

ON QUASI MODULES AT INFINITY FOR VERTEX ALGEBRAS

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ABSTRACT. A theory of quasi modules at infinity for (weak) quantum vertex algebras including vertex algebras was previously developed in [Li6]. In this current paper, we develop this theory of quasi modules at infinity for vertex algebras further. Among the main results, we obtain a commutator formula for general quasi modules at infinity and we establish a category equivalence between the category of quasi modules at infinity for a certain family of vertex algebras and the category of lowest weight modules for a certain family of Lie algebras.

1. INTRODUCTION

In order to associate quantum vertex algebras to certain algebras such as centerless double Yangians and the Lie algebra of pseudo-differential operators on the circle, a theory of what were called quasi modules at infinity for quantum vertex algebras was developed in [Li6]. The notion of (weak) quantum vertex algebra, which was formulated in [Li3] (cf. [EK]), is a natural generalization of the notions of vertex algebra and vertex super-algebra. Let V be a weak quantum vertex algebra. Note that for a (quasi) V -module W , each vector $v \in V$ is represented by a formal series $Y_W(v, x) \in \text{Hom}(W, W((x)))$. In contrast, for a (quasi) V -module at infinity M each vector $v \in V$ is represented by a formal series $Y_M(v, x) \in \text{Hom}(M, M((x^{-1})))$. Normally, associative or Lie algebras are associated with vertex algebras or more general quantum vertex algebras through their “highest weight” modules. A matter of fact is that certain algebras such as the Lie algebra of pseudo-differential operators on the circle only admit “lowest weight” modules; they do *not* admit nontrivial highest weight modules. This was the main motivation for developing the theory of quasi modules at infinity. Indeed, this theory enabled us to associate quantum vertex algebras to the aforementioned algebras through their lowest weight modules.

Let V be a weak quantum vertex algebra. A V -module at infinity is a vector space W equipped with a linear map

$$\begin{aligned} Y_W(\cdot, x) : \quad V &\rightarrow \text{Hom}(W, W((x^{-1}))) \\ v &\mapsto Y_W(v, x), \end{aligned}$$

satisfying the conditions that $Y_W(\mathbf{1}, x) = 1_W$ and that for any $u, v \in V$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1^{-1}, x_2^{-1}))),$$

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$$(1.1) \quad x_0^k Y_W(Y(u, x_0)v, x_2) = ((x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2)) |_{x_1=x_2+x_0}.$$

For the definition of a quasi V -module at infinity, the second condition above is replaced with the condition that for any $u, v \in V$, there is a nonzero polynomial $p(x_1, x_2)$ such that

$$p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1^{-1}, x_2^{-1}))),$$

$$p(x_0 + x_2, x_2)Y_W(Y(u, x_0)v, x_2) = (p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2)) |_{x_1=x_2+x_0}.$$

It was proved therein that if V is a vertex algebra, the second condition in the definition of a module at infinity amounts to the following opposite Jacobi identity

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(v, x_2)Y_W(u, x_1) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(u, x_1)Y_W(v, x_2) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2). \end{aligned}$$

Consequently, for a vertex algebra V , the notion of V -module at infinity is the same as that of right V -module (see [HL], [Li2], [X]). But, for a *quasi* V -module at infinity one has only a *quasi* opposite Jacobi identity, which is weaker than the opposite Jacobi identity.

A notion of quasi module for vertex algebras was previously introduced in [Li4], in order to associate vertex algebras to certain Lie algebras. Meanwhile, a notion of vertex Γ -algebra was also introduced with Γ a group, where a *vertex Γ -algebra* is a vertex algebra V equipped with two group homomorphisms

$$L : \Gamma \rightarrow GL(V) \quad \text{and} \quad \phi : \Gamma \rightarrow \mathbb{C}^\times,$$

satisfying the condition that $L(g)\mathbf{1} = \mathbf{1}$,

$$L(g)Y(v, x) = Y(L(g)v, \phi(g)x)L(g) \quad \text{for } g \in \Gamma, v \in V.$$

In [Li6], a notion of quasi V -module at infinity for a vertex Γ -algebra V was studied. For simplicity, consider the special case with Γ a group of linear functions $g(x) = \alpha x + \beta$ with $\alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}$ (with respect to function composition). In this case, a quasi V -module at infinity is a quasi module at infinity (W, Y_W) for V viewed as a vertex algebra, which also satisfies

$$Y_W(L(g)v, x) = Y_W(v, g^{-1}(x)) \quad \text{for } g \in \Gamma, v \in V.$$

In the present paper, we focus our attention on quasi modules at infinity for vertex algebras and we develop the theory of quasi modules for vertex Γ -algebras further. As the main results of this paper, we obtain a commutator formula for quasi modules at infinity for a general vertex Γ -algebra V , which to a certain extent is analogous to the twisted vertex operator commutator formula (see [FLM]). Just as the twisted commutator formula is important in the study of twisted modules, this commutator formula is important in the study of quasi modules at infinity for vertex Γ -algebras. As an application of the commutator formula, we establish a version of the converse of Theorem 5.14 in [Li6]. (Roughly speaking, this theorem asserts that each restricted module for a certain Lie algebra with some parameters is naturally a quasi module at infinity for a certain vertex Γ -algebra.) In addition, we generalize a technical result of [Li6] (Lemma 5.11), filling in a gap in the proof of the aforementioned theorem.

This paper is organized as follows: In Section 2, we refine or extend several results of [Li6] and we present a commutator formula. In Section 3, we present a complete proof of Theorem 5.14 of [Li6] and establish a converse.

