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ORIGINAL PAPER

On the structure of convex sets with symmetries

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Abstract Asymmetry of a compact convex body $\mathcal{L} \subset \mathbf{R}^n$ viewed from an interior point $\mathcal{O}$ can be measured by considering how far $\mathcal{L}$ is from its inscribed simplices that contain $\mathcal{O}$. This leads to a measure of symmetry $\sigma(\mathcal{L}, \mathcal{O})$. The interior of $\mathcal{L}$ naturally splits into regular and singular sets, where the singular set consists of points $\mathcal{O}$ with largest possible $\sigma(\mathcal{L}, \mathcal{O})$. In general, to calculate the singular set of a compact convex body is difficult. In this paper we determine a large subset of the singular set in centrally symmetric compact convex bodies truncated by hyperplane cuts. As a function of the interior point $\mathcal{O}$, $\sigma(\mathcal{L}, .)$ is concave on the regular set. It is natural to ask to what extent does concavity of $\sigma(\mathcal{L}, .)$ extend to the whole (interior) of $\mathcal{L}$. It has been shown earlier that in dimension two, $\sigma(\mathcal{L}, .)$ is concave on $\mathcal{L}$. In this paper, we show that in dimensions greater than two, for a centrally symmetric compact convex body $\mathcal{L}$, $\sigma(\mathcal{L}, .)$ is a non-concave function provided that $\mathcal{L}$ has a codimension one simplicial intersection. This is the case, for example, for the n-dimensional cube, $n \geq 3$. This non-concavity result relies on the fact that a centrally symmetric compact convex body has no regular points.

Keywords Convex set · Distortion

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1 Introduction and statement of results

In this paper, we use basic concepts and results in the theory of convex sets and functions [2,5]. Let $\mathcal{E}$ be a Euclidean vector space of dimension n. (As usual, we take $\mathcal{E} = \mathbf{R}^n$.) For an arbitrary subset $\mathcal{K}$ of $\mathcal{E}$ we define $[\mathcal{K}]$ and $\langle\mathcal{K}\rangle$ the *convex hull* and the *affine span* of $\mathcal{K}$. For $\mathcal{K}$ finite, say $\mathcal{K} = \{B_0, \ldots, B_m\}$, $[\mathcal{K}]$ is a convex *polytope*. This polytope is an *m-simplex* if $B_0, \ldots, B_m$ are *affinely independent*, or equivalently, if $\dim[\mathcal{K}] = \dim\langle\mathcal{K}\rangle = m$. A convex

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set $\mathcal{L} \subset \mathcal{E}$ with nonempty interior is called a *convex body*. Every convex set is a convex body in its affine span.

Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body and $\mathcal{O}$ an interior point of $\mathcal{L}$. Given $C \in \mathcal{L}$, $C \neq \mathcal{O}$, we define the *opposite* C^o of C (with respect to $\mathcal{O}$) to be the unique boundary point of $\mathcal{L}$ with $\mathcal{O}$ in the interior of the line segment $[C, C^o]$. For C a boundary point, the ratio $\Lambda_{\mathcal{L}}(C, \mathcal{O}) = \Lambda(C, \mathcal{O})$ of lengths that $\mathcal{O}$ splits $[C, C^o]$ is called the *distortion* of C with respect to $\mathcal{O}$. A *configuration* of $\mathcal{L}$ (with respect to $\mathcal{O}$) is a multi-set of boundary points $\{C_0, \dots, C_n\}$ (with possible repetitions) such that $\mathcal{O}$ is in the convex hull $[C_0, \dots, C_n]$. We let $\mathcal{C}(\mathcal{L}, \mathcal{O})$ denote the set of all configurations of $\mathcal{L}$. With this, we define

$$\sigma(\mathcal{L}, \mathcal{O}) = \inf_{\{C_0,\dots,C_n\} \in \mathcal{C}(\mathcal{L},\mathcal{O})} \sum_{i=0}^{n} \frac{1}{1 + \Lambda(C_i, \mathcal{O})}. \tag{1}$$

A configuration at which the infimum is attained is called *minimal*. Since $\mathcal{L}$ is compact, minimal configurations exist since a minimizing sequence of configurations subconverges in $\mathcal{C}(\mathcal{L}, \mathcal{O})$. Minimal configurations are by no means unique.

A configuration $\{C_0, \dots, C_n\} \in \mathcal{C}(\mathcal{L}, \mathcal{O})$ is called *simplicial* if $[C_0, \dots, C_n]$ is a simplex with $\mathcal{O}$ in its interior. The set of all simplicial configurations is denoted by $\mathcal{S}(\mathcal{L}, \mathcal{O})$. Since $\mathcal{S}(\mathcal{L}, \mathcal{O}) \subset \mathcal{C}(\mathcal{L}, \mathcal{O})$ is dense (in the obvious topology), the infimum in (1) can be restricted to $\mathcal{S}(\mathcal{L}, \mathcal{O})$.

Let $\{C_0, \dots, C_n\}$ be a simplicial configuration. We then have

$$1 = \sum_{i=0}^{n} \frac{1}{1 + \Lambda_{[C_0,\dots,C_n]}(C_i, \mathcal{O})} \leq \sum_{i=0}^{n} \frac{1}{1 + \Lambda_{\mathcal{L}}(C_i, \mathcal{O})}, \tag{2}$$

where the equality follows by an easy exercise in the use of projectivities from the vertices C_i, $i = 0, \dots, n$, of the simplex $[C_0, \dots, C_n]$, and the inequality follows from the definition of the distortion. We obtain the lower bound

$$1 \leq \sigma(\mathcal{L}, \mathcal{O}). \tag{3}$$

It is clear from (2) via convexity that the lower bound is attained if and only if $\mathcal{L}$ is a simplex. For any $C \in \partial\mathcal{L}$, we have

$$\frac{1}{1 + \Lambda(C, \mathcal{O})} + \frac{1}{1 + \Lambda(C^o, \mathcal{O})} = 1. \tag{4}$$

Applying this to the elements of a configuration and taking the corresponding infima, we obtain the upper bound

$$\sigma(\mathcal{L}, \mathcal{O}) \leq \frac{n+1}{2}. \tag{5}$$

For $n \geq 2$, it is not hard to see that the upper bound is attained if and only if $\mathcal{L}$ is centrally symmetric with respect to $\mathcal{O}$. (The 'if' part is obvious; for the 'only if' part, see [6].)

