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TORSION AND DEFORMATION OF CONTACT METRIC STRUCTURES ON 3-MANIFOLDS

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Abstract. S.-S. Chern raised the question of determining those compact 3-manifolds M admitting a contact metric structure whose characteristic vector field generates a one-parameter group of isometries. S. Tachibana showed that the first betti number of these spaces must be even, and H. Sato proved that the second homotopy group of M is zero unless M is homotopy equivalent to $S^1 \times S^2$. A. Weinstein pointed out that M is a Seifert fibre space over an orientable surface. In this paper, it is shown as a consequence of a more general theorem that if, in addition, the scalar curvature is suitably bounded below by a negative constant, then the metric may be deformed to a metric of positive constant sectional curvature. Thus, if the manifold is simply connected it is diffeomorphic with the 3-sphere.

1. Introduction. Lutz and Martinet [6] showed that every compact and oriented 3-manifold M possesses a contact structure, that is, M carries a globally defined 1-form ω with $\omega \wedge d\omega \neq 0$ everywhere. One can associate with ω a vector field X_0 (determined by $\omega(X_0) = 1$ and $d\omega(X_0, \cdot) = 0$), a linear transformation field φ (which is a complex structure on $B = \ker \omega$, and has kernel $\boldsymbol{R}X_0$) and a Riemannian metric g (with respect to which φ is skew-symmetric and $\omega = g(X_0, \cdot)$). The resulting *contact metric structure* $(\varphi, X_0, \omega, g)$ is said to be *K-contact* if X_0 is a Killing field with respect to g. Chern and Hamilton [3] introduced the torsion invariant $c = |\tau|$, where $\tau = L_{X_0} g$ is the Lie derivative of g with respect to X_0, and conjectured that for fixed ω, with X_0 inducing a Seifert foliation, there exists a complex structure $\varphi | B$ on B such that the 'Dirichlet energy'

$$E(\tau) = \frac{1}{2} \int_M c^2 \operatorname{vol}(M, g)$$

is critical over all CR-structures. Should this conjecture be true, $\nabla_{X_0} \tau$ must vanish, or equivalently, the sectional curvature of all planes at a

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given point perpendicular to B are equal (cf. [3]). The torsion τ is then said to be *critical.*

We now state our main result.

THEOREM. *Let M be a compact oriented 3-manifold with contact metric structure $(\varphi, X_0, \omega, g)$ and critical torsion. If there exists a constant a, $0 < a < 1$, such that $c < 2a$ and*

$$(1) \qquad |\sigma|^2 < 2\left(a^2 - \frac{c^2}{4}\right)\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}c\right) ,$$

where $\sigma = (\iota_{X_0}S)|B$, S denotes the Ricci tensor and r the scalar curvature, then M admits a contact metric of positive Ricci curvature. If, in addition, M is simply connected, it is diffeomorphic with the 3-sphere.

COROLLARY. *Let M be a compact oriented 3-manifold with K-contact metric structure $(\varphi, X_0, \omega, g)$. If $r > -2$, then M admits a contact metric of positive Ricci curvature.*

If the torsion invariant c is critical, the Webster curvature (cf. [3]) $W = (r+4)/8$ is independent of c, and the condition $r > -2$ is equivalent to $W > 1/4$.

An analogous result restricting the Ricci curvature of g was obtained in [4].

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2. Contact manifolds. A $(2n+1)$-dimensional C^∞ manifold is called a *contact manifold* if it carries a global 1-form ω with the property that $\omega \wedge (d\omega)^n \neq 0$ everywhere. It has an underlying *almost contact metric structure* $(\varphi, X_0, \omega, g)$, that is,

$$\omega(X_0) = 1,\ \varphi X_0 = 0,\ \varphi^2 = -I + \omega \otimes X_0,\ \omega = g(X_0, \cdot),\ g(\varphi X, Y) = -g(X, \varphi Y) ,$$

where I is the identity transformation. Moreover,

$$g(X, \varphi Y) = d\omega(X, Y) .$$

If the almost complex structure J on $M \times \boldsymbol{R}$ defined by $J(X, fd/dt) = (\phi X - fX_0, \omega(X)d/dt)$, where f is a real-valued function, is integrable, the contact structure is said to be *normal.* In this case, X_0 is a Killing vector field, that is $\tau = 0$. Conversely, if $n = 1$, and X_0 is a Killing field, then M is normal.

