolev norm of order r. Consequently (by a result in [12]) Δ_b is *hypoelliptic* i.e. if u is a distribution solution to $\Delta_b u = f$ with f smooth then u is smooth. The pseudo-Einstein equation $R_{1\bar{1}}(\epsilon) = 0$ i.e.

$$\Delta_b u - 4|Lu|^2 = 0 \tag{20}$$

is then a second order nonlinear subelliptic PDE for which we wish to solve both the Dirichlet problem on a bounded domain $\Omega \subset \mathbb{H}_1$ and determine solutions u such that $u(x) \to -\infty$ as $|x| \to +\infty$. Here $|x| = \left(|z|^4 + t^2\right)^{1/4}$ (the Heisenberg norm) for any $x = (z, t) \in \mathbb{H}_1$.

4. Subelliptic harmonic maps of $(\mathbb{H}_1, \theta_0)$ into $(\mathbb{R}, e^{4t}\, dt^2)$

Our main result in this section is

Theorem 1. *Let $\Omega \subset \mathbb{H}_1$ be a bounded domain and $\Phi \in M^1(\Omega)$ such that $\Phi(\overline{\Omega})$ is a bounded set. There is $u \in C^\infty(\Omega)$ such that $u = \Phi$ on $\partial\Omega$ and $\left(1 + \epsilon\, e^{2u}\right)\theta_0$ is a solution to $R_{1\bar{1}} = 0$ in Ω to order $O(\epsilon)$.*

The boundary condition $u = \Phi$ on $\partial\Omega$ is understood as $u - \Phi \in M^1_0(\Omega)$. The Folland–Stein (Sobolev type) spaces $M^1(\Omega)$ and $M^1_0(\Omega)$ (cf. [13]) will be introduced shortly. The proof of Theorem 1 relies on identifying (20) as the subelliptic harmonic map equation, for maps from $(\mathbb{H}_1, \theta_0)$ into $\mathbb{R}$ carrying a particular (flat) Riemannian metric. Indeed let us endow $\mathbb{R}$ with the Riemannian metric $h = e^{4t}\, dt \otimes dt$. The energy of a C^2 map $u : \mathbb{H}_1 \to \mathbb{R}$ of the pseudohermitian manifold $(\mathbb{H}_1, \theta_0)$ into the Riemannian manifold $(\mathbb{R}, h)$ is (cf. [8])

$$E_\Omega(u) = \frac{1}{2} \int_\Omega \mathrm{Tr}_{G_{\theta_0}} \left(\Pi_H\, u^* h \right)\, \Psi_0$$

where $\Omega \subset \mathbb{H}_1$ is a relatively compact domain and $\Pi_H B$ denotes the restriction of the bilinear form B to $H(\mathbb{H}_1) \otimes H(\mathbb{H}_1)$. Then

$$\mathrm{Tr}_{G_{\theta_0}} \left(\Pi_H\, u^* h \right) = \sum_{a=1}^{2} (u^* h)(E_a, E_a) = \sum_{a=1}^{2} e^{4u}\, E_a(u)^2,$$
$$\Psi_0 = \theta_0 \wedge d\theta_0 = 4\, dx \wedge dy \wedge dt,$$

so that

$$E_\Omega(u) = \int_\Omega e^{4u} \left\{ X(u)^2 + Y(u)^2 \right\}\, d\mathbf{x}$$

where $d\mathbf{x}$ is the Lebesgue measure on $\mathbb{R}^3$ (compare to (4.5) in [3], p. 4639). A function $u \in C^2(\mathbb{H}_1, \mathbb{R})$ is a *critical point* of E_Ω if

$$\frac{d}{ds}\{E_\Omega\,(u_s)\}_{s=0} = 0$$

for any smooth 1-parameter variation $\{u_s\}_{|s|<\delta} \subset C^2(\mathbb{H}_1, \mathbb{R})$ of u (i.e. $u_0 = u$) supported in Ω (i.e. $\mathrm{Supp}(\partial u_s/\partial s)_{s=0} \subset \Omega$). A function $u \in C^2(\mathbb{H}_1, \mathbb{R})$ is a *subelliptic harmonic map* of $(\mathbb{H}_1, \theta_0)$ into $(\mathbb{R}, h)$ if u is a critical point of E_Ω for any $\Omega \subset\subset \mathbb{H}_1$. A study of the existence and regularity of subelliptic harmonic maps was started in by J. Jost and C-J. Xu (cf. [3]) in a more general context [i.e. for maps from a domain in $\mathbb{R}^N$ carrying a fixed Hörmander system of vector fields, into a complete Riemannian manifold [of bounded (from above) sectional curvature]]. The Euler–Lagrange equations of the variational principle $\delta\, E_\Omega(u) = 0$ are (cf. e.g. (4.6) in [3], p. 4639)

$$-\,\Delta_b u + \sum_{a=1}^{2} \Gamma^1_{11}(u)\, E_a(u)^2 = 0 \tag{21}$$

where Γ^1_{11} are the Christoffel symbols of $h_{11} = e^{4t}$. Next

$$\Gamma^1_{11} = h^{11}\, \Gamma_{111} = \frac{1}{2}\, h^{11}\, \frac{\partial h_{11}}{\partial t} = 2$$

so that (21) is the equation to be solved. Therefore the variational treatment in [3] applies to (20). At this point we may solve the Dirichlet problem

$$\Delta_b u - 4|Lu|^2 = 0 \quad \text{in} \quad \Omega, \qquad u = \Phi \quad \text{on} \quad \partial\Omega. \tag{22}$$

