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Convex Sets with Large Distortion

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Abstract. Given a compact convex body $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$ with a fixed base point $\mathcal{O}$ in the interior of $\mathcal{L}$, an affine invariant $\sigma(\mathcal{L})$ can be defined that measures how distorted $\mathcal{L}$ is with respect to $\mathcal{O}$. The two extreme values of $\sigma(\mathcal{L})$ are 1 corresponding to a simplex, and $(\dim \mathcal{E} + 1)/2$ corresponding to a (centrally) symmetric $\mathcal{L}$. In this paper we study the structure of $\mathcal{L}$ when $\sigma(\mathcal{L}) < 1 + 1/(1 + \dim \mathcal{E})$. We construct a polytope that contains $\mathcal{L}$, study its combinatorial structure, and prove that $\mathcal{L}$ is between two simplices scaled in the ratio $1/(2 + \dim \mathcal{E} - (1 + \dim \mathcal{E})\sigma(\mathcal{L})) \div 1$. This, in turn, gives an upper bound on the volume of $\mathcal{L}$ in terms of $\sigma(\mathcal{L})$ and the inscribed simplex.

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1. Introduction

Let $\mathcal{L}$ be a compact convex body in a Euclidean vector space $\mathcal{E}$ and $\mathcal{O} \in \operatorname{int} \mathcal{L}$, a base point in the interior of $\mathcal{L}$. Given $C \in \partial\mathcal{L}$, the line passing through $\mathcal{O}$ and C intersects $\partial\mathcal{L}$ in another point. We call this the *opposite* of C with respect to $\mathcal{O}$ and denote it by C^o. Clearly, $(C^o)^o = C$.

The *distortion function* $\Lambda : \partial\mathcal{L} \to \mathbf{R}$ is defined by

$$\Lambda(C) = \frac{d(C, \mathcal{O})}{d(C^o, \mathcal{O})}, \quad C \in \partial\mathcal{L},$$

where $d(X, X') = |X - X'|$ is the Euclidean distance. Clearly, $\Lambda(C^o) = 1/\Lambda(C)$.

Let $\dim \mathcal{E} = m$. As an application of Helly's theorem [1], we have

$$\frac{1}{m} \leq \Lambda \leq m, \tag{1}$$

provided that the base point $\mathcal{O}$ is chosen appropriately. The extreme values are attained by a simplex.

A multiset $\{C_0, \ldots, C_m\}$ is called a *configuration* of $\mathcal{L}$ (relative to $\mathcal{O}$) if $\{C_0, \ldots, C_m\} \subset \partial\mathcal{L}$ and $\mathcal{O}$ is contained in the convex hull $[C_0, \ldots, C_m]$ of $C_0, \ldots, C_m$. Let $\mathcal{C}(\mathcal{L})$ denote the set of all configurations of $\mathcal{L}$. We define

$$\sigma(\mathcal{L}) = \inf_{\{C_0,\ldots,C_m\}\in\mathcal{C}(\mathcal{L})} \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} .$$

A configuration $\{C_0, \ldots, C_m\}$ is called *minimal* if

$$\sigma(\mathcal{L}) = \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} .$$

Minimal configurations always exist since $\mathcal{L}$ is compact.

In [5] we showed that

$$1 \le \sigma(\mathcal{L}) \le \frac{m+1}{2} . \tag{2}$$

In addition, $\sigma(\mathcal{L}) = 1$ iff $\mathcal{L}$ is an m-simplex. In this case a minimal configuration $\{C_0, \ldots, C_m\} \in \mathcal{C}(\mathcal{L})$ is unique and is given by the set of vertices of $\mathcal{L}$. Moreover, minimality

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} = 1 , \tag{3}$$

implies

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} C_i = \mathcal{O} . \tag{4}$$

Finally, for $m \ge 2$, $\sigma(\mathcal{L}) = (m+1)/2$ iff $\Lambda = 1$ on $\partial\mathcal{L}$, that is, iff $\mathcal{L}$ is symmetric (with respect to $\mathcal{O}$).

There are many results on the geometry and metric properties of $\mathcal{L}$ when it is symmetric [8]. In particular, a deep result of Dvoretzky [2, 3] reveals how close (slices of) symmetric compact convex bodies are to being spherical. He proved that, for given $\epsilon > 0$ and $k \in \mathbf{N}$, there exists $N = N(\epsilon, k) \in \mathbf{N}$ such that for any compact convex body $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$ of dimension $m \ge N$ that is symmetric with respect to $\mathcal{O}$ there exists a k-dimensional affine subspace $\mathcal{F}$, $\mathcal{O} \in \mathcal{F}$, such that $\mathcal{L} \cap \mathcal{F}$ is between two Euclidean balls of $\mathcal{F}$ with center at $\mathcal{O}$ and radii in the ratio $1 + \epsilon \div 1$.

In this paper we study the opposite case, that is, when $\mathcal{L}$ has large distortion. This work is motivated by our study of moduli spaces for spherical immersions, where the moduli are compact convex bodies and, at times, their slices by affine subspaces exhibit close to simplicial behavior. (See [5–7] for details.)

Our main result is the following:

Theorem 1. *Let $\mathcal{L} \subset \mathcal{E}$ be a compact convex body in a Euclidean vector space $\mathcal{E}$ of dimension m, and $\mathcal{O} \in \operatorname{int} \mathcal{L}$ a base point. Assume that the distortion function* *satisfies* $1/m \leq \Lambda \leq m$ *and*

$$\sigma(\mathcal{L}) < 1 + \frac{1}{m+1}\,. \tag{5}$$

Then $\mathcal{L}$ *is between two simplices* Δ *and* Δ^**:*

$$\Delta \subset \mathcal{L} \subset \Delta^*\,.$$

Δ *can be chosen as the convex hull of any minimal configuration of* $\mathcal{L}$*. The outer simplex* Δ^* *is obtained from* Δ *by central magnification from a point* $C^* \in \Delta$*, and the ratio of magnification is* $1/(m+2-(m+1)\sigma(\mathcal{L}))$*. In particular, we have the volume estimate*

$$\mathrm{vol}\,(\Delta)\;\leq\;\mathrm{vol}\,(\mathcal{L})\;\leq\;\frac{\mathrm{vol}\,(\Delta)}{(m+2-(m+1)\sigma(\mathcal{L}))^m}\,. \tag{6}$$

Remark 1. If $\sigma(\mathcal{L}) > 1$ (that is, if $\mathcal{L}$ is not a simplex) then the center of magnification is

$$C^* = \frac{1}{\sum_{i=0}^m \left(\frac{1}{1+\Lambda(C_i)} - \frac{1}{1+\bar{\Lambda}(C_i)}\right)} \sum_{i=0}^m \left(\frac{1}{1+\Lambda(C_i)} - \frac{1}{1+\bar{\Lambda}(C_i)}\right) C_i\,,$$

where $\Delta = [C_0,\ldots,C_m]$ and $\bar{\Lambda}(C_i)$ ($\geq \Lambda(C_i)$) is the distortion of $C_i \in \partial\Delta$ *of the simplex* Δ with respect to $\mathcal{O}$. (We will show that the convex hull of any minimizing configuration is a simplex with $\mathcal{O}$ in its interior; see Proposition 4 in Section 4.) If $\sigma(\mathcal{L}) = 1$ then we can choose $C^* = \mathcal{O}$.

