Associating quantum vertex algebras to Lie algebra $\mathfrak{gl}_{\infty}$

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Abstract

In this paper, we present a canonical association of quantum vertex algebras and their ϕ-coordinated modules to Lie algebra $\mathfrak{gl}_{\infty}$ and its 1-dimensional central extension. To this end we construct and make use of another closely related infinite-dimensional Lie algebra.

1 Introduction

It has been known that vertex algebras can be canonically associated to both twisted and untwisted affine Lie algebras. More specifically, for an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ and for any complex number ℓ, one has a vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ (cf. [FZ], [DL], [Lia], [Li1]), based on a distinguished level ℓ generalized Verma $\hat{\mathfrak{g}}$-module often called the vacuum module, while the category of $V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules is canonically isomorphic to the category of restricted $\hat{\mathfrak{g}}$-modules of level ℓ. On the other hand, it was known (see [Li2]; cf. [FLM]) that the category of restricted modules for a twisted affine algebra $\hat{\mathfrak{g}}[\sigma]$ with σ a finite-order automorphism of $\mathfrak{g}$ is canonically isomorphic to the category of $\bar{\sigma}$-twisted modules for the vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ (which was associated to the *untwisted* affine algebra), where $\bar{\sigma}$ is the corresponding automorphism of vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$.

In this paper, we study Lie algebra $\mathfrak{gl}_{\infty}$ in the content of quantum vertex algebras in the sense of [Li3], and as the main result we obtain a canonical association of quantum vertex algebras to the one-dimensional central extension $\widetilde{\mathfrak{gl}}_{\infty}$ of Lie algebra $\mathfrak{gl}_{\infty}$, which is somewhat similar to the association of vertex algebras and twisted modules to twisted affine Lie algebras.

By definition, $\mathfrak{gl}_{\infty}$ is the Lie algebra of doubly infinite complex matrices with only finitely many nonzero entries under the usual commutator bracket. A canonical base consists of matrices $E_{m,n}$ for $m, n \in \mathbb{Z}$, where $E_{m,n}$ denotes the matrix whose only nonzero entry is the (m, n)-entry which is 1. For the natural representation of $\mathfrak{gl}_{\infty}$ on $\mathbb{C}^{\infty}$ with the standard base denoted by $\{v_n \mid n \in \mathbb{Z}\}$, we have

$$E_{m,n} v_r = \delta_{n,r} v_m \quad \text{for } m, n, r \in \mathbb{Z}.$$

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On Lie algebra $\mathfrak{gl}_\infty$, there is a 2-cocycle ψ defined by

$$\psi(E_{i,j}, E_{j,i}) = 1 = -\psi(E_{j,i}, E_{i,j}) \quad \text{if } i \leq 0 \text{ and } j \geq 1,$$
$$\psi(E_{i,j}, E_{m,n}) = 0 \quad \text{otherwise.}$$

Using this 2-cocycle one obtains a 1-dimensional central extension

$$\widetilde{\mathfrak{gl}}_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}\mathbf{k}.$$

With various motivations, one of us (H.L) has extensively studied (see [Li3-5]) vertex algebra-like structures generated by fields (on a general vector space), that behave well, but are not necessarily mutually local (cf. [DL], [Li1]). (A result of [Li1] is that every set of mutually local fields on a general vector space canonically generates a vertex algebra.) In this study, a theory of (weak) quantum vertex algebras and their modules was developed, where the notion of quantum vertex algebra naturally generalizes the notion of vertex algebra and that of vertex superalgebra. In this theory, a key role is played by the notion of $\mathcal{S}$-locality due to Etingof-Kazhdan, which is a generalization of that of locality, and the essence is that every $\mathcal{S}$-local subset of fields on a vector space W generates a (weak) quantum vertex algebra with W as a natural module. (The pioneer work [EK] has been an important inspiration for the development of this theory.)

We next explain how quantum vertex algebras are associated to Lie algebra $\widetilde{\mathfrak{gl}}_\infty$. Note that in the association of vertex algebras to affine Lie algebras, a key role was played by the canonical generating functions (or fields), where roughly speaking, the associated vertex algebras are generated by the generating functions. As the starting point of this paper, for each $m \in \mathbb{Z}$ we form a generating function

$$E(m, x) = \sum_{n \in \mathbb{Z}} E_{m,m+n} x^{-n}.$$

Then the main defining relations of $\widetilde{\mathfrak{gl}}_\infty$ can be written as

$$[E(m, x_1), E(n, x_2)] = \left(\frac{x_1}{x_2}\right)^{m-n} (E(m, x_2) - E(n, x_1) + f(m, n)\mathbf{k})$$

for $m, n \in \mathbb{Z}$, where $f : \mathbb{Z}^2 \to \{-1, 0, 1\}$ is a function determined by the 2-cocycle ψ (see Section 2). The generating functions $E(m, x)$ ($m \in \mathbb{Z}$) are *not* mutually local, but for any (suitably defined) restricted $\widetilde{\mathfrak{gl}}_\infty$-module W, $E(m, x)$ for $m \in \mathbb{Z}$ form what was called in [Li5] an S_{trig}-local set of fields on W.

In order to associate quantum vertex algebras to certain algebras including quantum affine algebras, a theory of what were called ϕ-coordinated modules for a weak quantum vertex algebra was developed and a notion of $\mathcal{S}_{trig}$-locality was introduced in [Li5], where it was proved that every $\mathcal{S}_{trig}$-local subset of fields on a vector space W generates in a certain canonical way a (weak) quantum vertex algebra with W as a natural ϕ-coordinated module. Taking W to be a (suitably defined) restricted

$\widetilde{\mathfrak{gl}}_\infty$-module, one can show that $E(m,x)$ for $m\in\mathbb{Z}$ indeed form an S_{trig}-local set of fields on W. In view of this, weak quantum vertex algebras can be associated to Lie algebra $\widetilde{\mathfrak{gl}}_\infty$ *conceptually.*

To associate quantum vertex algebras to $\widetilde{\mathfrak{gl}}_\infty$ *explicitly*, we introduce an infinite dimensional Lie algebra $\widetilde{\mathfrak{gl}}_\infty^e$, which has generators $B(m,n)$ with $m,n\in\mathbb{Z}$, subject to relations

$$[B(m,x_1),B(n,x_2)]=e^{(m-n)(x_1-x_2)}\left(B(m,x_2)-B(n,x_1)+f(m,n)\mathbf{k}\right),$$

where $\mathbf{k}$ is a nonzero central element and $B(m,x)=\sum_{n\in\mathbb{Z}}B(m,n)x^{-n-1}$. For any complex number ℓ, we construct a universal "vacuum module" $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ of level ℓ for Lie algebra $\widetilde{\mathfrak{gl}}_\infty^e$. Then by using a result of [Li4] we show that there exists a canonical structure of a quantum vertex algebra on $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ and that a restricted $\widetilde{\mathfrak{gl}}_\infty^e$-module structure of level ℓ on a vector space W exactly amounts to a $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$-module structure on W. Furthermore, by using [Li5] we show that a restricted $\widetilde{\mathfrak{gl}}_\infty$-module structure of level ℓ on a vector space W exactly amounts to a ϕ-coordinated $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$-module structure on W.

In literature, there have been many interesting and important studies on Lie algebras $\mathfrak{gl}_\infty$ and $\widetilde{\mathfrak{gl}}_\infty$, including a remarkable relation with soliton equations, which was discovered and developed by Kyoto school (cf. [DKM], [DJKM]), and explicit bosonic and fermionic vertex operator realizations of the basic (level 1 irreducible) modules (see [JM], [K]). Closely related to Lie algebra $\mathfrak{gl}_\infty$, certain vertex algebras and their representations have been studied by several authors (see [FKRW], [KR], [Xu]). In those studies, the key idea is to use the embedding of certain Lie algebras into the completion $\overline{\mathfrak{gl}_\infty}$. The present study, which directly uses the Lie algebra $\widetilde{\mathfrak{gl}}_\infty$ itself, is different from those in nature.

This paper is organized as follows: In Section 2, we review Lie algebras $\mathfrak{gl}_\infty$ and $\widetilde{\mathfrak{gl}}_\infty$, and we introduce generating functions and reformulate their defining relations in terms of generating functions. In Section 3, we introduce Lie algebra $\widetilde{\mathfrak{gl}}_\infty^e$ and using this we construct a family of quantum vertex algebras $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ with a complex parameter ℓ. In Section 4, we present a canonical connection between restricted $\widetilde{\mathfrak{gl}}_\infty$-modules of level ℓ and ϕ-coordinated $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$-modules. In Section 5, we construct a family of irreducible $\widetilde{\mathfrak{gl}}_\infty^e$-modules.

2 Lie algebra $\widetilde{\mathfrak{gl}}_\infty$ and their restricted modules

This is a short preliminary section. In this section, we recall from [K] the Lie algebra $\mathfrak{gl}_\infty$ and its 1-dimensional central extension $\widetilde{\mathfrak{gl}}_\infty$, and we formulate a notion of restricted module and introduce generating functions.