$\sigma(\mathcal{L}, \mathcal{O})$ is invariant under similarity transformations, and is a continuous function on the space of compact convex bodies with specified interior point. Because of these properties and (3) and (5), $\sigma(\mathcal{L}, \mathcal{O})$ is a *measure of symmetry* on convex bodies in the sense of Grünbaum [1].

A minimizing sequence in $\mathcal{S}(\mathcal{L}, \mathcal{O})$, does not necessarily subconverge in $\mathcal{S}(\mathcal{L}, \mathcal{O})$. If it does not, we call $\mathcal{O}$ a *singular point*. The set of singular points is called the *singular set* of $\mathcal{L}$. In the opposite case, that is, if *every* minimal configuration is simplicial, we call $\mathcal{O}$ a *regular* *point*. The set of regular points $\mathcal{R}$ is called the *regular set* of $\mathcal{L}$. It is easy to see that we always have

$$\sigma(\mathcal{L}, \mathcal{O}) \leq \sigma_{n-1}(\mathcal{L}, \mathcal{O}) + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}, \tag{6}$$

where σ_{n-1} is defined as in (1) with n replaced by $n-1$. Now, according to a result in [6], $\mathcal{O}$ is a regular point if and only if sharp inequality holds in (6). In particular, the regular set is open (since σ and σ_{n-1} are continuous in $\mathcal{O}$ [7]), therefore, the singular set is relatively closed in $\mathcal{L}$. By definition, the convex hull $[C_0, \ldots, C_n]$ of a minimal configuration $\{C_0, \ldots, C_n\}$ for a regular point is a simplex with $\mathcal{O}$ contained in its interior. In view of (1), the distortion function $\Lambda(., \mathcal{O})$ has to assume a local maximum at each vertex C_i, $i = 0, \ldots, n$. In general, if $\Lambda(., \mathcal{O})$ assumes a local maximum at a boundary point C then $[C, C^o]$ is an *affine diameter* [1,3,4] in the sense that at the endpoints C and C^o there are *parallel* supporting hyperplanes to $\mathcal{L}$. (This follows from the local study of extrema of Λ in Sect. 7 of [7].) The following geometric picture emerges: Each regular point is the common intersection of $n+1$ (affinely independent) affine diameters.

Our first result asserts that the two extreme values of $\sigma(\mathcal{L}, .)$ in (3) and (5) correspond to the entire interior of $\mathcal{L}$ consisting of regular or singular points.

Theorem A *Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body and $\mathcal{O}$ and interior point of $\mathcal{L}$. If $\sigma(\mathcal{L}, \mathcal{O}) = 1$ then ($\mathcal{L}$ is a simplex and) the interior of $\mathcal{L}$ consists of regular points only. If $\sigma(\mathcal{L}, \mathcal{O}) = (n+1)/2$ then ($\mathcal{L}$ is centrally symmetric with respect to $\mathcal{O}$ and) the entire interior of $\mathcal{L}$ consists of singular points only.*

It seems reasonable to conjecture that the converse of both statements in Theorem A are true.

Conjecture 1 *A compact convex body with all interior points regular is a simplex. In fact, in [9], we showed that this conjecture is true under the additional assumption that $\mathcal{L}$ has at least n isolated extremum points on its boundary.*

In view of this, to remove this assumption, one needs to show that every extremum point is isolated. (By a well-known theorem of Minkowski (see [5]) the convex hull of the extremum points is the entire convex body. Therefore, there must be at least $n+1$ extremum points.)

Another observation related to Conjecture 1 is the following. Let $\mathcal{L}$ be a compact convex body with an isolated extremum point C on the boundary. If the interior of $\mathcal{L}$ consists of regular points only then, away from C, the boundary $\partial\mathcal{L}$ is not smooth. To give a quick proof of this, let $\mathcal{B}$ be the set of boundary points $B \neq C$ at which a supporting hyperplane is parallel to a supporting hyperplane at C. Assuming that $\partial\mathcal{L}$ is smooth away from C, the local conical structure of $\mathcal{L}$ at C shows that $\mathcal{B}$ has a nonempty relative interior $\mathcal{B}_0$ in $\partial\mathcal{L}$, and the convex hull $[\mathcal{B}_0]$ intersects the interior of $\mathcal{L}$. (See [9].) Each interior point $\mathcal{O}$ in this intersection is a singular point. Indeed, a minimizing simplicial configuration must have a vertex C_0 contained in $\mathcal{B}_0$. Since C_0 is an end point of an affine diameter that passes through $\mathcal{O}$, the antipodal of C_0 must be C. Since C_0 is a smooth point (and C is not), the distortion function cannot assume a local maximum at C_0.

It is also tempting to consider a limiting argument and try to derive a statement for the existence of singular points for compact convex bodies with smooth boundary. Being a measure of symmetry, $\sigma(\mathcal{L}, \mathcal{O})$ continuously depends on the boundary $\partial\mathcal{L}$ and the interior point $\mathcal{O}$. In a convergent sequence of compact convex bodies, the regular set, however, may display discontinuous behavior. The following example shows this.

