On the Moduli of Isotropic and Helical Minimal Immersions between Spheres

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Abstract

DoCarmo-Wallach theory and its subsequent refinements assert the rich abundance of spherical minimal immersions, minimal immersions of round spheres into round spheres. A spherical minimal immersion can be written as a conformal minimal immersion $f : S^m \to S_V$ with domain the Euclidean m-sphere S^m and range the unit sphere S_V of a Euclidean vector space V. Takahashi's theorem then implies that the conformality constant of f can only

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attain discrete values, λ_p/m, where λ_p is the pth eigenvalue of the Laplace operator (acting on functions on S^m), and then the components of f are spherical harmonics of order p on S^m. The dimension and complexity of the moduli $\mathcal{M}_m^p$ of degree p spherical minimal immersions of S^m increase rapidly with p (and m). In this paper we impose the additional condition of (partial) isotropy expressed in terms of the higher fundamental forms of such immersions. In view of Tsukada's rigidity theorem for fully isotropic minimal immersions, this is a convenient condition to reduce the moduli to smaller slices, $\mathcal{M}_m^{p;k}$ for isoptropy of order k, $2 \leq k \leq p$, and, at the same time, to retain some important examples. Sakamoto's study of "helical" minimal immersions gives a transparent geometric characterization of the moduli $\mathcal{M}_m^{p;k}$ (Theorem A) in terms of partial helicality, that is, (universal) constancy of initial sequences of curvatures of the image curves of geodesics under the respective spherical minimal immersions. As shown by the works of DeTurck and Ziller, a rich subclass of spherical minimal immersions is comprised by minimal $SU(2)$-orbits in spheres (of various dimensions). The main result of this paper (Theorem B) gives a full characterization of isotropic $SU(2)$-equivariant spherical minimal immersions of S^3 into the unit sphere of real and complex $SU(2)$-modules. As a specific example and immediate byproduct, we recover a result of Escher and Weingart which asserts that the icosahedral minimal immersion $Ico : S^3 \to S^{12}$ (giving a minimal embedding of the isosahedral manifold S^3/I^* by the binary icosahedral group I^* into S^{12}) is isotropic whereas its tetrahedral and octahedral cousins are not.

1 Preliminaries and Statement of the Results

1.1 Spherical Minimal Immersions and Moduli

$\mathbb{I}$ Minimal isometric immersions of round spheres into round spheres form a rich and subtle class of objects in differential geometry studied by many authors; see [3, 6, 7, 9, 10, 13, 17, 20, 21, 22, 23, 24, 26, 33, 34, 37, 38]; and, for a more complete list, the bibliography at the end of the second author's monograph [29]. Such immersions can be written as $f : S^m_\kappa \to S_V$ of a round m-sphere S^m_κ of (constant) curvature $\kappa > 0$ into the unit sphere S_V of a Euclidean vector space V (where $\kappa = 1$ is suppressed from the notation); or, scaling the domain sphere S^m_κ to unity, as minimal immersions $f : S^m \to S_V$ with *homothety constant* $1/\kappa$. By minimality, the components $\alpha \circ f$, $\alpha \in V^*$ (the dual of V), of f are necessarily eigenfunctions of the Laplacian $\triangle$ of S^m corresponding to the (same) *eigenvalue* $\lambda = m/\kappa$. Setting $\lambda = \lambda_p = p(p + m - 1)$, $p \geq 1$, the pth *eigenvalue*, and $\mathcal{H}^p_m \subset C^\infty(S^m)$, the corresponding *eigenspace* of spherical harmonics of order p on S^m, a homothetic minimal immersion $f : S^m \to S_V$ with homothety constant λ_p/m is called a *spherical minimal immersion of degree* p. (For the standard results recalled here and below, see [29, Appendix 2], or [37] as well as the summary in [34].)

It is well-known that spherical harmonics of order p are precisely the restrictions (to S^m) of harmonic homogeneous polynomials (of $m+1$ variables) of degree p. Thus, algebraically a spherical minimal immersion $f : S^m \to S_V$ of degree p can be viewed as a *harmonic p-homogeneous polynomial map* $\mathbb{R}^{m+1} \to V$ restricted to the ambient unit spheres.

A spherical minimal immersion $f : S^m \to S_V$ is *full* if the image of f in S_V is not contained in any great sphere of S_V, or equivalently, if the image of f in V spans the entire space V. In this case the map $V^* \ni \alpha \mapsto \alpha \circ f \in \mathcal{H}^p_m$ is a linear isomorphism of V^* onto a linear subspace $V_f \subset \mathcal{H}^p_m$, the *space of components of* f. Since V is Euclidean, we have $V \cong V^*$, so that V is naturally isomorphic with V_f.
Restricting to the linear span of the image, a spherical minimal immersion can always be made full.

The special orthogonal group $SO(m+1)$ acts on S^m by isometries, and thereby also on $\mathcal{H}^p_m$ via precomposition (with the inverse). This makes $\mathcal{H}^p_m$ a linear $SO(m+1)$ representation space. In addition, the L^2-scalar product on $\mathcal{H}^p_m$ is $SO(m+1)$-invariant, and with this $\mathcal{H}^p_m$ becomes an (irreducible) $SO(m+1)$-module. (For basic facts on representations of the (special) orthogonal group used here and below, we refer to [2] or the more modern treatment in [11, 14]; see also the summary in [37].)

The archetype of all spherical minimal immersions is the *Dirac delta* map $\delta_{m,p} : S^m \to S_{(\mathcal{H}^p_m)^*}$ defined by evaluating the spherical harmonics in $\mathcal{H}^p_m$ on points of S^m. Here $(\mathcal{H}^p_m)^*$ carries the natural $SO(m+1)$-module structure dual to $\mathcal{H}^p_m$. (Note that the L^2-scalar product on $\mathcal{H}^p_m$ needs to be scaled suitably so that the Dirac delta maps into the unit sphere.) The Dirac delta map is *equivariant* with respect to the action of $SO(m+1)$ on S^m and the $SO(m+1)$-module structure on $(\mathcal{H}^p_m)^*$.
As a homogeneous space $S^m = SO(m+1)/SO(m)$ is isotropy irreducible, that is, the isotropy group $SO(m+1)_o = SO(m) \oplus [1] \cong SO(m)$ at the base point $o = (0, \dots, 0, 1)$, say, acts on the tangent space $T_o(S^m) \cong \mathbb{R}^m$ irreduciby (actually transitively, by the ordinary action of $SO(m)$ on $\mathbb{R}^m$). It follows that $\delta_{m,p}$ is homothetic, hence a spherical minimal immersion of degree p.
With respect to an orthonormal basis $\{\chi_i\}_{i=0}^{n(m,p)} \subset \mathcal{H}^p_m$, $n(m,p) + 1 = \dim \mathcal{H}^p_m$, the Dirac-delta immersion is (classically) expanded as $\delta_{m,p}(x) = \sum_{i=0}^{n(m,p)} \chi_i(x)\chi_i$, $x \in S^m$, and therefore, it can also be defined as the spherical minimal immersion $\delta_{m,p} : S^m \to S^{n(m,p)}$ whose components form an orthonormal basis in $\mathcal{H}^p_m$. With this definition, it is also referred to as the *standard minimal immersion* of S^m of degree p.

Beyond the classical *Veronese immersions* $\delta_{2,p} : S^2 \to S^{2p}$, $p \geq 2$, and various generalizations (for $m \geq 3$ and $p = 2, 3$), it is well-known that there is an enormous variety of spherical minimal immersions. According to the DoCarmo-Wallach theory, for $m \geq 3$ and $p \geq 4$, the set of full spherical minimal immersions $f : S^m \to S_V$ modulo congruence on the ranges can be parametrized by a (non-trivial) compact convex body $\mathcal{M}_m^p$ in a linear subspace $\mathcal{F}_m^p$ of the symmetric square $S^2(\mathcal{H}_m^p)$. The convex body $\mathcal{M}_m^p$ is called the *moduli* for spherical minimal immersions $f : S^m \to S_V$ of degree p. (For the original work of DoCarmo and Wallach, see [9] as well as [37].)