For any vector field $V \in \mathfrak{X}(\mathbb{H}_1)$ let V^* be its *formal adjoint* i.e.

$$\int_{\mathbb{H}_1} V^*(f)\,\varphi\,d\mathbf{x} = \int_{\mathbb{H}_1} f\,V(\varphi)\,d\mathbf{x}$$

for any $f \in C^1(\mathbb{H}_1)$ and $\varphi \in C_0^\infty(\mathbb{H}_1)$. A function $u \in L^1_{\rm loc}(\Omega)$ is *weakly differentiable* along $H(\mathbb{H}_1)$ if for each $V \in \{X, Y\}$ there exist functions $u_V \in L^1_{\rm loc}(\Omega)$ such that

$$\int_\Omega u_V\,\varphi\,d\mathbf{x} = \int_\Omega u\,V^*(\varphi)\,d\mathbf{x},$$

for any $\varphi \in C_0^\infty(\Omega)$. Such u_V are uniquely determined almost everywhere and denoted by $V(u) = u_V$. Let $M^1(\Omega)$ be the space of all $u \in L^2(\Omega)$ admitting weak L^2 derivatives along $H(\mathbb{H}_1)$ i.e. $X(u),\ Y(u) \in L^2(\Omega)$. Then $M^1(\Omega)$ is a separable Hilbert space with the scalar product of associated norm

$$\|u\|^2_{M^1} = \int_\Omega u^2\,d\mathbf{x} + \int_\Omega \left[(Xu)^2 + (Yu)^2\right]\,d\mathbf{x}. \tag{23}$$

Although Eq. (5) is nonlinear, there is a natural concept of weak solution in the Folland–Stein space $M^1(\Omega)$. Let $M^1_0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the norm (23). A function $u \in M^1(\Omega)$ is a *weak solution* to $\Delta_b u - 4|Lu|^2 = 0$ if

$$\int_\Omega \{X(u)X(\varphi) + Y(u)Y(\varphi)\}\,d\mathbf{x} = 2\int_\Omega \left\{X(u)^2 + Y(u)^2\right\}\varphi\,d\mathbf{x} \tag{24}$$

for any $\varphi \in M^1_0(\Omega) \cap L^\infty(\Omega)$. Since $V^* = -V$ for each $V \in \{X, Y\}$ one has

$$\Delta_b u = \frac{1}{2}\left\{X^*\,X\,u + Y^*\,Y\,u\right\}.$$

Also $|Lu|^2 = \frac{1}{4}\left\{X(u)^2 + Y(u)^2\right\}$ hence any weak solution of class C^2 is [integrating by parts in (24)] a strong solution to (20).

To solve the Dirichlet problem (22) we rely on a result by J. Jost and C-J. Xu, [3]. Precisely let S be a complete m-dimensional Riemannian manifold of sectional curvature $\mathrm{Sect}(S) \le \kappa^2$ for some $\kappa > 0$, which may be covered by a single coordinate chart $\chi = \left(t^1, \ldots, t^m\right) : S \to \mathbb{R}^m$, so that the Sobolev space $M^1(\Omega, S)$ is unambiguously defined i.e. it consists of all maps $u : \Omega \to S$ such that $t^j \circ u \in M^1(\Omega)$ for any $1 \le j \le m$. Let $\Phi \in M^1(\Omega, S) \cap C(\overline{\Omega}, S)$ be a map such that $\Phi(\overline{\Omega}) \subset B(p, \mu)$ for some $p \in S$ and some $0 < \mu < \min\{\pi/(2\kappa), i(p)\}$ where $i(p)$ is the injectivity radius of p. Also $B(p, \mu)$ is the metric ball of center p and radius μ [with respect to the distance function associated to the given Riemannian metric on S]. Then (cf. Main Theorem in [3], p. 4639) there is a unique map $u \in M^1(\Omega, S) \cap L^\infty(\Omega, S)$ with $u|_{\partial\Omega} = \Phi, u(\overline{\Omega}) \subset B(p, \mu)$, minimizing E_Ω among all such maps, and this map u is a weak solution to (20). Also u possesses the same regularity properties at the interior of Ω as solutions to linear hypoelliptic systems. Moreover if $\partial\Omega$ is smooth and noncharacteristic for $\{X, Y\}$, and Φ is smooth, then the corresponding boundary regularity of u follows.