2. QUASI MODULES AT INFINITY FOR VERTEX ALGEBRAS

In this section, we first recall some basic results from [Li6] and we then refine several results and establish a commutator formula for quasi modules at infinity.

We begin with some basic formal variable notations. First of all, throughout this paper, vector spaces will be over \mathbb{C} (the field of complex numbers) unless it is stated otherwise. We shall use the formal variable notations and conventions as established in [FLM] and [FHL]. In particular, for a vector space W , $W[[x, x^{-1}]]$ and $W[[x_1^{\pm 1}, x_2^{\pm 1}]]$ denote the spaces of doubly infinite formal series with coefficients in W , while $W((x))$ and $W((x_1, x_2))$ denote the spaces of lower truncated infinite formal series. Recall also the formal delta functions:

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n,$$

$$x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) = \sum_{n \in \mathbb{Z}} x_1^{-n-1} x_2^n.$$

Denote by $\mathbb{C}(x)$ the field of rational functions in variable x and by $\mathbb{C}(x_1, x_2)$ the field of rational functions in x_1 and x_2 . We have canonical field embeddings:

$$\begin{aligned} \iota_{x_1, x_2} : \quad & \mathbb{C}(x_1, x_2) \rightarrow \mathbb{C}((x_1))((x_2)), \\ \iota_{x_1^{-1}, x_2} : \quad & \mathbb{C}(x_1, x_2) \rightarrow \mathbb{C}((x_1^{-1}))((x_2)), \end{aligned}$$

which extend the ring embeddings of $\mathbb{C}[x_1, x_2]$ into $\mathbb{C}((x_1))((x_2))$ and $\mathbb{C}((x_1^{-1}))((x_2))$, respectively.

Throughout this paper, we set

$$(2.1) \quad G = \{\alpha x + \beta \mid \alpha, \beta \in \mathbb{C}, \alpha \neq 0\} \subset \mathbb{C}[x],$$

which is considered as a group with respect to function composition. (Note that G consists of all the linear transformations that preserve ∞ .) For $g(x) = \alpha x + \beta \in G$, we have $g^{-1}(x) = \alpha^{-1}x - \alpha^{-1}\beta$. We have a canonical group homomorphism

$$(2.2) \quad \Phi : G \rightarrow \mathbb{C}^\times, \quad g(x) = \alpha x + \beta \mapsto \alpha.$$

Let $g(x) = \alpha x + \beta \in G$. For $n \in \mathbb{Z}$, we view $g(x)^n$ as an element of $\mathbb{C}((x^{-1}))$:

$$g(x)^n = (\alpha x + \beta)^n = \sum_{i \geq 0} \binom{n}{i} \alpha^{n-i} \beta^i x^{n-i}.$$

It is understood that

$$(2.3) \quad \delta\left(\frac{g(x_2)}{x_1}\right) = \sum_{n \in \mathbb{Z}} x_1^{-n} g(x_2)^n = \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} \binom{n}{i} \alpha^{n-i} \beta^i x_1^{-n} x_2^{n-i}.$$

For a vector space W over \mathbb{C} , set

$$(2.4) \quad \mathcal{E}^o(W) = \text{Hom}(W, W((x^{-1}))) \subset (\text{End}W)[[x, x^{-1}]].$$

Vector space $\mathcal{E}^o(W)$ (over \mathbb{C}) is also naturally a module over $\mathbb{C}((x^{-1}))$. The identity operator, denoted by 1_W , is a special element of $\mathcal{E}^o(W)$.

Let $a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1} \in \mathcal{E}^o(W)$ and let $g(x) = \alpha x + \beta \in G$. As a convention we define

$$a(g(x)) = \sum_{n \in \mathbb{Z}} a_n (\alpha x + \beta)^{-n-1} = \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} \binom{-n-1}{i} \alpha^{-n-i-1} \beta^i a_n x^{-n-i-1},$$

which still lies in $\mathcal{E}^o(W)$.

For $g \in G$, define $L_g \in \text{End}(\mathcal{E}^o(W))$ by

$$(2.5) \quad L_g(a(x)) = a(g^{-1}(x)) \quad \text{for } g \in G, a(x) \in \mathcal{E}^o(W).$$

This makes $\mathcal{E}^o(W)$ a (left) G -module¹.

Definition 2.1. Let V be a vertex algebra. A *quasi V -module at infinity* is a vector space W equipped with a linear map

$$\begin{aligned} Y_W(\cdot, x) : \quad V &\rightarrow \text{Hom}(W, W((x^{-1}))) \subset (\text{End}W)[[x, x^{-1}]] \\ v &\mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{with } v_n \in \text{End}W), \end{aligned}$$

satisfying the conditions that $Y_W(\mathbf{1}, x) = 1_W$ (the identity operator on W) and that for any $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$(2.6) \quad p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1^{-1}, x_2^{-1})))$$

and

$$(2.7) \quad (p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2)) |_{x_1=x_2+x_0} = p(x_0 + x_2, x_2) Y_W(Y(u, x_0)v, x_2).$$

Remark 2.2. Note that for $A(x_1, x_2) \in \text{Hom}(W, W((x_1^{-1}, x_2^{-1})))$ with W a vector space,

$$A(x_2 + x_0, x_2) \quad \text{exists in } (\text{Hom}(W, W((x_2^{-1})))) [[x_0]].$$

The condition (2.6) in Definition 2.1 guarantees that the expression on the left-hand side of (2.7) exists. On the other hand, assume that (W, Y_W) is a quasi module at infinity for a vertex algebra V and let $u, v \in V$. Then (2.7) holds for any nonzero polynomial $p(x_1, x_2)$ such that (2.6) holds.