The group $SO(m + 1)$ acts on the set of all spherical minimal immersions by precomposition, and this action naturally carries over to the moduli $\mathcal{M}_m^p$. This latter action, in turn, is the restriction of the $SO(m + 1)$-module structure on $S^2(\mathcal{H}_m^p)$ (extended from that of $\mathcal{H}_m^p$) with $\mathcal{F}_m^p$ being an $SO(m+1)$-submodule. The complexification of $\mathcal{F}_m^p$ decomposes as

$$\mathcal{F}_m^p \otimes_{\mathbb{R}} \mathbb{C} \cong \sum_{(u,v)\in\triangle_2^p;\ u,v \text{ even}} V_{m+1}^{(u,v,0,\ldots,0)}, \tag{1}$$

where $\triangle_2^p \subset \mathbb{R}^2$ is a closed convex triangle with vertices $(4, 4)$, (p, p), $(2p-4, 4)$. Here $V_{m+1}^{(u_1,\ldots,u_d)}$, $d = [(m + 1)/2]$, denotes the complex irreducible $SO(m + 1)$-module with highest weight vector $(u_1, \ldots , u_d)$ relative to the standard maximal torus in $SO(m+1)$. (The case $m = 3$ is special; due to non-self-conjugacy, the symbol $V_3^{(u,v)}$ means the direct sum $V_3^{(u,v)} \oplus V_3^{(u,-v)}$.) Since the dimension of the irreducible components in (1) can be explicitly calculated by the Weyl-dimension formula, say, as a byproduct, we obtain the exact dimension $\dim \mathcal{M}_m^p = \dim \mathcal{F}_m^p$ of the moduli. (The fact that the right-hand side in (1) is a lower bound for $\mathcal{F}_m^p \otimes_{\mathbb{R}} \mathbb{C}$ is the main result of the DoCarmo-Wallach theory. The equality, the so-called exact dimension conjecture of DoCarmo-Wallach, was proved by the second author in [32]; see also [29, Chapter 3], and also a subsequent different proof in [38].)

The first non-trivial case of the 18-dimensional moduli $\mathcal{M}_3^4$ of the quartic spherical minimal immersions of domain S^3 was completely described by Ziller and the second author in [34]. (Note that, prior to the resolution of the exact dimension conjecture, Y. Mutō in [21] verified that $\dim \mathcal{M}_3^4 = 18$ by extensive but explicit computations.)

1.2 Isotropy and Helicality

The dimension as well as the subtlety of the moduli $\mathcal{M}_m^p$ increase rapidly with $m \geq 3$ and $p \geq 4$. To reduce the complexity and to pin down some interesting part of the moduli one needs to impose further geometric restrictions on the spherical minimal immersions. As we will see below, two competing natural geometric properties of spherical minimal immersions are "isotropy" and "helicality."

Let $f : S^m \to S_V$ be a spherical minimal immersion of degree p. For $k = 1, \ldots, p$, let $\beta_k(f)$ denote the *kth fundamental form of* f, and $\mathcal{O}^k_f$ the *kth osculating bundle of* f. (See [37] and also [13].) (For $k = 1$, we set $\beta_1(f) = f_*$, the differential of f, and $\mathcal{O}^1_f = T(S^m)$ regarded as a subbundle of the pull-back $f^*T(S_V)$. For $k \geq 2$, the kth osculating bundle $\mathcal{O}^k_f$ is a subbundle of the normal bundle $\mathcal{N}_f$ of f.) The higher fundamental forms and osculating bundles are defined on a (maximal) open dense set $D_f \subset S^m$. On D_f, the kth fundamental form is a bundle map $\beta_k(f) : S^k(T(S^m)) \to \mathcal{O}^k_f$, which is fibrewise *onto*. The higher fundamental forms are defined inductively as

$$\begin{aligned}\beta_k(f)(X_1, \ldots, X_k) &= (\nabla^\perp_{X_k}\beta_{k-1}(f))(X_1, \ldots, X_{k-1})^{\perp_{k-1}}, \\ & \qquad X_1, \ldots, X_k \in T_x(S^m),\ x \in D^{k-1}_f,\end{aligned} \tag{2}$$

where $\nabla^\perp$ is the natural connection on the normal bundle $\mathcal{N}_f$, and $\perp_{k-1}$ is the orthogonal projection with kernel $\mathcal{O}^0_{f;x} \oplus \mathcal{O}^1_{f;x} \oplus \ldots \oplus \mathcal{O}^{k-1}_{f;x}$ ($\mathcal{O}^0_{f;x} = \mathbb{R} \cdot f(x)$), and D^k_f is the set of points $x \in D^{k-1}_f$ at which the image $\mathcal{O}^k_{f;x}$ of $\beta_k(f)$ has maximal dimension. We set $D_f = \bigcap_{k=0}^p D^k_f$. The largest k for which $\beta_k(f)$ does not vanish (identically) is called the *geometric degree* d_f of f.

Due to equivariance, for the Dirac delta immersion $\delta_{m,p} : S^m \to S_{(\mathcal{H}^p_m)^*}$, at the base point $o = (0, \ldots, 0, 1)$, the branching

$$\mathcal{H}^p_m|_{SO(m)} = \mathcal{H}^0_{m-1} \oplus \mathcal{H}^1_{m-1} \oplus \ldots \oplus \mathcal{H}^p_{m-1}$$

(as $SO(m)$-modules) corresponds (isomorphically) to the decomposition of osculating spaces

$$\mathcal{O}^0_{\delta_{m,p};o} \oplus \mathcal{O}^1_{\delta_{m,p};o} \oplus \ldots \oplus \mathcal{O}^p_{\delta_{m,p};o}.$$

Thus, the geometric degree of $\delta_{m,p}$ is equal to p, and, for any spherical minimal immersion $f : S^m \to S_V$, we also have $d_f \leq d_{\delta_{m,p}} = p$.

A spherical minimal immersion $f : S^m \to S_V$ is called *isotropic of order* k, $2 \leq k \leq p$, if, for $2 \leq l \leq k$, we have

$$\begin{aligned}&\langle \beta_l(f)(X_1, \ldots, X_l), \beta_l(f)(X_{l+1}, \ldots, X_{2l})\rangle \\ &\qquad = \langle \beta_l(\delta_{m,p})(X_1, \ldots, X_l), \beta_l(\delta_{m,p})(X_{l+1}, \ldots, X_{2l})\rangle, \\ &\qquad\qquad X_1, \ldots, X_{2l} \in T_x(S^m),\ x \in D_f.\end{aligned} \tag{3}$$

This condition implies that, for $2 \leq l \leq k$, the osculating bundles $\mathcal{O}^l_f$ of f are isomorphic with those of the Dirac delta immersion $\delta_{m,p}$. In particular, for a spherical minimal immersion $f : S^m \to S_V$ that is isotropic of order k, we have the *lower bound*

$$\dim V \geq \dim(\mathcal{H}^0_{m-1} \oplus \mathcal{H}^1_{m-1} \oplus \ldots \oplus \mathcal{H}^k_{m-1}) = \dim \mathcal{H}^k_m.$$

The moduli $\mathcal{M}_m^{p;k}$ parametrizing the spherical minimal immersions $f : S^m \to S_V$ of degree p that are isotropic of order k is a linear slice of the moduli $\mathcal{M}_m^p$ by an $SO(m+1)$-submodule $\mathcal{F}_m^{p;k} \subset \mathcal{F}_m^p$. We have the decomposition

$$\mathcal{F}_m^{p;k} \otimes_{\mathbb{R}} \mathbb{C} \cong \sum_{(u,v)\in\triangle_{k+1}^p;\ u,v \text{ even}} V_{m+1}^{(u,v,0,\ldots,0)}, \tag{4}$$

where the closed convex triangle $\triangle_k^p \subset \mathbb{R}^2$, $k = 2, 3, \ldots, [p/2]$ has vertices $(2k, 2k)$, (p, p), and $2(p-k), 2k)$. As before, this gives the exact dimension of the moduli: $\dim \mathcal{M}_m^{p;k} = \dim \mathcal{F}_m^{p;k}$. (These results have been proved by Gauchman and the second author, for $m \geq 4$, in [13]; and the case $m = 3$ has been completed in [29].)