We recall that the injectivity radius of $p \in S$ is the largest radius of a ball (in the tangent space $T_p(S)$, centered at 0) on which the exponential map $\exp_p$ is injective. The geodesics equation for the Riemannian metric $e^{4t}\,dt \otimes dt$ on $\mathbb{R}$ is

$$\gamma''(s) + \Gamma^1_{11}\,\gamma'(s)^2 = 0$$

i.e. the Bernoulli equation $\gamma'' + 2\gamma'^2 = 0$ hence $\left\{\gamma^\pm_{a,b} : a, b \in \mathbb{R}\right\}$ with

$$\gamma^+_{a,b} : \left(-\frac{a}{2}, +\infty\right) \to \mathbb{R}, \quad \gamma^-_{a,b} : \left(-\infty, -\frac{a}{2}\right) \to \mathbb{R},$$
$$\gamma^\pm_{a,b}(s) = \log\left[\pm(2s + a)\right] + b,$$

is the family of all geodesics of $(\mathbb{R},\ e^{4t}dt^2)$. Any such geodesic may be reparametrized to be defined on the whole of $\mathbb{R}$ hence $(\mathbb{R},\ e^{4t}dt^2)$ is complete. Moreover the injectivity radius of any $t \in \mathbb{R}$ is $i(t) = +\infty$. For instance if $t \in \mathbb{R}$ and $v \in T_t(\mathbb{R})$ is a tangent vector $v = \lambda\,(\partial/\partial t)_t$ with $\lambda > 0$ then

$$\gamma^+_{2/\lambda, t - \log(2/\lambda)} : \left(-\frac{1}{\lambda}, +\infty\right) \to \mathbb{R}$$

is the unique geodesic of initial data (t, v). Consequently

$$\exp_t v = \gamma^+_{2/\lambda, t-\log(2/\lambda)}(1) = t + \log\lambda$$

which is well defined and remains injective for any $\lambda > 0$. Let $t_0 \in \mathbb{R}$ and let $\mu > 0$ be arbitrarily chosen [as $\mathrm{Sect}(\mathbb{R}, e^{4t}\,dt^2) = 0$ any constant $\kappa > 0$ will do] and let us set $R = \log\sqrt{1 + 2\mu e^{-2t_0}}$. Also let $\Phi \in M^1(\Omega)$ such that $\Phi(\overline{\Omega}) \subset I_R(t_0)$ where $I_R(t_0) = \{t \in \mathbb{R} : |t - t_0| < R\}$. The distance function associated to the Riemannian metric $e^{4t}dt^2$ is

$$d(s, t) = \frac{1}{2}\left\{e^{2t} - e^{2s}\right\}, \quad t > s,$$

hence $I_R(t_0) = B(t_0, \mu)$. By the result of J. Jost and C-J. Xu mentioned above there exists a weak solution $u \in M^1(\Omega)$ to (20) such that

$$u - \Phi \in M^1_0(\Omega) \tag{25}$$

and $u(\overline{\Omega}) \subset I_R(t_0)$. Moreover (by Theorem 2 in [3], p. 4644) the weak solution u to the Dirichlet problem (22) is continuous i.e. $u \in C(\Omega)$. Finally we need to apply a result by C-J. Xu and C. Zuily (cf. [14]). Let us consider the PDE

$$\sum_{j,k=1}^{2} X_j^* \left(a^{jk}(x, u(x)) X_k(u)\right) = f(x,\ u(x),\ Xu(x),\ Yu(x)) \tag{26}$$

where $X_1 = X, X_2 = Y$ and $\left[a^{jk}(x, u)\right]_{1\le j,k\le 2}$ is positive definite. By a result in [14] (cf. Theorem 1.1, p. 323) if a^{jk} and f are C^∞ functions and

$$|f(x, u, \mathbf{p})| \le a\|\mathbf{p}\|^2 + b, \quad (x, u, \mathbf{p}) \in \Omega\times\mathbb{R}\times\mathbb{R}^2, \tag{27}$$

for some $a, b \in \mathbb{R}$, then any weak solution $u \in M^1(\Omega)$ to (26) which is continuous in Ω is also C^∞ in Ω. Eq. (20) may be written as

$$X^*(Xu) + Y^*(Yu) = f(x, u, Xu, Yu)$$

with $f(x, u, \mathbf{p}) = 2\|\mathbf{p}\|^2$ hence (20) enters the class of quasilinear PDEs (26) [with $a^{jk} = \delta^{jk}$ and f satisfying (27) with $a = 2$ and $b = 0$]. Hence the solution u to (22), (25) is smooth in Ω.