Note that it can well happen that C^* is not in the interior of Δ; for example, it could be one of the vertices of Δ. (This is the case when equality holds for all but one of the indices $i = 0,\ldots,m$ in $\bar{\Lambda}(C_i) \geq \Lambda(C_i)$.)

Remark 2. The inclusion $\Delta \subset \mathcal{L}$ is obvious. The inclusion $\mathcal{L} \subset \Delta^*$ is technical; in fact most part of this paper is devoted to the proof of this. The estimates in Theorem 1 are sharp: equalities hold iff $\mathcal{L}$ is a simplex.

Remark 3. The critical value $1 + 1/(m+1)$ for $\sigma(\mathcal{L})$ is the best possible. In fact, according to Example 1 in [5], the pentagon $\mathcal{L}$ in $\mathbf{R}^2$ with vertices

$$(1,-1),(1,1),(0,2),(-1,1),(-1,-1)$$

has $\sigma(\mathcal{L}) = 4/3$ but there are minimal configurations whose convex hulls are reduced to line segments. Therefore, one cannot expect Theorem 1 to hold without the restriction in (5).

Remark 4. The conditions (1) and (5) are independent. Indeed, if $\mathcal{L} = [C_0,\ldots,C_m]$ is a simplex and $\mathcal{O}$ approaches to one of its vertices C_i within the interior of $\mathcal{L}$ then $\Lambda(C_i)$ approaches to zero, and, for $j = 0,\ldots,m$, $j \neq i$, $\Lambda(C_j)$ approaches to ∞. On the other hand, $\sigma(\mathcal{L}) = 1$ independently of $\mathcal{O}$.

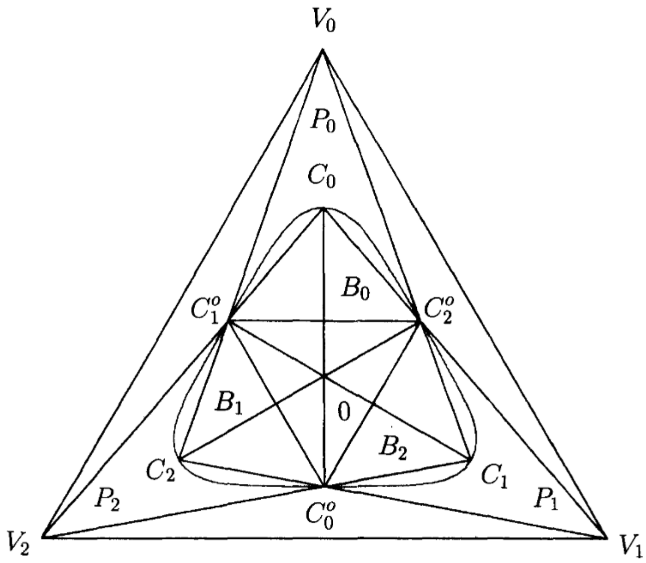

Figure 1

We will prove a stronger version of Theorem 1 where the minimal configuration $\{C_0, \ldots, C_m\}$ will be replaced by a configuration $\{C_0, \ldots, C_m\}$ with convex hull Δ that satisfies

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} < 1 + \frac{1}{m+1},$$

and in the estimates $\sigma(\mathcal{L})$ is replaced by the sum on the left-hand side.

For simplicity, we will always assume that the base point $\mathcal{O}$ is the origin. The key point in the proof is to write $\mathcal{L}$ as the union of the antipodal simplex $[C_0^0, \ldots, C_m^o]$ and $(m+1)$ 'bulges' B_i, $i = 0, \ldots, m$. The bulge B_i is the part of $\mathcal{L}$ contained in the positive cone spanned by $\{C_0^o, \ldots, \widehat{C_i^o}, \ldots, C_m^o\}$ and truncated by the i-th face $[C_0^o, \ldots, \widehat{C_i^o}, \ldots, C_m^o]$. We imbed each bulge B_i into a polytope P_i (Theorem 2 in Section 3). (See Figures 1–2 for $m = 2, 3$.)

If V_i denotes the outermost vertex of P_i then we will show that $\mathcal{L}$ is contained in the m-simplex $[V_0, \ldots, V_m]$ (Theorem 3 in Section 4). Finally, another estimate will yield $[V_0, \ldots, V_m] \subset [C_0^*, \ldots, C_m^*]$, where $\{C_0^*, \ldots, C_m^*\}$ are the vertices of the outer simplex Δ^* magnified from Δ with center at C^*.

P_i has an interesting combinatorial structure; in particular, its vertices can be parametrized by subsets of $\{0, \ldots, \widehat{i}, \ldots, m\}$ and the larger the cardinality of the subset is the further the vertex is from the origin. (See Section 3 and Figure 4 for $m = 3$.)

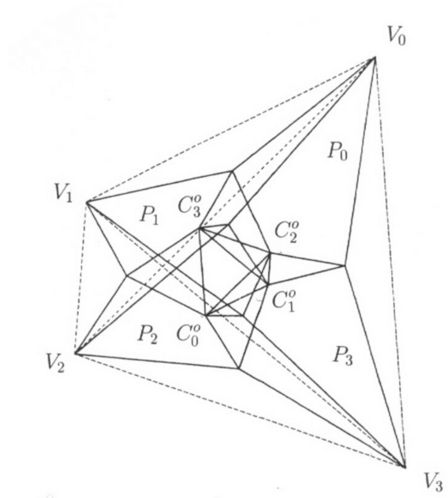

FIGURE 2

2. Notation and preliminaries

Let $\mathcal{E}$ be a Euclidean vector space. Given a subset $\mathcal{S} \subset \mathcal{E}$, we denote by $[\mathcal{S}]$ the convex hull, $\langle\mathcal{S}\rangle$ the affine span and $L(\mathcal{S}) = \langle\mathcal{S}, 0\rangle$ the linear span of $\mathcal{S}$. (For general reference for convexity, see [1,4].) Then $[\mathcal{S}]$ is a convex body in $\langle\mathcal{S}\rangle$. We will always consider a convex body in its affine span, and concepts like relative boundary, relative interior, etc. are understood within the affine span. For simplicity, we will use the terms boundary, interior, etc. within their relative context. If $\mathcal{S}$ is finite then $[\mathcal{S}]$ is a convex polytope in $\langle\mathcal{S}\rangle$. Moreover, $\dim\langle\mathcal{S}\rangle \leq |\mathcal{S}| - 1$ with equality iff $[\mathcal{S}]$ is a simplex.