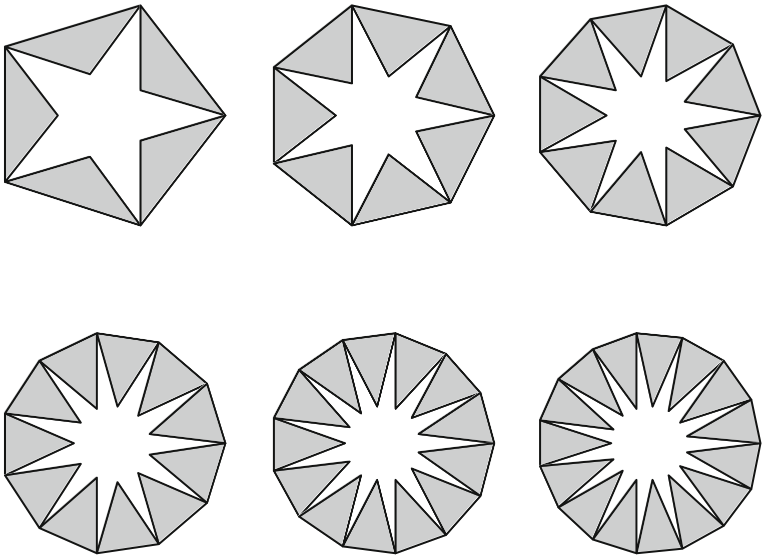

Fig. 1 Regular and singular sets in odd sided regular polygons

Example Let $\mathcal{P}_\ell = \{\ell\} \subset \mathbf{R}^2$, $\ell \geq 3$, be a regular ℓ-sided polygon. By the first statement of Theorem A, for $\ell = 3$, the interior of the triangle $\mathcal{P}_3$ consists of regular points only. By the second statement of Theorem A, for $\ell = 2m$ even, the interior of $\mathcal{P}_{2m} = \{2m\}$ consists of singular points only. In contrast, for $\ell = 2m+1$ odd, in Sect. 2 we will show that the regular set of $\mathcal{P}_{2m+1} = \{2m+1\}$ is the interior of the star-polygon $\{\frac{2m+1}{m}\}$. Figure 1 depicts the first six cases. (The shaded regions correspond to the singular sets.) It is interesting to note that the ratio of the areas of $\{\frac{2m+1}{m}\}$ and $\{2m+1\}$ tends to $2/3$ as $m \to \infty$ whereas the limiting polygon is a disk with no regular points. More specifically, in each polygon $\mathcal{P}_{2m+1}$ the open central disk D of radius $1/3$ is contained in the regular set $\mathcal{R}$, whereas in the limiting disk each point of D becomes singular.

Although natural, the following conjecture seems much more difficult.

Conjecture 2 *A compact convex body with all interior points singular is centrally symmetric.*

The last statement of Theorem A is a much simplified case of a more general result that gives an insight as well as a variety of examples to the structure of the set of regular points.

Theorem B *Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body obtained from a centrally symmetric compact convex body $\mathcal{L}_0 \subset \mathbf{R}^n$ by truncation with an affine hyperplane $\mathcal{K} \subset \mathbf{R}^n$. Assume that the center of symmetry $\mathcal{O}_0$ of $\mathcal{L}_0$ is contained in the interior of $\mathcal{L}$. Let $\mathcal{F} = \mathcal{K} \cap \mathcal{L}_0$, and $\mathcal{F}'$ the set of boundary points of $\mathcal{L}$ away from $\mathcal{F}$ at which a supporting hyperplane passes through parallel to $\mathcal{K}$. Then every interior point of $\mathcal{L}$ away from the convex hull $[\mathcal{F}, \mathcal{F}']$ is singular.*

In many applications (for example, if $\mathcal{L}_0$ is strictly convex), $\mathcal{F}'$ reduces to a point. In this case, Theorem B asserts that the set of regular points $\mathcal{R}$ is contained in the interior of a cone whose base is the slice $\mathcal{F}$ cut out from $\mathcal{L}_0$ by the truncating hyperplane $\mathcal{K}$.

As the example above shows, the interior of the convex hull $[\mathcal{F}', \mathcal{F}']$ is not necessarily contained in $\mathcal{R}$, so that the converse of Theorem B is false. For $n = 2$, a partial converse is as follows:

Theorem C *Let $\mathcal{L} \subset \mathbf{R}^2$ be a compact convex body with a maximal line segment $\mathcal{F} = [C_0, C_1]$ on the boundary of $\mathcal{L}$. Assume that the set $\mathcal{F}'$ defined in Theorem B consists of a single point C_0. Then the set*

$$\mathcal{T} = \{\mathcal{O} \in \operatorname{int}[C_0, C_1, C_2] \mid \Lambda(C_0, \mathcal{O}) \leq \max(\Lambda(C_1, \mathcal{O}), \Lambda(C_2, \mathcal{O}))\}$$

is contained in the regular set $\mathcal{R}$.

Remark The set $\mathcal{T}$ has nonempty interior. Indeed, if $M_1 \in [C_0, C_1]$ and $M_2 \in [C_0, C_2]$ are the midpoints of the respective line segments then

$$[C_0, M_1, M_2] \cap \operatorname{int}[C_0, C_1, C_2] \subset \mathcal{T}.$$

In addition to the behavior of points with regard to the infimum in (1), it is equally interesting to study the properties of $\sigma(\mathcal{L}, .)$ as a function of the interior point $\mathcal{O}$. In [7] we proved that $\sigma(\mathcal{L}, .)$ is continuous, and

$$\lim_{d(\mathcal{O},\partial\mathcal{L})\to 0} \sigma(\mathcal{L}, \mathcal{O}) = 1. \tag{7}$$

The latter implies that $\sigma(\mathcal{L}, .)$ extends continuously (to 1) to the boundary of $\mathcal{L}$.
In [7] we also proved that $\sigma(\mathcal{L}, .)$ is concave on the regular set $\mathcal{R}$. Using this, and a detailed study of this function at singular points, we also showed that, for $n = 2$, $\sigma(\mathcal{L}, .)$ is a concave on the *entire* interior of $\mathcal{L}$. In contrast, in [8] we constructed a four-dimensional cone such that near the base $\sigma(\mathcal{L}, .)$ was not concave. Although concavity still holds for three-dimensional cones, the question remained unsettled in dimension 3.

The last result of this paper resolves this problem negatively in any dimension ≥ 3. In fact, for centrally symmetric compact convex bodies the situation is much simpler. For example, as the theorem below shows, for the n-dimensional cube, $n \geq 3$, $\sigma(\mathcal{L}, .)$ is not concave.