We begin with Lie algebra $\mathfrak{gl}_\infty$. By definition, $\mathfrak{gl}_\infty$ consists of all doubly infinite complex matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with only finitely many nonzero entries. A canonical base of $\mathfrak{gl}_\infty$ consists of matrices $E_{i,j}$ for $i,j\in\mathbb{Z}$, where $E_{i,j}$ denotes the matrix whose only nonzero entry is the (i,j)-entry which is 1. We have

$$[E_{m,n}, E_{r,s}] = \delta_{n,r}E_{m,s} - \delta_{m,s}E_{r,n} \tag{2.1}$$

for $m,n,r,s \in \mathbb{Z}$. Let $\mathbb{C}^\infty$ denote an infinite-dimensional vector space (over $\mathbb{C}$) with a designated base $\{v_n \mid n \in \mathbb{Z}\}$. Then $\mathfrak{gl}_\infty$ naturally acts on $\mathbb{C}^\infty$ by

$$E_{i,j}v_k = \delta_{j,k}v_i \quad \text{for } k \in \mathbb{Z}.$$

Define $\deg E_{i,j} = j-i$ for $i,j \in \mathbb{Z}$ to make $\mathfrak{gl}_\infty$ a $\mathbb{Z}$-graded Lie algebra, where the degree-n homogeneous subspace $(\mathfrak{gl}_\infty)_{(n)}$ for $n \in \mathbb{Z}$ is linearly spanned by $E_{m,m+n}$ for $m \in \mathbb{Z}$. Using this $\mathbb{Z}$-grading one obtains a triangular decomposition

$$\mathfrak{gl}_\infty = \mathfrak{gl}_\infty^+ \oplus \mathfrak{gl}_\infty^0 \oplus \mathfrak{gl}_\infty^-,$$

where $\mathfrak{gl}_\infty^\pm = \sum_{\pm(j-i)>0} \mathbb{C}E_{i,j}$ and $\mathfrak{gl}_\infty^0 = \sum_i \mathbb{C}E_{i,i}$, which is a Cartan subalgebra. All the trace-zero matrices form a subalgebra $\mathfrak{sl}_\infty$, which is isomorphic to the affine Kac-Moody Lie algebra $\mathfrak{g}'(A)$ of type A_∞ (see [K]). We have $\mathfrak{gl}_\infty = \mathfrak{sl}_\infty \oplus \mathbb{C}E_{0,0}$, where $\mathfrak{gl}_\infty$ extends $\mathfrak{sl}_\infty$ by $E_{0,0}$ viewed as a derivation.

On Lie algebra $\mathfrak{gl}_\infty$, there is a 2-cocycle ψ defined by

$$\begin{aligned} &\psi(E_{i,j}, E_{j,i}) = 1 = -\psi(E_{j,i}, E_{i,j}) \quad \text{if } i \leq 0 \text{ and } j \geq 1, \\ &\psi(E_{i,j}, E_{m,n}) = 0 \quad \text{otherwise.} \end{aligned} \tag{2.2}$$

Using this 2-cocycle, one obtains a 1-dimensional central extension of $\mathfrak{gl}_\infty$, which is denoted by $\widetilde{\mathfrak{gl}}_\infty$. That is,

$$\widetilde{\mathfrak{gl}}_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}\mathbf{k}, \tag{2.3}$$

where $\mathbf{k}$ is a nonzero central element and

$$[E_{m,n}, E_{r,s}] = \delta_{n,r}E_{m,s} - \delta_{m,s}E_{r,n} + \psi(E_{m,n}, E_{r,s})\mathbf{k} \tag{2.4}$$

for $m,n,r,s \in \mathbb{Z}$.

For convenience, we define a function $f : \mathbb{Z}^2 \to \{-1,0,1\}$ by

$$\begin{aligned} &f(m,n) = 1 = -f(n,m) \quad \text{if } m \leq 0 \text{ and } n \geq 1, \\ &f(m,n) = 0 \quad \text{otherwise.} \end{aligned} \tag{2.5}$$

In terms of function f we have

$$\psi(E_{i,j}, E_{m,n}) = \delta_{i,n}\delta_{j,m}f(i,j) \quad \text{for } i,j,m,n \in \mathbb{Z}. \tag{2.6}$$

For $m \in \mathbb{Z}$, set

$$E(m,x) = \sum_{n\in\mathbb{Z}} E_{m,m+n}x^{-n}. \tag{2.7}$$

Lemma 2.1. *The relations (2.4) can be equivalently written as*

$$[E(m,x_1),E(n,x_2)] = \left(\frac{x_1}{x_2}\right)^{m-n} (E(m,x_2) - E(n,x_1) + f(m,n)\mathbf{k}) \tag{2.8}$$

for $m,n \in \mathbb{Z}$.

Proof. It is straightforward:

$$\begin{aligned}
& [E(m,x_1),E(n,x_2)] \\
= & \sum_{r,s\in\mathbb{Z}} \left([E_{m,m+r},E_{n,n+s}] + \delta_{m+r,n}\delta_{m,n+s} f(m,m+r)\mathbf{k}\right) x_1^{-r}x_2^{-s} \\
= & \sum_{r,s\in\mathbb{Z}} \left(\delta_{m+r,n}E_{m,n+s} - \delta_{m,n+s}E_{n,m+r}\right) x_1^{-r}x_2^{-s} + f(m,n)\left(\frac{x_1}{x_2}\right)^{m-n}\mathbf{k} \\
= & \left(\frac{x_1}{x_2}\right)^{m-n} (E(m,x_2) - E(n,x_1) + f(m,n)\mathbf{k})
\end{aligned}$$

for $m,n \in \mathbb{Z}$. □

As an immediate consequence of Lemma 2.1 we have:

Corollary 2.2. *The function* $f : \mathbb{Z}^2 \to \{-1,0,1\}$ *defined in (2.5) satisfies relations*

$$f(m,n) = -f(n,m), \qquad f(m,n) + f(n,r) = f(m,r) \tag{2.9}$$

for $m,n,r \in \mathbb{Z}$.

We formulate the following standard notions:

Definition 2.3. A $\widetilde{\mathfrak{gl}}_\infty$-module W is said to be of *level* $\ell \in \mathbb{C}$ if $\mathbf{k}$ acts on W as scalar ℓ, and W is said to be *restricted* if for every $m \in \mathbb{Z}$ and for $w \in W$, $E_{m,n}w = 0$ for n sufficiently large.

For a vector space W, set

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x,x^{-1}]]. \tag{2.10}$$

We see that a $\widetilde{\mathfrak{gl}}_\infty$-module W is restricted if and only if $E(m,x) \in \mathcal{E}(W)$ for $m \in \mathbb{Z}$.

It can be readily seen that the natural $\mathfrak{gl}_\infty$-module $\mathbb{C}^\infty$, which is a $\widetilde{\mathfrak{gl}}_\infty$-module of level 0, is restricted. On the other hand, highest weight $\widetilde{\mathfrak{gl}}_\infty$-modules are also restricted modules, where a highest weight $\widetilde{\mathfrak{gl}}_\infty$-module with highest weight $\lambda \in H^*$ is a module W with a vector w such that

$$\begin{aligned}
& E_{m,m}w = \lambda_m w \quad \text{ for } m \in \mathbb{Z}, \\
& E_{m,n}w = 0 \quad \text{ for } m,n \in \mathbb{Z} \text{ with } m < n, \\
& W = U(\widetilde{\mathfrak{gl}}_\infty)w,
\end{aligned}$$

where $H = \mathrm{span}\{E_{n,n} \mid n \in \mathbb{Z}\}$ and $\lambda_m = \lambda(E_{m,m})$.

3 Lie algebra $\widetilde{\mathfrak{gl}}_\infty^e$ and quantum vertex algebras

In this section, we introduce an infinite-dimensional Lie algebra $\widetilde{\mathfrak{gl}}_\infty^e$, which is intrinsically related to $\widetilde{\mathfrak{gl}}_\infty$, and for any complex number ℓ, we associate a quantum vertex algebra $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ to $\widetilde{\mathfrak{gl}}_\infty^e$ and associate a $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$-module to each restricted $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ.

We first recall from [Li3] the notion of weak quantum vertex algebra.