If $H \subset \mathcal{E}$ is a hyperplane and $\mathcal{S}$, $\mathcal{T}$ are subsets of $\mathcal{E}$ then we say that $\mathcal{S}$ and $\mathcal{T}$ are *on the same side* of H if $\mathcal{S} \cup \mathcal{T}$ is contained in one of the closed half-spaces with boundary H.

We begin with a finite set of points $\{C_i \,|\, i \in \mathcal{I}\} \subset \partial\mathcal{L}$ indexed by a set $\mathcal{I} \subset \mathbf{Z}$. (Unless stated otherwise, all sets will be assumed nonempty.)

For simplicity we write $\lambda_i = \Lambda(C_i)$, $i \in \mathcal{I}$, and assume that

$$\sum_{i\in\mathcal{I}} \frac{1}{1+\lambda_i} < 1\,. \tag{7}$$

For a (nonempty) subset $I \subset \mathcal{I}$, we define

$$\sigma_I = \sum_{i\in I} \frac{1}{1+\lambda_i} \tag{8}$$

and

$$V_I = -\frac{1}{1-\sigma_I}\sum_{i\in I}\frac{1}{1+\lambda_i}C_i = \frac{1}{1-\sigma_I}\sum_{i\in I}\frac{\lambda_i}{1+\lambda_i}C_i^o\,. \tag{9}$$

Note that, for $i \in \mathcal{I}$, $V_{\{i\}} = C_i^o$.

If $I = \{i_1, \dots, i_k\}$, we also write $\sigma_I = \sigma(C_{i_1}, \dots, C_{i_k})$ and $V_I = V(C_{i_1}, \dots, C_{i_k})$.

Lemma 1. *For $i \in I \subset \mathcal{I}$, $|I| \geq 2$, we have $V_{I\setminus\{i\}} \in [C_i, V_I]$.*

Proof. Splitting off i from I in the sum in (9), we obtain

$$V_{I\setminus\{i\}} = \frac{1}{1-\sigma_{I\setminus\{i\}}}\frac{1}{1+\lambda_i}C_i + \frac{1-\sigma_I}{1-\sigma_{I\setminus\{i\}}}V_I\,.$$

Since the coefficients are positive and by (8):

$$1-\sigma_I + \frac{1}{1+\lambda_i} = 1-\sigma_{I\setminus\{i\}}\,,$$

the lemma follows. □

Lemma 2. *Let $I, J \subset \mathcal{I}$ be nonempty subsets.*

(i) *If $I \cap J \neq \emptyset$ then there exist $0 < t \leq s < 1$ such that*

$$(1-t)V_I + tV_J = (1-s)V_{I\cup J} + sV_{I\cap J}\,.$$

(ii) *If I and J are disjoint then then there exist $r > 1$ and $0 < t < 1$ such that*

$$r\big((1-t)V_I + tV_J\big) = V_{I\cup J}\,.$$

Proof. We consider the convex combination

$$(1-t)V_I + tV_J = \frac{1-t}{1-\sigma_I}\sum_{i\in I}\frac{\lambda_i}{1+\lambda_i}C_i^o + \frac{t}{1-\sigma_J}\sum_{j\in J}\frac{\lambda_j}{1+\lambda_j}C_j^o\,.$$

Setting

$$t = \frac{1-\sigma_J}{2-\sigma_I-\sigma_J}$$

the coefficients in front of the two sums above become equal. With this, we have

$$(1-t)V_I + tV_J = \frac{1}{2-\sigma_I-\sigma_J}\left(\sum_{k\in I\cup J}\frac{\lambda_k}{1+\lambda_k}C_k^o + \sum_{l\in I\cap J}\frac{\lambda_l}{1+\lambda_l}C_l^o\right).$$

Since $\sigma_I + \sigma_J = \sigma_{I\cup J} + \sigma_{I\cap J}$, we obtain

$$(1-t)V_I + tV_J = (1-s)V_{I\cup J} + sV_{I\cap J}\,,$$

where

$$s = \frac{1-\sigma_{I\cap J}}{2-\sigma_{I\cup J}-\sigma_{I\cap J}}\,,$$

and the second term is absent if $I \cap J = \emptyset$. The rest is clear. □

Remark. (i) and (ii) can be stated together if we set $V_\emptyset = 0$. There is no advantage in accepting this in the future.

3. The covering polytope

We use the assumptions and the notations of the previous section, and, in addition, we will consider only subsets $I \subset \mathcal{I}$ for which $\{C_i \,|\, i \in I\}$ are linearly independent. In particular, we have $0 \notin \langle \{C_i\}_{i\in I} \rangle$.

We will be working in the linear span

$$L_I = L\big(\{C_i\}_{i\in I}\big) = L\big(\{C_i^o\}_{i\in I}\big)\,.$$

We define

$$T_I = \left\{ \sum_{i\in I} \mu_i C_i^o \;|\; \sum_{i\in I} \mu_i \geq 1,\, \mu_i \geq 0,\, \forall i \in I \right\}. \tag{10}$$

Clearly, $T_I \subset L_I$ is a truncated cone. In addition, by (9)–(10), for $|I| \geq 2$, T_I contains V_I in its interior since

$$\frac{1}{1-\sigma_I} \sum_{i\in I} \frac{\lambda_i}{1+\lambda_i} = \frac{1}{1-\sigma_I} \sum_{i\in I} \left(1 - \frac{1}{1+\lambda_i}\right) = \frac{|I| - \sigma_I}{1-\sigma_I} > 1\,. \tag{11}$$

Moreover, for $J \subset I$, we have $T_J = T_I \cap L_J$, in particular, $\{V_J\}_{J\subset I} \subset T_I$.

We define $P_I \subset L_I$ inductively (with respect to $|I|$) as follows. For $I = \{i\}$, we set $P_{\{i\}} = \{V_{\{i\}}\} = \{C_i^o\}$, and, for $|I| \geq 2$, we define P_I as the convex hull of V_I and $\cup_{i\in I} P_{I\setminus\{i\}}$.

It is clear that $P_I \subset T_I$. In addition, for $J \subset I$ we have

$$P_J = P_I \cap L_J\,.$$

As usual, for $I = \{i_1, \ldots, i_k\}$, we will also use the notations $L_I = L(C_{i_1}, \ldots, C_{i_k})$, $T_I = T(C_{i_1}, \ldots, C_{i_k})$, and $P_I = P(C_{i_1}, \ldots, C_{i_k})$.

Example 1. For $I = \{i, j\}$, we have

$$P(C_i, C_j) = \big[V(C_i, C_j), V(C_i), V(C_j)\big]\,,$$

where $V(C_i) = C_i^o$, $V(C_j) = C_j^o$, and

$$V(C_i, C_j) = \frac{1}{1 - \frac{1}{1+\lambda_i} - \frac{1}{1+\lambda_j}} \left(\frac{\lambda_i}{1+\lambda_i} C_i^o + \frac{\lambda_j}{1+\lambda_j} C_j^o \right).$$

Thus, $P(C_i, C_j)$ is a triangle. (See Figure 3.)