Theorem D *Let $\mathcal{L} \subset \mathbf{R}^n$ be a centrally symmetric compact convex body with $\mathcal{O}_0$ the center of symmetry. The measure of symmetry $\sigma(\mathcal{L}, .)$ attains its unique absolute maximum at $\mathcal{O}_0$. If $\sigma(\mathcal{L}, .)$ is a concave function then it is linear on all line segments with $\mathcal{O}_0$ as one end-point. In particular, if $n \geq 3$ and $\mathcal{L}$ contains a codimension one simplicial intersection then $\sigma(\mathcal{L}, .)$ is not concave.*

2 Proofs

Proof of Theorem A By [7], we need only to prove the last statement: If $\mathcal{L}$ is centrally symmetric then every interior point of $\mathcal{L}$ is singular.

Let $\mathcal{O}_0$ be the center of symmetry for $\mathcal{L}$. Then $\mathcal{O}_0$ is an interior point of $\mathcal{L}$ and the distortion $\Lambda(., \mathcal{O}_0)$ is identically one. Clearly, any configuration in $\mathcal{C}(\mathcal{L}, \mathcal{O}_0)$ is minimal, and $\mathcal{O}_0$ is a singular point.

Let $\mathcal{O}$ be another interior point, $\mathcal{O} \neq \mathcal{O}_0$, and assume that $\mathcal{O}$ is regular. In what follows, the opposite of a boundary point C *with respect to* $\mathcal{O}$ will be denoted by C^o. Let $\{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, \mathcal{O})$ be a minimal configuration for $\sigma(\mathcal{L}, \mathcal{O})$. By regularity, $[C_0, \ldots, C_n]$ is an n-simplex with $\mathcal{O}$ in its interior. At each C_i, $i = 0, \ldots, n$, the distortion $\Lambda(., \mathcal{O})$ attains a relative maximum. As noted above, the line segment $[C_i, C_i^o]$ is an affine diameter. This means that there exist parallel supporting hyperplanes $\mathcal{H}_i$ and $\mathcal{H}_i^o$ passing through C_i and C_i^o, respectively.

Let $A \in \partial\mathcal{L}$ be the opposite of $\mathcal{O}$ with respect to $\mathcal{O}_0$. We claim that $[A, C_i] \subset \partial\mathcal{L}$, provided that $C_i \notin \langle\mathcal{O}, \mathcal{O}_0\rangle$.

For simplicity, we suppress the subscript, assume that $\Lambda(., \mathcal{O})$ attains a local maximum at a point $C \notin \langle\mathcal{O}, \mathcal{O}_0\rangle$ of a minimal configuration, and the affine diameter $[C, C^o]$ has parallel supporting hyperplanes $\mathcal{H}$ and $\mathcal{H}^o$ at the endpoints. Let $C_0^o \in \partial\mathcal{L}$ be the opposite of C *with respect to* $\mathcal{O}_0$. By central symmetry at $\mathcal{O}_0$, the hyperplane parallel to $\mathcal{H}$ and passing through C_0^o must support $\mathcal{L}_0$. Hence, it must coincide with $\mathcal{H}^o$. We obtain that $[C^o, C_0^o] \subset \partial\mathcal{L}$. We now define a sequence of points $\{A_j\}_{j\geq 1} \subset \partial\mathcal{L}$ as follows. Let A_1 be the opposite of C^o with respect to $\mathcal{O}_0$. With A_j defined, we let A_{j+1} be the opposite of A_j^o with respect to $\mathcal{O}_0$.

We now claim that $[A_j, C] \subset \partial\mathcal{L}$, $j \geq 1$. First, since $[C^o, C_0^o]$ is on the boundary, taking antipodals with respect to $\mathcal{O}_0$, symmetry implies that $[A_1, C] \subset \partial\mathcal{L}$. Proceeding inductively, assume that, for some $j \geq 1$, the line segment $[A_j, C]$ is on the boundary of $\mathcal{L}$. Consider a point C' moving continuously from C to A_j along $[A_j, C]$. Since $\mathcal{H}$ and $\mathcal{H}^o$ are parallel, we have

$$\Lambda(C', \mathcal{O}) \geq \Lambda(C, \mathcal{O}). \tag{8}$$

Hence, for every specific C', we can replace C by C' in the configuration $\{C_0, \ldots, C_n\}$ as long as the condition $\mathcal{O} \in [C_0, \ldots, C_n]$ stays intact. In this case, by minimality, we also have $\Lambda(C', \mathcal{O}) = \Lambda(C, \mathcal{O})$. We now observe that this replacement does keep the condition $\mathcal{O} \in [C_0, \ldots, C_n]$ intact for *any* C' since otherwise we would get a minimizing configuration with $\mathcal{O}$ on the boundary of its convex hull, contradicting to the regularity of $\mathcal{O}$. We obtain that the distortion $\Lambda(., \mathcal{O})$ is constant along $[A_j, C]$, or equivalently, $[A_j^o, C^o]$ is parallel to $[A_j, C]$. This means that $[A_j^o, C^o]$ is contained in $\mathcal{H}^o$, therefore lies on the boundary of $\mathcal{L}$. Reflecting $[A_j^o, C^o]$ to the center of symmetry $\mathcal{O}_0$, we see that $[A_{j+1}, A_1] \subset \mathcal{H}$ is also on the boundary of $\mathcal{L}$. Since the entire construction takes place in the plane $\langle\mathcal{O}, \mathcal{O}_0, C\rangle$, we see that C, A_1, A_{j+1} are collinear and $[A_{j+1}, C] \subset \partial\mathcal{L}$. The claim follows.

The sequence $\{A_j\}_{j\geq 1}$ converges to A. (In fact, as elementary computation shows, the distance between A_j and A is geometric.) As a byproduct, we also see that $\langle\mathcal{O}, \mathcal{O}_0\rangle$ cannot be parallel to $\mathcal{H}$. Taking the limit, we obtain that $[A, C]$ is on the boundary of $\mathcal{L}$.

As in the proof, we can replace C by A in the minimal configuration as long as $C \notin \langle\mathcal{O}, \mathcal{O}_0\rangle$.