Definition 3.1. A *weak quantum vertex algebra* is a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot,x): \quad & V \to \mathrm{Hom}(V, V((x))) \subset (\mathrm{End}V)[[x,x^{-1}]] \\ & v \mapsto Y(v,x)=\sum_{n\in\mathbb{Z}} v_n x^{-n-1} \ \ (\text{where } v_n \in \mathrm{End}V), \end{aligned}$$

called the *adjoint vertex operator map*, and a vector $\mathbf{1}\in V$, called the *vacuum vector*, satisfying the following conditions: For $v\in V$,

$$Y(\mathbf{1},x)v=v, \quad Y(v,x)\mathbf{1}\in V[[x]] \ \text{ and } \ \lim_{x\to 0} Y(v,x)\mathbf{1}=v,$$

and for $u,v\in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x)\in \mathbb{C}((x)) \quad \text{for } i=1,\dots,r$$

such that

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) \\ & \qquad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\sum_{i=1}^{r} f_i(x_2-x_1)Y(v^{(i)},x_2)Y(u^{(i)},x_1) \\ = \ & x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y(Y(u,x_0)v,x_2). \end{aligned} \tag{3.1}$$

For a weak quantum vertex algebra V, following [EK] let

$$Y(x): V\otimes V \to V((x))$$

be the canonical linear map associated to the vertex operator map $Y(\cdot,x)$.

A *rational quantum Yang-Baxter operator* on a vector space U is a linear map

$$\mathcal{S}(x): \ U\otimes U \to U\otimes U\otimes \mathbb{C}((x)),$$

satisfying

$$\mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z)\mathcal{S}^{23}(z)=\mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z)\mathcal{S}^{12}(x)$$

(the *quantum Yang-Baxter equation*), where for $1 \leq i < j \leq 3$,

$$\mathcal{S}^{ij}(x) : V \otimes V \otimes V \to V \otimes V \otimes V \otimes \mathbb{C}((x))$$

denotes the canonical extension of $\mathcal{S}(x)$. It is said to be *unitary* if

$$\mathcal{S}(x)\mathcal{S}^{21}(-x) = 1,$$

where $\mathcal{S}^{21}(x) = \sigma\mathcal{S}(x)\sigma$ with σ denoting the flip operator on $U \otimes U$.

Definition 3.2. A *quantum vertex algebra* is a weak quantum vertex algebra V equipped with a unitary rational quantum Yang-Baxter operator $\mathcal{S}(x)$ on V, satisfying the following conditions:

$$\mathcal{S}(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v \quad \text{ for } v \in V, \tag{3.2}$$

$$[\mathcal{D} \otimes 1, \mathcal{S}(x)] = -\frac{d}{dx}\mathcal{S}(x), \tag{3.3}$$

$$Y(u,x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \text{for } u, v \in V, \tag{3.4}$$

$$\mathcal{S}(x_1)(Y(x_2) \otimes 1) = (Y(x_2) \otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1 + x_2), \tag{3.5}$$

where $\mathcal{D}$ is the linear operator on V defined by $\mathcal{D}(v) = v_{-2}\mathbf{1}$ for $v \in V$. We denote a quantum vertex algebra by a pair $(V, \mathcal{S})$.

Note that this very notion is a slight modification of the same named notion in [Li3] and [Li4] with extra axioms (3.2) and (3.5).

The following notion was due to Etingof and Kazhdan (see [EK]):

Definition 3.3. A weak quantum vertex algebra V is said to be *non-degenerate* if for every positive integer n, the linear map

$$Z_n : V^{\otimes n} \otimes \mathbb{C}((x_1)) \cdots ((x_n)) \to V((x_1)) \cdots ((x_n)),$$

defined by

$$Z_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f) = fY(v^{(1)}, x_1) \cdots Y(v^{(n)}, x_n)\mathbf{1}$$

for $v^{(1)}, \ldots, v^{(n)} \in V,\ f \in \mathbb{C}((x_1)) \cdots ((x_n))$, is injective.

The following is a reformulation of a result in [Li3] (cf. [EK]):

Proposition 3.4. *Let V be a weak quantum vertex algebra. Assume that V is non-degenerate. Then there exists a linear map $\mathcal{S}(x) : V \otimes V \to V \otimes V \otimes \mathbb{C}((x))$, which is uniquely determined by*

$$Y(u,x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \textit{for } u, v \in V.$$

Furthermore, $(V, \mathcal{S})$ carries the structure of a quantum vertex algebra and the following relation holds

$$[1 \otimes \mathcal{D}, \mathcal{S}(x)] = \frac{d}{dx}\mathcal{S}(x). \tag{3.6}$$

Remark 3.5. Note that a quantum vertex algebra was defined as a pair $(V, \mathcal{S})$. In view of Proposition 3.4, the term "a non-degenerate quantum vertex algebra" without reference to a quantum Yang-Baxter operator is unambiguous. If a weak quantum vertex algebra V is of countable dimension over $\mathbb{C}$ and if V as a V-module is irreducible, then by Corollary 3.10 of [Li4], V is non-degenerate. In view of this, the term "irreducible quantum vertex algebra" is also unambiguous.

Definition 3.6. Let V be a weak quantum vertex algebra. A *V-module* is a vector space W equipped with a linear map

$$Y_W(\cdot, x) : V \to \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x, x^{-1}]]$$

satisfying the conditions that

$$Y_W(\mathbf{1}, x) = 1_W \quad \text{(where } 1_W \text{ denotes the identity operator on } W\text{)}$$

and that for $u, v \in V,\ w \in W$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2) w. \tag{3.7}$$

As we need, we here briefly review the general construction of weak quantum vertex algebras and their modules from [Li3]. Let W be a vector space. Recall that

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x, x^{-1}]].$$

A subset U of $\mathcal{E}(W)$ is said to be *$\mathcal{S}$-local* if for any $u(x), v(x) \in U$, there exist

$$u^{(i)}(x), v^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

(finitely many) such that

$$(x_1 - x_2)^k u(x_1) v(x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1) v^{(i)}(x_2) u^{(i)}(x_1) \tag{3.8}$$

for some nonnegative integer k.

Notice that the relation (3.8) implies

$$(x_1 - x_2)^k u(x_1) v(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))). \tag{3.9}$$

Now, let $u(x), v(x) \in \mathcal{E}(W)$. Assume that there exists a nonnegative integer k such that (3.9) holds. Define $u(x)_n v(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}(u(x), x_0) v(x) = \sum_{n \in \mathbb{Z}} u(x)_n v(x) x_0^{-n-1}$$

by

$$Y_{\mathcal{E}}(u(x), x_0) v(x) = x_0^{-k} \left((x_1 - x)^k u(x_1) v(x) \right) |_{x_1 = x + x_0}. \tag{3.10}$$

(It was proved that the expression on the right hand side is independent of the choice of k.) If $u(x), v(x)$ are from an $\mathcal{S}$-local subset U, assuming relation (3.8) we have

$$\begin{aligned} & u(x)_n v(x) \\ = \ & \mathrm{Res}_{x_1}\left((x_1-x)^n u(x_1)v(x) - (-x+x_1)^n \sum_{i=1}^{r} f_i(x-x_1)v^{(i)}(x)u^{(i)}(x_1)\right). \end{aligned} \quad (3.11)$$

The following result was obtained in [Li3]:

Theorem 3.7. *Every $\mathcal{S}$-local subset U of $\mathcal{E}(W)$ generates a weak quantum vertex algebra $\langle U\rangle$ with W as a faithful module where $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U\rangle$.*

Next, as one of the main ingredients we introduce a new infinite-dimensional Lie algebra.

Proposition 3.8. *Let E be a vector space with basis $\{e_m \mid m \in \mathbb{Z}\}$. Set*

$$\widetilde{\mathfrak{gl}}^e_\infty = E \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{k},$$

where $\mathbf{k}$ is a symbol. Then there exists a Lie algebra structure on $\widetilde{\mathfrak{gl}}^e_\infty$ such that

$$\begin{aligned} & [\mathbf{k}, \widetilde{\mathfrak{gl}}^e_\infty] = 0, \\ & [B_t(m, x_1), B_t(n, x_2)] = e^{(m-n)(x_1-x_2)}\left(B_t(m, x_2) - B_t(n, x_1) + f(m, n)\mathbf{k}\right) \end{aligned} \quad (3.12)$$

for $m, n \in \mathbb{Z}$, where

$$B_t(m, x) = \sum_{n\in\mathbb{Z}} (e_m \otimes t^n)x^{-n-1}. \quad (3.13)$$

Proof. For $m \in \mathbb{Z}$, we have

$$B_t(m, x) = \sum_{n\in\mathbb{Z}} (e_m \otimes t^n)x^{-n-1} = e_m \otimes x^{-1}\delta\left(\frac{t}{x}\right).$$

Furthermore, for $g(t) \in \mathbb{C}((t))$ we have

$$g(x)B_t(m, x) = e_m \otimes g(t)x^{-1}\delta\left(\frac{t}{x}\right) \in \widetilde{\mathfrak{gl}}^e_\infty[[x, x^{-1}]]$$

and

$$e_m \otimes g(t) = \mathrm{Res}_x g(x)B_t(m, x). \quad (3.14)$$

Define a bilinear operation $[\cdot, \cdot]$ on $\widetilde{\mathfrak{gl}}^e_\infty$ by

$$\left[\mathbf{k}, \widetilde{\mathfrak{gl}}^e_\infty\right] = 0 = \left[\widetilde{\mathfrak{gl}}^e_\infty, \mathbf{k}\right],$$

$$
\begin{aligned}
& [e_m \otimes g(t), e_n \otimes h(t)] \\
= \; & \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} g(x_1)h(x_2)e^{(m-n)(x_1-x_2)} \left(B_t(m,x_2) - B_t(n,x_1) + f(m,n)\mathbf{k}\right)
\end{aligned} \tag{3.15}
$$

for $m, n \in \mathbb{Z}$, $g(t), h(t) \in \mathbb{C}((t))$. Indeed, for any fixed integers m, n, it is $\mathbb{C}$-linear in both $g(t)$ and $h(t)$. It can be readily seen that skew symmetry holds as the function $f(m,n)$ on $\mathbb{Z} \times \mathbb{Z}$ is skew symmetric. We next show that Jacobi identity also holds.