We have the following result which strengthens Lemma 5.1 of [Li6]:

Proposition 2.3. *Let (W, Y_W) be a quasi module at infinity for a vertex algebra V . Then for any $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that*

$$(2.8) \quad p(x_1, x_2) Y_W(v, x_2) Y_W(u, x_1) = p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2)$$

and

$$(2.9) \quad \begin{aligned} &x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) p(x_1, x_2) Y_W(v, x_2) Y_W(u, x_1) \\ &\quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) p(x_1, x_2) Y_W(Y(u, x_0)v, x_2). \end{aligned}$$

¹An action of G on $\mathcal{E}^o(W)$ was defined in [Li6] by $R_g(a(x)) = a(g(x))$ for $g \in G$, $a(x) \in \mathcal{E}^o(W)$. In fact, this gives a *right* action instead of a *left* action because G is nonabelian.

Proof. Lemma 5.1 in [Li6] asserts that there exists a nonzero polynomial $p(x_1, x_2)$ such that (2.8) holds, from which we see that (2.6) holds. From Remark 2.2, (2.7) holds. Then (2.9) follows immediately from Lemma 4.2 therein. \square

Definition 2.4. Let W be a vector space. A subset U of $\mathcal{E}^o(W)$ is said to be *quasi local* if for any $a(x), b(x) \in U$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$(2.10) \quad p(x_1, x_2)[a(x_1), b(x_2)] = 0.$$

The following notion was due to [GKK]:

Definition 2.5. Let W be a vector space and let Γ be a subgroup of G . A subset U of $\mathcal{E}^o(W)$ is said to be Γ -*local* if for any $a(x), b(x) \in U$, (2.10) holds with $p(x_1, x_2)$ a product of $x_1 - g(x_2)$ with $g \in \Gamma$ (not necessarily multiplicity-free).

Note that the relation (2.10), which can be written as

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2)b(x_2)a(x_1),$$

implies

$$(2.11) \quad p(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1^{-1}, x_2^{-1}))).$$

A subset U of $\mathcal{E}^o(W)$ is said to be *quasi compatible* if for any $a(x), b(x) \in U$, there exists a nonzero polynomial $p(x_1, x_2)$ such that (2.11) holds. Furthermore, for a subgroup Γ of G we define Γ -*compatibility* correspondingly.

Now, let $a(x), b(x) \in \mathcal{E}^o(W)$. Assume that there exists a nonzero polynomial $p(x_1, x_2)$ such that (2.11) holds. We define

$$a(x)_n b(x) \in \mathcal{E}^o(W) \quad \text{for } n \in \mathbb{Z}$$

in terms of generating function

$$Y_{\mathcal{E}^o}(a(x), x_0)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) x_0^{-n-1}$$

by

$$Y_{\mathcal{E}^o}(a(x), x_0)b(x) = \iota_{x^{-1}, x_0} \left(\frac{1}{p(x + x_0, x)} \right) (p(x_1, x)a(x_1)b(x))|_{x_1=x+x_0},$$

where $\iota_{x^{-1}, x_0} : \mathbb{C}(x, x_0) \rightarrow \mathbb{C}((x^{-1}))((x_0))$ is the unique extension of the embedding of $\mathbb{C}[x, x_0]$ into $\mathbb{C}((x^{-1}))((x_0))$.

The following result was obtained in [Li6] (Theorem 5.4):

Theorem 2.6. *Let W be a vector space and let U be any quasi local subset of $\mathcal{E}^o(W)$. Then U generates a vertex algebra $\langle U \rangle$ with $Y_{\mathcal{E}^o}$ as the vertex operator map and with 1_W as the vacuum vector, and W is a quasi module at infinity with*

$$Y_W(\alpha(x), z) = \alpha(z) \quad \text{for } \alpha(x) \in \langle U \rangle.$$

The following was also essentially obtained in [Li6]:

Proposition 2.7. *Let V be a vertex algebra and let W be a vector space equipped with a linear map $Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x^{-1})))$ with $Y_W(\mathbf{1}, x) = 1_W$. Set*

$$V_W = \{Y_W(v, x) \mid v \in V\} \subset \mathcal{E}^o(W).$$

Then (W, Y_W) is a quasi V -module at infinity if and only if V_W is quasi local, $(V_W, Y_{\mathcal{E}^o}, 1_W)$ carries the structure of a vertex algebra, and Y_W is a vertex algebra homomorphism from V to V_W .

Proof. Assume that (W, Y_W) is a quasi V -module at infinity. By Proposition 2.3, V_W is a quasi local subspace of $\mathcal{E}^o(W)$. From [Li6] (the first half of the proof of Lemma 5.9) we have

$$(2.12) \quad Y_{\mathcal{E}^o}(Y_W(u, x), x_0)Y_W(v, x) = Y_W(Y(u, x_0)v, x) \quad \text{for } u, v \in V.$$

We also have $Y_W(\mathbf{1}, x) = 1_W$. Then it follows that $(V_W, Y_{\mathcal{E}^o}, 1_W)$ carries the structure of a vertex algebra and Y_W is a homomorphism of vertex algebras. The other direction is clear from Theorem 2.6. \square

The following result generalizes Lemma 5.11 of [Li6]:

Lemma 2.8. *Let W be a vector space and let $a(x), b(x) \in \mathcal{E}^o(W)$. Assume*

$$(2.13) \quad [a(x_1), b(x_2)] = \sum_{j=0}^r A_j(x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) \\ + \sum_{i=1}^k \sum_{j=0}^s B_{ij}(x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right),$$

where $A_j(x), B_{ij}(x) \in \mathcal{E}^o(W)$, $g_i(x) \in G$ with $g_i(x) \neq x$ for $1 \leq i \leq k$. Then

$$(2.14) \quad \begin{aligned} a(x)_j b(x) &= -A_j(x) \quad \text{for } 0 \leq j \leq r, \\ a(x)_j b(x) &= 0 \quad \text{for } j > r. \end{aligned}$$

Proof. Set

$$q(x_1, x) = \prod_{i=1}^k ((x_1 - x)^{r+1} - (g_i(x) - x)^{r+1})^{s+1} \in \mathbb{C}[x_1, x].$$

Noticing that

$$q(x_1, x) = \bar{q}(x_1, x) \prod_{i=1}^k (x_1 - g_i(x))^{s+1}$$

for some $\bar{q}(x_1, x) \in \mathbb{C}[x_1, x]$ and that

$$(x_1 - g_i(x))^{s+1} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right) = 0$$

for $1 \leq i \leq k$, $0 \leq j \leq s$, we have

$$q(x_1, x_2) \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right) = 0.$$

Then we get

$$(2.15) \quad q(x_1, x_2)[a(x_1), b(x_2)] = \sum_{j=0}^r q(x_1, x_2) A_j(x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{x_2}{x_1} \right).$$

Note that (2.15) implies

$$(x_1 - x_2)^{r+1} q(x_1, x_2)[a(x_1), b(x_2)] = 0.$$

From the definition of $Y_{\mathcal{E}^o}$ we have

$$x_0^{r+1} q(x + x_0, x) Y_{\mathcal{E}^o}(a(x), x_0) b(x) = ((x_1 - x)^{r+1} q(x_1, x) a(x_1) b(x)) |_{x_1=x+x_0}.$$

Combining the two identities above with Lemma 4.2 of [Li6], we obtain

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x}{x_0} \right) (x_1 - x)^{r+1} q(x_1, x) b(x) a(x_1) \\ & - x_0^{-1} \delta \left(\frac{x - x_1}{-x_0} \right) (x_1 - x)^{r+1} q(x_1, x) a(x_1) b(x) \\ = & x^{-1} \delta \left(\frac{x_1 - x_0}{x} \right) (x_1 - x)^{r+1} q(x_1, x) Y_{\mathcal{E}^\circ}(a(x), x_0) b(x). \end{aligned}$$

Applying $\text{Res}_{x_0} x_0^{-r-1}$ to both sides, we get

$$-q(x_1, x)[a(x_1), b(x)] = \sum_{i \geq 0} a(x)_i b(x) \frac{1}{i!} q(x_1, x) \left(\frac{\partial}{\partial x} \right)^i x_1^{-1} \delta \left(\frac{x}{x_1} \right),$$

which is a finite sum. Combining this with (2.15) we obtain

$$(2.16) \quad \sum_{i \geq 0} (A_i(x) + a(x)_i b(x)) \frac{1}{i!} q(x_1, x) \left(\frac{\partial}{\partial x} \right)^i x_1^{-1} \delta \left(\frac{x}{x_1} \right) = 0,$$

where we set $A_i(x) = 0$ for $i > r$. Notice that we have

$$q(x_1, x) = (x_1 - x)^{r+1} P(x_1, x) + Q(x),$$

where $P(x_1, x)$ is a polynomial and

$$Q(x) = (-1)^{k(s+1)} \prod_{i=1}^k (g_i(x) - x)^{(r+1)(s+1)}.$$

Since

$$(x_1 - x)^{r+1} \left(\frac{\partial}{\partial x} \right)^j x_1^{-1} \delta \left(\frac{x}{x_1} \right) = 0 \quad \text{for } 0 \leq j \leq r,$$

(2.16) reduces to

$$\sum_{i \geq 0} (A_i(x) + a(x)_i b(x)) \frac{1}{i!} Q(x) \left(\frac{\partial}{\partial x} \right)^i x_1^{-1} \delta \left(\frac{x}{x_1} \right) = 0.$$

From [Li1] (Lemma 2.1.4), we get

$$(A_i(x) + a(x)_i b(x)) Q(x) = 0 \quad \text{for } i \geq 0.$$

Since $g_i(x) \neq x$ for $1 \leq i \leq k$, we have $Q(x) \neq 0$. Then our assertions follow immediately. \square

As a generalization and a corollary of Lemma 2.8 we have:

Proposition 2.9. *Let W be a vector space and let $a(x), b(x) \in \mathcal{E}^\circ(W)$. Assume*

$$(2.17) \quad [a(x_1), b(x_2)] = \sum_{i=1}^k \sum_{j=0}^r A_{ij}(x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right),$$

where $A_{ij}(x) \in \mathcal{E}^\circ(W)$ and $g_i(x) \in G$ distinct for $1 \leq i \leq k$. Then

$$(2.18) \quad \begin{aligned} & \Phi(g_i) a(g_i(x))_j b(x) = -A_{ij}(x) \quad \text{for } 0 \leq j \leq r, \\ & \Phi(g_i) a(g_i(x))_j b(x) = 0 \quad \text{for } j > r, \end{aligned}$$

where $\Phi : G \rightarrow \mathbb{C}^\times$ was defined before by $\Phi(g) = \alpha$ for $g(x) = \alpha x + \beta \in G$.