We introduce the φ-torsion ψ which is closely related to τ. It is defined by $\psi(X, Y) = g((L_{X_0}\varphi)X, Y)$, and is known to be symmetric (cf. [2]).

PROPOSITION 1. (i) $\tau(X_0, \cdot) = \psi(X_0, \cdot) = 0$,

(ii) $\psi(X, Y) = -\tau(X, \varphi Y)$, *or equivalently*, $\tau(X, Y) = \psi(X, \varphi Y)$, $X, Y \in C^\infty(TM)$.

(iii) φ *is symmetric with respect to both* τ *and* ψ,

(iv) $\tau(\varphi X, \varphi Y) = -\tau(X, Y)$ *and* $\psi(\varphi X, \varphi Y) = -\psi(X, Y)$, $X, Y \in C^\infty(TM)$,

(v) $\operatorname{trace} \tau = \operatorname{trace} \psi = 0$,

(vi) $\tau(X, Y) = \psi(\varphi^{1/2} X, \varphi^{1/2} Y)$, $X, Y \in C^\infty(TM)$,

(vii) $|\tau| = |\psi| (=c)$.

PROOF. (i) For contact metric structures, $\nabla_{X_0} X_0 = 0$ (cf. [2]). Hence,

$$\tau(X_0, X) = (L_{X_0} g)(X_0, X) = X_0 \cdot g(X_0, X) - g(X_0, [X_0, X]) = g(X_0, \nabla_X X_0)$$
$$= \frac{1}{2} X \cdot g(X_0, X_0) = 0 , \quad X \in C^\infty(TM) .$$

The statement for ψ follows from $(L_{X_0}\varphi) X_0 = 0$.

(ii)
$$\begin{aligned} \tau(X, \varphi Y) &= (L_{X_0} g)(X, \varphi Y) = X_0 \cdot g(X, \varphi Y) \\ &\quad - g([X_0, X], \varphi Y) - g(X, [X_0, \varphi Y]) \\ &= X_0 \cdot g(X, \varphi Y) - g([X_0, X], \varphi Y) \\ &\quad - g(X, \varphi[X_0, Y]) - \psi(X, Y) . \end{aligned}$$

On the other hand, $(d\omega)(X, Y) = g(X, \varphi Y)$, so

$$\begin{aligned} (L_{X_0}(d\omega))(X, Y) &= X_0 \cdot (d\omega)(X, Y) - d\omega([X_0, X], Y) - d\omega(X, [X_0, Y]) \\ &= X_0 \cdot g(X, \varphi Y) - g([X_0, X], \varphi Y) - g(X, \varphi[X_0, Y]) \end{aligned}$$

which vanishes since $L_{X_0}(d\omega) = 0$.

(iii) Follows directly from (ii) since τ and ψ are symmetric in their arguments.

(iv) By repeated application of (ii), we obtain

$$\tau(\varphi X, \varphi Y) = -\psi(\varphi X, Y) = -\psi(Y, \varphi X) = -\tau(Y, X) = -\tau(X, Y) .$$

A similar proof holds for ψ.

(v) Choosing a φ-basis $\{E^i, \varphi E^i, X_0\}_{i=1}^n$,

$$\operatorname{trace} \tau = \sum_{i=1}^n \tau(E^i, E^i) + \sum_{i=1}^n \tau(\varphi E^i, \varphi E^i) + \tau(X_0, X_0) = 0$$

by (i) and (iv).

(vi) By (i), we may assume that $X, Y \in C^\infty(B)$, $B = \ker \omega$. Since $\varphi^{1/2} = (I + \varphi)/\sqrt{2}$ on B,

$$\psi(\varphi^{1/2} X, \varphi^{1/2} Y) = \frac{1}{2} \psi(X + \varphi X, Y + \varphi Y) = \psi(X, \varphi Y) = \tau(X, Y)$$

by (ii)-(iv).

(vii) Follows from (vi) since $\varphi^{1/2}$ is an isometry on B.

The integrability tensor $N^{(1)}$ occurring in the normality condition for contact metric structures in [2] is given by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2d\omega(X, Y)X_0\ , \quad X, Y \in C^\infty(TM)\ ,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. For fixed $X \in C^\infty(TM)$, we consider the 2-tensor μ_X on M defined by

$$\mu_X(Y, Z) = g(N^{(1)}(X, Y), \varphi Z)\ , \quad Y, Z \in C^\infty(TM)\ .$$

Clearly, $\mu_X(\cdot, X_0) = 0$ and

$$(2) \qquad g((\nabla_X\varphi)Y, Z) = \frac{1}{2}\mu_Y(Z, X) + g(Y, X)\omega(Z) - g(Z, X)\omega(Y)\ , \qquad X, Y, Z \in C^\infty(TM)$$

(see [2]).