From definition, (3.12) holds for $m, n \in \mathbb{Z}$. Furthermore, by the standard formal variable convention we have

$$
\begin{aligned}
& [g(x_1)B_t(m,x_1), h(x_2)B_t(n,x_2)] \\
= \; & g(x_1)h(x_2)e^{(m-n)(x_1-x_2)} \left(B_t(m,x_2) - B_t(n,x_1) + f(m,n)\mathbf{k}\right)
\end{aligned} \tag{3.16}
$$

for $g(t), h(t) \in \mathbb{C}((t))$.

Let $m, n, r \in \mathbb{Z}$. We have

$$
\begin{aligned}
& [[B_t(m,x_1), B_t(n,x_2)], B_t(r,x_3)] \\
= \; & e^{(m-n)(x_1-x_2)} [B_t(m,x_2) - B_t(n,x_1) + f(m,n)\mathbf{k}, B_t(r,x_3)] \\
= \; & e^{(m-n)(x_1-x_2)} e^{(m-r)(x_2-x_3)} \left(B_t(m,x_3) - B_t(r,x_2) + f(m,r)\mathbf{k}\right) \\
& - e^{(m-n)(x_1-x_2)} e^{(n-r)(x_1-x_3)} \left(B_t(n,x_3) - B_t(r,x_1) + f(n,r)\mathbf{k}\right) \\
= \; & e^{(m-n)x_1 + (n-r)x_2 + (r-m)x_3} \left(B_t(m,x_3) - B_t(r,x_2) + f(m,r)\mathbf{k}\right) \\
& - e^{(m-r)x_1 + (n-m)x_2 + (r-n)x_3} \left(B_t(n,x_3) - B_t(r,x_1) + f(n,r)\mathbf{k}\right),
\end{aligned}
$$

$$
\begin{aligned}
& [B_t(m,x_1), [B_t(n,x_2), B_t(r,x_3)]] \\
= \; & e^{(n-r)(x_2-x_3)} e^{(m-n)(x_1-x_3)} \left(B_t(m,x_3) - B_t(n,x_1) + f(m,n)\mathbf{k}\right) \\
& - e^{(n-r)(x_2-x_3)} e^{(m-r)(x_1-x_2)} \left(B_t(m,x_2) - B_t(r,x_1) + f(m,r)\mathbf{k}\right) \\
= \; & e^{(m-n)x_1 + (n-r)x_2 + (r-m)x_3} \left(B_t(m,x_3) - B_t(n,x_1) + f(m,n)\mathbf{k}\right) \\
& - e^{(m-r)x_1 + (n-m)x_2 + (r-n)x_3} \left(B_t(m,x_2) - B_t(r,x_1) + f(m,r)\mathbf{k}\right),
\end{aligned}
$$

$$
\begin{aligned}
& [B_t(n,x_2), [B_t(m,x_1), B_t(r,x_3)]] \\
= \; & e^{(m-r)x_1 + (n-m)x_2 + (r-n)x_3} \left(B_t(n,x_3) - B_t(m,x_2) + f(n,m)\mathbf{k}\right) \\
& - e^{(m-n)x_1 + (n-r)x_2 + (r-m)x_3} \left(B_t(n,x_1) - B_t(r,x_2) + f(n,r)\mathbf{k}\right).
\end{aligned}
$$

Recall from Corollary 2.2 that

$$
f(m,n) + f(n,r) = f(m,r), \quad f(m,r) + f(n,m) = f(n,r).
$$

(Note that the second relation follows from the first one as $f(m,n) = -f(n,m)$.) Then we see that the Jacobi identity holds. $\square$

Remark 3.9. For $n \in \mathbb{Z}$, set $U_n = E \otimes t^n \mathbb{C}[[t]]$. This gives a descending sequence $\{U_n\}_{n \in \mathbb{Z}}$ with $\cap_{n \in \mathbb{Z}} U_n = 0$. Equip vector space $\widetilde{\mathfrak{gl}}^e_\infty$ with the topological vector space structure associated to $\{U_n\}_{n\in\mathbb{Z}}$. Let $u \in E,\ g(t) \in \mathbb{C}((t))$. From definition (see (3.15)) we have

$$[u \otimes g(t), U_n] \subset U_n \quad \text{for } n \geq 0, \tag{3.17}$$

which implies that the Lie bracket is continuous. Thus $\widetilde{\mathfrak{gl}}_\infty^e$ is a topological Lie algebra where the elements $\mathbf{k}$ and $e_m \otimes t^r$ for $m, r \in \mathbb{Z}$ form a topological basis.

Remark 3.10. For $n \in \mathbb{Z}$, set

$$\widetilde{\mathfrak{gl}}_\infty^e[n] = \begin{cases} E \otimes t^n\mathbb{C}[[t]] & \text{if } n \geq 1 \\ E \otimes t^n\mathbb{C}[[t]] + \mathbb{C}\mathbf{k} & \text{if } n \leq 0. \end{cases} \tag{3.18}$$

This defines a descending filtration of $\widetilde{\mathfrak{gl}}_\infty^e$. Using (3.15) one can show that

$$\left[\widetilde{\mathfrak{gl}}_\infty^e[m], \widetilde{\mathfrak{gl}}_\infty^e[n]\right] \subset \widetilde{\mathfrak{gl}}_\infty^e[m+n]$$

for $m, n \in \mathbb{Z}$. Thus, $\widetilde{\mathfrak{gl}}_\infty^e$ equipped with this filtration becomes a filtered Lie algebra. (But, Lie algebra $\widetilde{\mathfrak{gl}}_\infty^e$ is not $\mathbb{Z}$-graded in the obvious way.)

For $m, r \in \mathbb{Z}$, let $B(m, r)$ denote the operator corresponding to $e_m \otimes t^r$ on a $\widetilde{\mathfrak{gl}}_\infty^e$-module. For $m \in \mathbb{Z}$, set

$$B(m, x) = \sum_{n\in\mathbb{Z}} B(m, n)x^{-n-1}. \tag{3.19}$$

We define a notion of restricted $\widetilde{\mathfrak{gl}}_\infty^e$-module in the usual way. That is, a $\widetilde{\mathfrak{gl}}_\infty^e$-module W is said to be *restricted* if for any $m \in \mathbb{Z},\ w \in W$, $B(m, n)w = 0$ for n sufficiently large, or equivalently $B(m, x) \in \mathcal{E}(W)$ for $m \in \mathbb{Z}$. We also assume continuity.

Set

$$\mathcal{B}^+ = E \otimes \mathbb{C}[[t]] \subset \widetilde{\mathfrak{gl}}_\infty^e.$$

Let $m, n \in \mathbb{Z},\ g(t), h(t) \in \mathbb{C}[[t]]$. Noticing that $e^{(m-n)(x_1-x_2)} \in \mathbb{C}[[x_1, x_2]]$, from (3.15) we get

$$[e_m \otimes g(t), e_n \otimes h(t)] = 0.$$

Thus, $\mathcal{B}^+$ is an abelian subalgebra of $\widetilde{\mathfrak{gl}}_\infty^e$. Denote by $\mathcal{B}^-$ the subspace of $\widetilde{\mathfrak{gl}}_\infty^e$, linearly spanned by $e_m \otimes t^r$ for $m, r \in \mathbb{Z}$ with $r < 0$. We have

$$\widetilde{\mathfrak{gl}}_\infty^e = \mathcal{B}^+ \oplus \mathbb{C}\mathbf{k} \oplus \mathcal{B}^-$$

as a vector space. (Note that $\mathcal{B}^-$ is not a subalgebra.)