Proof. Let $g(x) = \alpha x + \beta \in G$ (with $\alpha \in \mathbb{C}^\times$, $\beta \in \mathbb{C}$). We have

$$\begin{aligned} x_1^{-1} \delta \left(\frac{g(x_2)}{x_1} \right) &= x_1^{-1} \delta \left(\frac{\alpha x_2 + \beta}{x_1} \right) = \alpha^{-1} x_2^{-1} \delta \left(\frac{x_1 - \beta}{\alpha x_2} \right) \\ &= \alpha^{-1} x_2^{-1} \delta \left(\frac{\alpha^{-1} x_1 - \alpha^{-1} \beta}{x_2} \right) = \Phi(g)^{-1} x_2^{-1} \delta \left(\frac{g^{-1}(x_1)}{x_2} \right). \end{aligned}$$

Using this, from (2.17) we have

$$[a(g(x_1)), b(x_2)] = \sum_{i=1}^k \sum_{j=0}^r \Phi(g_i)^{-1} A_{ij}(x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_2^{-1} \delta \left(\frac{g_i^{-1} g(x_1)}{x_2} \right).$$

Taking $g(x) = g_i(x)$ with $1 \leq i \leq k$, by Lemma 2.8 we obtain

$$a(g_i(x))_j b(x) = -\Phi(g_i)^{-1} A_{ij}(x) \quad \text{for } 0 \leq j \leq r$$

and $a(g_i(x))_j b(x) = 0$ for $j > r$, as desired. \square

The following technical result generalizes a result of [GKK]:

Lemma 2.10. *Let W be a vector space, let*

$$A(x_1, x_2) \in \text{Hom}(W, W((x_2^{-1}))[[x_1, x_1^{-1}]]) ,$$

and let

$$p(x_1, x_2) = \prod_{i=1}^r (x_1 - g_i(x_2))^{k_i},$$

where $g_1(x), \dots, g_r(x)$ are distinct elements of G and k_1, \dots, k_r are positive integers.

Then $p(x_1, x_2)A(x_1, x_2) = 0$ if and only if

$$(2.19) \quad A(x_1, x_2) = \sum_{i=1}^r \sum_{j=0}^{k_i-1} A_{ij}(x_2) \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right)$$

for some $A_{ij}(x) \in \mathcal{E}^o(W)$.

Proof. The “if” part is clear. For the “only if” part, we first consider the case $r = 1$. Note that for any $h(x) \in G$, $A(x_1, h(x_2))$ exists in $\text{Hom}(W, W((x_2^{-1}))[[x_1, x_1^{-1}]])$. Let us simply use $g(x)$ for $g_1(x)$. We see that

$$(x_1 - g(x_2))^k A(x_1, x_2) = 0$$

if and only if

$$(x_1 - x_2)^k A(x_1, g^{-1}(x_2)) = 0.$$

It was known (see [K2], [DLM]) that the latter is equivalent to

$$(2.20) \quad A(x_1, g^{-1}(x_2)) = \sum_{j=0}^{k-1} B_j(x_2) \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{x_2}{x_1} \right)$$

for some $B_j(x) \in (\text{End}W)[[x, x^{-1}]]$. For $0 \leq j \leq k-1$, since

$$\text{Res}_{x_1} (x_1 - x_2)^j \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) = j!,$$

we have

$$B_j(x_2) = \frac{1}{j!} \text{Res}_{x_1} (x_1 - x_2)^j A(x_1, g^{-1}(x_2)).$$

As $A(x_1, g^{-1}(x_2)) \in \text{Hom}(W, W((x_2^{-1}))[[x_1, x_1^{-1}]])$, we get $B_j(x) \in \mathcal{E}^o(W)$. It is clear that (2.20) is equivalent to

$$(2.21) \quad A(x_1, x_2) = \sum_{j=0}^{k-1} B_j(g(x_2)) \Phi(g)^{-j} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g(x_2)}{x_1} \right)$$

under the condition that $B_j(x) \in \mathcal{E}^o(W)$ for $0 \leq j \leq k-1$.

We next consider the general case. For $1 \leq i \leq r$, set

$$p_i(x_1, x_2) = \frac{p(x_1, x_2)}{(x_1 - g_i(x_2))^{k_i}} \in \mathbb{C}[x_1, x_2].$$

Note that $p_i(x_1, x_2)$ with $1 \leq i \leq r$, viewed as polynomials in x_1 with coefficients in $\mathbb{C}(x_2)$, are relatively prime. Then there exist $q_i(x_1, x_2) \in \mathbb{C}(x_2)[x_1]$ for $1 \leq i \leq r$ such that

$$1 = p_1(x_1, x_2)q_1(x_1, x_2) + \cdots + p_k(x_1, x_2)q_k(x_1, x_2).$$

Recall the field embedding $\iota_{x, \infty} : \mathbb{C}(x) \rightarrow \mathbb{C}((x^{-1}))$. This gives an algebra embedding of $\mathbb{C}(x_2)[x_1]$ into $\mathbb{C}((x_2^{-1}))[x_1]$. Denote by $\bar{q}_i(x_1, x_2)$ the image of $q_i(x_1, x_2)$ in $\mathbb{C}((x_2^{-1}))[x_1]$. We have

$$1 = p_1(x_1, x_2)\bar{q}_1(x_1, x_2) + \cdots + p_k(x_1, x_2)\bar{q}_k(x_1, x_2)$$

in $\mathbb{C}((x_2^{-1}))[x_1]$, so that

$$(2.22) \quad A(x_1, x_2) = \sum_{i=1}^k p_i(x_1, x_2)\bar{q}_i(x_1, x_2)A(x_1, x_2).$$