PROPOSITION 2. (i) $\mu_{X_0} = -\psi$,
(ii) $\mu_X(\varphi Y, \varphi Z) = -\mu_X(Y, Z)$, $Y, Z \in C^\infty(B)$, $B = \ker \omega$,
(iii) trace $\mu_{X_0} = 0$.

PROOF. (i) For $Y, Z \in C^\infty(TM)$,

$$\begin{aligned}\mu_{X_0}(Y, Z) &= g([\varphi, \varphi](X_0, Y), \varphi Z) = g(\varphi^2[X_0, Y], \varphi Z) - g(\varphi[X_0, \varphi Y], \varphi Z)\\ &= g(\varphi[X_0, Y], Z) - g([X_0, \varphi Y], Z) + \omega([X_0, \varphi Y])g(X_0, Z)\\ &= -g((L_{X_0}\varphi)Y, Z) + \omega((L_{X_0}\varphi)Y)g(X_0, Z) = -\psi(Y, Z)\ ,\end{aligned}$$

since $\omega((L_{X_0}\varphi)Y) = g(X_0, (L_{X_0}\varphi)Y) = \tau(X_0, Y) = 0$ by (i) of Proposition 1.

(ii) By the previous step and (iv) of Proposition 1, we may assume that $X \in C^\infty(B)$. Then,

$$\begin{aligned}\mu_X(\varphi Y, \varphi Z) + \mu_X(Y, Z) &= -g([\varphi, \varphi](X, \varphi Y), Z) + g([\varphi, \varphi](X, Y), \varphi Z)\\ &= 0\ .\end{aligned}$$

(iii) As in (v) of Proposition 1, we choose a φ-basis and apply (i) and (ii).

3. Proof of the Theorem. We first replace g by the new metric

$$(3) \qquad \tilde{g} = ag + b\omega \otimes \omega\ ,$$

where $a, b \in \boldsymbol{R}$ with $a > 0$, $a + b > 0$. Then, the corresponding Ricci tensors $\tilde{S}$ and S are related by the formula

$$(4) \qquad \tilde{S} = S - \frac{2b}{a}g + \frac{2b}{a^2}[(2n + 1)a + nb]\omega \otimes \omega$$

$$+\frac{b}{a+b}\psi+\frac{b}{2(a+b)}\nabla_{X_0}\tau \; .$$

To see this, let W be the tensor field defined by $W^i_{jk}=\tilde{\Gamma}^i_{jk}-\Gamma^i_{jk}$. Then, by (3),

$$W^i_{jk}=-\frac{b}{a}(\varphi^i_{\cdot j}\omega_k+\varphi^i_{\cdot k}\omega_j)+\frac{b}{2(a+b)}X^i_0\tau_{jk} \; ,$$

where $\tau_{jk}=\nabla_j\omega_k+\nabla_k\omega_j$ (see [4]). Now,

$$\begin{aligned}\tilde{S}_{jk}-S_{jk}=\tilde{R}^i_{\cdot jki}-R^i_{\cdot jki}&=\nabla_i W^i_{jk}-\nabla_k W^i_{ji}+W^i_{ri}W^r_{jk}-W^i_{rk}W^r_{ji}\\&=-\frac{b}{a}\{\omega_k\nabla_i\varphi^i_{\cdot j}+\omega_j\nabla_i\varphi^i_{\cdot k}+\varphi^i_{\cdot j}\nabla_i\omega_k+\varphi^i_{\cdot k}\nabla_i\omega_j\}\\&\quad+\frac{b}{2(a+b)}X^i_0\nabla_i\tau_{jk}+\frac{2nb^2}{a^2}\omega_j\omega_k-\frac{b^2}{a(a+b)}\psi_{jk} \; ,\end{aligned}$$

where we used div $X_0=\operatorname{trace}\nabla\omega=(1/2)\operatorname{trace}\tau=0$ (by (v) of Proposition 1), Proposition 1 (ii), as well as various well-known identities for contact metric structures. Since

