On Infinitesimal and Local Rigidity of Harmonic Maps between Spheres Defined by Spherical Harmonics (*).

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Sunto. – *Si studia la deformabilità infinitesimale e locale di mappe armoniche in sfere dimostrando che le immersioni minime standard $f\colon S^2 \to S^n$ (in particolare, la superficie di Veronese) sono localmente rigide. Si dà un esempio in cui la rigidità locale non implica la rigidità infinitesimale.*

1. – Introduction and preliminaries.

To any harmonic map $f\colon M \to S^n$ [4] of a compact oriented Riemannian manifold M of dimension m into the Euclidean n-sphere S^n there is associated a finite dimensional vector space $K(f)$ [12] consisting of all Jacobi fields along f whose generalized divergence vanishes, i.e. a vector field v along f belongs to $K(f)$ if and only if

$$\text{(i)}\quad \nabla^2 v = \operatorname{trace}\langle f_*, v\rangle f_* - 2e(f)v\,,$$

$$\text{(ii)}\quad \operatorname{div}_f v = \operatorname{trace}\langle f_*, \nabla v\rangle = 0$$

are satisfied, where ∇ and $\langle\,,\rangle$ denote the canonical connection and metric of the Riemannian-connected bundle $F \otimes \Lambda^*(T^*(M))$, $F = f^*(T(S^n))$, resp., f_* is the differential of f considered as a section of the bundle $F \otimes T^*(M)$ and $e(f)$ stands for the energy density of f. Identifying the Lie algebra of Killing vector fields on S^n with $so(n+1)$ we have $so(n+1)\circ f \subset PK(f)$ [11], where $PK(f) \subset K(f)$ denotes the linear subspace of all projectable vector fields along f. The harmonic map $f\colon M \to S^n$ is said to be infinitesimally rigid if $so(n+1)\circ f = PK(f)$ [11].

The variation space $V(f)$ of $f\colon M \to S^n$ defined by

$$V(f) = \{v \in K(f) \,|\, \|v\| = \text{const}\}$$

can be geometrically interpreted as the set of vector fields v along f for which $t \to f_t = \exp\circ(tv)$, $t \in \boldsymbol{R}$, is a variation of f through harmonic maps (i.e. $f_t\colon M \to S^n$

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is harmonic for all $t \in \boldsymbol{R}$) [10]. Equivalently, $v \in V(f)$ if and only if v is a Jacobi field along f such that $e(f_t) = e(f)$, $f_t = \exp \circ (tv)$, $t \in \boldsymbol{R}$, holds [10]. The harmonic map $f: M \to S^n$ is said to be locally rigid if for every $v \in V(f) \cap PK(f)$ there exists a one-parameter subgroup $(\varphi_t) \subset SO(n+1)$ of isometries of S^n such that $f_t = \exp \circ (tv) = \varphi_t \circ f$, $t \in \boldsymbol{R}$, is valid.

The aim of this note is to continue the earlier studies ([9], [10], [11], [12] and [13]) describing infinitesimal and local behaviour of harmonic maps from the point of view of rigidity. In Sec. 2, using Calabi's rigidity theorem [2] we prove that any full homothetic minimal immersion $f: S^2 \to S^n$ has zero variation space, in particular, is locally rigid. (This can also be considered as an extension of an earlier result, settled by elementary computation, for the Veronese surface $f: S^2 \to S^4$ [7].) Finally, in Sec. 3 we prove that the harmonic map $f: S^3 \to S^4$ arising from the Hopf-Whitehead construction, [14] or [8], p. 20, applied to the real tensor product $\mu: \boldsymbol{R}^2 \times \boldsymbol{R}^2 \to \boldsymbol{R}^4$ is locally rigid but non infinitesimally rigid showing that local rigidity cannot be considered as a local version of infinitesimal rigidity introduced above.

Throughout this note all manifolds, maps, etc. will be smooth and adopting the sign conventions of [6], we use the Report [4] and [5] as general references and background for the theory of harmonic maps.

We wish to thank *A.* Lee for giving a matrix theoretical approach for the last step in proving Theorem 2.

2. - Rigidity of homothetic minimal immersions $f: S^2 \to S^n$.

A subset $H \subset S^n$ is said to be full if $H \in \boldsymbol{R}^{n+1}$ is not contained in any proper linear subspace of $\boldsymbol{R}^{n+1}$. A map $f: M \to S^n$ is full if f has a full image in S^n. The aim of this section is to prove the following:

THEOREM 1. - Any full homothetic minimal immersion $f: S^2 \to S^n$ has zero variation space, in particular, is locally rigid.