Note that

$$(x_1 - g_i(x_2))^{k_i} p_i(x_1, x_2)\bar{q}_i(x_1, x_2)A(x_1, x_2) = 0$$

and $p_i(x_1, x_2)\bar{q}_i(x_1, x_2)A(x_1, x_2) \in \text{Hom}(W, W((x_2^{-1}))[[x_1, x_1^{-1}]])$. Then it follows immediately from (2.22) and the special case. \square

The following is a variation of the notion of Γ -vertex algebra, introduced in [Li6] (cf. [Li4])²:

Definition 2.11. Let Γ be a group. A *vertex Γ -algebra* is a vertex algebra V equipped with two group homomorphisms

$$L : \Gamma \rightarrow GL(V) \quad \text{and} \quad \phi : \Gamma \rightarrow \mathbb{C}^\times,$$

satisfying the condition that $L(g)\mathbf{1} = \mathbf{1}$ for $g \in \Gamma$ and

$$(2.23) \quad L(g)Y(u, x)v = Y(L(g)u, \phi(g)x)L(g)v \quad \text{for } g \in \Gamma, u, v \in V.$$

It is clear that for any group H equipped with a homomorphism from H to Γ , a vertex Γ -algebra V is naturally a vertex H -algebra. Define a *homomorphism of vertex Γ -algebras* from U to V to be a homomorphism θ of vertex algebras such that

$$\theta(L(g)u) = L(g)\theta(u) \quad \text{for } g \in \Gamma, u \in U.$$

²The notion of vertex Γ -algebra here with respect to the pair (Γ, ϕ) amounts to the notion of Γ -vertex algebra in [Li6] with respect to (Γ, ϕ^{-1}) .

Remark 2.12. Let V be a \mathbb{Z} -graded vertex algebra in the sense that V is a vertex algebra equipped with a \mathbb{Z} -grading $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$ such that

$$u_k v \in V_{(m+n-k-1)} \quad \text{for } u \in V_{(m)}, v \in V_{(n)}, m, n, k \in \mathbb{Z}.$$

Let $L(0)$ denote the linear operator on V , defined by

$$L(0)v = nv \quad \text{for } v \in V_{(n)}, n \in \mathbb{Z}.$$

Let Γ be any automorphism group of V , which preserves the \mathbb{Z} -grading, and let $\phi : \Gamma \rightarrow \mathbb{C}^\times$ be any linear character. For $g \in \Gamma$, set $L(g) = \phi(g)^{L(0)}g$. Then it can be readily seen that V becomes a vertex Γ -algebra.

The following is a modification of the same named notion introduced in [Li6]³:

Definition 2.13. Let V be a vertex Γ -algebra. A *quasi V -module at infinity* is a quasi module at infinity (W, Y_W) for V viewed as a vertex algebra, equipped with a group homomorphism

$$\Psi : \Gamma \rightarrow G,$$

satisfying the condition that $\phi = \Phi \circ \Psi$,

$$(2.24) \quad Y_W(L(g)v, x) = Y_W(v, \Psi(g)^{-1}(x)) \quad \text{for } g \in \Gamma, v \in V,$$

and $\{Y_W(v, x) \mid v \in V\}$ is $\Psi(\Gamma)$ -local.

Lemma 2.14. *In Definition 2.13, the condition that $\{Y_W(v, x) \mid v \in V\}$ is $\Psi(\Gamma)$ -local can be replaced with a weaker condition that for any $u, v \in V$, there exists a product $q(x_1, x_2)$ of linear polynomials $x_1 - g(x_2)$ with $g \in \Psi(\Gamma)$ such that*

$$(2.25) \quad q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1^{-1}, x_2^{-1}))).$$

Proof. Let $u, v \in V$. Assume that $q(x_1, x_2)$ is a product of linear polynomials $x_1 - g(x_2)$ with $g \in \Psi(\Gamma)$ such that (2.25) holds. From Proposition 2.3, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) = p(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1).$$

Then

$$p(x_1, x_2)(q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) - q(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1)) = 0.$$

From definition, $q(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1)$ lies in $\text{Hom}(W, W((x_2^{-1}))(x_1^{-1}))$. As (2.25) holds, $q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2)$ also lies in $\text{Hom}(W, W((x_2^{-1}))(x_1^{-1}))$. Consequently, (multiplying by $\iota_{x_2^{-1}, x_1^{-1}}(1/p(x_1, x_2))$) we get

$$q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) = q(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1).$$

This proves that $\{Y_W(v, x) \mid v \in V\}$ is $\Psi(\Gamma)$ -local. \square

Recall the (left) group action of G on $\mathcal{E}^o(W)$. We have (cf. [Li6], Theorem 5.10)⁴:

³The only change from the original definition is in (2.24), to incorporate the change of the right action R of G on $\mathcal{E}^o(W)$ to the left action L .

⁴This is the corrected version of Theorem 5.10 in [Li6] with the right action R of Γ replaced with the left action L .