We thus have the filtration

$$\mathcal{F}_m^p = \mathcal{F}_{m+1}^{p,1} \supset \mathcal{F}_{m+1}^{p,2} \supset \ldots \supset \mathcal{F}_{m+1}^{p;[p/2]-1},$$

where each term is obtained from the decomposition above by restriction to the respective triangle in the sequence

$$\triangle_2^p = \triangle_3^p \supset \ldots \supset \triangle_{[p/2]}^p,$$

As a byproduct, we obtain the following:

Corollary. *Let $f : S^m \to S_V$ be a spherical minimal immersion of degree p and order of isotropy k. If $p \leq 2k+1$ then f is congruent to the Dirac delta immersion $\delta_{m,p}$.*

In a series of papers [22, 23, 24] Sakamoto introduced and studied the concept of *helical minimal immersions*. The primary motivation for his study was originated in a construction in Besse [1] of a minimal immersion of a strongly harmonic manifold into a sphere which maps geodesics of the domain into curves of the range with (universally) constant curvatures; a property Sakamoto termed "helical." Locally harmonic manifolds in general have received much attention due to Lichnerowicz' conjecture that they are either flat or locally symmeric spaces of rank one. (This is now resolved for compact simply connected Riemannian manifolds by Z. I. Szabo; see [25].)

In [33] Tsukada made an extensive study of *isotropic* spherical minimal immersions $f : S^m \to S_V$ in which he defined *isotropy of order* k by requiring $\|\beta_l(f)(X, X, \ldots, X)\|$ to be *universal constants* Λ_l, $2 \leq l \leq k$, (depending only on m and p) for all unit vectors $X \in T_x(S^m)$, $x \in D_f$. (Most authors use lowercase letters λ_l, $2 \leq l \leq k$, for the *constants of isotropy*. The change to uppercase is to avoid confusion with the eigenvalues of the Laplacian on the domain as in Section 1.1.) This is a special case of (3) above with all arguments equal $X = X_1 = \ldots = X_{2l}$ (and of unit length). In a technical argument [33, Proposition 3.1] he showed that these two concepts of isotropies are the same. Using (a natural extension of) the DoCarmo-Wallach theory expounded above, he then established the right-hand side of (4) *as a lower bound* for $\mathcal{F}_m^{p;k} \otimes_{\mathbb{R}} \mathbb{C}$.

Tsukada's second goal was to prove that, for spherical domains, a helical spherical minimal immersion is *rigid* in the sense that it is congruent to the Dirac delta immersion of the respective degree; see [33, Theorem B]. By developing a formula for the Frenet frame of the image curve $\sigma = f \circ \gamma$ of a geodesic $\gamma : \mathbb{R} \to S^m$ with the curvatures of σ and the higher fundamental forms involving $\beta_k(f)(\sigma', \ldots, \sigma')$, $2 \leq k \leq d_f$, he showed that Sakamoto's helicality and the concept of isotropy are the same [33, Proposition 5.1]. Finally, he used the extended concept of DoCarmo-Wallach rigidity as in the corollary above.

In our present paper, continuing the DoCarmo-Wallach approach beyond Tsukada's rigidity, we will be interested in the geometric characterization of spherical minimal immersions parametrized by the moduli $\mathcal{M}_m^{p;k}$. Implicit in [33, Proposition 5.1], we propose the following concept of "partial helicality" in which we require only an initial sequence of curvatures to be (universal) constants. More precisely, a spherical minimal immersion $f : S^m \to S_V$ of degree p is called *helical up to order* k if, for any arc-length parametrized geodesic $\gamma : \mathbb{R} \to S^m$, the first $k-1$ curvatures of the image curve $\sigma = f \circ \gamma : \mathbb{R} \to S_V$ are non-zero *constants*, and these constants are universal in that they do not depend on the choice of γ but only on m and p. (Recall that the curvatures are obtained by taking higher order covariant derivatives of σ' along with a Gram-Schmidt orthogonalization process. Note also that the universal constants have been determined in [12].)

Now, the defining properties of the moduli $\mathcal{M}_m^{p;k}$ as well as the proof of Proposition 5.1 in [33] (with appropriate modifications) give the following:

Theorem A. *A spherical minimal immersion $f : S^m \to S_V$ of degree p is isotropic of order k if and only if it is helical up to order k. Therefore, the moduli $\mathcal{M}_m^{p;k}$ parametrizes the congruence classes of full spherical minimal immersions $f : S^m \to S_V$ of degree p which are helical up to order k, that is, they map geodesics of S^m to curves whose first $k-1$ curvatures are (non-zero) universal constants depending on p and m only.*

The applications of this theorem are severalfold. First, in the corollary above, "order of isotropy k" can be replaced by "helical up to order k;" and thereby Tsukada's rigidity follows. Second, as noted above, $\dim \mathcal{M}_m^{p;k}$ can be calculated explicitly. In the past helical minimal immersions have only been studied individually, and here we have a precise formula for the dimension of the moduli of such maps. Third, helicality is a much simpler condition than isotropy, therefore, in several instances, this condition can be checked by explicit calculation.

1.3 The Lowest Order Isotropy

The complexity of the condition of isotropy increases rapidly with the order. The lowest order of isotropy, isotropy of order two, has special significance because of the relative simplicity of the formula expressing the first curvature of the image curve of a geodesic under the immersion. In this short section we obtain an simple condition for isotropy of order two of a spherical minimal immersion that will be used subsequently.

For brevity, we will suppress the order, and refer to a spherical minimal immersion of degree p and order of isotropy two simply as an *isotropic spherical minimal immersion* (of degree p).

As noted above, the moduli parametrizing the (congruence classes of) full isotropic spherical minimal immersions is $\mathcal{M}_m^{p;2}$ which, by the corollary above, is non-trivial if and only if $p \geq 6$.

By definition, a spherical minimal immersion $f : S^m \to S_V$ is isotropic (of order two) if $\|\beta(f)(X,X)\|$ is a *universal* constant Λ for all unit vectors $X \in T_x(S^m)$, $x \in S^m$. It is sometimes convenient to specify (or calculate) the actual value of this constant.
It is well-known that, this holds if (and only if) the second fundamental form $\beta(f)$, is *pointwise isotropic*, that is, *for any* $x \in S^m$, $\beta(f)$ is isotropic on the tangent space $T_x(S^m)$ as a symmetric bilinear form in the classical sense (with $\|\beta(f)(X,X)\|$ being independent of the unit vector $X \in T_x(S^m)$). (See for example [33, Proposition 3.1].) Isotropy (at a point) can be conveniently reformulated in terms of the *shape operator* $\mathcal{A}(f)$ of $f : S^m \to S_V$ as

$$\mathcal{A}(f)_{\beta(f)(X,X)}X \wedge X = 0, \quad X \in T_x(S^m),\ x \in S^m. \tag{5}$$

Indeed, for $x \in S^m$, polarizing $\|\beta(f)(X,X)\|^2$, $X \in T_x(S^m)$, we see that $\beta(f)$ is isotropic on $T_x(S^m)$ if and only if

$$\langle \beta(f)(X,X), \beta(f)(X,Y)\rangle = \langle \mathcal{A}_{\beta(f)(X,X)}X, Y\rangle = 0$$

for all $X, Y \in T_x(S^m)$ with $\langle X, Y\rangle = 0$. (See also [23, (2.2)] or [6, Section 2].)