Since $[C_0, \ldots, C_n]$ is a simplex, there must be at least two points in the configuration away from the line $\langle\mathcal{O}, \mathcal{O}_0\rangle$. When we replace these two points with A we obtain a minimal configuration whose convex hull is not a simplex. This contradicts to regularity of $\mathcal{O}$. Theorem A follows. □

Proof of Theorem B We will use the notations introduced in Theorem B. In particular, $\mathcal{L}$ is truncated from a centrally symmetric $\mathcal{L}_0$ via truncation along a hyperplane $\mathcal{K}$. Without loss of generality, we may assume that the truncation is *proper* in the sense that $\mathcal{L} \neq \mathcal{L}_0$, or equivalently, the relative interior of $\mathcal{F}$ (the interior of $\mathcal{L}_0$ intersected with $\mathcal{K}$) is nonempty. (If the truncation is not proper then we land in the last statement of Theorem A already proved.)

Before the proof we make some preparations.

First, we observe that

$$\mathcal{O}_0 \in [\mathcal{F}, \mathcal{F}']. \tag{9}$$

Indeed, consider $\mathcal{L}_0$ and $\mathcal{K}$. By symmetry of $\mathcal{L}_0$, the center $\mathcal{O}_0$ is the midpoint of an affine diameter of $\mathcal{L}_0$ with supporting hyperplanes passing through its endpoints both parallel to $\mathcal{K}$.

On the one hand, this affine diameter has an endpoint in $\mathcal{F}'$. On the other hand, it intersects $\mathcal{K}$ and the intersection point must be in $\mathcal{F}$. Thus, (9) follows.

We now let $\mathcal{O} \in \operatorname{int} \mathcal{L} \backslash [\mathcal{F}, \mathcal{F}']$. By (9), $\mathcal{O} \neq \mathcal{O}_0$.

Second, we consider a boundary point $C \in \partial \mathcal{L}$ at which $\Lambda_{\mathcal{L}}(., \mathcal{O})$ assumes a local maximum. We claim that

$$C^o \in \mathcal{F} \Rightarrow C \in \mathcal{F}', \tag{10}$$

where (as in the previous proof) C^o is the opposite of C with respect to $\mathcal{O}$.

If C^o is in the relative interior of $\mathcal{F}$ then the *only* supporting hyperplane at C^o to $\mathcal{L}$ is $\mathcal{K}$, and, since $[C, C^o]$ is an affine diameter, the hyperplane passing through C and parallel to $\mathcal{K}$ must also support $\mathcal{L}$. Thus, $C \in \mathcal{F}'$. Assume now that C^o is on the relative boundary of $\mathcal{F}$, therefore also on $\partial \mathcal{L}_0$. Let $\mathcal{K}'$ be the hyperplane passing through C and parallel to $\mathcal{K}$. As before, we need to show that $\mathcal{K}'$ supports $\mathcal{L}$. Since $[C, C^o]$ is an affine diameter, there exist parallel supporting hyperplanes $\mathcal{H}$ and $\mathcal{H}^o$ passing through C and C^o, respectively. We may assume that $\mathcal{H} \neq \mathcal{K}'$ (since otherwise we are done). Since $\mathcal{H}$ and $\mathcal{K}'$ meet at C they also meet at a codimension two affine subspace. The same holds for the pair $\mathcal{H}^o$ and $\mathcal{K}$. We know that $\mathcal{H}$, $\mathcal{H}^o$ and $\mathcal{K}$ support $\mathcal{L}$, and need to show that $\mathcal{K}'$ also supports $\mathcal{L}$. Consider an oriented plane τ that contains $[C, C^o]$ and intersects $\mathcal{H}$ and $\mathcal{K}'$ transversally. Let $\alpha = \alpha_\tau(C)$ and $\alpha' = \alpha'_\tau(C)$ be the two asymptotic angles of the boundary of $\mathcal{L} \cap \tau$ at C (as defined in Sect. 7 of [7]). Similarly, let α^o and α'^o be the asymptotic angles at C^o. Since $\Lambda_{\mathcal{L}\cap\tau}(., \mathcal{O})$ has a local maximum at C, by [7] (Corollary 1 in Sect. 7), we have $\alpha \leq \alpha^o$ and $\alpha' \leq \alpha'^o$. Since $\mathcal{H}^o \cap \tau$ and $\mathcal{K} \cap \tau$ support $\mathcal{L} \cap \tau$ at C^o, the angular sector of this intersecting pair of supporting lines that contains C also contains the entire $\mathcal{L} \cap \tau$. The angle comparisons above show that the corresponding parallel angular sector formed by $\mathcal{H} \cap \tau$ and $\mathcal{K}' \cap \tau$ must also contain the entire $\mathcal{L} \cap \tau$. Hence, $\mathcal{K}' \cap \tau$ supports $\mathcal{L} \cap \tau$. Since τ was (generically) arbitrary, we obtain that $\mathcal{K}'$ supports $\mathcal{L}$ at C. Thus, $C \in \mathcal{F}'$.

Third, let A be as in the previous proof, that is, the unique intersection of the ray emanating from $\mathcal{O}$ and passing through $\mathcal{O}_0$ with the boundary of $\mathcal{L}$. Assume that $A^o \notin \mathcal{F} \cup \mathcal{F}'$. We need to perform a technical step that will make sure that if a simplicial minimal configuration point happens to be A^o then it can be moved away from A^o. In fact, we claim that if $\Lambda(., \mathcal{O})$ assumes a local maximum at A^o then it is locally constant near A^o.

To prove this, we first observe that, by the previous step (applied to $C = A^o$), we have $A \notin \mathcal{F}$. Thus the affine diameter $[A, A^o]$ is away from $\mathcal{K}$. We can now disregard $\mathcal{K}$ and consider $[A, A^o]$ in the centrally symmetric $\mathcal{L}_0$, with the note that the distortion $\Lambda_{\mathcal{L}_0}(., \mathcal{O})$ also assumes a local maximum at A^o.

We now need the simple fact that, *for* $\mathcal{L}_0$, *absolute maximum* of the distortion with respect to $\mathcal{O}$ occurs at A (and hence absolute minimum of the distortion occurs at A^o). A quick proof of this is as follows. Let C be *any* boundary point away from $\langle \mathcal{O}, \mathcal{O}^o \rangle$. Let A_1 be the opposite of C^o with respect to $\mathcal{O}_0$. By central symmetry with respect to $\mathcal{O}_0$, the lines $\langle A, A_1 \rangle$ and $\langle A^o, C^o \rangle$ are parallel. Let C' be the unique intersection point of $\langle A, A_1 \rangle$ and $\langle C, C^o \rangle$. Then $\mathcal{O}$ splits the segment $[C', C^o]$ in the ratio $\Lambda(A, \mathcal{O})$. By convexity of $\mathcal{L}$, C is contained in the line segment $[\mathcal{O}, C']$. Therefore, we have

$$\Lambda(A, \mathcal{O}) \geq \Lambda(C, \mathcal{O})$$

and the claim follows.