REMARK - There is a large supply of full homothetic minimal immersions $f: S^m \to S^n$ provided (partly) by the standard minimal immersions. Namely, if $\mathcal{H}_{\lambda(s)}$, $s = 2, 3, \ldots$, denotes the Euclidean vector space of spherical harmonics of order s on S^m, i.e. the eigenspace of the Laplacian Δ^{S^m} corresponding to the eigenvalue $\lambda(s) = s(s + m - 1)$ [1], with

$$\dim \mathcal{H}_{\lambda(s)} = n(s) + 1\,, \qquad n(s) = (2s + m - 1)\frac{(s + m - 2)!}{s!(m-1)!}$$

then fixing an orthonormal base $\{f^1, \ldots, f^{n(s)+1}\} \subset \mathcal{H}_{\lambda(s)}$ we have $\sum_{i=1}^{n(s)+1} (f^i)^2 = \text{const}$ [3] and hence, by a normalizing factor $N > 0$, the standard minimal immersion

$f: S^m \to S^{n(s)}$ is defined by $f(x) = (Nf^1(x), \ldots, Nf^{n(s)+1}(x))$, $x \in S^m$. Then [1] f is a full homothetic minimal immersion and different choices of the base give rise to maps that differ by performing isometries of the codomain $S^{n(s)}$. In contrast to Theorem 1 we proved in [13] that for $m \geqslant 3$ odd the standard minimal immersion $f: S^m \to S^{n(s)}$ is non locally rigid for all $s \geqslant 2$. Combining this with the rigidity theorem of M. Do Carmo - N. Wallach [3] to the effect that, for $s \leqslant 3$, full homothetic minimal immersions $f: S^m \to S^{n(s)}$ are standard we obtain, in case $s \leqslant 3$, the existence of a harmonic variation $v \in V(f)$ such that the deformed harmonic maps $f_t = \exp \circ (tv)$ will not be in general homothetic. Further, according to a result in [13], in case $s = 2$, the standard minimal immersion $f: S^m \to S^{n(2)}$ is infinitesimally rigid if and only if $m = 2$ and, moreover, local rigidity of the Veronese surface $f: S^2 \to S^4$ (i.e. case $n = n(2) = 4$ of Theorem 1) was proved in [7] by matrix computation.

The proof of Theorem 1 is preceded by the following:

LEMMA 1. – Let $H \subset S^n$ be a full subset and $\varphi: (-\varepsilon, \varepsilon) \to SO(n+1)$, $\varepsilon > 0$, a curve with $\varphi_0 = I_{n+1}$ ($=$ identity) such that for $y \in H$ the curve $t \to \varphi_t(y) \in S^n$, $|t| < \varepsilon$, is a geodesic segment parametrized by the arc-length. Then $X = d\varphi_t/dt|_{t=0} \in so(n+1)$ is a complex structure on $\boldsymbol{R}^{n+1}$, in particular, n is odd. Moreover, φ is a local one-parameter subgroup of $SO(n+1)$ and can then be extended to a (global) one-parameter subgroup all of whose trajectories are closed geodesics on S^n.

PROOF. – Identifying X, as usual, with the corresponding Killing vector field on S^n, for $y \in H$, we have

$$\varphi_t \cdot y = \varphi_t(y) = \exp(tX_y) = \cos t \cdot y + \sin t \cdot Xy\,, \qquad |t| < \varepsilon\,,$$

where the matrices φ_t and X are considered to act on the vector $y \in \boldsymbol{R}^{n+1}$ by the usual multiplication. As $H \subset S^n$ is full we get

$$\varphi_t = \cos t \cdot I_{n+1} + \sin t \cdot X\,, \qquad |t| < \varepsilon\,, \tag{1}$$

in particular, the orthogonality relation $\varphi_t \cdot \varphi_t^T = I_{n+1}$, with skew-symmetricity of X, implies

$$(\cos tI_{n+1} + \sin tX)(\cos tI_{n+1} - \sin tX) = I_{n+1}\,.$$

Differentiating twice at $t = 0$ we obtain $X^2 = -I_{n+1}$, i.e. X is a complex structure on $\boldsymbol{R}^{n+1}$. Further, for $s, t \in \boldsymbol{R}$ with $|s|$, $|t|$, $|s+t| < \varepsilon$, by (1), we get