5. Relationship to Einstein's gravity coupled to a sigma model

Equation (20) may also be derived from Einstein's gravity on the total space of the canonical circle bundle $S^1 \to \mathfrak{M} \to \mathbb{H}_1$, coupled to a nonlinear sigma-model. To make this statement precise, we recall that a complex-valued differential 2-form ω on $\mathbb{H}_1$ is a $(2, 0)$-*form* (or a form of *type* $(2, 0)$) if $T_{0,1}(\mathbb{H}_1) \rfloor \omega = 0$. That is, for any pseudohermitian structure θ on $\mathbb{H}_1$ a $(2, 0)$-form may be represented as

$$\omega = f\, dz \wedge \theta, \quad f \in C^\infty(\mathbb{H}_1, \mathbb{C}).$$

Let $\Lambda^{2,0}(\mathbb{H}_1) \to \mathbb{H}_1$ be the relevant bundle [a complex line bundle, such that each $(2, 0)$-form is a C^∞ section in $\Lambda^{2,0}(\mathbb{H}_1)$]. There is a natural free action of $\mathbb{R}_+ = \mathrm{GL}^+(1, \mathbb{R})$ (the positive reals) on $\Lambda^{(2,0)}(\mathbb{H}_1) \setminus \{\text{zero section}\}$. Let

$$\mathfrak{M} = \left[\Lambda^{2,0}(\mathbb{H}_1) \setminus \{\text{zero section}\}\right] / \mathbb{R}_+$$

be the quotient space and $\pi : \mathfrak{M} \to \mathbb{H}_1$ the projection. The circle S^1 acts freely on $\mathfrak{M}$ so that $\mathfrak{M}$ may be organized as the total space of a principle bundle, referred to as the *canonical circle bundle* over $\mathbb{H}_1$. The Heisenberg group is globally embeddable (as a CR manifold globally CR isomorphic to the boundary of the Siegel domain, cf. our § 1) hence the canonical circle bundle is trivial [i.e. $\mathfrak{M} \approx \mathbb{H}_1\times S^1$]. Let θ be a positively oriented contact from on $\mathbb{H}_1$ and let us consider the real 1-form

$$\sigma = \frac{1}{3}\left\{d\gamma + \pi^*\left(i\,\omega_1{}^1 - \frac{i}{2}\, g^{1\bar 1}\, dg_{1\bar 1} - \frac{\rho}{8}\,\theta\right)\right\}.$$

Here γ is a local fiber coordinate on $\mathfrak{M}$. By a result of C.R. Graham (cf. [15]) σ is a connection 1-form on the principal bundle $S^1 \to \mathfrak{M} \to \mathbb{H}_1$. Let $\tilde G_\theta$ be defined by $\tilde G_\theta = G_\theta$ on $H(\mathbb{H}_1) \otimes H(\mathbb{H}_1)$ and $\tilde G_\theta(\xi, V) = 0$ for any $V \in T(\mathbb{H}_1)$. By a result of J.M. Lee (cf. [16])

$$F_\theta = \pi^*\tilde G_\theta + 2\,(\pi^*\theta) \odot \sigma$$

is a Lorentzian metric on $\mathfrak{M}$ [the *Fefferman metric* of $(\mathbb{H}_1, \theta)$]. For each tangent vector field $V \in \mathfrak{X}(\mathbb{H}_1)$ let $V^\uparrow \in \mathfrak{X}(\mathfrak{M})$ be the horizontal lift of V with respect to σ i.e.

$$V^\uparrow \in \mathrm{Ker}(\sigma), \quad \pi_* V^\uparrow = V \circ \pi .$$

If S is the tangent to the S^1 action then $X_\theta = \xi^\uparrow - S$ is a globally defined timelike vector field on $\mathfrak{M}$ [a *time orientation* of the Lorentzian manifold $(C(\mathbb{H}_1), F_\theta)$] so that $(\mathfrak{M},\ F_\theta, X_\theta)$ is a space–time. By a result in [16] there is no $\Lambda \in \mathbb{R}$ such that $R_{\mu\nu} - \Lambda F_{\mu\nu} = 0$. Here

$$F_{\mu\nu} = L_\theta(\partial_\mu, \partial_\nu), \quad \partial_\nu = \partial/\partial x^\nu,$$

and $R_{\mu\nu}$ is the Ricci tensor of $(\mathfrak{M},\ F_\theta)$. Components are meant with respect to the local coordinate system

$$(x^\nu)_{0\le\mu,\nu\le 3} \equiv (\tilde x^j, \gamma), \quad \tilde x^j = x^j \circ \pi ,$$

where $(x^j)_{1\le j\le 3} \equiv (x,\ y,\ t)$, $z = x + iy$, are (globally defined) coordinates on $\mathbb{H}_1$ and $x^0 \equiv \gamma$. Let us consider the action

$$S_D\,(\theta, u) = \frac{1}{2}\int_D \left[-\frac{R}{2} + \frac{1}{\lambda^2}\, F^{\mu\nu} u_{|\mu} u_{|\nu} e^{4u}\right] d\,\mathrm{vol}(F_\theta)$$