We now return to the main line and recall that at A^o the distortion $\Lambda_{\mathcal{L}_0}(., \mathcal{O})$ assumes a *local maximum*. Since, at the same time, it is also an absolute minimum, the distortion has to be constant *near* A^o. Thus, if $C' \in \partial \mathcal{L}_0 \backslash \langle \mathcal{O}, \mathcal{O}_0 \rangle$ is close to A^o then, being locally constant, $\Lambda_{\mathcal{L}_0}(., \mathcal{O})$ also assumes a local maximum at C' with $\Lambda(A^o, \mathcal{O}) = \Lambda(C, \mathcal{O})$.

After these preparations we now turn to the proof of Theorem B. Assume, on the contrary, that $\mathcal{O}$ is a *regular point of* $\mathcal{L}$. Let $[C_0, \ldots, C_n]$ be a simplicial minimal configuration for $\sigma(\mathcal{L}, \mathcal{O})$. We will prove now that each configuration point can be replaced by another which is either equal to A or is contained in $\mathcal{F} \cup \mathcal{F}'$. Our argument here mimics the proof of Theorem A above. Let $C = C_i$ be a configuration point. As before, being a point in a simplicial minimal configuration for the regular point $\mathcal{O}$, $\Lambda(., \mathcal{O})$ assumes a local maximum at C. We may assume that $C \notin \mathcal{F} \cup \mathcal{F}'$ and $C \neq A$. If $C = A^o$ then, by what we just proved, we can move C away from A^o to a nearby point. Thus, retaining $C \notin \mathcal{F} \cup \mathcal{F}'$, we may assume that $C \notin \langle \mathcal{O}, \mathcal{O}_0 \rangle$. This is exactly the situation in the proof of Theorem A. Repeating the argument there (working in $\mathcal{L}_0$), the only problem is when the moving point C' or its opposite $C_0'^o$ (with respect to $\mathcal{L}_0$) leaves $\mathcal{L}$. When this happens, however, then C' first hits $\mathcal{F}$ and we are done, or $C_0'^o$ leaves $\mathcal{F}$ in which case (8) still holds in $\mathcal{L}_0$ and even more so in $\mathcal{L}$. Summarizing, C can be moved to A or to $\mathcal{F} \cup \mathcal{F}'$.

Finally, we show that the convex hull of $\mathcal{F} \cup \mathcal{F}'$ and the point A does *not* contain $\mathcal{O}$. With this we will get a contradiction to $\{C_0, \ldots, C_n\}$ being a configuration for $\sigma(\mathcal{L}, \mathcal{O})$, and the proof of Theorem B will be complete.

Recall that $\mathcal{O} \neq \mathcal{O}_0$, $\mathcal{O} \notin [\mathcal{F}, \mathcal{F}']$, $\mathcal{O}_0 \in [\mathcal{F}, \mathcal{F}']$, and $\mathcal{O}_0 \in [A, \mathcal{O}]$. Assume, on the contrary, that $\mathcal{O} \in [\mathcal{F}, \mathcal{F}', A]$. This means that

$$\mathcal{O} = \alpha A + \mu F, \quad \alpha + \mu = 1, \quad 0 < \alpha, \mu < 1, \quad F \in [\mathcal{F}, \mathcal{F}']. \tag{11}$$

Writing $\mathcal{O}_0 = \alpha_0 A + \lambda \mathcal{O}$ with $\alpha_0 + \lambda = 1$ and $0 < \alpha_0, \lambda < 1$, after eliminating A, (11) rewrites as

$$\mathcal{O} = \frac{\alpha}{\alpha_0 + \lambda\alpha} \mathcal{O}_0 + \frac{\mu\alpha_0}{\alpha_0 + \lambda\alpha} F.$$

For the coefficients on the right-hand side, we have

$$\frac{\alpha}{\alpha_0 + \lambda\alpha} + \frac{\mu\alpha_0}{\alpha_0 + \lambda\alpha} = \frac{\alpha + (1-\alpha)\alpha_0}{\alpha_0 + (1-\alpha_0)\alpha} = 1.$$

Since $\mathcal{O}_0 \in [\mathcal{F}, \mathcal{F}']$, we obtain $\mathcal{O} \in [\mathcal{F}, \mathcal{F}']$. This is a contradiction. $\square$

Proof of Theorem C Let $\mathcal{L} \subset \mathbf{R}^2$ be as in Theorem C and $\mathcal{O} \in \mathcal{T}$. We first claim that

$$\max_{[C_1, C_2]} \Lambda(., \mathcal{O}) = \max(\Lambda(C_1, \mathcal{O}), \Lambda(C_2, \mathcal{O})).$$

Let $C_1' \in [C_0^o, C_1]$. Then, with obvious notations, we have

$$\Lambda(C_1', \mathcal{O}) \leq \Lambda_{[C_0, C_1, C_0^o, C_1^o]}(C_1', \mathcal{O}) \leq \Lambda(C_1, \mathcal{O}),$$

where the second inequality is because at C_0 there is a supporting line to $\mathcal{L}$ parallel to (C_1, C_2). Similarly, for $C_2' \in [C_0^o, C_2]$, we have $\Lambda(C_2', \mathcal{O}) \leq \Lambda(C_2, \mathcal{O})$. The claim follows.

Let $\mathcal{G}$ be the opposite of $[C_1, C_2]$ (with respect to $\mathcal{O} \in \mathcal{T}$). Then $\mathcal{G}$ is a (connected) continuous curve on the boundary of $\mathcal{L}$ with endpoints C_1^o and C_2^o.