Example 2. For $I = \{i, j, k\}$, $P(C_i, C_j, C_k)$ is a polyhedron depicted in Figure 4. The 7 vertices of $P(C_i, C_j, C_k)$ are V_I, and $V_{\{i,j\}}, V_{\{j,k\}}, V_{\{k,i\}}$, and $V_{\{i\}}, V_{\{j\}}, V_{\{k\}}$. In Proposition 2 we will prove that there are 7 faces; the base $F_0 = [V_{\{i\}}, V_{\{j\}}, V_{\{k\}}]$, and

$$F_i = [V_{\{j,k\}}, V_{\{j\}}, V_{\{k\}}]\,, \quad F_j = [V_{\{k,i\}}, V_{\{k\}}, V_{\{i\}}]\,, \quad F_k = [V_{\{i,j\}}, V_{\{i\}}, V_{\{j\}}]\,,$$

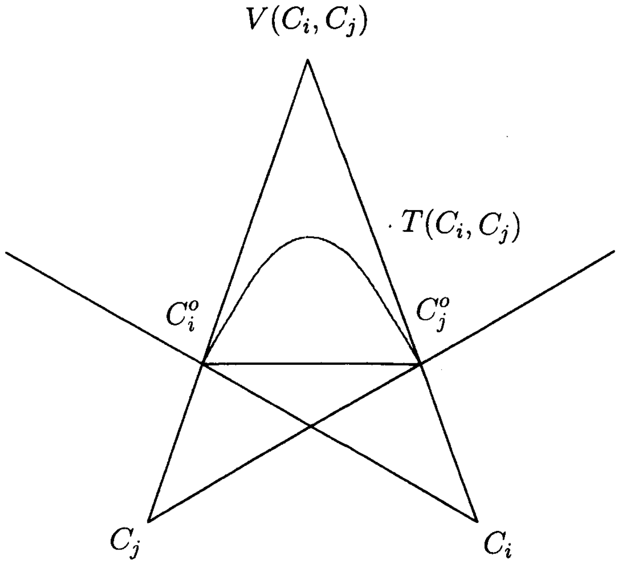

FIGURE 3

and

$$F^i = [V_I, V_{\{i,j\}}, V_{\{k,i\}}, V_{\{i\}}]\,,$$
$$F^j = [V_I, V_{\{j,k\}}, V_{\{i,j\}}, V_{\{j\}}]\,,$$
$$F^k = [V_I, V_{\{j,k\}}, V_{\{k,i\}}, V_{\{k\}}]\,.$$

We also see that P_I is not a simplex for $|I| \geq 3$.

Proposition 1. *P_I is a convex polytope in L_I with vertices V_J, $J \subset I$.*

Proof. An easy induction shows that P_I is the convex hull of the points V_J, $J \subset I$. (Here and in the inductions that follow, the first step of the induction is by a simple inspection of Examples 1–2. Therefore we will only discuss the general induction steps.) Thus, P_I is a convex polytope in L_I. We now show that each V_J, $J \subset I$, is an extremal point of P_I. We begin with V_I.

Consider the hyperplane

$$H_I = \frac{|I| - \sigma_I}{1 - \sigma_I} \big\langle \{C_i^o\}_{i \in I} \big\rangle\,.$$

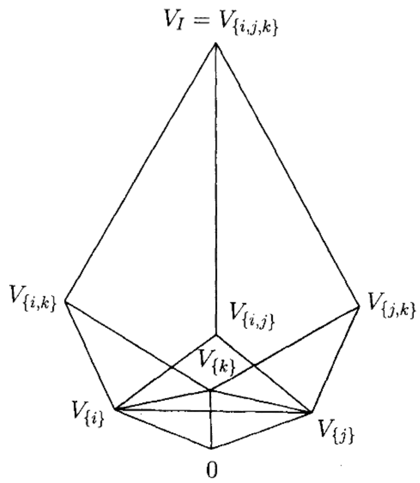

FIGURE 4

By (9) and (11), we have $V_I \in H_I$. We now claim that V_J, $J \subset I$, $J \neq I$, are contained in the same open half-space with boundary H_I. Comparing

$$V_J = \frac{1}{1-\sigma_J} \sum_{j \in J} \frac{\lambda_j}{1+\lambda_j} C_j^o \tag{12}$$

with (9) and (11), we need to show that

$$\frac{|J| - \sigma_J}{1 - \sigma_J} < \frac{|I| - \sigma_I}{1 - \sigma_I}\,.$$

This, however, is clear since $|I| > |J|$ and $\sigma_I > \sigma_J$. The claim follows.

Since P_I is the convex hull of V_J, $J \subset I$, we obtain that H_I is a supporting hyperplane of P_I at V_I, and

$$P_I \cap H_I = \{V_I\}\,.$$

Thus, V_I is a vertex of P_I.

We now assume that $|I| \geq 3$ and show by induction with respect to $|I|$ that V_J, $J \subset I$, are vertices of P_I. Let $i \in I$. By the induction hypothesis, $\{V_J\}_{J \subset I \setminus \{i\}}$ are vertices of $P_{I \setminus \{i\}}$. The polytope $P_{I \setminus \{i\}}$ is a face of P_I with supporting hyperplane extension $L_{I \setminus \{i\}}$. Hence $\{V_J\}_{J \subset I \setminus \{i\}}$ are also vertices of P_I. (If P is a convex polytope and H is a supporting hyperplane then any vertex of $P \cap H$ is also a vertex of P. This follows since the vertices are extremal points.) Thus, for each proper subset $J \subset I$, V_J is a vertex of P_I. Since V_I is also a vertex, we are done. Finally, V_J, $J \subset I$, are all the vertices of P_I since their convex hull is P_I. The proposition follows. $\square$

Proposition 2. *Let* $|I| \geq 3$. *Then* P_I *has* $2|I|+1$ *faces as follows:* $F_0 = [\{C_i^o\}_{i \in I}]$, *and*

$$F_i = P_{I \setminus \{i\}} = [\{V_J \mid i \notin J\}] , \quad i \in I ,$$

and

$$F^i = [\{V_J \mid i \in J\}] , \quad i \in I .$$

Proof. P_I is contained in the truncated cone T_I whose faces are F_0 and $T_{I \setminus \{i\}}$, $i \in I$. Clearly, F_0 is also a face of P_I. Since $F_i = P_{I \setminus \{i\}} = P_I \cap T_{I \setminus \{i\}}$, it is clear that F_i, $i \in I$, are also faces of P_I.