$$\varphi_s \cdot \varphi_t = (\cos sI_{n+1} + \sin sX)(\cos tI_{n+1} + \sin tX) = \\ = \cos(s+t)I_{n+1} + \sin(s+t)X = \varphi_{s+t}\,,$$

i.e. φ is a local one-parameter subgroup of $SO(n+1)$. Denoting by $\varphi\colon \boldsymbol{R}\to SO(n+1)$ the canonical extension, the Killing vector field X is clearly induced by φ and $(\nabla_X X)|H = 0$ holds. The connected components of Zero $(\nabla_X X)$, being the intersections of the eigenspaces in $\boldsymbol{R}^{n+1}$ of the matrix X^2 with S^n (cf. proof of Th. 2 in [12]), are totally geodesic submanifolds and so fullness of H implies that $\nabla_X X = 0$ on S^n, i.e. all the integral curves of φ are closed geodesics of S^n and the lemma follows.

PROOF OF THEOREM 1. – Suppose, on the contrary, that there exists a nonzero element $v\in V(f)$ and consider the deformed harmonic maps $f_t\colon S^2\to S^n$, $t\in\boldsymbol{R}$. As there is no holomorphic quadratic differential on S^2 [5] the map f_t is conformal for all $t\in\boldsymbol{R}$, i.e. there exists a scalar $\mu_t\colon S^2\to\boldsymbol{R}$ with $\|(f_t)_*X\|^2=\mu_t\|X\|^2$, $X\in\mathfrak{X}(S^2)$. Conservation of the energy density along a harmonic variation, mentioned in Sec. 1, yields

$$\mu_t=\tfrac{1}{2}\operatorname{trace}\|(f_t)_*\|^2=e(f_t)=e(f)=\mu_0\,,\qquad t\in\boldsymbol{R}\,,$$

and we obtain that the deformed harmonic maps $f_t\colon S^2\to S^n$, $t\in\boldsymbol{R}$, are homothetic (and hence minimal [4]) immersions with the same homothety constant μ_0. Further, fullness of f being expressed by open relations, there exists $\varepsilon>0$ such that

$$f_t\colon S^2\to S^n\quad\text{ is full for }\ |t|<\varepsilon\,.$$

Then [2] CALABI's rigidity theorem applies to the full homothetic minimal immersions f and f_t, $|t|<\varepsilon$, yielding the existence of an isometry $\varphi_t\in O(n+1)$ such that

$$f_t=\varphi_t\circ f\,,\qquad |t|<\varepsilon\,,\tag{2}$$

holds. As a linear transformation, $\varphi_t\colon\boldsymbol{R}^{n+1}\to\boldsymbol{R}^{n+1}$ is determined by its values on a base of $\boldsymbol{R}^{n+1}$, in particular, φ_t occuring in (2) is uniquely determined. We claim that the curve $\varphi\colon(-\varepsilon,\varepsilon)\to O(n+1)$ is smooth. Indeed, again by fullness of f, there exist $x_1,\dots,x_{n+1}\in S^2$ such that $\{f(x_1),\dots,f(x_{n+1})\}\subset\boldsymbol{R}^{n+1}$ is a base and, for $i=1,\dots,\dots,n+1$, the curve $t\to\varphi_t(f(x_i))=f_t(x_i)$, $|t|<\varepsilon$, being smooth, the matrix function $t\to\varphi_t\in O(n+1)$, $|t|<\varepsilon$, is also smooth. Now the preceding lemma applies (with $H=\operatorname{im} f$) yielding that n is odd. On the other hand, according to CALABI's rigidity theorem [2] any full homothetic minimal immersion $f\colon S^2\to S^n$ has even dimensional codomain which is a contradiction.

Thus the theorem is proved.

3. – An example of a locally rigid but non infinitesimally rigid harmonic map $f\colon S^3\to S^4$.