Once again the existence of a supporting line to $\mathcal{L}$ at C_0 parallel to (C_1, C_2) implies

$$\max_{\mathcal{G}} \Lambda(., \mathcal{O}) = \Lambda(C_0, \mathcal{O}).$$

By the definition of $\mathcal{T}$, we thus have

$$\max_{\mathcal{G}} \Lambda(., \mathcal{O}) \leq \max(\Lambda(C_1, \mathcal{O}), \Lambda(C_2, \mathcal{O})).$$

These observations imply that the maximum of $\Lambda(., \mathcal{O})$ is attained on one of the two connected components of $\overline{\partial\mathcal{L}\setminus([C_1, C_2] \cup \mathcal{G})}$.

Without loss of generality, we may assume that the maximum is attained at a point $\bar{C}$ which belongs to the component with endpoints C_1 and C_2^o. Since $\mathcal{O} \in [C_0, C_1, C_2]$ and (automatically) $\mathcal{O} \in [C_0, C_2, C_2^o]$, by convexity, we also have $\mathcal{O} \in [C_0, \bar{C}, C_2]$. Thus, $\{C_0, \bar{C}, C_2\}$ is a configuration. We thus have

$$\sigma(\mathcal{L}, \mathcal{O}) \leq \frac{1}{1+\Lambda(C_0, \mathcal{O})} + \frac{1}{1+\Lambda(C_2, \mathcal{O})} + \frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}.$$

To prove that $\mathcal{O}$ is a regular point, by the characterization of regular points using (6), we need to show that

$$\frac{1}{1+\Lambda(C_0, \mathcal{O})} + \frac{1}{1+\Lambda(C_2, \mathcal{O})} < 1 \tag{12}$$

(Note that (4) implies $\sigma_1(\mathcal{L}, \mathcal{O}) = 1$.) Rewriting (12), we have

$$\frac{1}{1+\Lambda(C_2, \mathcal{O})} < \frac{1}{1+\Lambda(C_0^o, \mathcal{O})}.$$

By the existence of a supporting line to $\mathcal{L}$ at C_0 parallel to (C_1, C_2), non-strict inequality certainly holds. Finally, if $\Lambda(C_2, \mathcal{O}) = \Lambda(C_0^o)$ then (C_0, C_2^o) is parallel to $(C_2, C_0^o) = (C_1, C_2)$. This means that C_0 is not unique in $\mathcal{F}'$. This contradicts to our assumption. Theorem C follows.

We now let $\mathcal{P} \subset \mathbf{R}^2$ be a convex plane polygon and $\mathcal{O}$ a regular point in the interior of $\mathcal{P}$. Then the convex hull of a minimal configuration is a triangle with $\mathcal{O}$ in its interior. Since $\Lambda(., \mathcal{O})$ assumes local maxima at the configuration points, we may assume that these configuration points are *vertices* of $\mathcal{P}$. (This follows easily from regularity, see also [9].) In what follows we will always assume that this is the case. As noted in Sect. 1, the three configuration points connected with their antipodals form three affine diameters passing through $\mathcal{O}$. Finally, as noted above, $\mathcal{O}$ is regular if and only if

$$\sigma(\mathcal{P}, \mathcal{O}) < 1 + \frac{1}{1+\max \Lambda(., \mathcal{O})}. \tag{13}$$

Here the maximum is taken either on the entire boundary or on the set of vertices of $\mathcal{P}$. (See the corollary in Sect. 7 of [6].)

Let V_0, V_1, V_2, V_3 be four consecutive vertices on the boundary of $\mathcal{P}$. Let W be the intersection of $[V_0, V_2]$ and $[V_1, V_3]$. We first make the following observation: Let α_1 and α_2 be the (interior) angles of $\mathcal{P}$ at V_1 and V_2, respectively. If $\alpha_1 + \alpha_2 > \pi$ then the interior of the triangle $[V_1, V_2, W]$ is contained in the singular set of $\mathcal{P}$.

Indeed, if $\mathcal{O}$ were a regular point in the interior of $[V_1, V_2, W]$ then a minimizing configuration would contain either V_1 or V_2. Clearly, $V_1^o \in [V_2, V_3]$ and $V_2^o \in [V_0, V_1]$. Due to the angle condition, however, $[V_1, V_1^o]$ and $[V_2, V_2^o]$ are not affine diameters.

As an immediate application of this observation, we can determine the set of regular points for a quadrilateral. Figure 2 depicts the three possible cases (with the singular sets shaded).

The case when $\mathcal{P}$ is a parallelogram is covered by Theorem A.

If $\mathcal{P}$ is a trapezoid (but not a parallelogram), then the fact that the shaded region in the middle of Fig. 2 is contained in the singular set follows from Theorem B since $\mathcal{P}$ can be considered as a truncated parallelogram by each of the non-parallel sides. Let the consecutive vertices of $\mathcal{P}$ be denoted by V_0, V_1, V_2, V_3 with $[V_1, V_2]$ corresponding to the shorter parallel side, and let $W = [V_0, V_2] \cap [V_1, V_3]$. Assume that $\mathcal{O} \in [V_0, W, V_3]$ (not in the

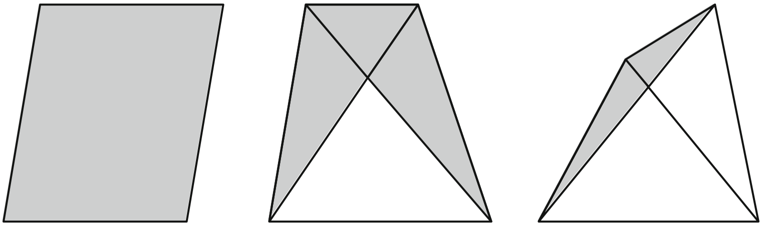

Fig. 2 Regular and singular sets in quadrilaterals

shaded region). We must show that $\mathcal{O}$ is regular. Indeed, a minimizing configuration must contain V_0 and V_3 and either of the top vertices V_1 or V_2. (Note that $\Lambda(V_1, \mathcal{O}) = \Lambda(V_2, \mathcal{O})$.) Now a simple case-by-case check (depending on at which vertex does $\Lambda(., \mathcal{O})$ assume its maximum) gives (13).