where $D \subset \mathfrak{M}$ is a relatively compact domain, $R = F^{\mu\nu} R_{\mu\nu}$ is the scalar curvature of F_θ, and λ^2 is a constant (cf. [7]). Here $u : \mathbb{H}_1 \to \mathbb{R}$ is a C^∞ function (one does not distinguish notationally between u and its vertical lift $u \circ \pi$) and $u_{|\mu} = \partial u / \partial x^\mu$. Also $[F^{\mu\nu}] = \left[F_{\mu\nu}\right]^{-1}$. By a result in [7] (cf. also [17], p. 111) the Euler–Lagrange equations of the variational principle $\delta S_D(\theta, u) = 0$ are

$$R_{\mu\nu} = \frac{1}{\lambda^2} u_{|\mu} u_{|\nu} e^{4u}, \quad -\Box u + F^{\mu\nu} u_{|\mu} u_{|\nu} \Gamma^1_{11}(u) = 0, \tag{28}$$

where

$$\Box u = -\frac{1}{\sqrt{-G}} (\sqrt{-G}\, F^{\mu\nu}\, u_{|\nu})_{|\mu}, \quad G = \det[F_{\mu\nu}],$$

[the wave operator (on functions) for F_θ] and Γ^1_{11} are the Christoffel symbols of $e^{4t} dt \otimes dt$. As $S^1 \subset \mathrm{Isom}(\mathfrak{M}, F_\theta)$ the wave operator is S^1-invariant so that $\Box$ pushes forward by π i.e. for any $u \in C^\infty(\mathbb{H}_1, \mathbb{R})$ there is a unique C^∞ function $(\pi_* \Box) u : \mathbb{H}_1 \to \mathbb{R}$ whose vertical lift is $\Box(u \circ \pi)$. By a result in [16]

$$\pi_* \Box = (2/3)\, \Delta_b\, .$$

By a result of E. Barletta (cf. e.g. [18])

$$\left[F_{\theta_0}(\partial_\mu, \partial_\nu)\right]^{-1} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & \frac{1}{2} & 0 & y \\ 0 & 0 & \frac{1}{2} & -x \\ 3 & y & -x & 2|z|^2 \end{pmatrix}. \tag{29}$$

Finally for $\theta = \theta_0$ [by (29) and $u_{|0} = 0$] the last Eq. in (28) projects (pushes forward by π) on $\mathbb{H}_1$ to give (20).

6. Solutions with Heisenberg spherical symmetry

Let us look for solutions to (5) of the form $u(x) = f(r)$ where $r = |x|$. We shall make use of the function $\psi(z, t) = |z|^2 - it$ (which is CR i.e. $\overline{L}\psi = 0$). A calculation shows that

$$\overline{L} u = \frac{f'(r)}{2r^3}\, z\, \psi$$

hence

$$\Delta_b u = -\left(L\overline{L} + \overline{L}L\right) u = -\frac{|z|^2}{2r^2}\left[f''(r) + \frac{3}{r} f'(r)\right],$$

$$4\, |Lu|^2 = \frac{f'(r)^2}{r^2}\, |z|^2,$$

and substitution into (5) leads to Bernoulli equation

$$f''(r) + \frac{3}{r} f'(r) = -2 f'(r)^2\, .$$

Let us set $g = 1/f'$ and solve $g' - (3/r) g = 2$ such as to get $g(r) = -r$ and then $f(r) = \log(1/r)$. Then

$$u(x) = \log(1/|x|) \tag{30}$$

is an exact solution to (5) (such that the corresponding perturbation term $\epsilon\theta$ decays to zero as $|x| \to +\infty$). As to the global properties of (30) we show that it is a weak solution to (5) on $\mathbb{H}_1$ in the sense of the following

Theorem 2. *For every $\epsilon > 0$ let us set $\psi_\epsilon(x) = |z|^2 + \epsilon^2 - it$ for any $x = (z, t) \in \mathbb{H}_1$. If*

$$r_\epsilon = |\psi_\epsilon|^{1/2}, \quad u_\epsilon = \log(1/r_\epsilon), \quad v_\epsilon = \Delta_b u_\epsilon - 4|Lu_\epsilon|^2,$$

then $\lim_{\epsilon \to 0^+} u_\epsilon = \log(1/r)$ and $\lim_{\epsilon \to 0^+} v_\epsilon = 0$ pointwise on $\mathbb{H}_1 \setminus \{0\}$. Moreover (i) $u \in L^1_{\mathrm{loc}}(\mathbb{H}_1)$, $u_\epsilon, v_\epsilon \in C^\infty(\mathbb{H}_1) \cap L^1(\mathbb{H}_1)$, (ii) *$u_\epsilon \to u$ as $\epsilon \to 0^+$ in distributional sense. Moreover* (iii) *v_ϵ satisfies*

$$v_\epsilon = \epsilon^{-2}\, v_1 \circ \delta_{1/\epsilon}, \quad \epsilon > 0,$$

where $\delta_s : \mathbb{H}_1 \to \mathbb{H}_1$ is the parabolic dilation $\delta_s(z, t) = (sz,\ s^2 t)$, $s > 0$. Consequently (iv) *$v_\epsilon \to 0$ as $\epsilon \to 0^+$ in distributional sense.*