The Hopf-Whitehead construction [8] applied to the real tensor product $\mu\colon\boldsymbol{R}^2\times\times\boldsymbol{R}^2\to\boldsymbol{R}^4$ gives rise to a (full) harmonic polynomial map $f\colon S^3\to S^4$ defined componentwise by spherical harmonics of order 2 as

$$f = (\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4, 2\varphi_{13}, 2\varphi_{14}, 2\varphi_{23}, 2\varphi_{24}), \tag{3}$$

where $\varphi_k(x) = x_k^2$, $\varphi_{ij}(x) = x_i x_j$, $x = (x_1, \ldots, x_4) \in \boldsymbol{R}^4$, $k = 1, \ldots, 4$, $1 \leqslant i < j \leqslant 4$. In this section we prove the following:

THEOREM 2. – For the harmonic map $f\colon S^3 \to S^4$ we have

$$\dim PK(f) = 11 \quad \text{and} \quad V(f) \cap PK(f) = \{0\},$$

in particular, as $\dim so(5) = 10$, f is non infinitesimally rigid but locally rigid.

The proof of Theorem 2 is broken up into two steps.

I. *Infinitesimal behaviour.* – Translating the vectors tangent to $S^4 \subset \boldsymbol{R}^5$ to the origin of $\boldsymbol{R}^5$ any vector field $v\colon S^3 \to T(S^4)$ along f gives rise to a vector-valued function $\hat{v}\colon S^3 \to \boldsymbol{R}^5$ with $\langle f, \hat{v}\rangle = 0$, where f is considered to take its values in $\boldsymbol{R}^5$. Then, by [7], $v \in K(f)$ if and only if

$$\Delta \hat{v} = 2e(f)\hat{v}$$

is satisfied, i.e. as $e(f) = 4$, the components $\hat{v}^r$, $r = 0, \ldots, 4$, are spherical harmonics of order 2 on S^3. Hence [1]

$$\hat{v}^r = \sum_{k=1}^{4} a_k^r \varphi_k + \sum_{i<j} b_{ij}^r \varphi_{ij}, \qquad r = 0, \ldots, 4,$$

holds for some $a_k^r, b_{ij}^r \in \boldsymbol{R}$, $k = 1, \ldots, 4$, $1 \leqslant i < j \leqslant 4$, such that $\sum_{k=1}^{4} a_k^r = 0$.

As the projectable elements of $K(f)$ are to be determined we state the following:

LEMMA 2. – A scalar $\mu\colon S^3 \to \boldsymbol{R}$ of the form

$$\mu = \sum_{k=1}^{4} a_k \varphi_k + \sum_{i<j} b_{ij}\varphi_{ij}, \qquad \sum_{k=1}^{4} a_k = 0,$$

with $a_k, b_{ij} \in \boldsymbol{R}$, $k = 1, \ldots, 4$, $1 \leqslant i < j \leqslant 4$, is projectable along f (i.e. $f(x) = f(x')$, $x, x' \in S^3$, implies $\mu(x) = \mu(x')$) if and only if

$$a_1 = a_2 = -a_3 = -a_4 \quad \text{and} \quad b_{12} = b_{34} = 0 \tag{4}$$

are valid.

PROOF. – The first coordinate function $\varphi_1+\varphi_2-\varphi_3-\varphi_4$ of f suggests to use É. CARTAN's isoparametric coordinates of degree 2, i.e. we write

$$x=(x_1,x_2,x_3,x_4)=(\cos t\cos\varphi,\ \cos t\sin\varphi,\ \sin t\cos\psi,\ \sin t\sin\psi)\in S^3\,,$$

where $0\leqslant t,\ \varphi,\ \psi<2\pi$. Then we have

$$f(x)=\big(\cos(2t),\ \sin(2t)\cos\varphi\cos\psi,\ \sin(2t)\cos\varphi\sin\psi,\ \sin(2t)\sin\varphi\cos\psi,\ \sin(2t)\sin\varphi\sin\psi\big)\,,$$

in particular, the focal varieties of S^3 parametrized by $(0,\varphi,0)$ and $(\pi/2,0,\psi)$, $0\leqslant\varphi$, $\psi<2\pi$, are mapped by f to $(1,0,0,0,0)$ and $(-1,0,0,0,0)$, respectively. Assuming that μ is projectable we obtain

$$\mu(\cos\varphi,\sin\varphi,0,0)=\text{const}\quad\text{and}\quad\mu(0,0,\cos\psi,\sin\psi)=\text{const}\,.$$

Expanding these equations into Fourier polynomials the relations (4) are easily obtained. The converse being obvious the statement follows.