Finally, let $\mathcal{P}$ be a general quadrilateral whose opposite sides are not parallel. We view $\mathcal{P}$ as the intersection of two angular domains. The fact that the shaded regions in Fig. 2 are contained in the singular set follows by the observation above (applied twice). That the non-shaded region consists of regular points follows by a case-by-case verification.

We now turn to the example in Sect. 1. For simplicity, we may assume that $\mathcal{P}_{2m+1} = \{2m+1\}$ has vertices $V_k = (\cos(2\pi k/(2m+1)), \sin(2\pi k/(2m+1)))$, $k = 0, \ldots, 2m$. Let $\mathcal{O}$ be any interior point of $\mathcal{P}_{2m+1}$ and assume that, for some $j = 0, \ldots, 2m$, $[V_j, V_j^o]$ is an affine diameter (passing through $\mathcal{O}$). Comparing supporting lines, we see that V_j^o must be in the opposite side to V_j, that is, we have $V_j^o \in [V_{j+m}, V_{j+m+1}]$. Thus, $\mathcal{O}$ must be contained in the triangle $[V_j, V_{j+m}, V_{j+m+1}]$. The intersection of any *three* of these triangles (corresponding to three different indices $j = 0, \ldots, 2m$) is contained in the star-polygon $\{\frac{2m+1}{m}\}$. Thus, any interior point $\mathcal{O}$ of $\mathcal{P}_{2m+1}$ complementary to this star-polygon must be singular.

It remains to show that the interior points of the star-polygon are regular. Let

$$X_k = [V_k, V_{m+k}] \cap [V_{k+1}, V_{m+k+1}],$$

where the indices are counted modulo $2m+1$. We now let $C_0 = V_k$, $C_1 = V_{m+k}$ and $C_2 = V_{m+k+1}$ and apply Theorem C. We obtain that

$$\mathcal{T} = [V_k, X_k, 0] \cup [V_k, X_{m+k}, 0].$$

The union of these for $k = 0, \ldots, 2m$ is the star polygon $\{\frac{2m+1}{m}\}$ and we are done.

To calculate the area $\mathcal{A}_{2m+1}$ of the singular set, the complement of the star-polygon $\{\frac{2m+1}{m}\}$ in $\mathcal{P}_{2m+1}$, is elementary. A simple computation gives

$$\mathcal{A}_{2m+1} = (2m+1)\sin\left(\frac{2m}{2m+1}\pi\right)\left(\frac{1}{2\cos\left(\frac{2m}{2m+1}\pi\right) - 1} - \cos\left(\frac{2m}{2m+1}\pi\right)\right).$$

The limit as $m \to \infty$ is clearly $2\pi/3$.

The minimum distance of the singular set from the center is

$$\frac{1}{1 - 2\cos\left(\frac{2m}{2m+1}\pi\right)}.$$

As $m \to \infty$, this decreases to $1/3$. Thus, the regular set of $\mathcal{P}_{2m+1}$ contains the open disk with center at the origin and radius $1/3$. In the limit, each point of this disk will turn singular. $\square$

Proof of Theorem D We now return to the centrally symmetric compact convex body $\mathcal{L} \subset \mathbf{R}^n$. As usual, let $\mathcal{O} \neq \mathcal{O}_0$ be an interior point, and A its opposite. Note first that during the course of proving Theorem B we showed that $\Lambda(., \mathcal{O})$ assumes its absolute maximum at A. Writing $\mathcal{O}$ as $\mathcal{O}_\lambda = (1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1$, where $\mathcal{O}_1 = A^o \in \partial\mathcal{L}$ and $0 \leq \lambda < 1$, we thus we have

$$\frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)} = \frac{1-\lambda}{2}.$$

Consider the configuration $\{A, \ldots, A, A^o\}$, where A is repeated n-times. Evaluating the sum in (1) on this configuration, we obtain

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) \leq 1 + (n-1)\frac{1-\lambda}{2}. \tag{14}$$

On the other hand, equality holds for $\lambda = 0$ (by central symmetry with respect to $\mathcal{O}_0$), and also for $\lambda = 1$ in the limiting sense by (7).

Assume now that $\sigma(\mathcal{L}, .)$ is concave. Since it is concave on each line segment, equality must hold in (14) and the function $\lambda \mapsto \sigma(\mathcal{L}, \mathcal{O}_\lambda)$ must be linear. Clearly, the configuration $\{A, \ldots, A, A^o\}$ is minimal.

Assume, in addition, that $\mathcal{L}$ has a codimension one simplicial intersection across $\mathcal{O}_\lambda$. By Theorem A, $\mathcal{O}_\lambda$ is a singular point so that equality holds in (6). We thus have

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \sigma_{n-1}(\mathcal{L}, \mathcal{O}_\lambda) + \frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)} = \sigma_{n-1}(\mathcal{L}, \mathcal{O}_\lambda) + \frac{1-\lambda}{2}. \tag{15}$$

Moreover, just like in (3), $1 \leq \sigma_{n-1}$ [6]. In our situation, equality holds since there is a simplicial intersection across $\mathcal{O}_\lambda$. We obtain that

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = 1 + \frac{1-\lambda}{2}.$$

Comparing this with (14) (with the equality sign), we obtain $n = 2$. Thus, for $n \geq 3$, $\sigma(\mathcal{L}, .)$ cannot be concave. Theorem C follows. □

Example Let $\mathcal{L} \subset \mathbf{R}^n$ be an n-dimensional cube. Let $\mathcal{O}_0$ be the center of symmetry and $\mathcal{O}_1$ a vertex of $\mathcal{L}$. With the notations above, we see that for $1 - 2/n \leq \lambda < 1$, the (parallel) vertex figures at $\mathcal{O}_1$ provide $(n-1)$-dimensional simplicial intersections of $\mathcal{L}$ passing through $\mathcal{O}_\lambda$. Thus, $\sigma(\mathcal{L}, \mathcal{O}_\lambda) = (3-\lambda)/2$ is a linear function for $1 - 2/n \leq \lambda \leq 1$. On the other hand, $\sigma(\mathcal{L}, \mathcal{O}_0) = (n+1)/2$. In particular, $\sigma(\mathcal{L}, .)$ cannot be concave.

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