We now show that F^i, $i \in I$, is a face of P_I. By definition, $V_{\{i\}} \in F^i$, and $V_{\{i,j\}} \in F^i$, $j \in I \setminus \{i\}$. It is easy to see that $V_{\{i,j\}} - V_{\{i\}}$, $j \in I \setminus \{i\}$, are linearly independent. We claim that, for any $J \subset I$ with $i \in J$, we have

$$V_J \in \left\langle V_{\{i\}}, \{V_{\{i,j\}}\}_{j \in J \setminus \{i\}} \right\rangle .$$

We need to show that

$$V_J = \alpha_i V_{\{i\}} + \sum_{j \in J \setminus \{i\}} \alpha_j V_{\{i,j\}} \tag{13}$$

and

$$\alpha_i + \sum_{j \in J \setminus \{i\}} \alpha_j = 1 . \tag{14}$$

Expanding the right-hand side in (13), and using that $V_{\{i\}} = C_i^o$, we obtain

$$V_J = \left(\alpha_i + \sum_{j \in J \setminus \{i\}} \frac{\alpha_j}{1 - \sigma_{\{i,j\}}} \frac{\lambda_i}{1 + \lambda_i} \right) C_i^o + \sum_{j \in J \setminus \{i\}} \frac{\alpha_j}{1 - \sigma_{\{i,j\}}} \frac{\lambda_j}{1 + \lambda_j} C_j^o . \tag{15}$$

Equating the coefficients with those of V_J in (12), we obtain

$$\alpha_i = \frac{2 - |J|}{1 - \sigma_J} \frac{\lambda_i}{1 + \lambda_i} \quad \text{and} \quad \alpha_j = \frac{1 - \sigma_{\{i,j\}}}{1 - \sigma_J} , \quad j \in J \setminus \{i\} .$$

Moreover, we have (14) since

$$(2 - |J|) \frac{\lambda_i}{1 + \lambda_i} + \sum_{j \in J \setminus \{i\}} (1 - \sigma_{\{i,j\}}) = 1 - \sigma_J ,$$

where

$$\sum_{j \in J \setminus \{i\}} \sigma_{\{i,j\}} = \sum_{j \in J \setminus \{i\}} \left(\frac{1}{1 + \lambda_i} + \frac{1}{1 + \lambda_j} \right) = \frac{|J| - 2}{1 + \lambda_i} + \sigma_J .$$

The claim follows. We obtain that

$$\dim \left\langle V_{\{i\}}, \{V_{\{i,j\}}\}_{j \in I \setminus \{i\}} \right\rangle = \dim \langle F^i \rangle = |I| - 1 .$$

Finally, to conclude that F^i is a face of P_I, it remains to show that P_I is on one side of the hyperplane $\langle F^i \rangle$. Let V_K be a vertex of P_I not listed in F^i, that is,

$i \notin K$. Let $J = K \cup \{i\}$. Applying Lemma 2 (ii) to the disjoint subsets K and $\{i\}$, we obtain that

$$r\big((1-t)V_K + tV_{\{i\}}\big) = V_J\,,$$

for some $r > 1$ and $0 < t < 1$. Since $V_{\{i\}}, V_J \in F^i$, this means that V_K and the origin 0 are on the same side of $\langle F^i \rangle$. Thus, F^i is a face of P_I.

It remains to prove that F_0, and F_j, F^j, $j \in I$, are *all* the faces of P_I. To do this, we consider the hyperplane extensions of these faces:

$$\begin{aligned} H_0 &= \langle F_0 \rangle = \big\langle \{C_j^o\}_{j\in I} \big\rangle\,, \\ H_j &= \langle F_j \rangle = L_{I\setminus\{j\}} = L\big(\{C_i^o\}_{i\in I\setminus\{j\}}\big)\,, \\ H^j &= \langle F^j \rangle = \langle \{V_J \,|\, j \in J\} \rangle\,. \end{aligned}$$

Each of these hyperplanes is the boundary of a half-space that contains P_I. Let $\bar{P}_I$ denote the intersection of these half-spaces. Clearly, $P_I \subset \bar{P}_I \subset T_I$. It remains to show that $P_I = \bar{P}_I$, or equivalently, that the vertices of $\bar{P}_I$ are the same as the vertices of P_I (given in Proposition 1). To do this, we consider the vertices of $\bar{P}_I$ as (nonredundant) intersections of the hyperplanes above. To obtain a vertex, we need to take at least $|I|$ hyperplanes as $\dim L_I = |I|$. We split the discussion into two cases according to whether H_0 is participating in the intersection or not.

Case (i): Assume that H_0 is participating in the intersection. We first show that, for each $i \in I$, $H_0 \cap H^i$ intersects F_0 at the single point C_i^o so that the remaining part of the intersection is redundant (that is, disjoint from $\bar{P}_I$). Let $X \in H^i$. As above, we have

$$X = \alpha_i V_{\{i\}} + \sum_{j\in I\setminus\{i\}} \alpha_j V_{\{i,j\}} \tag{16}$$

and

$$\alpha_i + \sum_{j\in I\setminus\{i\}} \alpha_j = 1\,. \tag{17}$$

Expanding the right-hand side of (16) as in (15) we obtain

$$X = \left(\alpha_i + \sum_{j\in I\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}}\frac{\lambda_i}{1+\lambda_i}\right) C_i^o + \sum_{j\in I\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}}\frac{\lambda_j}{1+\lambda_j} C_j^o\,. \tag{18}$$

Now, $X \in F_0$ iff $\alpha_j \geq 0$, $j \in I \setminus \{i\}$, and

$$\alpha_i + \sum_{j\in I\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}}\frac{\lambda_i}{1+\lambda_i} \geq 0$$

and

$$\alpha_i + \sum_{j\in I\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}}\frac{\lambda_i}{1+\lambda_i} + \sum_{j\in I\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}}\frac{\lambda_j}{1+\lambda_j} = 1\,.$$

This last equality reduces to

$$\alpha_i + \sum_{j\in I\setminus\{i\}} \alpha_j \frac{2-\sigma_{\{i,j\}}}{1-\sigma_{\{i,j\}}} = 1\,.$$

Combining this with (17), we obtain

$$\sum_{j\in I\setminus\{i\}} \alpha_j \frac{1}{1-\sigma_{\{i,j\}}} = 0\,.$$

Since the coefficients are nonnegative, this is possible only if $\alpha_j = 0$, $j \in I \setminus \{i\}$. By (17), $\alpha_i = 1$ and $X = V_{\{i\}} = C_i^o$ follows.

Thus, besides H_0, the only participating hyperplanes we need to consider are H_j, $j \in I$. There must be at least $|I| - 1$ of these. On the other hand, there cannot be $|I|$ of these as their intersection is the redundant origin $0 \notin T_I$. Hence, there exists $i \in I$, such that the participating hyperplanes are H_0 and H_j, $j \in I \setminus \{i\}$. The intersection of these is clearly C_i^o. Case (i) follows.

Remark. Although in our proof some of the conclusions will be used below, the referee pointed out the following elegant and direct proof of Case (i). The vertices of $\bar{P}_I$ contained in H_0 are exactly the vertices of the polytope $\bar{P}_I \cap H_0$, since $\bar{P}_I$ is on one side of H_0. By $P_I \subset \bar{P}_I \subset T_I$ and $P_I \cap H_0 = T_I \cap H_0 = F_0$, $\bar{P}_I \cap H_0 = F_0$. So the vertices of $\bar{P}_I$ in H_0 are the vertices of F_0, which are C_i^o, $i \in I$.