Proof. One has

$$L\psi_\epsilon = 2\bar{z}, \quad \bar{L}\psi_\epsilon = 0, \tag{31}$$

(in particular ψ_ϵ is a CR function on $\mathbb{H}_1$). Starting from $r_\epsilon^4 = \psi_\epsilon \overline{\psi}_\epsilon$ one derives $Lr_\epsilon = (\bar{z}/2)\, r_\epsilon^{-3}\, \overline{\psi}_\epsilon$ and hence

$$Lu_\epsilon = -\frac{\bar{z}}{2\psi_\epsilon}\,. \tag{32}$$

Let us apply $\bar{L}$ to (32) and use (31) and $Lz = 1$. We obtain $\bar{L}Lu_\epsilon = -(1/2)\,\psi_\epsilon^{-1}$ and hence

$$\Delta_b u_\epsilon = -\left(L\bar{L} + \bar{L}L\right) u_\epsilon = \left(|z|^2 + \epsilon^2\right) r_\epsilon^{-4}\,. \tag{33}$$

Also [by (32)]

$$|Lu_\epsilon| = (1/2)\,|z|\,r_\epsilon^{-2}\,. \tag{34}$$

Finally (33)–(34) yield

$$\Delta_b u_\epsilon - 4|Lu_\epsilon|^2 = \epsilon^2\, r_\epsilon^{-4}\,. \tag{35}$$

Let $K \subset\subset \mathbb{H}_1$ be a compact subset. Then

$$\begin{aligned}\int_K |u(x)|\, d\mathbf{x} &= \int_{K\cap\{r\le 1\}} |u(x)|\, d\mathbf{x} + \int_{K\cap\{r>1\}} |u(x)|\, d\mathbf{x} \\ &= -\int_{K\cap\{r\le 1\}} \log r\, d\mathbf{x} + A\end{aligned}$$

where $A = \int_{K\cap\{r>1\}} \log r\, d\mathbf{x}$. On the other hand (cf. e.g. [1])

$$|x| \le 1 \Longrightarrow \|x\| \le |x| \le \|x\|^{1/2}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^3$, hence

$$\begin{aligned}-\int_{K\cap\{r\le 1\}} \log r\; d\mathbf{x} &= \frac{1}{2}\int_{K\cap\{r\le 1\}} \log r^{-2}\, d\mathbf{x} \\ &\le \frac{1}{2}\int_{K\cap\{\|x\|\le 1\}} \log \|x\|^{-2}\, d\mathbf{x} \\ &\le \frac{1}{2}\int_0^1 d\rho \int_{\|x\|=\rho} \left(\log \rho^{-2}\right)\, d\sigma(x) \\ &= 2\pi \int_0^1 \rho^2 \log \rho^{-2}\, d\rho = 4\pi/9\end{aligned}$$

so that $u \in L^1(K)$. The functions u_ϵ and v_ϵ are continuous and bounded, hence integrable on $\mathbb{H}_1$. We need the following

Lemma 1. *Let $\varphi \in C_0^\infty(\mathbb{H}_1)$ and let us set $K = \text{Supp}(\varphi)$. Let $0 < \delta < 1$ be fixed and let $f \in C(B_1)$ given by $f(z,t) = -|z|^2 + \sqrt{1-t^2}$ where B_R is the Heisenberg ball of center 0 and radius $R > 0$ i.e. $B_R = \{x \in \mathbb{H}_1 : |x| < R\}$. Let $\epsilon_\delta > 0$ be given by $\epsilon_\delta^2 = \inf_{x\in B_\delta} f(x)$. If $g : \mathbb{H}_1 \to \mathbb{R}$ is the function*

$$g(x) = \begin{cases} \dfrac{1}{4}\log\left[\left(\epsilon_\delta^2 + |z|^2\right)^2 + t^2\right], & \text{for } |x| > 1, \\ \dfrac{1}{2}\log\left(1+\epsilon_\delta^2\right) + \log|x|^{-1}, & \text{for } \delta < |x| \le 1, \\ \log|x|^{-1} & \text{for } |x| \le \delta, \end{cases}$$

then $g \in L^1(K)$ and

$$|u_\epsilon \varphi| \le g(x), \quad x \in K, \quad 0 < \epsilon < \epsilon_\delta.$$

In particular

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{H}_1} u_\epsilon\, \varphi\, dx = \int_{\mathbb{H}_1} u\, \varphi\, dx \tag{36}$$

for any $\varphi \in C_0^\infty(\mathbb{H}_1)$.