$$\begin{aligned}\varphi^i_{\cdot j}\nabla_i\omega_k+\varphi^i_{\cdot k}\nabla_i\omega_j&=\varphi^i_{\cdot j}\tau_{ik}-\varphi^i_{\cdot j}\nabla_k\omega_i+\varphi^i_{\cdot k}\tau_{ij}-\varphi^i_{\cdot k}\nabla_j\omega_i\\&=\varphi^i_{\cdot j}\tau_{ik}+\omega_i\nabla_k\varphi^i_{\cdot j}+\varphi^i_{\cdot k}\tau_{ij}+\omega_i\nabla_j\varphi^i_{\cdot k}\\&=-2\psi_{jk}+\omega_i(\nabla_k\varphi^i_{\cdot j}+\nabla_j\varphi^i_{\cdot k}) \; ,\end{aligned}$$

we obtain

$$\begin{aligned}\tilde{S}_{jk}-S_{jk}=&-\frac{b}{a}\{\omega_k\nabla_i\varphi^i_{\cdot j}+\omega_j\nabla_i\varphi^i_{\cdot k}+\omega_i(\nabla_k\varphi^i_{\cdot j}+\nabla_j\varphi^i_{\cdot k})\}\\&+\frac{b}{2(a+b)}\nabla_{X_0}\tau_{jk}+\frac{2nb^2}{a^2}\omega_j\omega_k+\frac{2b}{a}\Big(1-\frac{b}{2(a+b)}\Big)\psi_{jk} \; .\end{aligned}$$

To simplify the terms in $\{\cdots\}$, we use (2) and the properties of μ_X given in Proposition 2. Thus,

$$\begin{aligned}\{\cdots\}&=\frac{1}{2}\omega_k\operatorname{trace}\mu_{\partial/\partial x^j}+\frac{1}{2}\omega_j\operatorname{trace}\mu_{\partial/\partial x^k}-\frac{1}{2}\mu_{X_0}(\partial/\partial x^j,\partial/\partial x^k)\\&\quad-\frac{1}{2}\mu_{X_0}(\partial/\partial x^k,\partial/\partial x^j)+2g_{jk}-2(2n+1)\omega_j\omega_k\\&=\psi_{jk}+2g_{jk}-2(2n+1)\omega_j\omega_k \; .\end{aligned}$$

To see this, we first re-write formula (2):

$$(2')\qquad g((\nabla_{\partial/\partial x^i}\varphi)\partial/\partial x^j,\partial/\partial x^k)=\frac{1}{2}\mu_{\partial/\partial x^j}(\partial/\partial x^k,\partial/\partial x^i)+g_{ij}\omega_k-g_{ik}\omega_j \; ,$$

that is,

$$g_{lk}\nabla_i\varphi^l_{\cdot j} = \frac{1}{2}\mu_{jki} + g_{ij}\omega_k - g_{ik}\omega_j ,$$

where $\mu_{jki} = \mu_{\partial/\partial x^j}(\partial/\partial x^k, \partial/\partial x^i)$, from which

$$\nabla_i\varphi^r_{\cdot j} = \frac{1}{2}g^{rs}\mu_{jsi} + g_{ij}X_0^r - \delta_i^r\omega_j .$$

It follows that

$$\omega_k\nabla_i\varphi^i_{\cdot j} = \frac{1}{2}\omega_k g^{is}\mu_{\partial/\partial x^j}(\partial/\partial x^s, \partial/\partial x^i) - 2n\omega_j\omega_k = \frac{1}{2}\omega_k \operatorname{trace}\mu_{\partial/\partial x^j} - 2n\omega_j\omega_k ,$$

and

$$\omega_i\nabla_k\varphi^i_{\cdot j} = \omega_i\Big(\frac{1}{2}g^{is}\mu_{jsk} + g_{kj}X_0^i - \delta_k^i\omega_j\Big) = \frac{1}{2}X_0^s\mu_{jsk} + g_{kj} - \omega_k\omega_j ,$$

from which $\{\cdots\}$ follows. This yields (4).