Case (ii): We first show that for the same $i \in I$, the hyperplanes H_i and H^i cannot participate together in the intersection, in particular, that there are exactly $|I|$ participating hyperplanes, one for each index in I. Let $X \in H^i \cap H_i$. Write X as in (16) with (17). Expanding as in (18), $X \in H_i$ forces the coefficient of C_i^o to vanish:

$$\alpha_i + \sum_{j\in I\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}} \frac{\lambda_i}{1+\lambda_i} = 0\,.$$

This, combined with (17) gives

$$\sum_{j\in I\setminus\{i\}} \left(1 - \frac{1}{1-\sigma_{\{i,j\}}} \frac{\lambda_i}{1+\lambda_i}\right) \alpha_j = 1\,.$$

The coefficient of α_j is negative since $(1-\sigma_{\{i\}})/(1-\sigma_{\{i,j\}}) > 1$. For nonredundancy, X must be in T_I, in particular, $\alpha_j \geq 0$, $j \in I \setminus \{i\}$. This is a contradiction.

Let $J \subset I$ parametrize the participating hyperplanes H^j, $j \in J$, and let its complement, $K = I \setminus J$, parametrize the participating hyperplanes H_k, $k \in K$. As above we may assume that J is nonempty. By definition, V_J is contained in all these hyperplanes. It remains to show that

$$\bigcap_{j\in J} H^j \cap \bigcap_{k\in K} H_k = \{V_J\}\,.$$

Let X be in the intersection. First, since $X \in \bigcap_{k\in K} H_k$, it can be written as

$$X = \sum_{j\in J} \mu_j C_j^o \,, \tag{19}$$

where (for nonredundancy) we may assume that $\sum_{j\in J} \mu_j \geq 1$ and $\mu_j \geq 0$, $j \in J$. Now, fix $i \in J$, so that $X \in H^i$. We thus have (16)–(18). Comparing these with (19), we see that $\alpha_k = 0$ for $k \in K$, so that in (16)–(18) I can be replaced by J. Moreover, comparing coefficients, we obtain

$$\mu_i = \alpha_i + \frac{\lambda_i}{1+\lambda_i} \sum_{j\in J\setminus\{i\}} \frac{\alpha_j}{1-\sigma_{\{i,j\}}}$$

and

$$\mu_j = \frac{\alpha_j}{1-\sigma_{\{i,j\}}} \frac{\lambda_j}{1+\lambda_j} \,, \quad j \in J \setminus \{i\} \,. \tag{20}$$

Solving for α_i, we also get

$$\alpha_i = \mu_i - \frac{\lambda_i}{1+\lambda_i} \sum_{j\in J\setminus\{i\}} \frac{1+\lambda_j}{\lambda_j} \mu_j \,. \tag{21}$$

Expressing the α's in terms of the μ's using (20)–(21), after a simple computation, (17) reduces to

$$\mu_i - \sum_{j\in J\setminus\{i\}} \frac{\mu_j}{\lambda_j} = 1 \,.$$

We now vary $i \in J$ and consider this as a system of equations for μ_j, $j \in J$. We see that

$$\mu_i \left(1 + \frac{1}{\lambda_i}\right) = c \,,$$

where c is a constant, independent of i. The value of the constant can be determined by substitution:

$$c = \frac{1}{1-\sigma_J} \,.$$

We obtain that

$$\mu_i = \frac{1}{1-\sigma_J} \frac{\lambda_i}{1+\lambda_i}$$

and $X = V_J$. Case (ii) follows.

The proof of the proposition is complete. □

Remark. Let $j \in I$. Then, for $i \in I \setminus \{j\}$, $V_{I\setminus\{i\}} \in H^j$. Since $V_I \in H^j$, by Lemma 1, we also have $C_i \in H^j$. We obtain that $H^j = \langle V_I, \{C_i\}_{i\in I\setminus\{j\}}\rangle$. Thus, apart from the base $\langle \{C_i\}_{i\in I}\rangle$, H^j, $j \in I$, are the bounding hyperplanes of the simplex $[V_I, \{C_i\}_{i\in I}]$ in L_I. The following simple picture emerges: P_I is the intersection of this simplex with the truncated cone T_I.

For $I \subset \mathcal{I}$, we define $B_I = T_I \cap \mathcal{L}$. For $I = \{i_1, \ldots, i_k\}$, we also write $B_I = B(C_{i_1}, \ldots, C_{i_k})$. The main result of this section is the following:

Theorem 2. $B_I \subset P_I$.

Remark. B_I is the 'bulge' for the linear slice $\mathcal{L} \cap L_I$ over $[\{C_i^o\}_{i\in I}]$ discussed in Section 1.

Proof. We will proceed by induction with respect to $|I|$. The theorem holds for $|I| = 2$ by inspection of Example 1.

We will show that B_I is on the same side of the hyperplane extension of each face of the covering polytope P_I. Let H be a hyperplane extension of a face F of P_I. By Proposition 2, $H = H_0$ or $H = H_i = L_{I\setminus\{i\}}$, or $H = H^i$ for some $i \in I$. Since $B_I \subset T_I$, and the hyperplane extensions of the faces of T_I are H_0 and H_i, $i \in I$, we may assume that $H = H^j$ for some $j \in I$.

It is enough to show that the interior of B_I is on the same side of H as the origin 0. Let $X \in \operatorname{int} B_I$. Let $i \in I \setminus \{j\}$. Then $j \in I \setminus \{i\}$ so that $V_{I\setminus\{i\}} \in F \subset H$. By Lemma 1, we also have $C_i \in H$.

$C_i \notin T_I$ since $C_i = -\lambda_i C_i^o$. Since T_I is convex, the line segment $[X, C_i]$ intersects the boundary of T_I at a unique point $X_i \in \partial T_I$.

C_i and C_i^o are at opposite sides of the hyperplane $L_{I\setminus\{i\}}$. Thus C_i and X are also on opposite sides of $L_{I\setminus\{i\}}$. Hence the line segment $[X, C_i]$ intersects $L_{I\setminus\{i\}}$ at a unique point $Y_i \in L_{I\setminus\{i\}}$. Note that, by convexity, $X_i, Y_i \in \mathcal{L}$.

Case (i): $X_i = Y_i$. Since $X_i \in \partial T_I \cap L_{I\setminus\{i\}}$, we also have $X_i \in B_{I\setminus\{i\}}$. By the induction hypothesis, $B_{I\setminus\{i\}} \subset P_{I\setminus\{i\}}$, so that $X_i \in P_{I\setminus\{i\}}$. Consider $H \cap L_{I\setminus\{i\}}$. This is a hyperplane extension of a face of $P_{I\setminus\{i\}}$ in $L_{I\setminus\{i\}}$ and it contains $V_{I\setminus\{i\}}$. We see that X_i and 0 are on the same side of $H \cap L_{I\setminus\{i\}}$ in $L_{I\setminus\{i\}}$. Thus X_i and 0 are on the same side of H. Since $X_i \in [C_i, X]$ and $C_i \in H$, we obtain that X and 0 are on the same side of H.