Proof. One has $|u_\epsilon \varphi| \le \Gamma\ |u_\epsilon|$ where $\Gamma = \sup_K |\varphi|$. We distinguish three cases, as follows. If (I) $|x| > 1$ then r_ϵ^2 is $|x|^4$ plus a positive term of order $O(\epsilon^2)$ hence $r_\epsilon > 1$. It follows that

$$|u_\epsilon(x)| = \log r_\epsilon \le \frac{1}{4} \log\left[\left(\epsilon_\delta^2 + |z|^2\right)^2 + t^2\right] = g(x)$$

for any $0 < \epsilon < \epsilon_\delta$. If (II) $|x| \le \delta$ then $r_\epsilon < 1$ for any $0 < \epsilon < \epsilon_\delta$. Indeed if $|x| < 1$ then the quadratic polynomial $P(x, s) \in \mathfrak{M}[s]$

$$P(x, s) = s^2 + 2|z|^2 s + |x|^4 - 1$$

(whose coefficients are nonnegative continuous functions on $\mathbb{H}_1$) has determinant $4\left(1 - t^2\right) > 0$ and hence two real roots $s_1 < 0$ and $s_2 = f(x) > 0$. Then $r_\epsilon < 1$ is equivalent to $P(x, \epsilon^2) < 0$ holding for $0 < \epsilon^2 < f(x)$ because of $\epsilon < \epsilon_\delta = \left[\inf_{|x|\le\delta} f(x)\right]^{\frac{1}{2}}$. In case (II) one then has

$$|u_\epsilon(x)| = \frac{1}{4}\ \log\left[\left(\epsilon^2 + |z|^2\right)^2 + t^2\right]^{-1} \le \frac{1}{4}\ \log\left(|z|^4 + t^2\right)^{-1} = \frac{1}{4}\ \log|x|^{-4} = g(x).$$

Finally if (III) $\delta < |x| \le 1$ then $|x| \le 1$ yields $r_\epsilon^4 \le \left(\epsilon^2 + 1\right)^2$ hence

$$\begin{aligned} |u_\epsilon(x)| &= \frac{1}{4}\begin{cases} \log r_\epsilon^4 & \text{for } r_\epsilon \ge 1, \\ -\log r_\epsilon^4 & \text{for } r_\epsilon < 1, \end{cases} \\ &\le \frac{1}{4}\begin{cases} 2\ \log\left(1 + \epsilon_\delta^2\right) & \text{for } r_\epsilon \ge 1, \\ \log|x|^{-4} & \text{for } r_\epsilon < 1, \end{cases} \le \frac{1}{2}\ \log\left(1 + \epsilon_\delta^2\right) + \log|x|^{-1} = g(x). \end{aligned}$$

The proof of the integrability of g on $K = \mathrm{Supp}(\varphi)$ is similar to that of u (and hence omitted). Finally (36) follows by applying Lebesgue's dominated convergence theorem.

Let us go back to the proof of Theorem 2. One has

$$v_\epsilon(z, t) = \frac{\epsilon^2}{\left||z|^2 + \epsilon^2 - it\right|^2}, \quad (z, t) \in \mathbb{H}_1,$$

hence

$$v_\epsilon = \epsilon^{-2}\, v_1 \circ \delta_{1/\epsilon}\,. \tag{37}$$

The function v_1 is continuous and bounded on $\mathbb{H}_1$ hence $v_1 \in L^1(\mathbb{H}_1)$. Let us set $C = \int_{\mathbb{H}_1} v_1(x)\, dx$. Then [by applying (37) followed by a change of variables $z' = \epsilon^{-1} t$ and $t' = \epsilon^{-2} t$]

$$\int_{\mathbb{H}_1} v_\epsilon\, d\mathbf{x} = \epsilon^{-2} \int_{\mathbb{H}_1} \left(v_1 \circ \delta_{1/\epsilon}\right)\, d\mathbf{x} = C\epsilon^2\,.$$

Finally

$$\left|\int_{\mathbb{H}_1} v_\epsilon \varphi\, d\mathbf{x}\right| \le \Gamma \int_{\mathbb{H}_1} v_\epsilon\, d\mathbf{x} = \Gamma C \epsilon^2 \to 0$$