As expected, higher order isotropy is more complex. For completeness, we briefly indicate the formula analogous to (5). Let $X \in T_x(S^m)$, $x \in D_f$, and denote by ζ_k a (locally defined) section of the osculating bundle $\mathcal{O}_f^k$. (We use the notations in Section 1.2, and tacitly assume that we work over $D_f \subset S^m$ so that all osculating bundles are well-defined.) We define T^k by

$$T_X^k(\zeta_{k-1}) = \left(\nabla_X^\perp \zeta_{k-1}\right)^{\mathcal{O}_f^k},$$

where $\nabla^\perp$ is the connection of the normal bundle $\mathcal{N}_f$ and the osculating bundle in the superscript indicates orthogonal projection. By definition, we have

$$\beta_k(f)(X_1, \dots, X_k) = T_{X_1}^k(\beta_{k-1}(f)(X_2, \dots, X_k))$$

for (locally defined) vector fields $X_1, \dots, X_k$ on D_f.
Let S_X^{k-1} be the adjoint of T_X^k (with respect to the bundle metrics on the respective osculating bundles induced by the Riemannian metric on S_V). Clearly, we have

$$S_X^{k-1}(\zeta_k) = -\left(\nabla_X^\perp \zeta_k\right)^{\mathcal{O}_f^{k-1}}.$$

Now, polarizing $\|\beta_k(f)(X, \dots, X)\|^2$ as before, we obtain that, for $x \in S^m$, $\beta_k(f)$ is isotropic on $T_x(S^m)$ if and only if

$$\langle \beta_k(f)(X, \dots, X), \beta_k(f)(X, \dots, X, Y)\rangle = 0$$

whenever $X, Y \in T_x(S^m)$ with $\langle X, Y\rangle = 0$. We now calculate

$$\begin{aligned}
&\langle \beta_k(f)(X, \dots, X), \beta_k(f)(X, \dots, X, Y)\rangle \\
&\qquad = \langle \beta_k(f)(X, \dots, X), T_X^k \beta_{k-1}(f)(X, \dots, X, Y)\rangle \\
&\qquad = \langle S_X^{k-1}\beta_k(f)(X, \dots, X), \beta_{k-1}(f)(X, \dots, X, Y)\rangle \\
&\qquad = \langle S_X^2 S_X^3 \cdots S_X^{k-1}\beta_k(f)(X, \dots, X), \beta(f)(X, Y)\rangle \\
&\qquad = \langle \mathcal{A}(f)_{S_X^2 S_X^3 \cdots S_X^{k-1}\beta_k(f)(X, \dots, X)} X, Y\rangle.
\end{aligned}$$

Summarizing, we obtain that, for $x \in S^m$, $\beta_k(f)$, $k \geq 3$, is isotropic on $T_x(S^m)$ if and only if, we have

$$\mathcal{A}(f)_{S_X^2 S_X^3 \cdots S_X^{k-1}\beta_k(f)(X, \dots, X)} X \wedge X = 0, \quad X \in T_x(S^m).$$

Remark. Another approach for order k isotropy in general was derived by Hong and Houh in [15, Theorem 2.3]. The first $k-1$ curvatures are constant if and only if, for $2 \leq l \leq 2k-1$, we have

$$\mathcal{A}(f)_{(D^{l-2}\beta(f))(X, \dots, X)} X \wedge X = 0, \quad X \in T_x(S^m),\ x \in S^m,$$

where D is the covariant differentiation on $T(M) \oplus \mathcal{N}_f$ with $\mathcal{N}_f$ being the normal bundle of f. (Note that, in this case, $\mathcal{A}_{(D^{l-2}\beta(f))(X \dots, X)} X = 0$ for l odd.)

These conditions are formulated in terms of the notion of *contact number* of Euclidean submanifolds. See [4, 5] for details for pseudo-Euclidean submanifolds. The first author generalized this notion for the case of affine immersions in projectively flat space; see [19].

The condition of isotropy in (5), in turn, can further be expressed in terms of higher derivatives of the image curves $\sigma = f \circ \gamma$ under f, where $\gamma : \mathbb{R} \to S^m$ runs through all arc-length parametrized geodesics.

Proposition. *Let $f : S^m \to S_V$ be a spherical minimal immersion of degree p. For a unit vector $X \in T_x(S^m)$, let $\gamma_X : \mathbb{R} \to S^m$ be the (arc-length parametrized) geodesic such that $\gamma_X(0) = x$ and $\gamma'_X(0) = X$, and set $\sigma_X = f \circ \gamma_X : \mathbb{R} \to S_V$. Then $f : S^m \to S_V$ is isotropic (of order two) if and only if, for any $x \in S^m$, and $X, Y \in T_x(S^m)$ with $\langle X, Y\rangle = 0$, we have*

$$\langle \sigma'''_X(0), \sigma'_Y(0)\rangle = 0. \tag{6}$$

Here $\sigma_X^{(k)}$, $k \geq 1$, is the kth derivative of σ_X as a vector-valued function (with values in V) and viewed as a vector field along the curve σ_X.
If $f : S^m \to S_V$ is an isotropic spherical minimal immersion of degree p then, for the isotropy constant Λ, we have

$$\langle \sigma'''_X(0), \sigma'_X(0)\rangle = -\Lambda^2 - \frac{\lambda_p}{m}, \quad \|X\| = 1,\ X \in T_x(S^m),\ x \in S^m. \tag{7}$$

The proof of the proposition will be given in Section 2.

1.4 $SU(2)$-Equivariant Minimal Immersions

The moduli $\mathcal{M}_m^p$ parametrizing the congruence classes of full spherical minimal immersions $f : S^m \to S_V$ of degree p is non-trivial if and only if $m \geq 3$ and $p \geq 4$. The lowest dimension of the domain S^m for non-trivial moduli is $m = 3$. This case is special since the acting isometry group $SO(4)$ has the almost product structure

$$SO(4) = SU(2) \cdot SU(2)'. \tag{8}$$

This splitting can be obtained by identifying $\mathbb{R}^4$ and $\mathbb{C}^2$ in the usual way: $\mathbb{R}^4 \ni (x, y, u, v) \mapsto (z, w) = (x + \imath y, u + \imath v) \in \mathbb{C}^2$. With this identification, the group $SU(2)$ of special unitary matrices with parametrization

$$\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}, \quad |a|^2 + |b|^2 = 1,\ a, b \in \mathbb{C}, \tag{9}$$

becomes a subgroup of $SO(4)$.
This also shows that $SU(2) = S^3$, where the latter is the unit sphere in $\mathbb{C}^2$ (and $(a, b) \in S^3 \subset \mathbb{C}^2$ corresponds to the typical element in (9)). Finally, $S^3 = SU(2)$ can also be viewed as the unit sphere of the skew-field of quaternions $\mathbb{H}$, and, under this identification, $(a, b) \in S^3 \subset \mathbb{C}^2$ corresponds to the unit quaternion $g = a + \jmath b \in S^3 \subset \mathbb{H}$.

The orthogonal matrix $\gamma = \mathrm{diag}\,(1, 1, 1, -1) \in O(4)$ (or, in complex coordinates, $\gamma : z \mapsto z$, $w \mapsto \bar{w}$, $(z, w) \in \mathbb{C}^2$) conjugates $SU(2)$ to the subgroup

$$SU(2)' = \gamma SU(2)\gamma \subset SO(4), \quad \gamma^{-1} = \gamma.$$

Both subgroups $SU(2)$ and $SU(2)'$ are normal in $SO(4)$ and we have $SU(2) \cap SU(2)' = \{\pm I\}$, so that (8) follows.

In view of the splitting in (8), it is natural to consider spherical minimal immersions $f : S^3 \to S_V$ of degree p that are $SU(2)$-*equivariant*, that is, there exists a homomorphism $\rho_f : SU(2) \to SO(V)$ such that

$$f \circ L_g = \rho_f(g) \circ f, \quad g \in S^3, \tag{10}$$

where L_g is left-multiplication by the unit quaternion g.
The homomorphism ρ_f (associated to $SU(2)$-equivariance) defines an $SU(2)$-module structure on the Euclidean vector space V. Moreover, the natural isomorphism between V and the space of components $V_f \subset \mathcal{H}_3^p$ (through the dual V^*) is $SU(2)$-equivariant, and we obtain that V is an $SU(2)$-submodule of the restriction $\mathcal{H}_3^p|_{SU(2)}$.

In general, the irreducible complex $SU(2)$-modules are parametrized by their dimension, and they can be realized as submodules appearing in the (multiplicity one) decomposition of the $SU(2)$-module of complex homogeneous polynomials $\mathbb{C}[z, w]$ in two variables. For $p \geq 0$, the pth submodule W_p, $\dim_{\mathbb{C}} W_p = p + 1$, comprises the homogeneous polynomials of degree p. With respect to the L^2-scalar product (suitably scaled) the standard orthonormal basis for W_p is $\{z^{p-q}w^q/\sqrt{(p-q)!q!}\}_{q=0}^p$. For p odd, W_p is irreducible *as a real* $SU(2)$*-module.* For p even, the fixed point set R_p of the complex anti-linear self map $z^q w^{p-q} \mapsto (-1)^q z^{p-q} w^q$, $q = 0, \ldots, p$, of W_p is an irreducible real submodule with $W_p = R_p \otimes_{\mathbb{R}} \mathbb{C}$.
For the space of *complex-valued* spherical harmonics $\mathcal{H}_3^p$ of order p, we have

$$\mathcal{H}_3^p = W_p \otimes W_p',$$

as complex $SO(4)$-modules, where W_p' is the $SU(2)'$-module obtained from the $SU(2)$-module W_p via conjugation by γ, and the tensor product is understood by the local product structure in (8). Restricting to $SU(2)$, we obtain

$$\mathcal{H}_3^p = (p + 1)W_p,$$

as complex $SU(2)$-modules.
For *real-valued* spherical harmonics, for p even, this gives

$$\mathcal{H}_3^p = (p+1)R_p.$$

Similarly, for p odd, we have

$$\mathcal{H}_3^p = \frac{p+1}{2} W_p$$

as real $SU(2)$-modules.