Let ℓ be any complex number. Letting $\mathcal{B}^+$ act on $\mathbb{C}$ trivially and letting $\mathbf{k}$ act as scalar ℓ, we obtain a $(\mathcal{B}^+ \oplus \mathbb{C}\mathbf{k})$-module denoted by $\mathbb{C}_\ell$. Then form an induced module

$$V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0) = U(\widetilde{\mathfrak{gl}}_\infty^e) \otimes_{U(\mathcal{B}^+\oplus\mathbb{C}\mathbf{k})} \mathbb{C}_\ell. \tag{3.20}$$

In view of the P-B-W theorem, $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$ as a vector space is isomorphic to $S(\mathcal{B}^-)$ (the symmetric algebra over $\mathcal{B}^-$). Set $\mathbf{1} = 1 \otimes 1 \in V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$, and then set

$$b^{(m)} = B(m,-1)\mathbf{1} \in V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0) \quad \text{for } m \in \mathbb{Z}. \tag{3.21}$$

Note that $B(m,k)\mathbf{1} = 0$ for $m \in \mathbb{Z},\ k \in \mathbb{N}$. Let $m,n \in \mathbb{Z},\ k \in \mathbb{N}$. From the Lie bracket relation (3.12) we get

$$[B(m,k), B(n,x_2)] = -e^{(n-m)x_2} \sum_{j\geq 0} \frac{(m-n)^j}{j!} B(n,k+j). \tag{3.22}$$

It follows from this relation and induction that $B(m,k) = 0$ on $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$ for $m \in \mathbb{Z},\ k \geq 0$. In particular, $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$ is a restricted $\widetilde{\mathfrak{gl}}_{\infty}^{e}$-module.

As one of the main results in this section, we have:

Theorem 3.11. *For every complex number ℓ, there exists a weak quantum vertex algebra structure on $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$, which is uniquely determined by the conditions that* $\mathbf{1}$ *is the vacuum vector and that*

$$Y(b^{(m)},x) = B(m,x) \quad \textit{for } m \in \mathbb{Z}.$$

Furthermore, $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$ is a non-degenerate quantum vertex algebra and the following relations hold:

$$b_k^{(m)} b^{(n)} = 0 \quad \textit{for } m,n \in \mathbb{Z},\ k \in \mathbb{N} \tag{3.23}$$

and

$$\begin{aligned} & Y(b^{(m)},x_1)Y(b^{(n)},x_2) - Y(b^{(n)},x_2)Y(b^{(m)},x_1) \\ = \ & e^{(m-n)(x_1-x_2)}\left(Y(b^{(m)},x_2) - Y(b^{(n)},x_1) + f(m,n)\ell\right) \end{aligned} \tag{3.24}$$

for $m,n \in \mathbb{Z}$.

Proof. Let W be any restricted $\widetilde{\mathfrak{gl}}_{\infty}^{e}$-module of level ℓ. Set

$$U_W = \{B(m,x) \mid m \in \mathbb{Z}\} \cup \{1_W\} \subset \mathcal{E}(W).$$

Writing the defining relation (3.12) as

$$\begin{aligned} B(m,x_1)B(n,x_2) &= B(n,x_2)B(m,x_1) \\ &\quad + e^{(m-n)(x_1-x_2)}\left(B(m,x_2) - B(n,x_1) + f(m,n)\ell\right), \end{aligned} \tag{3.25}$$

we see that U_W is an $\mathcal{S}$-local subset of $\mathcal{E}(W)$. Then U_W generates a weak quantum vertex algebra $\langle U_W \rangle$ inside $\mathcal{E}(W)$ with W as a canonical module. With (3.25), by Proposition 3.13 of [Li3] (cf. Proposition 2.12, [Li4]), we have

$$\begin{aligned} & Y_{\mathcal{E}}(B(m,x),x_1)Y_{\mathcal{E}}(B(n,x),x_2) = Y_{\mathcal{E}}(B(n,x),x_2)Y_{\mathcal{E}}(B(m,x)x_1) \\ & \quad + e^{(m-n)(x_1-x_2)}\left(Y_{\mathcal{E}}(B(m,x),x_2) - Y_{\mathcal{E}}(B(n,x),x_1) + f(m,n)\ell\right) \end{aligned}$$

for $m, n \in \mathbb{Z}$. This shows that $\langle U_W \rangle$ is a $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ with $B(m, z)$ acting as $Y_{\mathcal{E}}(B(m, x), z)$ for $m \in \mathbb{Z}$. Furthermore, we have

$$B(m, x)_k 1_W = 0 \quad \text{for } m \in \mathbb{Z},\ k \in \mathbb{N}.$$

It follows from the construction of $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ that there exists a $\widetilde{\mathfrak{gl}}_\infty^e$-module homomorphism θ from $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ to $\langle U_W \rangle$ with $\theta(\mathbf{1}) = 1_W$.

Take $W = V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$. Then it follows from Theorem 2.9 of [Li4] that there is a weak quantum vertex algebra structure on $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ with all the desired properties.

As for non-degeneracy, we shall use a result of [Li4]. For $n \geq 0$, let $V[n]$ denote the subspace of $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ linearly spanned by vectors

$$b_{k_1}^{(m_1)} \cdots b_{k_r}^{(m_r)} \mathbf{1}$$

where $0 \leq r \leq n,\ m_i, k_i \in \mathbb{Z}$ for $i = 1, \dots, r$. (Note that $b_k^{(m)} = B(m, k)$.) Since $\{b^{(m)} \mid m \in \mathbb{Z}\}$ generates $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ as a weak quantum vertex algebra, this defines an ascending filtration of $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$. Denote the associated graded weak quantum vertex algebra by K. By Proposition 3.15 of [Li4], K is a commutative vertex algebra. Furthermore, it follows from the P-B-W basis that $K = S(\mathcal{B}^-)$. Then, by Theorem 3.19 of [Li4], $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ is non-degenerate.

Since $Y(b^{(m)}, x) = B(m, x)$ for $m \in \mathbb{Z}$, the last assertion is clear. For $m, n \in \mathbb{Z}$, from the last assertion we have

$$\begin{aligned}
&Y(b^{(m)}, x_1) Y(b^{(n)}, x_2) = Y(b^{(n)}, x_2) Y(b^{(m)}, x_1) \\
&+ e^{(m-n)(x_1 - x_2)} \left(Y(b^{(m)}, x_2) Y(\mathbf{1}, x_1) - Y(\mathbf{1}, x_2) Y(b^{(n)}, x_1) + \ell f(m, n) Y(\mathbf{1}, x_2) Y(\mathbf{1}, x_1) \right),
\end{aligned}$$

noticing that $Y(\mathbf{1}, x) = 1$. Set

$$\begin{aligned}
&P = Y(b^{(n)}, x_2) Y(b^{(m)}, x_1) \\
&+ e^{(m-n)(x_1 - x_2)} \left(Y(b^{(m)}, x_2) Y(\mathbf{1}, x_1) - Y(\mathbf{1}, x_2) Y(b^{(n)}, x_1) + \ell f(m, n) Y(\mathbf{1}, x_2) Y(\mathbf{1}, x_1) \right).
\end{aligned}$$

For $k \geq 0$, from the $\mathcal{S}$-Jacobi identity for weak quantum vertex algebras we have

$$Y(b_k^{(m)} b^{(n)}, x_2) = \mathrm{Res}_{x_1} (x_1 - x_2)^k \left(Y(b^{(m)}, x_1) Y(b^{(n)}, x_2) - P \right) = 0.$$

It then follows that $b_k^{(m)} b^{(n)} = 0$ for $m, n \in \mathbb{Z},\ k \in \mathbb{N}$. $\square$

Furthermore, we have:

Theorem 3.12. *For any restricted $\widetilde{\mathfrak{gl}}_\infty^e$-module W of level ℓ, there exists a $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$-module structure Y_W, which is uniquely determined by*

$$Y_W(b^{(m)}, x) = B(m, x) \quad \text{for } m \in \mathbb{Z}.$$

On the other hand, for any $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$-module (W, Y_W), W is a restricted $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ with

$$B(m, x) = Y_W(b^{(m)}, x) \quad \text{for } m \in \mathbb{Z}.$$

Proof. Set $U_W = \{B(m,x) \mid m \in \mathbb{Z}\} \cup \{1_W\} \subset \mathcal{E}(W)$. From the proof of Theorem 3.11, we have a weak quantum vertex algebra $\langle U_W \rangle$ generated by U_W and W is a $\langle U_W \rangle$-module with $Y_W(\alpha(x),z) = \alpha(z)$ for $\alpha(x) \in \langle U_W \rangle$. Furthermore, $\langle U_W \rangle$ is a $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ with $B(m,z) = Y_{\mathcal{E}}(B(m,x),z)$ for $m \in \mathbb{Z}$ and there exists a $\widetilde{\mathfrak{gl}}_\infty^e$-module homomorphism θ from $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ to $\langle U_W \rangle$ with $\theta(\mathbf{1}) = 1_W$. We have

$$\theta(b^{(m)}) = \theta(B(m,-1)\mathbf{1}) = B(m,x)_{-1}1_W = B(m,x)$$

and

$$\theta(Y(b^{(m)},z)v) = \theta(B(m,z)v) = Y_{\mathcal{E}}(B(m,x),z)\theta(v) = Y_{\mathcal{E}}(\theta(b^{(m)}),z)\theta(v)$$

for $m \in \mathbb{Z}$, $v \in V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$. As $\{b^{(m)} \mid m \in \mathbb{Z}\}$ generates $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ as a weak quantum vertex algebra, it follows that θ is a homomorphism of weak quantum vertex algebras. Consequently, W becomes a $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$-module with

$$Y_W(v,x) = \theta(v)(x) \in \langle U_W \rangle \subset \mathcal{E}(W)$$

for $v \in V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$. In particular, we have $Y_W(b^{(m)},x) = \theta(b^{(m)}) = B(m,x)$ for $m \in \mathbb{Z}$.