as $\epsilon \to 0^+$. □

7. Conclusions and final comments

We linearized the pseudo-Einstein equation $R_{1\bar{1}} = 0$ on the Heisenberg group $\mathbb{H}_1$ about the canonical Tanaka–Webster flat contact structure by setting $\theta_\epsilon = \theta_0 + \epsilon\theta$ ($\epsilon \ll 1$). Once θ_0 is fixed the function space $C^\infty(\mathbb{H}_1, \mathbb{R})$ parametrizes the space of all positively oriented contact forms on $\mathbb{H}_1$ i.e. $\theta = e^{2u}\theta_0$ for some smooth function $u : \mathbb{H}_1 \to \mathbb{R}$. The equation obtained from $R_{1\bar{1}} = 0$ under the above perturbation is of course linear in θ yet the corresponding equation on the parameter space $\Delta_b u - 4|Lu|^2 = 0$ is but quasi-linear. The larger program outlined in [3] is to formally replace the ordinary Laplacian Δ (in quasi-linear elliptic systems of variational origin, such as the harmonic maps system) by a Hörmander operator $H = -\sum_{j=1}^m X_j^* X_j$ associated to a given Hörmander system $\{X_j : 1 \le j \le m\}$ of vector fields (on an open set in $\mathbb{R}^n$) and exploit the fact that, although H is not elliptic, it is at least hypoelliptic (as is Δ). The following findings in the present paper are new:

(i) The principal part in the linearized pseudo-Einstein equations turns out to be the sublaplacian Δ_b [which is the Hörmander operator associated to the Hörmander system $\{X, Y\}$, where $\overline{L} = \frac{1}{2}(X + iY)$ is the Lewy operator] thus relating to the J. Jost and C-J. Xu program (cf. *op. cit.*);

(ii) Equation (5) is recognized as the subelliptic harmonic maps equation for maps u from the pseudohermitian manifold $(\mathbb{H}_1, \theta_0)$ into the Riemannian manifold $(\mathbb{R},\ e^{4t}\, dt^2)$ [cf. [8] where maps of the sort were termed *pseudoharmonic* and whose local manifestations (with respect to a local orthonormal frame of the Levi distribution) are J. Jost and C-J. Xu's subelliptic harmonic maps];

(iii) While the pseudohermitian analog $R_{1\bar{1}} = 0$ to the gravitational field equations is postulated, the associated linearized equation is shown to be the projection on $\mathbb{H}_1$ of

$$-\Box u + F^{\mu\nu} u_{|\mu} u_{|\nu} \Gamma^1_{11}(u) = 0 \tag{38}$$

[where $\Box$ is the wave operator i.e. the Laplace–Beltrami operator associate to the Fefferman metric of $(\mathbb{H}_1, \theta_0)$, a Lorentzian metric living on the total space of the canonical circle bundle over $\mathbb{H}_1$] and (38) is part of the Euler–Lagrange system of the variational principle $\delta\, S_D(\theta, u) = 0$.

(iv) We solve (5) on a bounded region $\Omega \subset \mathbb{H}_1$ by making use of techniques available in subelliptic theory. This is somewhat unsatisfactory for, should one produce a perturbation contact form θ responsible for modeling a weak gravitational field decaying to zero far from the gravitating body, one should solve the Dirichlet problem for Eq. (5) with conditions at infinity i.e. with $u(x) \to -\infty$ as $|x| \to +\infty$. This inadequacy is compensated for by producing an explicit solution u to (5), possessing Heisenberg symmetry and such that $u(x) \to -\infty$ as $|x| \to +\infty$. As it turns out, the solution u is singular at the origin yet locally integrable on $\mathbb{H}_1$ and a weak solution to (5). Thus

$$\left[1 + \frac{\epsilon}{\sqrt{|z|^4 + t^2}}\right] \{dt + i\,(z\, d\bar{z} - \bar{z}\, dz)\}$$

is a solution to $R_{1\bar{1}} = 0$ to order $O(\epsilon)$, decaying to θ_0 at space infinity;

(v) The particular solution to the linearized field equation corresponding to our finding $u = \log(1/r)$ in § 6 is $\theta = |x|^{-2}\theta_0$ which is to some surprise the contact form considered in [19]. Let G_s be the discrete transformation group of $\mathbb{H}_1$ generated by the parabolic dilation $\delta_s : \mathbb{H}_1 \to \mathbb{H}_1$ (with $s > 0$ fixed). By a result in [19], G_s acts freely on $\mathbb{H}_1 \setminus \{0\}$ as a properly discontinuous group of transformations hence the quotient $\mathbb{H}_1(s) \equiv (\mathbb{H}_1 \setminus \{0\})\,/G_s$ is a C^∞ manifold. Also $\mathbb{H}_1(s) \approx \Sigma^2 \times S^1$ (where $\Sigma^2 = \partial B_1$ is the Heisenberg sphere of center 0 and radius 1) so that $\mathbb{H}_1(s)$ is compact. The real 1-form $|x|^{-2}\theta_0 \in \Omega^1(\mathbb{H}_1 \setminus \{0\})$ is G_s invariant, hence descends to a globally defined contact form $\theta(s)$ on $\mathbb{H}_1(s)$. Again by a result in [19], $\theta(s)$ is a pseudo-Einstein contact form with nonzero pseudohermitian scalar curvature and nonzero pseudohermitian torsion (built in [19] as a contact analog to the Boothby metric on a complex Hopf manifold, cf. [20]).

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