Returning to our $SU(2)$-equivariant spherical minimal immersion $f : S^3 \to S_V$, we see that the $SU(2)$-module V is isomorphic with a multiple of R_p for p even, and a multiple of W_p for p odd. As a byproduct, we also obtain that the dimension $\dim V$ is *divisible* by $p+1$ if p is even, and by $2(p+1)$ if p is odd.

The $SU(2)$-equivariant spherical minimal immersions are parametrized by the *$SU(2)$-equivariant moduli* $(\mathcal{M}_3^p)^{SU(2)}$, the fixed point set of $SU(2)$ acting on the moduli $\mathcal{M}_3^p$. It is a compact convex body in the fixed point set $(\mathcal{F}_3^p)^{SU(2)}$ which, in view of the splitting (8), is an $SU(2)'$-module. We have

$$(\mathcal{F}_3^p)^{SU(2)} = \sum_{q=2}^{[p/2]} R'_{4q}, \tag{11}$$

as real $SU(2)'$-modules. In particular, we have the dimension formula

$$\dim(\mathcal{M}_3^p)^{SU(2)} = \dim(\mathcal{F}_3^p)^{SU(2)} = \left(2\left[\frac{p}{2}\right]+5\right)\left(\left[\frac{p}{2}\right]-1\right), \ \ p \geq 4. \tag{12}$$

(This dimension formula was first heuristically derived in [7]. Subsequently, the decomposition in (11) was proved in [34] yielding (12). For more details, see also [29].)

To seek explicit examples of $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_V$, it is natural to consider the simplest case when $V = W_p$ (regardless the parity of p). This has been initiated by K. Mashimo [17], and it is usually referred to as the *equivariant construction.*

More explicitly, given a (nonzero) polynomial

$$\xi = \sum_{q=0}^{p} c_q z^{p-q} w^q \in W_p, \tag{13}$$

we consider the orbit map $f_\xi : S^3 \to W_p$, $f_\xi(g) = g \cdot \xi = \xi \circ g^{-1}$, $g \in SU(2)$, through ξ. It maps into a unit sphere S_{W_p} if and only if

$$\|\xi\|^2 = \sum_{q=0}^{p} (p-q)!q!|c_q|^2 = 1. \tag{14}$$

Assuming this, we obtain a map $f_\xi : S^3 \to S_{W_p}$ which is obviously $SU(2)$-equivariant.

Note that, if p is even and $\xi \in R_p$, then $f_\xi : S^3 \to S_{R_p}$ with range the irreducible *real* $SU(2)$-submodule $R_p \subset W_p$.

Since $SU(2)$ acts transitively on S^3, homothety needs to be imposed only on the tangent space $T_1(S^3)$, say. A simple computation then gives that f_ξ is homothetic with homothety constant $\lambda_p/3 = p(p+2)/3$ if and only if

$$\sum_{q=0}^{p-2}(p-q)!(q+2)!c_q\bar{c}_{q+2} = 0, \tag{15}$$

$$\sum_{q=0}^{p-1}(p-q)!(q+1)!(p-2q-1)c_q\bar{c}_{q+1} = 0, \tag{16}$$

$$\sum_{q=0}^{p}(p-q)!q!(p-2q)^2|c_q|^2 = \frac{p(p+2)}{3}. \tag{17}$$

(For more details, see [7, 8] or [34, 29].

Examples. The quartic ($p = 4$) minimal immersion $\mathcal{I} : S^3 \to S_{W_4} = S^9$, the $SU(2)$-orbit map of the polynomial $\xi = (\sqrt{6}/24)(z^4 - w^4) + (\sqrt{2}/4)z^2w^2 \in W_4$, is archetypal in understanding the structure of the moduli $(\mathcal{M}_3^4)^{SU(2)}$ and thereby $\mathcal{M}_3^4$; see [34]. Moreover, the sextic ($p = 6$) tetrahedral minimal immersion $Tet : S^3 \to S_{R_6} = S^6$, the $SU(2)$-orbit map of the polynomial $\xi = (1/(4\sqrt{15}))zw(z^4 - w^4) \in R_6 \subset W_6$, is a famous example because it realizes the minimum range dimension among all non-standard spherical minimal immersions of S^3. (For more details, and for an extensive list of $SU(2)$-equivariant spherical minimal immersions, see [7, 8, 29].)

1.5 Isotropic $SU(2)$-Equivariant Spherical Minimal Immersions

$SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_V$ of degree p that are isotropic of order k are parametrized by the $SU(2)$-*equivariant moduli* $(\mathcal{M}_3^{p,k})^{SU(2)}$. It is a compact convex body in the fixed point set $(\mathcal{F}_3^{p;k})^{SU(2)}$ which, in view of the splitting (8), is an $SU(2)'$-module. We have

$$(\mathcal{F}_3^{p,2})^{SU(2)} = \sum_{q=k+1}^{[p/2]} R'_{4q},$$

as real $SU(2)'$-modules. In particular, we have the dimension formula

$$\dim(\mathcal{M}_3^{p;k})^{SU(2)} = \dim(\mathcal{F}_3^{p;k})^{SU(2)} = \left(2\left[\frac{p}{2}\right] + 2k + 3\right)\left(\left[\frac{p}{2}\right] - k\right), \quad p \geq 2k+2. \tag{18}$$

In view of the simple characterization of (order two) isotropic spherical minimal immersions in the proposition (in Section 1.3), as a final task we now give a complete characterization of isotropic $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_{W_p}$ of degree p.

Theorem B. *Let $f = f_\xi : S^3 \to S_{W_p}$ be an $SU(2)$-equivariant spherical minimal immersion of order p. Setting $f = f_\xi$ with $\xi \in W_p$ satisfying (13)-(17), $f = f_\xi$ is isotropic (of order two) if and only if the following system of equations holds*

$$\sum_{q=0}^{p-4} (p-q)!(q+4)! c_q \bar{c}_{q+4} = 0, \tag{19}$$

$$\sum_{q=0}^{p-3} (p-q)!(q+3)!(p-2q-3) c_q \bar{c}_{q+3} = 0, \tag{20}$$

$$\sum_{q=0}^{p-2} (p-q)!(q+2)!(p-2q-2)^2 c_q \bar{c}_{q+2} = 0, \tag{21}$$

$$\sum_{q=0}^{p-1} (p-q)!(q+1)!(p-2q-1)^3 c_q \bar{c}_{q+1} = 0, \tag{22}$$

$$\sum_{q=0}^{p} (p-q)!q!(p-2q)^4 |c_q|^2 = \frac{p(p+2)(3p(p+2)-4)}{15}. \tag{23}$$

Finally, in this case, for the constant of isotropy Λ, we have

$$\Lambda^2 = \frac{p(p+2)(p(p+2)-3)}{5}. \tag{24}$$

To exhibit specific examples of isotropic $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_{W_p}$ thus amounts to solve the system of equations (19)-(23) along with (14)-(17).

The proof of Theorem B is very technical, and it is deferred to (most part of) the next section. The system of equations (19)-(23) exhibits specific symmetries, and it is easy to make a specific conjecture as to what the analogous system should be for higher order isotropy.

1.6 Examples

The archetypal $SU(2)$-equivariant spherical minimal immersions are the *tetrahedral*, *octahedral*, and *icosahedral* minimal immersions. As recognized by DeTurck and Ziller in [7, 8], they are the $SU(2)$-orbits of Felix Klein's minimum degree absolute invariants of the tetrahedral, T, octahedral, O, and isosahedral, I, groups in $R_{2d} \subset W_{2d}$, for $d = 3, 4, 6$. As such they realize minimal embeddings of the tetrahedral, S^3/T^*, octahedral, S^3/O^*, and isocahedral, S^3/I^*, manifolds, where the asterisk indicates the respective binary groups. (For more details, see [29, Section 1.5].)