By Lemma 2 a vector field v along f belongs to $PK(f)$ if and only if there exist $a^r,\ b^r_{ij}\in\boldsymbol{R}$, $1\leqslant i<j\leqslant 4$, $r=0,\dots,4$, such that

$$\hat{v}^r=a^r(\varphi_1+\varphi_2-\varphi_3-\varphi_4)+b^r_{13}\varphi_{13}+b^r_{14}\varphi_{14}+b^r_{23}\varphi_{23}+b^r_{24}\varphi_{24}\,,\qquad r=0,\dots,4\,, \tag{5}$$

holds or equivalently

$$\hat{v}=\tfrac{1}{2}A\cdot f\,, \tag{6}$$

where the r-th row of A is $(2a^r,b^r_{13},b^r_{14},b^r_{23},b^r_{24})$, $r=0,\dots,4$, and in (6) the matrix A acts on the vector f (given in (3)) by the usual multiplication. Hence, to compute $\dim PK(f)$, we have to determine the vector space of functions $\hat{v}\colon S^3\to\boldsymbol{R}^5$ of the form (6) satisfying the equation $\langle f,\hat{v}\rangle=\frac{1}{2}\langle f,A\cdot f\rangle=0$. The scalar $\langle f,A\cdot f\rangle$ is a fourth-order homogeneous polynomial whose coefficients have to vanish.

Computing these coefficients we obtain that $\langle f,A\cdot f\rangle=0$ holds if and only if A, with new variables, has the form

$$A=\begin{bmatrix}0&\alpha_1&\alpha_2&\alpha_3&\alpha_4\\-\alpha_1&0&\beta_1&\beta_2&\gamma_1\\-\alpha_2&-\beta_1&0&\gamma_2&\beta_3\\-\alpha_3&-\beta_2&-\gamma_3&0&\beta_4\\-\alpha_4&-\gamma_4&-\beta_3&-\beta_4&0\end{bmatrix} \tag{7}$$

with

$$\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 \,. \tag{8}$$

(The only relation to be taken into account among the spherical harmonics involved is $\varphi_{24}\varphi_{13} = \varphi_{23}\varphi_{14}$ which results (8), apart from this A is skew-symmetric.) In particular, $\dim PK(f) = 11$ which completes the proof of the first step.

II. *Local behaviour*. – Assuming $v \in V(f) \cap PK(f)$, with $\|v\| = 1$, the function $\hat{v}$ has the form (6)-(7) such that

$$\sum_{r=0}^{4} (\hat{v}^r)^2 = \langle \tfrac{1}{2} A \cdot f, \tfrac{1}{2} A \cdot f \rangle = 1$$

is valid on S^3. All the functions $\hat{v}^r$, $r = 0, \ldots, 4$, can also be considered as second-order homogeneous (harmonic) polynomials on $\boldsymbol{R}^4$, i.e. the last equation translates into

$$\langle A \cdot f, A \cdot f \rangle = \langle A^T A \cdot f, f \rangle = 4(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)^2 \tag{9}$$

which is valid on $\boldsymbol{R}^4$. Denoting by

$$H(c) = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & c \\ 0 & 0 & 4 & -c & 0 \\ 0 & 0 & -c & 4 & 0 \\ 0 & c & 0 & 0 & 4 \end{bmatrix}, \quad c \in \boldsymbol{R}, \tag{10}$$

and taking into account (3) it follows that (9) is equivalent to the relation $A^T A = H(c)$ for some $c \in \boldsymbol{R}$. To accomplish the proof of Theorem 2 we need to show the following:

LEMMA 3. – There are no constants $\alpha_i, \beta_i, \gamma_i \in \boldsymbol{R}$ with $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ such that the matrix A in (8) satisfies

$$A^T A = H(c) \tag{11}$$

for some $c \in \boldsymbol{R}$.

REMARK. – Writing out (11) componentwise we obtain an overdetermined system of 14 quadratic equations for the variables $\alpha_i, \beta_i, \gamma_i \in \boldsymbol{R}$, $i = 1, \ldots, 4$. Using elementary computation a tedious discussion shows that this system has no solution proving Lemma 3. Nevertheless, to reduce the amount of computations involved we first use a matrix theoretical approach due to A. LEE.