Case (ii): $X_i \neq Y_i$. We first claim that $X_i \in F_0$. Indeed, since $X_i \in \partial T_I$, the only other possibility in this case would be $X_i \in L_{I\setminus\{k\}}$ for some $k \in I \setminus \{i\}$. Write $X = \sum_{l\in I} \mu_l C_l^o$ with $\sum_{l\in I} \mu_l > 0$ and $\mu_l > 0$, $l \in I$. (Recall that X is in the interior of B_I.) Then, by the definition of X_i, for some $0 < t < 1$, we have

$$X_i = (1-t)X + tC_i = (1-t)\sum_{l\in I} \mu_l C_l^o - t\lambda_i C_i^o\,.$$

If $X_i \in L_{I\setminus\{k\}}$ then $(1-t)\mu_k = 0$, a contradiction. The claim follows, and $X_i \in F_0$.

Since C_i^o and 0 are on the same side of H so is X_i. As before, X and 0 are on the same side of H. The theorem follows. $\square$

Theorem 2 can be interpreted in terms of the bulging function $\beta : F_0 \to \mathbf{R}$ defined, for $X \in F_0$, as the largest number such that $\beta(X)X \in \partial\mathcal{L}$.

Proposition 3. *$1/\beta$ is a convex function.*

Proof. Let $X_0, X_1 \in F_0$, $X_0 \neq X_1$, and write $X_t = (1-t)X_0 + tX_1$, $0 \leq t \leq 1$. Let $Y_0 = \beta(X_0)X_0$ and $Y_1 = \beta(X_1)X_1$. Then, $Y_0, Y_1 \in \partial\mathcal{L}$. Let $r \geq 1$ and $0 \leq s \leq 1$ be such that $rX_t = (1-s)Y_0 + sY_1$. A simple computation then shows that

$$r = (1-s)\beta(X_0) + s\beta(X_1)$$

and

$$s = \frac{t\beta(X_0)}{(1-t)\beta(X_1) + t\beta(X_0)}\,.$$

Substituting, we obtain

$$r = \frac{\beta(X_0)\beta(X_1)}{(1-t)\beta(X_1) + t\beta(X_0)}\,.$$

Finally, convexity of $\mathcal{L}$ says that $r \leq \beta(X_t)$ with equality iff $[Y_0, Y_1] \subset \partial\mathcal{L}$. Putting everything together, we arrive at

$$(1-t)\frac{1}{\beta(X_0)} + t\frac{1}{\beta(X_1)} \geq \frac{1}{\beta((1-t)X_0 + tX_1)}\,.$$

The proposition follows. □

We can also define the bulging function $\beta_I : F_0 \to \mathbf{R}$ for the covering polytope P_I analogously for $X \in F_0$, as the largest number with $\beta_I(X)X \in \partial P_I$. Theorem 2 then says that $1 \leq \beta \leq \beta_I$. The proof of Proposition 3 also shows that $1/\beta_I$ is convex and *piecewise linear*. For a vertex $V_J \in P_I$, $J \subset I$, we have

$$\beta_I\left(\frac{1-\sigma_J}{|J|-\sigma_J}V_J\right) = \frac{|J|-\sigma_J}{1-\sigma_J}\,.$$

The smallest value of $1/\beta_I$ is thus

$$\min_{F_0}\frac{1}{\beta_I} = \min_{J\subset I}\frac{1-\sigma_J}{|J|-\sigma_J} = \frac{1-\sigma_I}{|I|-\sigma_I}\,.$$

We obtain the following:

Corollary. *The maximum bulging of the slice $\mathcal{L} \cap L_I$ over F_0 is*

$$\max_{F_0}\beta \leq \max_{F_0}\beta_I = \beta\left(\frac{1-\sigma_I}{|I|-\sigma_I}V_I\right) = \frac{|I|-\sigma_I}{1-\sigma_I}\,.$$

4. The outer simplex

Let $\mathcal{L} \subset \mathcal{E}$ be a compact convex body in a Euclidean vector space $\mathcal{E}$ of dimension $m \geq 2$, such that $\mathcal{O} = 0 \in \operatorname{int}\mathcal{L}$ is a base point. Assume that $\mathcal{L}$ satisfies (1) and (5). We now assume that the set of points $\{C_0, \ldots, C_m\} \subset \partial\mathcal{L}$ of Section 2 is a configuration of $\mathcal{L}$ ($\mathcal{O} \in [C_0, \ldots, C_m]$) with corresponding index-set $\mathcal{I} = \{0, \ldots, m\}$. Letting

$$\sigma = \sum_{i=0}^{m}\frac{1}{1+\lambda_i}\,,$$

we assume that

$$\sigma < 1 + \frac{1}{m+1}\,. \tag{22}$$

Proposition 4. $[C_0, \ldots, C_m]$ *is an* m*-simplex with* $\mathcal{O}$ *in its interior.*

Proof. If $[C_0, \ldots, C_m]$ were not a simplex or $\mathcal{O}$ were not in the interior of the simplex $[C_0, \ldots, C_m]$ then there would exist $0 \leq i \leq m$ such that $\{C_0, \ldots, \widehat{C_i}, \ldots, C_m\}$ would be a subconfiguration in the sense that $\mathcal{O} \in [C_0, \ldots, \widehat{C_i}, \ldots, C_m]$. By a result of [5] analogous to (2), we have

$$\sum_{j=0;\, j\neq i}^{m} \frac{1}{1+\lambda_j} \geq 1\,.$$

Thus, by (1), we have

$$\sigma \geq 1 + \frac{1}{1+\lambda_i} \geq 1 + \frac{1}{m+1}\,.$$

This contradicts to (22). The proposition follows. $\square$

For $0 \leq i \leq m$, we let

$$\sigma_i = \sum_{j=0;\, j\neq i}^{m} \frac{1}{1+\lambda_j}\,.$$

By (1) and (22), we have

$$\sigma_i + \frac{1}{1+m} \leq \sigma_i + \frac{1}{1+\lambda_i} = \sigma < 1 + \frac{1}{1+m}$$

so that $\sigma_i < 1$. In addition, since $[C_0, \ldots, C_m]$ is an m-simplex with 0 in its interior, $\{C_0, \ldots, \widehat{C_i}, \ldots, C_m\}$ is linearly independent. The construction of Section 3 applies with $I = \mathcal{I} \setminus \{i\}$. (Note that (7) is satisfied.). We obtain the polytope $P_i = P(C_0, \ldots, \widehat{C_i}, \ldots, C_m)$ containing the bulge $B_i = B(C_0, \ldots, \widehat{C_i}, \ldots, C_m)$ (Theorem 2) and the vertex $V_i = V(C_0, \ldots, \widehat{C_i}, \ldots, C_m)$. By Theorem 2 in Section 3, we have

$$\mathcal{L} = [C_0^o, \ldots, C_m^o] \cup \bigcup_{i=0}^{m} B_i \subset [C_0^o, \ldots, C_m^o] \cup \bigcup_{i=0}^{m} P_i\,. \tag{23}$$

The main result of this section is the following:

Theorem 3. *We have*

$$\mathcal{L} \subset [V_0, \ldots, V_m]\,. \tag{24}$$

Before the proof we introduce a technique that compares the geometry of $[V_0, \ldots, V_m]$ with the geometry of the inscribed simplex $[C_0, \ldots, C_m]$. Throughout, we let $0 \leq i \leq m$.