We note first that the tetrahedral minimal immersion $Tet : S^3 \to S_{R_6} = S^6$ cannot be isotropic for reasons of dimension since, for any isotropic $SU(2)$-equivariant spherical minimal immersion $f : S^3 \to S_V$, we have $\dim V \geq \dim \mathcal{H}_3^2 = 9$ (Section 1.2).

This dimension restriction does not exclude the octahedral minimal immersion $Oct : S^3 \to S_{R_8} = S^8$ to be isotropic; however, it is the $SU(2)$-orbit of the octahedral invariant $\xi = c_0(z^8 + 14z^4w^4 + w^8) \in R_8$, $c_0 = 1/(96\sqrt{21})$, which does not satisfy (19) or (23). Hence the octahedral minimal immersion is not isotropic.

The fact that the icosahedral minimal immersion $\mathcal{I} : S^3 \to S_{R_{12}} = S^{12}$ is isotropic has been proved by Escher and Weingart in [10] using basic representation theoretical tools. (See also [29, Remark 2 in 4.5].) Here it follows directly from Theorem B by simple substitution using the explicit form of Klein's icosahedral invariant $\xi = c_1(z^{11}w + 11z^6w^6 - zw^{11}) \in R_{12}$, $c_1 = 1/(3600\sqrt{11})$.

It is natural to expect that there are no isotropic spherical minimal immersions with ranges R_8 or R_{10}, and therefore the icosahedral minimal immersion is the minimum (co)dimension isotropic spherical minimal immersion. Over the reals, (13)-(17) and (19)-(23) represent 15 quadratic equations, for R_8, in 9 variables, and, for R_{10}, in 11 variables. Even though in both cases we have highly overdetermined systems, to show non-existence of solutions is a major technical problem (even with computer algebra systems). Note that even for R_{12} the system (13)-(17) and (19)-(27) is slightly overdetermined (15 equations in 13 variables); however, an "accidental" coincidence of some coefficients results in the existence of solutions. In this line of notes we finally remark that, for R_{10}, an extensive case-by-case computation shows that if an isotropic $SU(2)$-equivariant spherical minimal immersion $f_\xi : S^3 \to S_{R_{10}}$ exists then, for the coefficients c_q, $q = 0, \ldots, 5$, of $\xi \in R_{10}$ in (13), one of the products $c_2c_3c_4c_5$, $c_1c_3c_4c_5$, $c_1c_2c_3c_4$, $c_0c_2c_3c_4$ cannot vanish.
It is also natural to ask if the icosahedral minimal immersion is unique (up to isometries of the domain and the range) among all isotropic $SU(2)$-equivariant spherical minimal immersions with range R_{12}.

A slight change in the coefficients may result in a radically different spherical mini-

mal immersion. For example, $\xi = c_1(z^{11}w + 11\imath z^6w^6 - zw^{11})$ (with c_1 as above) belongs to W_{12} (and not R_{12}), and the corresponding (full) isotropic $SU(2)$-equivariant spherical minimal immersion $f_\xi : S^3 \to S_{W_{12}} = S^{25}$ has the binary dihedral group D_5^* as its invariance group, and it gives a minimal embedding of the dihedral manifold S^3/D_5^* into S^{25}.
The isocahedral minimal immersion above and this last example are in the complete list of DeTurck and Ziller of all spherical minimal embeddings of 3-dimensional space forms. (See [7, 8] and also [29, 1.5].) Using Theorem B, a simple case-by-case check shows that these are the only isotropic spherical minimal immersions in this list.

We have $W_{12} = 2R_{12}$ as real $SU(2)$-modules, so that the previous example immediately raises the problem of minimal multiplicity; that is, for given $p \geq 6$ even, what is the *minimal* $1 \leq k \leq p+1$ such that an isotropic $SU(2)$-equivariant spherical minimal immersion $f : S^3 \to S_{kR_p}$ exists. Using deeper representation theoretical tools, the second author in [30] showed the existence of isotropic $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_{4R_6}$ and $f : S^3 \to S_{6R_8}$.

Isotropic $SU(2)$-equivariant spherical minimal immersions with range W_p abound for $p \geq 11$. As the simplest example, letting $c_q = 0$ for $q \not\equiv 0 \pmod 5$, $q = 0, \ldots, 11$, (13)-(17) and (19)-(23) give

$$|c_0|^2 = \frac{1}{2^9 \cdot 3^5 \cdot 5^4 \cdot 11}, \quad |c_5|^2 = \frac{11}{2^7 \cdot 3^3 \cdot 5^4}, \quad |c_{10}|^2 = \frac{1}{2^9 \cdot 3^5 \cdot 5^4}.$$

Setting $\xi = c_0z^{11} + c_5z^6w^5 + c_{10}zw^{10} \in W_{11}$ we obtain isotropic $SU(2)$-equivariant spherical minimal immersions $f_\xi : S^3 \to S_{W_{11}} = S^{23}$.

For a somewhat more symmetric example in W_{12}, once again letting $c_q = 0$ for $q \not\equiv 0 \pmod 5$, $q = 0, \ldots, 12$, by (13)-(17) and (19)-(23), we have

$$|c_0|^2 = \frac{2^5}{12! \cdot 5^2 \cdot 7}, \qquad |c_5|^2 = \frac{2 \cdot 3 \cdot 11}{5! \cdot 7! \cdot 5^2 \cdot 7}, \qquad |c_{10}|^2 = \frac{11}{2! \cdot 10! \cdot 5^2}.$$

Setting $\xi = c_0z^{12} + c_5z^7w^5 + c_{10}z^2w^{10} \in W_{12}$, we obtain isotropic $SU(2)$-equivariant spherical minimal immersions $f_\xi : S^3 \to S_{W_{12}} = S^{25}$.

2 Proofs

Proof of Proposition. We let ∇ denote the Levi-Civita covariant differentiation on S^m and D the covariant (ordinary) differentiation on the Euclidean vector space V. Letting $\iota : S_V \to V$ denote the inclusion, we have

$$D_XY = \nabla_XY + \beta(f)(X, Y) - \langle X, Y\rangle\iota, \tag{25}$$

for any locally defined vector fields X, Y on S^m. As usual, we identify locally defined vector fields with their images under any immersions (such as $f : S^m \to S_V$ and $\iota \circ f : S^m \to V$, etc.). With this, for any unit tangent vector $X \in T_x(S^m)$, $x \in S^m$, we have

$$D_{\sigma'_X} \sigma_X^{(k)} = \sigma_X^{(k+1)}, \quad k \geq 0, \tag{26}$$

as vector fields along σ_X. Using (25)-(26), we now calculate

$$\sigma''_X = D_{\sigma'_X} \sigma'_X = \beta(f)(\sigma'_X, \sigma'_X) - (\lambda_p/m)\, \sigma_X,$$

where $\nabla_{\sigma'_X} \sigma'_X = 0$ since γ_X is a geodesic. Using this, we have

$$\begin{aligned} \sigma'''_X &= D_{\sigma'_X} \sigma''_X = D_{\sigma'_X} \beta(f)(\sigma'_X, \sigma'_X) - (\lambda_p/m)\, \sigma'_X \\ &= \nabla^\perp_{\sigma'_X} \beta(f)(\sigma'_X, \sigma'_X) - \mathcal{A}(f)_{\beta(f)(\sigma'_X, \sigma'_X)} \sigma'_X - (\lambda_p/m)\, \sigma'_X, \end{aligned}$$

where $\nabla^\perp$ denotes the covariant differentiation of the normal bundle $\mathcal{N}_f$ of $f : S^m \to S_V$. For unit tangent vectors $X, Y \in T_x(S^m)$, $x \in S^m$, this gives

$$\langle \sigma'''_X(0), \sigma'_Y(0) \rangle = -\langle \mathcal{A}(f)_{\beta(f)(X,X)} X, Y \rangle - (\lambda_p/m) \langle X, Y \rangle.$$

The equivalence of (5) and (6) is now clear.
Setting $X = Y \in T_x(S^m)$, $x \in S^m$, with $\|X\| = 1$, we obtain

$$\langle \sigma'''_X(0), \sigma'_X(0) \rangle = -\|\beta(f)(X, X)\|^2 - \frac{\lambda_p}{m} = -\Lambda^2 - \frac{\lambda_p}{m}.$$

The last statement and thereby the proposition follows.