PROOF. – Denote by $M(p,q)$, $p, q \in \boldsymbol{N}$, the vector space of $(p\times q)$ matrices and $I_p \in M(p,p)$ the identity. We write, with obvious notations,

$$A = \begin{bmatrix} B & U \\ -V^T & C \end{bmatrix} \quad \text{and} \quad H(c) = \begin{bmatrix} 4I_3 & c[-e_3 e_2] \\ c[-e_3^T e_2^T] & 4I_2 \end{bmatrix}$$

where $B \in so(3)$, $C \in so(2)$ and $\{e_1, e_2, e_3\} \subset \boldsymbol{R}^3$ is the canonical base. In terms of these decompositions (11) is equivalent to the system

$$B^T B + VV^T = 4I_3\,, \tag{12}$$

$$B^T U - VC = c[-e_3 e_2]\,, \tag{13}$$

$$U^T U + C^T C = 4I_2\,. \tag{14}$$

As $B \in so(3)$ the matrices B and $B^T B$ are singular and have a joint eigenvector $0 \neq X \in \boldsymbol{R}^3$ corresponding to the zero eigenvalue. Thus, $B^T BX = BX = 0$ and so, by (12), we obtain $VV^T X = 4X$, i.e. X is an eigenvector of the matrix $VV^T \in M(3,3)$ with eigenvalue 4. Further, V being of size (3×2), rank $(VV^T) =$ rank $V \leqslant 2$ and hence there exists $0 \neq Y \in \boldsymbol{R}^3$ such that $VV^T Y (= V^T Y) = 0$ and $\langle X, Y\rangle = 0$. Again by (12) we get $B^T BY = 4Y$, i.e. Y is an eigenvector of $B^T B$ with eigenvalue 4. As B is skew, this eigenvalue must have multiplicity 2 which implies the existence of a vector $0 \neq Z \in \boldsymbol{R}^3$ with $\langle X, Z\rangle = 0$ such that $\{X, Y, Z\} \subset \boldsymbol{R}^3$ is a base and Span $\{Y, Z\} \subset \boldsymbol{R}^3$ is the eigenspace of $B^T B$ corresponding to the eigenvalue 4. Applying (12) to Z we get $B^T BZ + VV^T Z = 4Z + VV^T Z = 4Z$, i.e. Span $\{Y, Z\} \subset \boldsymbol{R}^3$ is the nullspace of VV^T.

In particular, rank $(VV^T) =$ rank $V = 1$, i.e. the coloumns of V are linearly dependent. We may suppose that the first coloumn $(\alpha_3, \beta_2, \gamma_3)$ of V is nonzero since the other case can be treated similarly. Then there exists $p \in \boldsymbol{R}$ such that

$$\alpha_4 = p\alpha_3\,, \qquad \gamma_4 = p\beta_2\,, \qquad \beta_3 = p\gamma_3 \tag{15}$$

hold. On the other hand, the vector $(-\beta_1, \alpha_2, -\alpha_1)$ is in the nullspace of B and nonzero since $B \neq 0$. So, we may choose X as

$$X = (-\beta_1, \alpha_2, -\alpha_1)\,.$$

Then, by (12), we get

$$VV^T X = (1+p^2)(-\alpha_3\beta_1 + \alpha_2\beta_2 - \alpha_1\gamma_3)(\alpha_3, \beta_2, \gamma_3) = 4(-\beta_1, \alpha_2, -\alpha_1) = 4X\,,$$

i.e. putting $q = \frac{1}{4}(1+p^2)(-\alpha_3\beta_1 + \alpha_2\beta_2 - \alpha_1\gamma_3)$ $(\neq 0)$ we obtain

$$\beta_1 = -q\alpha_3\,, \qquad \alpha_2 = q\beta_2\,, \qquad \alpha_1 = -q\gamma_3 \tag{16}$$

and hence

$$(1+p^2)(\alpha_3^2+\beta_2^2+\gamma_3^2)=4\,.$$

Further, a direct computation shows that (13) is equivalent to the system

$$(1+p^2)c(\alpha_1,\alpha_2)=4q\beta_4(p,-1)\,. \tag{17}$$

Moreover, from (11) it follows that

$$\alpha_1\alpha_4+\alpha_2\alpha_3+(\beta_1-\beta_4)(\beta_2-\beta_3)=0$$

and multiplying this with c and using (15)-(16)-(17) we get

$$\beta_4^2(p^2-1)=0\,. \tag{18}$$

Again, by making use of (15)-(16), we can write (11) componentwise is terms of the variables $\alpha_3, \beta_2, \beta_4, \gamma_1, \gamma_2, \gamma_3, p, q\in \boldsymbol{R}$ and, by (18), an easy discussion of the possible cases leads to contradiction.

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