On the other hand, let (W, Y_W) be a $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$-module. With (3.24), by Corollary 5.4 of [Li3], we have

$$\begin{aligned} & Y_W(b^{(m)},x_1)Y_W(b^{(n)},x_2) - Y_W(b^{(n)},x_2)Y_W(b^{(m)},x_1) \\ = \; & e^{(m-n)(x_1-x_2)}\left(Y_W(b^{(m)},x_2) - Y_W(b^{(n)},x_1) + f(m,n)\ell\right) \end{aligned}$$

for $m,n \in \mathbb{Z}$. That is, W becomes a $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ with $B(m,x) = Y_W(b^{(m)},x)$ for $m \in \mathbb{Z}$. It is restricted as $Y_W(b^{(m)},x) \in \mathcal{E}(W)$ for $m \in \mathbb{Z}$ from definition. $\square$

4 Lie algebra $\widetilde{\mathfrak{gl}}_\infty$ and quantum vertex algebra $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$

In this section, we relate restricted $\widetilde{\mathfrak{gl}}_\infty$-modules of level ℓ with the quantum vertex algebra $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ in terms of ϕ-coordinated modules in the sense of [Li5]. More specifically, we show that a level-ℓ restricted $\widetilde{\mathfrak{gl}}_\infty$-module structure on a vector space W exactly amounts to a ϕ-coordinated module structure for the quantum vertex algebra $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$.

We first recall from [Li5] some basic notions and results on ϕ-coordinated modules for weak quantum vertex algebras. Set

$$\phi = \phi(x,z) = xe^z \in \mathbb{C}[[x,z]],$$

which is fixed throughout this section.

Definition 4.1. Let V be a weak quantum vertex algebra. A *ϕ-coordinated V-module* is a vector space W, equipped with a linear map

$$Y_W(\cdot,x) : V \to \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x,x^{-1}]],$$

satisfying the conditions that

$$Y_W(\mathbf{1},x)=1_W$$

and that for $u,v\in V$, there exists a nonnegative integer k such that

$$(x_1-x_2)^k Y_W(u,x_1)Y_W(v,x_2)\in \mathrm{Hom}(W,W((x_1,x_2))) \tag{4.1}$$

and

$$(e^{x_0}-1)^k Y_W(Y(u,x_0)v,x_2)=\left((x_1/x_2-1)^k Y_W(u,x_1)Y_W(v,x_2)\right)|_{x_1=x_2e^{x_0}}. \tag{4.2}$$

Let $\mathbb{C}(x)$ denote the field of rational functions in x. That is, $\mathbb{C}(x)$ is the fraction field of the polynomial algebra $\mathbb{C}[x]$. On the other hand, let $\mathbb{C}_*(x)$ denote the fraction field of the algebra $\mathbb{C}[[x]]$. There exists an algebra map $\iota:\mathbb{C}_*(x)\to\mathbb{C}((x))$, which is uniquely determined by

$$\iota(q(x))=q(x)\quad \text{ for } q(x)\in\mathbb{C}[[x]].$$

For $f(x)\in\mathbb{C}(x)$, $\iota(f(x))$ is simply the formal Laurent series expansion of $f(x)$ at $x=0$. For $f(x)\in\mathbb{C}(x)$, we have

$$(\iota f)(x_1/x_2)=\iota_{x_2,x_1}f(x_1/x_2).$$

Let W be a vector space. Recall that

$$\mathcal{E}(W)=\mathrm{Hom}(W,W((x)))\subset(\mathrm{End}W)[[x,x^{-1}]].$$

Let $a(x),b(x)\in\mathcal{E}(W)$. Assume that there exists a nonzero $p(x)\in\mathbb{C}[x]$ such that

$$p(x_1/x_2)a(x_1)b(x_2)\in\mathrm{Hom}(W,W((x_1,x_2))). \tag{4.3}$$

Define $a(x)_n^e b(x)\in\mathcal{E}(W)$ for $n\in\mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}^e(a(x),z)b(x)=\sum_{n\in\mathbb{Z}}a(x)_n^e b(x)z^{-n-1}$$

by

$$Y_{\mathcal{E}}^e(a(x),z)b(x)=\iota(1/p(e^z))\left(p(x_1/x)a(x_1)b(x)\right)|_{x_1=xe^z}, \tag{4.4}$$

where $p(x)$ is any nonzero polynomial such that (4.3) holds. (It was proved in [Li5] that $p(e^z)\neq 0$ in $\mathbb{C}[[z]]$ for any nonzero polynomial $p(x)$.)

Definition 4.2. A subset U of $\mathcal{E}(W)$ is said to be $\mathcal{S}_{trig}$*-local* if for any $u(x),v(x)\in U$, there exist

$$u^{(i)}(x),v^{(i)}(x)\in U,\quad f_i(x)\in\mathbb{C}(x)\quad(i=1,\dots,r)$$

(finitely many) such that

$$(x_1-x_2)^k u(x_1)v(x_2)=(x_1-x_2)^k\sum_{i=1}^r(\iota f_i)(x_1/x_2)v^{(i)}(x_2)u^{(i)}(x_1) \tag{4.5}$$

for some nonnegative integer k.

Notice that relation (4.5) implies that

$$(x_1/x_2 - 1)^k u(x_1)v(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))).$$

The following result was obtained in [Li5]:

Theorem 4.3. *Let W be a vector space and let U be an $\mathcal{S}_{trig}$-local subset of $\mathcal{E}(W)$. Then U generates a weak quantum vertex algebra $\langle U \rangle_e$ with W as a faithful ϕ-coordinated module where $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U \rangle_e$.*

Now, we are in a position to present the main result of this paper.

Theorem 4.4. *Let ℓ be any complex number. For every restricted $\widetilde{\mathfrak{gl}}_\infty$-module W of level ℓ, there exists a ϕ-coordinated $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$-module structure Y_W on W, which is uniquely determined by*

$$Y_W(b^{(m)}, x) = E(m, x) \quad \textit{for } m \in \mathbb{Z}.$$

On the other hand, for every ϕ-coordinated $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$-module (W, Y_W), W is a restricted $\widetilde{\mathfrak{gl}}_\infty$-module of level ℓ with

$$E(m, x) = Y_W(b^{(m)}, x) \quad \textit{for } m \in \mathbb{Z}.$$

Proof. Set

$$U_W = \{E(m, x) \mid m \in \mathbb{Z}\} \cup \{1_W\} \subset \mathcal{E}(W).$$

It can be readily seen from (2.8) that U_W is an $\mathcal{S}_{trig}$-local subset of $\mathcal{E}(W)$. By Theorem 4.3, U_W generates a weak quantum vertex algebra $\langle U_W \rangle_e$, where 1_W serves as the vacuum vector and the vertex operator map is denoted by $Y_{\mathcal{E}}^e(\cdot, x)$. Furthermore, W becomes a ϕ-coordinated module with $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U_W \rangle_e$.

With the relations (2.8) (with $\mathbf{k} = \ell$), by Proposition 5.3 of [Li5], we have

$$\begin{aligned} & Y_{\mathcal{E}}^e(E(m, x), x_1) Y_{\mathcal{E}}^e(E(n, x), x_2) - Y_{\mathcal{E}}^e(E(n, x), x_2) Y_{\mathcal{E}}^e(E(m, x), x_1) \\ = \; & e^{(m-n)(x_1 - x_2)} \left(Y_{\mathcal{E}}^e(E(m, x), x_2) - Y_{\mathcal{E}}^e(E(n, x), x_1) + f(m, n)\ell 1_W \right) \end{aligned}$$

for $m, n \in \mathbb{Z}$. This shows that $\langle U_W \rangle_e$ becomes a restricted $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ with $B(m, z)$ acting as $Y_{\mathcal{E}}^e(E(m, x), z)$ for $m \in \mathbb{Z}$. Furthermore, just as in the proof of Theorem 3.11, we have that $\langle U_W \rangle_e$ as a $\widetilde{\mathfrak{gl}}_\infty^e$-module is cyclic on 1_W and

$$E(m, x)_k^e 1_W = 0 \quad \text{ for } m \in \mathbb{Z},\ k \in \mathbb{N}.$$

It follows from the construction of the $\widetilde{\mathfrak{gl}}_\infty^e$-module $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ that there exists a $\widetilde{\mathfrak{gl}}_\infty^e$-module homomorphism θ from $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ to $\langle U_W \rangle_e$ with $\theta(\mathbf{1}) = 1_W$. For $m \in \mathbb{Z}$, we have

$$\theta(b^{(m)}) = \theta(B(m, -1)\mathbf{1}) = B(m, -1)\theta(\mathbf{1}) = E(m, x)_{-1}^e 1_W = E(m, x).$$