For the proof of Theorem B, we first need to develop several computational tools. In the Lie algebra $su(2)$ we take the standard (orthonormal) basis:

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}.$$

The *unit sphere* $S_{su(2)} \subset su(2)$ can then be parametrized by spherical coordinates as

$$\begin{aligned} U = U(\theta, \varphi) &= \cos\theta \cos\varphi \cdot X + \sin\theta \cos\varphi \cdot Y + \sin\varphi \cdot Z \\ &= \begin{bmatrix} \imath \sin\varphi & e^{\imath\theta} \cos\varphi \\ -e^{-\imath\theta} \cos\varphi & -\imath \sin\varphi \end{bmatrix} \in S_{su(2)}, \quad \theta, \varphi \in \mathbb{R}. \end{aligned}$$

(For simplicity, unless needed, we suppress the angular variables.) An important feature of the spherical coordinates to be used in the sequel is that, for *given* $\theta, \varphi \in \mathbb{R}$, the vectors $U(\theta, \varphi)$, $U(\theta + \pi/2, 0)$, and $U(\theta, \varphi + \pi/2)$ form an orthonormal basis of

$su(2)$ (which, for $\theta = \varphi = 0$ reduces to the standard basis above).
Moreover, since $U^2 = -I$, we have

$$U^{2l} = (-1)^l I \quad \text{and} \quad U^{2l+1} = (-1)^l U, \quad l \geq 1.$$

Hence, for the *exponential map* $\exp : su(2) \to SU(2)$, we obtain

$$\begin{aligned} \exp(t \cdot U) &= \sum_{j=0}^{\infty} \frac{1}{j!}(tU)^j = \sum_{l=0}^{\infty}(-1)^l \frac{t^{2l}}{(2l)!}U^{2l} + \sum_{l=0}^{\infty}(-1)^l \frac{t^{2l+1}}{(2l+1)!}U^{2l+1} \qquad (27) \\ &= \cos t \cdot I + \sin t \cdot U = \begin{bmatrix} \cos t + \imath \sin\varphi \sin t & e^{\imath\theta}\cos\varphi \sin t \\ -e^{-\imath\theta}\cos\varphi\sin t & \cos t - \imath \sin\varphi \sin t \end{bmatrix}, \quad t \in \mathbb{R}. \end{aligned}$$

Recall from Section 1.4 the equivariant construction which associates to a unit vector $\xi \in W_p$, $p \geq 4$, the orbit map $f_\xi : S^3 \to S_{W_p}$ defined by

$$f_\xi(g) = g \cdot \xi = \xi \circ g^{-1}, \quad g \in SU(2).$$

Here $SU(2) = S^3$, the unit sphere in $\mathbb{C}^2 = \mathbb{H}$, with typical element in (9) being identified with the unit quaternion $g = a + \jmath b \in S_{\mathbb{H}}$. For the inverse, we have

$$g^{-1} = g^* = (\bar{a}, -b) = (a + \jmath b)^{-1} = \bar{a} - \jmath b.$$

Using the realization W_p as an $SU(2)$-submodule of $\mathbb{C}[z, w]$, we obtain the explicit representation

$$\begin{aligned} f_\xi(g)(z, w) &= \xi(g^{-1}(z, w)) = \xi((\bar{a} - \jmath b)(z + \jmath w)) \\ &= \xi((\bar{a}z + \bar{b}w) + \jmath(-bz + aw)) \\ &= \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad g = (a, b) = a + \jmath b \in S^3. \end{aligned}$$

Let $U \in T_1(S^3) = T_I(SU(2)) = su(2)$ be a unit vector, and consider the geodesic $\gamma_U : \mathbb{R} \to S^3$, $\gamma_U(0) = 1$ and $\gamma_U'(0) = U$, as in the proposition in Section 1.3. Letting $U = U(\theta, \varphi)$, $\theta, \varphi \in \mathbb{R}$, by (27), we have

$$\gamma_U(t) = (\cos t + \imath \sin\varphi \sin t, -e^{-\imath\theta}\cos\varphi \sin t) \in S^3, \quad t \in \mathbb{R}.$$

By the proposition again, we let $\sigma_U = f_\xi \circ \gamma_U : \mathbb{R} \to S_{W_p}$ be the image curve under f_ξ. By the explicit representation above, we obtain

$$\sigma_U(t) = \xi(a(t), b(t)), \quad t \in \mathbb{R}, \qquad (28)$$

where

$$\begin{aligned}
a(t) = a(t,\theta,\varphi) : &= (\cos t - \imath \sin\varphi \sin t) z - (e^{\imath\theta} \cos\varphi \sin t) w \\
&= z \cdot \cos t + (-\imath \sin\varphi \cdot z - e^{\imath\theta} \cos\varphi \cdot w) \sin t, \\
b(t) = b(t,\theta,\varphi) : &= (e^{-\imath\theta} \cos\varphi \sin t) z + (\cos t + \imath \sin\varphi \sin t) w \\
&= w \cdot \cos t + (e^{-\imath\theta} \cos\varphi \cdot z + \imath \sin\varphi \cdot w) \sin t.
\end{aligned}$$

It is a simple but crucial fact that, for given $\theta, \varphi \in \mathbb{R}$, the pair $(a(t), b(t))$, $t \in \mathbb{R}$, satisfies the system of differential equations

$$\begin{aligned}
\frac{da}{dt} &= -\imath \sin\varphi \cdot a(t) - e^{\imath\theta} \cos\varphi \cdot b(t), \\
\frac{db}{dt} &= e^{-\imath\theta} \cos\varphi \cdot a(t) + \imath \sin\varphi \cdot b(t).
\end{aligned}$$

with initial conditions $a(0) = z$, $b(0) = w$. (Note that the coefficient matrix is in $SU(2)$.)
We now expand $\xi \in W_p$ as in (13). Evaluating this on the pair $(a(t), b(t))$, $t \in \mathbb{R}$, by (28), we obtain

$$\sigma_U(t) = \sum_{q=0}^{p} c_q \, a(t)^{p-q} \, b(t)^q, \quad t \in \mathbb{R}.$$

(It will be convenient to define $c_q = 0$ for the out-of-range indices $q < 0$ and $q > p$.) Taking derivatives and using the system of differential equations above, a simple induction gives the following:

Lemma 1. *Given $\theta, \varphi \in \mathbb{R}$, for any $k \in \mathbb{N}$, we have*

$$\sigma_U^{(k)}(t) = \sum_{q=0}^{p} c_q^{(k)} \, a(t)^{p-q} \, b(t)^q, \quad t \in \mathbb{R},$$

where the coefficients $c_q^{(k)} = c_q^{(k)}(\theta, \varphi)$ are given by

$$\begin{aligned}
c_q^{(k)} \;\; = \;\; & e^{-\imath\theta} \cos\varphi \cdot (q+1) c_{q+1}^{(k-1)} - \imath \sin\varphi \cdot (p - 2q) c_q^{(k-1)} \\
& - e^{\imath\theta} \cos\varphi \cdot (p - q + 1) c_{q-1}^{(k-1)}, \quad q = 0, \ldots, p.
\end{aligned} \tag{29}$$

Here $c_q^{(0)} = c_q$, $q \in \mathbb{Z}$, and $c_q^{(k)} = 0$ for the out-of-range indices $q < 0$ and $q > p$.