Theorem 2.15. *Let W be a vector space, let Γ be a subgroup of G , and let U be a Γ -local subset of $\mathcal{E}^o(W)$. Then $\Gamma \cdot U$ is Γ -local and the vertex algebra $\langle \Gamma \cdot U \rangle$ generated by $\Gamma \cdot U$ is a vertex Γ -algebra where $\phi = \Phi$ and L is given by $L(g) = L_g$ for $g \in \Gamma$, that is*

$$L(g)a(x) = a(g^{-1}(x)) \quad \text{for } g \in \Gamma, a(x) \in \langle \Gamma \cdot U \rangle.$$

Furthermore, W is a quasi module at infinity for $\langle \Gamma \cdot U \rangle$ with $Y_W(a(x), z) = a(z)$ for $a(x) \in \langle \Gamma \cdot U \rangle$ and with $\Psi = 1$.

Proof. As $L_g = R_{g^{-1}}$ on $\mathcal{E}^o(W)$ for $g \in G$, Lemma 3.13 of [Li6] asserts that

$$L_g Y_{\mathcal{E}^o}(a(x), x_0) b(x) = Y_{\mathcal{E}^o}(L_g a(x), \Phi(g)x_0) L_g b(x)$$

for $g \in \Gamma$, $a(x), b(x) \in \langle \Gamma \cdot U \rangle$. This together with Theorem 2.6 confirms the first assertion on the vertex Γ -algebra structure. As for the structure of a quasi module at infinity, we have

$$Y_W(L_g a(x), x_0) = Y_W(a(g^{-1}(x)), x_0) = a(g^{-1}(x_0)) = Y_W(a(x), g^{-1}(x_0))$$

for $g \in \Gamma$, $a(x) \in \langle \Gamma \cdot U \rangle$. On the other hand, it follows from Proposition 3.14 of [Li6] and induction that $\langle \Gamma \cdot U \rangle$ is Γ -local. This confirms the second assertion that W is a quasi module at infinity for $\langle \Gamma \cdot U \rangle$ viewed as a vertex Γ -algebra with $Y_W(a(x), x_0) = a(x_0)$ for $a(x) \in \langle \Gamma \cdot U \rangle$. \square

The following refinement of Proposition 2.7 is straightforward:

Proposition 2.16. *Let V be a vertex Γ -algebra, let $\Psi : \Gamma \rightarrow G$ be a group homomorphism such that $\Phi \circ \Psi = \phi$, and let W be a vector space equipped with a linear map $Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x^{-1})))$ with $Y_W(\mathbf{1}, x) = 1_W$. Set*

$$V_W = \{Y_W(v, x) \mid v \in V\} \subset \mathcal{E}^o(W).$$

Then (W, Y_W) is a quasi V -module at infinity if and only if V_W is $\Psi(\Gamma)$ -local, $(V_W, Y_{\mathcal{E}^o}, 1_W)$ carries the structure of a vertex $\Psi(\Gamma)$ -algebra, and $Y_W : V \rightarrow V_W$ is a homomorphism of vertex Γ -algebras, where V_W is viewed as a vertex Γ -algebra through the homomorphism $\Psi : \Gamma \rightarrow \Psi(\Gamma) \subset G$.

As the main result of this section, we have the following analog of the twisted vertex operator commutator formula (see [FLM]):

Theorem 2.17. *Let V be a vertex Γ -algebra with group homomorphisms $L : \Gamma \rightarrow GL(V)$; $\phi : \Gamma \rightarrow \mathbb{C}^\times$, let (W, Y_W) be a quasi V -module at infinity with group homomorphism $\Psi : \Gamma \rightarrow G$, and let $u, v \in V$. Then there exist finitely many $\sigma_1, \dots, \sigma_r \in \Gamma$ such that $\Psi(\sigma_i)$ ($i = 1, \dots, r$) are distinct in G and*

$$(2.26) \quad [Y_W(u, x_1), Y_W(v, x_2)] \\ = - \sum_{i=1}^r \sum_{j \in \mathbb{N}} \phi(\sigma_i)^{-1} Y_W((L(\sigma_i)u)_j v, x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\Psi(\sigma_i)^{-1}(x_2)}{x_1} \right).$$

Furthermore, if $\Psi : \Gamma \rightarrow G$ is one-to-one, then

$$(2.27) \quad [Y_W(u, x_1), Y_W(v, x_2)] \\ = - \sum_{\sigma \in \Gamma, j \in \mathbb{N}} \phi(\sigma)^{-1} Y_W((L(\sigma)u)_j v, x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\Psi(\sigma)^{-1}(x_2)}{x_1} \right),$$

which is a finite sum.

Proof. From definition, there exist distinct $g_1(x), \dots, g_r(x) \in \Psi(\Gamma) \subset G$ and positive integers k_1, \dots, k_r such that

$$(2.28) \quad \left(\prod_{i=1}^r (x_1 - g_i(x_2))^{k_i} \right) [Y_W(u, x_1), Y_W(v, x_2)] = 0.$$