Furthermore, we have

$$\theta(Y(b^{(m)},z)v)=\theta(B(m,z)v)=Y_{\mathcal{E}}^{e}(E(m,x),z)\theta(v)=Y_{\mathcal{E}}^{e}(\theta(b^{(m)}),z)\theta(v)$$

for $m\in\mathbb{Z},\ v\in V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$. Since $\{b^{(m)}\mid m\in\mathbb{Z}\}$ generates $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$ as a weak quantum vertex algebra, it follows that θ is a homomorphism of weak quantum vertex algebras. With W as a canonical ϕ-coordinated module for $\langle U_W\rangle_e$, through the homomorphism θ, W becomes a ϕ-coordinated $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$-module, where

$$Y_W(b^{(m)},x)=Y_W(\theta(b^{(m)}),x)=E(m,x)\quad \text{ for } m\in\mathbb{Z}.$$

The uniqueness follows from the fact that $\{b^{(m)}\mid m\in\mathbb{Z}\}$ generates $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$ as a weak quantum vertex algebra.

On the other hand, let (W,Y_W) be a ϕ-coordinated $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$-module. For $m,n\in\mathbb{Z}$, recalling (3.24), from [Li5] (Proposition 5.9) we have

$$\begin{aligned}
&Y_W(b^{(m)},x_1)Y_W(b^{(n)},x_2)-Y_W(b^{(n)},x_2)Y_W(b^{(m)},x_1)\\
&\quad-\left(\frac{x_1}{x_2}\right)^{m-n}\big(Y_W(b^{(m)},x_2)-Y_W(b^{(n)},x_1)+\ell f(m,n)\big)\\
=&\sum_{k\geq 0}Y_W(b_k^{(m)}b^{(n)},x_2)\frac{1}{k!}\left(x_2\frac{\partial}{\partial x_2}\right)^{k}\delta\left(\frac{x_2}{x_1}\right).
\end{aligned}$$

Recall that for quantum vertex algebra $V_{\widetilde{\mathfrak{gl}}_{\infty}^{e}}(\ell,0)$, the following relations hold:

$$b_k^{(m)}b^{(n)}=0\quad \text{ for } k\geq 0.$$

Then

$$\begin{aligned}
&Y_W(b^{(m)},x_1)Y_W(b^{(n)},x_2)-Y_W(b^{(n)},x_2)Y_W(b^{(m)},x_1)\\
=&\left(\frac{x_1}{x_2}\right)^{m-n}\big(Y_W(b^{(m)},x_2)-Y_W(b^{(n)},x_1)+\ell f(m,n)\big).
\end{aligned}$$

Thus W becomes a $\widetilde{\mathfrak{gl}}_{\infty}$-module of level ℓ with $E(m,x)$ acting as $Y_W(b^{(m)},x)$ for $m\in\mathbb{Z}$, and clearly it is a restricted module. $\square$

5 Constructing $\widetilde{\mathfrak{gl}}_{\infty}^{e}$-modules

In this section, we give a construction of $\widetilde{\mathfrak{gl}}_{\infty}^{e}$-modules from $\widetilde{\mathfrak{gl}}_{\infty}$-modules of a certain type, including the natural module $\mathbb{C}^{\infty}$. We also discuss a generalized-Verma-module construction of $\widetilde{\mathfrak{gl}}_{\infty}^{e}$-modules.

First, we introduce a special family of restricted $\widetilde{\mathfrak{gl}}_{\infty}$-modules, including the natural module $\mathbb{C}^{\infty}$ (of level 0).

Definition 5.1. Let C_{fin} denote the category of $\widetilde{\mathfrak{gl}}_\infty$-modules W such that

$$E(m,x)w \in W[x,x^{-1}] \quad \text{for } m \in \mathbb{Z},\ w \in W. \tag{5.1}$$

It can be readily seen that indeed C_{fin} contains the natural module $\mathbb{C}^\infty$.

The following is the main reason for formulating category C_{fin}:

Proposition 5.2. *Let W be a $\widetilde{\mathfrak{gl}}_\infty$-module of level $\ell \in \mathbb{C}$ in category C_{fin}. Then W becomes a restricted $\widetilde{\mathfrak{gl}}^e_\infty$-module of level ℓ with*

$$B(m,x) = E(m,e^x) = \sum_{n\in\mathbb{Z}} E_{m,m+n}e^{-nx}$$

for $m \in \mathbb{Z}$. Furthermore, if W is an irreducible $\widetilde{\mathfrak{gl}}_\infty$-module, then W viewed as a $\widetilde{\mathfrak{gl}}^e_\infty$-module is also irreducible. On the other hand, let W_1, W_2 be $\widetilde{\mathfrak{gl}}_\infty$-modules in category C_{fin} and let θ be a linear map from W_1 to W_2. Then θ is a homomorphism of $\widetilde{\mathfrak{gl}}^e_\infty$-modules if and only if θ is a homomorphism of $\widetilde{\mathfrak{gl}}_\infty$-modules.

Proof. First of all, notice that for any $p(x) \in \mathbb{C}[x,x^{-1}]$, $p(e^x)$ exists in $\mathbb{C}[[x]]$. From assumption we have $E(m,x)w \in W[x,x^{-1}]$ for $m \in \mathbb{Z},\ w \in W$. Then for each $m \in \mathbb{Z}$, $E(m,e^x)$ is a well defined element of $(\text{End}W)[[x]]$. For $m \in \mathbb{Z}$, set

$$\bar{B}(m,x) = E(m,e^x) = \sum_{n\in\mathbb{Z}} E_{m,m+n}e^{-nx}.$$

Writing $\bar{B}(m,x) = \sum_{n\in\mathbb{Z}} \bar{B}(m,n)x^{-n-1}$, we have $\bar{B}(m,r) = 0$ for $r \geq 0$ and

$$\bar{B}(m,-k-1) = \sum_{n\in\mathbb{Z}} \frac{1}{k!}(-n)^k E_{m,m+n} \tag{5.2}$$

for $k \geq 0$. Furthermore, we have

$$[\bar{B}(m,x_1), \bar{B}(n,x_2)] = e^{(m-n)(x_1-x_2)}\left(\bar{B}(m,x_2) - \bar{B}(n,x_1) + f(m,n)\ell\right).$$

Thus W becomes a $\widetilde{\mathfrak{gl}}^e_\infty$-module of level ℓ with $B(m,x) = \bar{B}(m,x)$ for $m \in \mathbb{Z}$, which is a restricted module as $\bar{B}(m,x)$ involves only nonnegative powers of x.

Regarding the assertion on irreducibility, let $w \in W,\ m \in \mathbb{Z}$. For $k \geq 0$, we have

$$(-1)^k k!\bar{B}(m,-k-1)w = \sum_{n\in\mathbb{Z}} n^k E_{m,m+n}w. \tag{5.3}$$

Note that the expression on the right hand side is a finite sum. Considering $k \geq 1$, by solving the system of equations we can express $E_{m,m+n}w$ for $n \neq 0$ in terms of $\bar{B}(m,-k-1)w$ with $k \geq 1$. We also have (with $k = 0$)

$$\bar{B}(m,-1)w = \sum_{n\in\mathbb{Z}} E_{m,m+n}w.$$

Using this, we can express $E_{m,m}w$ in terms of $\bar{B}(m,-1)w$ and $E_{m,m+n}w$ for n nonzero. Consequently, we can express $E_{m,m+n}w$ for every $n \in \mathbb{Z}$ in terms of $\bar{B}(m,-k-1)w$ with $k \geq 0$. It then follows immediately that W viewed as a $\widetilde{\mathfrak{gl}}_\infty^e$-module is still irreducible.

The last assertion is also clear. $\square$

In the following, we exhibit some irreducible $\widetilde{\mathfrak{gl}}_\infty$-modules of level 0, or equivalently $\mathfrak{gl}_\infty$-modules, in category C_{fin}. Note that category C_{fin} is closed under direct sum and tensor product. As $\mathbb{C}^\infty$ is in C_{fin}, it follows that the tensor algebra $T(\mathbb{C}^\infty)$ is naturally a $\mathfrak{gl}_\infty$-module in C_{fin}, on which $\mathfrak{gl}_\infty$ acts by derivations.