Let $\bar{\lambda}_i$ denote the distortion of C_i *of the simplex* $[C_0, \dots, C_m]$ (relative to $\mathcal{O}$). Clearly,

$$\lambda_i \leq \bar{\lambda}_i \,. \tag{25}$$

By (3)–(4) we also have

$$\sum_{i=0}^{m} \frac{1}{1+\bar{\lambda}_i} = 1\,, \quad \text{and} \quad \sum_{i=0}^{m} \frac{1}{1+\bar{\lambda}_i} C_i = 0\,. \tag{26}$$

We substitute these into the defining formula for V_i:

$$V_i = -\frac{1}{1-\sigma_i} \sum_{j=0;\, j\neq i}^{m} \frac{1}{1+\lambda_j} C_j\,,$$

and obtain

$$V_i = \frac{1}{\frac{1}{1+\lambda_i} - \epsilon} \left(\frac{1}{1+\lambda_i} C_i - \sum_{j=0}^{m} \epsilon_j C_j \right), \tag{27}$$

where

$$\epsilon_j = \frac{1}{1+\lambda_j} - \frac{1}{1+\bar{\lambda}_j} \geq 0$$

(cf. (25)), and

$$\epsilon = \sum_{j=0}^{m} \epsilon_j = \sigma - 1\,.$$

Note that $\epsilon = 0$ iff $\bar{\lambda}_i = \lambda_i$ for all $0 \leq i \leq m$ iff $\mathcal{L}$ is the simplex $[C_0, \dots, C_m] = [V_0, \dots, V_m]$. From now on we assume that this is not the case, so that $\epsilon > 0$. We define

$$C^* = \frac{1}{\epsilon} \sum_{j=0}^{m} \epsilon_j C_j \in [C_0, \dots, C_m]\,.$$

With this (27) can be written as

$$V_i - C^* = \frac{1}{1-\sigma_i} \frac{1}{1+\lambda_i} (C_i - C^*)\,. \tag{28}$$

From this it is immediately clear that $[V_0, \dots, V_m]$ is an m-simplex.

Lemma 3. *We have* $C_i \in [V_0, \dots, V_m]$.

Proof. Eliminating the denominators in (28), multiplying by ϵ_i and summing up with respect to $i = 0, \dots, m$, the definition of C^* gives

$$\sum_{i=0}^{m} \epsilon_i (1-\sigma_i)(1+\lambda_i)(V_i - C^*) = \sum_{i=0}^{m} \epsilon_i (C_i - C^*) = 0\,.$$

The coefficients are nonnegative and their sum is positive. This means that $C^* \in [V_0, \dots, V_m]$. Finally, (28) can be written as

$$\left(\frac{1}{1+\lambda_i} - \epsilon\right) V_i + \epsilon C^* = \frac{1}{1+\lambda_i} C_i \,.$$

This means that $C_i \in [V_i, C^*] \subset [V_0, \dots, V_m]$. The lemma follows. $\square$

Proof of Theorem 3. Recall that $V_i = V(C_0, \dots, \widehat{C_i}, \dots, C_m)$. Applying Lemma 1 inductively (first to $I = \{0, \dots, \widehat{i}, \dots, m\}$) and using Lemma 3, we see that, for $J \subset I$, we have $V_J \in [V_0, \dots, V_m]$. In the last step of the induction we obtain $C_j^o = V_{\{j\}} \in [V_0, \dots, V_m]$, and so

$$[C_0^o, \dots, C_m^o] \subset [V_0, \dots, V_m] \,.$$

Proposition 1 also gives

$$P_i \subset [V_0, \dots, V_m] \,, \quad i = 0, \dots, m \,.$$

Comparing these with (23) the theorem follows. $\square$

Proof of Theorem 1. The coefficient in (28) can be estimated as

$$\frac{1}{1-\sigma_i}\frac{1}{1+\lambda_i} = 1 + \frac{\sigma - 1}{\frac{1}{1+\lambda_i} + 1 - \sigma} \le 1 + \frac{\sigma - 1}{\frac{1}{1+m} + 1 - \sigma} = \frac{1}{m + 2 - (m+1)\sigma} \,. \tag{29}$$

We now define

$$C_i^* = \frac{1}{m + 2 - (m+1)\sigma}(C_i - C^*) + C^* \,, \quad i = 0, \dots, m \,, \tag{30}$$

and

$$\Delta^* = [C_0^*, \dots, C_m^*] \,.$$

By (28)–(30), we have

$$[V_0, \dots, V_m] \subset \Delta^* \,.$$

This, combined with Theorem 3, gives the first statement of Theorem 1. The second statement is clear from (30). Theorem 1 follows. $\square$

The discussion at the end of Section 3 carries over to the setting above to obtain the maximum bulging of $\mathcal{L}$ over the inscribed *opposite* simplex $[C_0^o, \dots, C_m^o] \subset \mathcal{L}$. We denote by $\beta_i : [C_0^o, \dots, \widehat{C_i^o}, \dots, C_m^o] \to \mathbf{R}$ the bulging function of the i-th face. By the corollary of Section 3, we have

$$\begin{aligned}\max_{0 \le i \le m} \max_{[C_0^o, \dots, \widehat{C_i^o}, \dots, C_m^o]} \beta_i &\le \max_{0 \le i \le m} \frac{m - \sigma_i}{1 - \sigma_i} = 1 + \frac{m-1}{1 - \sigma + \min_{0 \le i \le m} \frac{1}{1+\lambda_i}} \\ &\le 1 + \frac{m^2 - 1}{m + 2 - (m+1)\sigma} \,.\end{aligned}$$

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References

[1] M. Berger, Geometry I–II, Springer, New York, 1994.

[2] A. Dvoretzky, Some results on convex bodies and Banach spaces, in: Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem; Pergamon, Oxford (1961) 123–160.

[3] A. Dvoretzky, A theorem on convex bodies and applications to Banach spaces, Proc. Nat. Acad. Sci. U.S.A. **45** (1959) 223–226.

[4] B. Grünbaum, Convex Polytopes, Springer, New York, 2003.

[5] G. Toth, Simplicial intersections of a convex set and moduli for spherical minimal immersions, Michigan Math. J. Vol. **52** (2004) 341–359.

[6] G. Toth, Finite Möbius Groups, Minimal Immersions of Spheres, and Moduli, Springer, New York, 2002.

[7] G. Toth, Infinitesimal rotations of isometric minimal immersions between spheres, Amer. J. Math. **122** (2000) 117–152.

[8] F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

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