Now, consider the case $n = 1$, and assume that τ is critical, i.e. $\nabla_{X_0}\tau = 0$. Then, choosing $b = a^2 - a$, (4) reduces to

$$(5) \qquad \tilde{S} = S + 2(1-a)g + 2(a-1)(a+2)\omega\otimes\omega + \frac{a-1}{a}\psi .$$

To ensure that $\tilde{S} > 0$ we determine, at each point $x \in M$, the entries of the matrix of the r.h.s. of (5) with respect to a suitable φ-basis $\{E, \varphi E, X_0\}$ of T_xM, and compute the respective subdeterminants along the main diagonal. First, assume that $\sigma_x \neq 0$ and choose $E \in \ker\sigma_x$, $|E| = 1$, such that $\sigma(\varphi E) = |\sigma|$. Then,

$$\tilde{S}(X_0, X_0) = S(X_0, X_0) - 2(1-a^2) = 2\Big(a^2 - \frac{c^2}{4}\Big)$$

since

$$S(X_0, X_0) = 2 - \operatorname{trace}\Big(\frac{1}{2}L_{X_0}\varphi\Big)^2 = 2\Big(1 - \frac{c^2}{4}\Big)$$

by [2]. Since τ is critical,

$$g(R(E, X_0)X_0, E) = g(R(\varphi E, X_0)X_0, \varphi E) .$$

This implies that $S(E, E) = S(\varphi E, \varphi E)$, and by polarization, $S(E, \varphi E) = 0$. It follows that

$$S(E, E) = S(\varphi E, \varphi E) = \frac{r}{2} + \frac{c^2}{4} - 1 .$$

Hence,

$$\tilde{S} = \begin{bmatrix} \tilde{S}(E, E) & \dfrac{a-1}{a}\psi(E, \varphi E) & 0 \\ \dfrac{a-1}{a}\psi(E, \varphi E) & \tilde{S}(\varphi E, \varphi E) & |\sigma| \\ 0 & |\sigma| & 2\left(a^2 - \dfrac{c^2}{4}\right) \end{bmatrix},$$

where

$$\tilde{S}(E, E) = \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}\psi(E, E)$$

and

$$\tilde{S}(\varphi E, \varphi E) = \frac{r}{2} + \frac{c^2}{4} + 1 - 2a + \frac{1-a}{a}\psi(E, E)\ ,$$

Now, we claim that $c < 2a$ together with (1) ensures that $\tilde{S} > 0$ at $x \in M$. Indeed, since $c^2 = \psi(E, E)^2 + \psi(E, \varphi E)^2$, the subdeterminants along the main diagonal of $\tilde{S}$ can be estimated as

$$\begin{aligned} \tilde{S}(E, E) &= \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}\psi(E, E) \\ &\geqq \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}c > 0\ , \end{aligned}$$

$$\begin{aligned} &\tilde{S}(E, E)\tilde{S}(\varphi E, \varphi E) - \left(\frac{a-1}{a}\right)^2\psi(E, \varphi E)^2 \\ &\quad = \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - \left(\frac{1-a}{a}\right)^2 c^2 > 0\ , \end{aligned}$$

and

$$\begin{aligned} \det \tilde{S} &= 2\left(a^2 - \frac{c^2}{4}\right)\left\{\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - \left(\frac{1-a}{a}\right)^2 c^2\right\} \\ &\quad - |\sigma|^2\left\{\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}\psi(E, E)\right\} \\ &\geqq \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a + \frac{1-a}{a}c\right) \\ &\quad \times \left\{2\left(a^2 - \frac{c^2}{4}\right)\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}c\right) - |\sigma|^2\right\} > 0\ . \end{aligned}$$

For $\sigma_x = 0$ we choose an arbitrary φ-basis, and apply the above argument. Finally, the last statement is a consequence of Hamilton [5].

Remark. It is not difficult to see that

$$\sigma = -\frac{1}{2}(\delta\psi)\circ\varphi|B$$

and

$$\iota_{X_0}\delta\psi = 0 ,$$

where $\delta: S^2T^*M \to T^*M$ is the Berger-Ebin differential operator (cf. [1]) given by $(\delta\psi)X = \text{trace}\,\nabla\psi(X, \cdot; \cdot)$, $X \in C^\infty(TM)$, and S^2 is the symmetric square. Clearly, $\sigma = 0$, if and only if $\delta\psi = 0$. This is the case for K-contact metric structures. In general, by the Berger-Ebin decomposition theorem, we have the orthogonal splitting

$$\psi = \psi_0 + L_Z g ,$$

where $Z \in C^\infty(TM)$ and $\delta\psi_0 = 0$. Thus, $\delta\psi = 0$ means that in the space $\mathscr{M}$ of all Riemannian metrics on M, the tangent vector $\psi \in T_g\mathscr{M}$ is perpendicular to the orbit of g under the group of diffeomorphisms of M.

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