Example 5.3. Consider the symmetric algebra $S(\mathbb{C}^\infty)$, which is naturally a $\mathfrak{gl}_\infty$-module in C_{fin}. Identify $S(\mathbb{C}^\infty)$ with polynomial algebra $\mathbb{C}[x_n \mid n \in \mathbb{Z}]$ on which $\mathfrak{gl}_\infty$ acts by

$$E_{m,n} = x_m \frac{\partial}{\partial x_n} \quad \text{ for } m, n \in \mathbb{Z}.$$

Define $\deg x_n = 1$ for $n \in \mathbb{Z}$ to make $\mathbb{C}[x_n \mid n \in \mathbb{Z}]$ a $\mathbb{Z}$-graded algebra, and for $r \in \mathbb{N}$, let A_r denote the degree-r homogeneous subspace, which is linearly spanned by the monomials of total degree r. One can show that A_r $(r \in \mathbb{Z})$ are non-isomorphic irreducible submodules.

Example 5.4. On the other hand, the exterior algebra $\Lambda(\mathbb{C}^\infty)$ is also a $\mathfrak{gl}_\infty$-module in C_{fin}, on which $\mathfrak{gl}_\infty$ acts by derivations. Note that $\Lambda(\mathbb{C}^\infty)$ is an $\mathbb{N}$-graded algebra with $\mathbb{C}^\infty$ of degree 1. Decompose $\Lambda(\mathbb{C}^\infty)$ into homogeneous subspaces $\Lambda(\mathbb{C}^\infty) = \oplus_{r\in\mathbb{N}}\Lambda^r(\mathbb{C}^\infty)$. We have that $\Lambda^0(\mathbb{C}^\infty) = A_0 = \mathbb{C}$ and $\Lambda^1(\mathbb{C}^\infty) = A_1 = \mathbb{C}^\infty$. It is straightforward to show that $\Lambda_r(\mathbb{C}^\infty)$ with $r \in \mathbb{Z}$ are non-isomorphic irreducible submodules. Furthermore, one can show that A_m (from Example 5.3) and $\Lambda^n(\mathbb{C}^\infty)$ for $m, n \geq 2$ are non-isomorphic irreducible $\mathfrak{gl}_\infty$-modules.

Example 5.5. We here consider a generalization of Example 5.3. Let S be any finite subset of $\mathbb{Z}$ and let $\alpha : S \to \mathbb{C}$ be a function such that $\alpha_j \notin \mathbb{Z}$ for $j \in S$. Set

$$V(S,\alpha) = \left(\prod_{j\in S} x_j^{\alpha_j}\right) \mathbb{C}\left[x_j, x_j^{-1} \mid j \in S\right] \otimes \mathbb{C}\left[x_n \mid n \in \mathbb{Z}\backslash S\right],$$

which is a $\mathfrak{gl}_\infty$-module with $E_{m,n} = x_m \frac{\partial}{\partial x_n}$ for $m, n \in \mathbb{Z}$. It can be readily seen that $V(S,\alpha)$ is in category C_{fin}. Define $\deg x_n^\lambda = \lambda$ for $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, to make $V(S,\alpha)$ a $\mathbb{C}$-graded space. Then each homogeneous subspace of $V(S,\alpha)$ is a submodule. One can show that each homogeneous subspace as a $\mathfrak{gl}_\infty$-module is irreducible.

Remark 5.6. Note that C_{fin} does not contain any nontrivial highest weight modules nor nontrivial lowest weight modules. On the other hand, C_{fin} does not contain $\mathfrak{sl}_\infty$ viewed as an irreducible $\mathfrak{gl}_\infty$-submodule of the adjoint module either. All the examples given above are among what were called intermediate series modules.

It seems that every $\widetilde{\mathfrak{gl}}_\infty$-module in category C_{fin} is of level zero. Here we prove a weak version.

Lemma 5.7. *Let W be a $\widetilde{\mathfrak{gl}}_\infty$-module satisfying the condition that for any $w \in W$, there exists a finite subset S of $\mathbb{Z}$ such that $E_{p,q}w = 0$ for all $p, q \in \mathbb{Z}$ with $p, q \notin S$. Then W is of level 0.*

Proof. We may assume $W \neq 0$. Let w be any nonzero vector in W. By assumption, there exists a finite subset S of $\mathbb{Z}$ such that $E_{p,q}w = 0$ for all $p, q \in \mathbb{Z}$ with $p, q \notin S$. As S is finite, there exist a negative integer m and a positive integer n such that $m, n \notin S$. Then

$$E_{m,n}w = E_{n,m}w = E_{m,m}w = E_{n,n}w = 0,$$

from which we get

$$0 = E_{m,n}(E_{n,m}w) - E_{n,m}(E_{m,n}w) = E_{m,m}w - E_{n,n}w + \psi(E_{m,n}, E_{n,m})\mathbf{k}w = \mathbf{k}w.$$

Therefore $\mathbf{k}$ acts trivially on W. $\square$

Remark 5.8. Let W be any $\widetilde{\mathfrak{gl}}_\infty$-module of level ℓ in category C_{fin}. By Proposition 5.2, W becomes a $\widetilde{\mathfrak{gl}}_\infty^e$-module of level ℓ. From the construction, one sees that for any $m, n \in \mathbb{Z}$ with $n \geq 0$, $B(m, n)$ acts on W trivially. It follows from the construction of $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ that for any $w \in W$, there exists a unique $\widetilde{\mathfrak{gl}}_\infty^e$-module homomorphism from $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell, 0)$ to W, sending $\mathbf{1}$ to w.

Example 5.9. From Example 5.3, we have non-isomorphic irreducible $\mathfrak{gl}_\infty$-modules (or equivalently, $\widetilde{\mathfrak{gl}}_\infty$-modules of level 0) A_r ($r \in \mathbb{N}$) in category C_{fin}. By Proposition 5.2, we obtain non-isomorphic irreducible $\widetilde{\mathfrak{gl}}_\infty^e$-modules of level 0. The action of $\widetilde{\mathfrak{gl}}_\infty^e$ on $\mathbb{C}[x_n \mid n \in \mathbb{Z}]$ is determined by

$$B(m, x)x_k = x_m e^{(m-k)x} \quad \text{for } m, k \in \mathbb{Z}$$

with $B(m, n)$ ($m, n \in \mathbb{Z}$) acting as derivations. For $r \in \mathbb{N}$, denote by A_r^e the corresponding irreducible $\widetilde{\mathfrak{gl}}_\infty^e$-module. By Remark 5.8, all irreducible modules A_r^e for $r \in \mathbb{N}$ are homomorphism images of $V_{\widetilde{\mathfrak{gl}}_\infty^e}(0, 0)$. In view of this, irreducible vacuum $\widetilde{\mathfrak{gl}}_\infty^e$-modules of level 0 are not unique.

Remark 5.10. There are two problems naturally arisen, one of which is to classify irreducible $\widetilde{\mathfrak{gl}}_\infty$-modules in category C_{fin} and the other is to classify all irreducible vacuum $\widetilde{\mathfrak{gl}}_\infty^e$-modules of a fixed level ℓ. We hope to address these problems in a later publication.

We end this section with a generalized-Verma-module construction. Recall that $\widetilde{\mathfrak{gl}}_\infty^e[0] = E \otimes \mathbb{C}[[t]] + \mathbb{C}\mathbf{k}$ is an abelian subalgebra. Let $\ell \in \mathbb{C}$ and let $\lambda : \mathbb{Z} \to \mathbb{C}$ be a function. Define a $\widetilde{\mathfrak{gl}}_\infty^e[0]$-module structure on $\mathbb{C}$ with $E \otimes t\mathbb{C}[[t]]$ acting trivially, with $B(n,0)$ acting as λ_n for $n \in \mathbb{Z}$, and with $\mathbf{k}$ acting as ℓ. We denote this module by $\mathbb{C}_{\ell,\lambda}$. Then form an induced module

$$M(\ell,\lambda) = U(\widetilde{\mathfrak{gl}}_\infty^e) \otimes_{U(\widetilde{\mathfrak{gl}}_\infty^e[0])} \mathbb{C}_{\ell,\lambda}. \tag{5.4}$$

Noticing that $\widetilde{\mathfrak{gl}}_\infty^e = \widetilde{\mathfrak{gl}}_\infty^e[0] \oplus (E \otimes t^{-1}\mathbb{C}[t^{-1}])$ as a vector space, in view of the P-B-W theorem we have

$$M(\ell,\lambda) = S(E \otimes t^{-1}\mathbb{C}[t^{-1}])$$

as a vector space. Notice that in case $\lambda = 0$, we have $M(\ell,0) = V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$.

Remark 5.11. Unlike the case for affine Lie algebras, it is not clear whether $M(\ell,\lambda)$ has a unique maximal submodule. (Indeed, we have seen that maximal submodules of $M(\ell,0)$ $\left(= V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)\right)$ (with $\lambda = 0$) are not unique as there are many non-isomorphic irreducible vacuum modules.) It is also natural to ask under what condition $M(\ell,\lambda)$ is irreducible. An immediate conjecture is that $V_{\widetilde{\mathfrak{gl}}_\infty^e}(\ell,0)$ is irreducible for any generic level ℓ.

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