We now assume that $f_\xi : S^3 \to S_{W_p}$ is a spherical minimal immersion, that is, the coefficients of ξ in the expansion (13) satisfy (14)-(17). Our task is to give a

necessary and sufficient condition for f_ξ to be isotropic (of order two). Since f_ξ is $SU(2)$-equivariant, the vanishing of the scalar products in (6) of the proposition need to hold only for unit vectors in the tangent space $T_1(S^3) = su(2)$. Thus, our setting above applies.
We now let

$$U_1 := U(\theta, \varphi), \quad U_2 := U(\theta + \pi/2, 0), \quad U_3 := U(\theta, \varphi + \pi/2), \quad \theta, \varphi \in \mathbb{R}.$$

We observe that, for given $\theta, \varphi \in \mathbb{R}$, $\{U_1, U_2, U_3\} \subset T_1(S^3)$ is an orthonormal basis. Due to the arbitrary position of U_1 (given by the arbitrary choices of θ and φ), and linearity in the first derivative in (6), the proposition in Section 1.3 gives the following:

Lemma 2. *Let $f_\xi : S^3 \to S_{W_p}$ be an $SU(2)$-equivariant spherical immerison. Then f_ξ is isotropic if and only if , for any $\theta, \varphi \in \mathbb{R}$, we have*

$$\langle \sigma'''_{U_1}(0), \sigma'_{U_2}(0) \rangle = \langle \sigma'''_{U_1}(0), \sigma'_{U_3}(0) \rangle = 0. \tag{30}$$

In this case, for the constant of isotropy Λ, we have $\langle \sigma'''_{U_1}(0), \sigma'_{U_1}(0) \rangle = -\Lambda^2 - 1$.

Before the proof of Theorem B, we need a convenient scalar product on $W_p \subset \mathbb{C}[z, w]$, or, more generally, on the space of *complex* spherical harmonics $\mathcal{H}_3^p$. As usual, we identify $\mathcal{H}_3^p$ with the space of complex-valued degree p harmonic homogeneous polynomials on $\mathbb{C}^2 = \mathbb{R}^4$. To define this scalar product, we will regard a complex polynomial χ in the complex variables $z, w \in \mathbb{C}$ as a *real* polynomial in the variables $z, w, \bar{z}, \bar{w}$. Then, for $\chi_1, \chi_2 \in \mathcal{H}_3^p$, we define the scalar product on $\mathcal{H}_3^p$ by

$$\langle \chi_1, \chi_2 \rangle = \Re \left(\chi_1 \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) \bar{\chi}_2 \right),$$

where χ_1 acts on $\bar{\chi}_2$ as a *polynomial differential operator.* (This form of the scalar product on $\mathcal{H}_3^p$ has been used in [7, 8, 34].) Note that, with respect to this scalar product, $\{z^{p-q} w^q / \sqrt{(p-q)!q!}\}_{q=0}^p$ is an orthonormal basis of W_p as stated in Section 1.4.

Proof of Theorem B. The explicit calculation of the two scalar products in (30) are technically very similar with a slight difference that the vanishing of the first implies only (19)-(22) (and not (23)). Hence we will treat only the second scalar product in (30).

Using Lemma 1, for fixed $\theta, \varphi \in \mathbb{R}$, we have:

$$\langle \sigma'''_{U_1}(0), \sigma'_{U_3}(0) \rangle = \Re \left(\sum_{q=0}^{p} (p-q)!q! \cdot c_q^{(3)}(\theta, \varphi) \, \overline{c_q^{(1)}(\theta, \varphi + \pi/2)} \right) = \sum_{k=-4}^{4} e^{k\imath\theta} B_k, \tag{31}$$

where the last exponential sum is obtained by repeated application of the recurrence in (29) and the careful tracking of the exponential factors $e^{k\imath\theta}$, $k=-4,\ldots,4$. In this last sum each B_k, $k=-4,\ldots,4$, is independent of the variable θ. In particular, the scalar product on the left-hand side of (31) vanishes for all $\theta,\varphi\in\mathbb{R}$ if and only if the (Fourier) coefficents B_k, $k=-4,\ldots,4$, vanish for all $\varphi\in\mathbb{R}$.

Expanding the factors $c_q^{(3)}(\theta,\varphi)\,\overline{c_q^{(1)}(\theta,\varphi+\pi/2)}$, $q=0,\ldots,p$, in (31) in terms of the coefficients c_q, $q=0,\ldots,p$, requires fairly involved computations. It turns out that the expressions

$$e^{k\imath\theta}B_k+e^{-k\imath\theta}B_{-k},\quad k=0,\ldots,4, \tag{32}$$

are the least cumbersome to determine. (For $k=0$, this reduces to $2B_0$ which we included here.)

We begin with the simplest case $k=4$. As noted above, a technical computation gives

$$e^{4\imath\theta}B_4+e^{-4\imath\theta}B_{-4}=2\cos^3\varphi\sin\varphi\sum_{q=0}^{p-4}(p-q)!(q+4)!\cdot\Re\left(e^{4\imath\theta}c_q\bar{c}_{q+4}\right),$$

Cleary, this vanishes for all $\theta,\varphi\in\mathbb{R}$ if and only if (19) holds.

The cases $k0,=1,\ldots,3$ are similar but more involved. We will discuss only the case $k=1$. Once again, a technical computation gives

$$\begin{aligned}
e^{\imath\theta}B_1+e^{-\imath\theta}B_{-1}=\frac{\cos^4\varphi}{2}&\sum_{q=0}^{p-1}(p-q)!(q+1)!\times\\
\times\left[3(p-2q-1)^3+2(4-(p+1)^2)(p-2q-1)\right]&\cdot\Im\left(e^{\imath\theta}c_q\bar{c}_{q+1}\right)\\
-\frac{3\cos^2\varphi\sin^2\varphi}{2}&\sum_{q=0}^{p-1}(p-q)!(q+1)!\times\\
\times\left[7(p-2q-1)^3-\left(3(p+1)^2-20\right)(p-2q-1)\right]&\cdot\Im\left(e^{\imath\theta}c_q\bar{c}_{q+1}\right)\\
+2\sin^4\varphi&\sum_{q=0}^{p-1}(p-q)!(q+1)!\times\\
\times\left[(p-2q-1)^3+(p-2q-1)\right]&\cdot\Im\left(e^{\imath\theta}c_q\bar{c}_{q+1}\right).
\end{aligned}$$

Due to (16), the second term (with common factor $(p-2q-1)$) in each square bracket cancels. With this the simplified expression vanishes for all $\theta,\varphi\in\mathbb{R}$ if and only if (22) holds. (Note that we recover (22) three times corresponding to each sum above.)

The cases $k = 3$ and $k = 2$ are similar and they yield (20) and (21), respectively. Finally, we treat the case $k = 0$. We have

$$B_0 = \frac{\cos^3\varphi \sin\varphi}{8} \sum_{q=0}^{p} (p-q)!q! \times$$
$$\times \Big[15(p-2q)^4 + 18(2-p(p+2))(p-2q)^2 + 3p^2(p+2)^2 - 8p(p+2) \Big] |c_q|^2$$
$$- \frac{\cos\varphi \sin^3\varphi}{2} \sum_{q=0}^{p} (p-q)!q! \times$$
$$\times \Big[5(p-2q)^4 - (3p(p+2)-16)(p-2q)^2 - 4p(p+2) \Big] |c_q|^2. \tag{33}$$

(We keep the factor $p(p+2)$ intact as it is the pth eigenvalue of the Laplacian on S^3.) Now, $B_0 = 0$ for all $\theta, \varphi \in \mathbb{R}$ if and only if each of the two sums above vanish separately. We split the first as

$$15 \sum_{q=0}^{p} (p-q)!q!(p-2q)^4 |c_q|^2 + 18(2-p(p+2)) \sum_{q=0}^{p} (p-q)!q!(p-2q)^2 |c_q|^2$$
$$+ (3p^2(p+2)^2 - 8p(p+2)) \sum_{q=0}^{p} (p-q)!q! |c_q|^2 = 0$$

By (17) and (14), the second and third sums are equal to $p(p+2)/3$ and 1, respectively. Rearranging, we obtain (23). The second sum in (33) gives the same result.

Finally, to determine the constant of isotropy Λ, in view of the last statement of the proposition, we need to calculate

$$\langle \sigma'''_{U_1}(0), \sigma'_{U_1}(0) \rangle = \Re \left(\sum_{q=0}^{p} (p-q)!q! \cdot c_q^{(3)}(\theta,\varphi)\, \overline{c_q^{(1)}(\theta,\varphi)} \right).$$

Once again expanding, akin to the previous computations, we obtain

$$\langle \sigma'''_{U_1}(0), \sigma'_{U_1}(0) \rangle = -\frac{p(p+2)(3p(p+2)-4)}{15}.$$

Combining this with (7), the last statement of Theorem B follows.

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