In view of Lemma 2.10, we have

$$(2.29) \quad [Y_W(u, x_1), Y_W(v, x_2)] = \sum_{i=1}^r \sum_{j=0}^{k_i-1} A_{ij}(x_2) \frac{1}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right)$$

for some $A_{ij}(x) \in \mathcal{E}^o(W)$. By Proposition 2.9, we get

$$A_{ij}(x) = -\Phi(g_i) Y_W(u, g_i(x))_j Y_W(v, x),$$

where $Y_W(u, g_i(x))$ and $Y_W(v, x)$ are viewed as elements of $\mathcal{E}^o(W)$. Let $\sigma_1, \dots, \sigma_r \in \Gamma$ such that $\Psi(\sigma_i) = g_i^{-1}(x)$ for $1 \leq i \leq r$. As $Y_W(L(\sigma_i)u, x) = Y_W(u, g_i(x))$ and $\Phi(g_i) = \Phi(\Psi(\sigma_i^{-1})) = \phi(\sigma_i)^{-1}$, we have

$$A_{ij}(x) = -\phi(\sigma_i)^{-1} Y_W(L(\sigma_i)u, x)_j Y_W(v, x).$$

Recall from Proposition 2.7 that

$$(2.30) \quad Y_{\mathcal{E}^o}(Y_W(a, x), x_0) Y_W(b, x) = Y_W(Y(a, x_0)b, x) \quad \text{for } a, b \in V.$$

Using this we obtain

$$\begin{aligned} & [Y_W(u, x_1), Y_W(v, x_2)] \\ &= -\text{Res}_{x_0} \sum_{i=1}^r \Phi(g_i) Y_{\mathcal{E}^o}(Y_W(L(\sigma_i)u, x_2), x_0) Y_W(v, x_2) e^{x_0 \frac{\partial}{\partial x_2}} x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right) \\ &= -\text{Res}_{x_0} \sum_{i=1}^r \phi(\sigma_i)^{-1} Y_W(Y(L(\sigma_i)u, \Phi(g_i)^{-1}x_0)v, x_2) e^{x_0 \frac{\partial}{\partial x_2}} x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right) \\ &= -\text{Res}_{x_0} \sum_{i=1}^r \phi(\sigma_i)^{-1} Y_W(Y(L(\sigma_i)u, \phi(\sigma_i)x_0)v, x_2) e^{x_0 \frac{\partial}{\partial x_2}} x_1^{-1} \delta \left(\frac{g_i(x_2)}{x_1} \right), \end{aligned}$$

as desired.

Suppose Ψ is one-to-one. Let $\sigma \in \Gamma$ with $\sigma \neq \sigma_i$ for $1 \leq i \leq r$. Set $g(x) = \Psi(\sigma^{-1}) \in G$. Then $g(x) \neq g_i(x)$ for $1 \leq i \leq r$. Combining (2.29) with Proposition 2.9, we have

$$Y_W(u, g(x))_j Y_W(v, x) = 0 \quad \text{for } j \geq 0,$$

i.e.,

$$Y_{\mathcal{E}^o}(Y_W(u, g(x)), x_0) Y_W(v, x) \in \mathcal{E}^o(W)[[x_0]].$$

Using Proposition 2.7 we have

$$\begin{aligned} Y_W(Y(L(\sigma)u, x_0)v, x) &= Y_{\mathcal{E}^o}(Y_W(L(\sigma)u, x), x_0) Y_W(v, x) \\ &= Y_{\mathcal{E}^o}(Y_W(u, g(x)), x_0) Y_W(v, x). \end{aligned}$$

Consequently, $Y_W(Y(L(\sigma)u, x_0)v, x)$ involves only nonnegative powers of x_0 . Thus

$$\text{Res}_{x_0} Y_W(Y(L(\sigma)u, x_0)v, x_2) e^{x_0 \frac{\partial}{\partial x_2}} x_1^{-1} \delta \left(\frac{g(x_2)}{x_1} \right) = 0.$$

Then the second assertion follows immediately. \square

The following technical result is a modification of Lemma 5.9 in [Li6] with a slightly different proof:

Lemma 2.18. *Let V be a vertex Γ -algebra, let $\Psi : \Gamma \rightarrow G$ be a group homomorphism with $\Phi \circ \Psi = \phi$, and let (W, Y_W) be a quasi module at infinity for V viewed as a vertex algebra. Assume that $U_W = \{Y_W(u, x) \mid u \in U\}$ is $\Psi(\Gamma)$ -local and*

$$Y_W(L(g)u, x) = Y_W(u, \Psi(g)^{-1}(x)) \quad \text{for } g \in \Gamma, u \in U,$$

where U is a Γ -submodule which generates V as a vertex algebra. Then (W, Y_W) is a quasi V -module at infinity.

Proof. From assumption, U_W is a $\Psi(\Gamma)$ -local subset of $\mathcal{E}^o(W)$. For $g \in \Gamma$, $u \in U$, we have

$$(2.31) \quad Y_W(L(g)u, x) = Y_W(u, \Psi(g)^{-1}(x)) = L_{\Psi(g)}(Y_W(u, x)).$$

It follows that U_W is stable under the subgroup $\Psi(\Gamma)$ of G . By Theorem 2.15, we have a vertex $\Psi(\Gamma)$ -algebra $\langle U_W \rangle$ with W as a quasi module at infinity. Since (W, Y_W) is a module at infinity for V viewed as a vertex algebra, in view of Proposition 2.7, Y_W is a homomorphism of vertex algebras from V to $\langle U_W \rangle$.

Suppose that

$$Y_W(L(g)u, x) = Y_W(u, \Psi(g)^{-1}(x)) \quad \text{and} \quad Y_W(L(g)v, x) = Y_W(v, \Psi(g)^{-1}(x))$$

for some $g \in \Gamma$, $u, v \in V$. Then we have

$$\begin{aligned} & Y_W(L(g)Y(u, x_0)v, x) \\ &= Y_W(Y(L(g)u, \phi(g)x_0)L(g)v, x) \\ &= Y_{\mathcal{E}^o}(Y_W(L(g)u, x), \phi(g)x_0)Y_W(L(g)v, x) \\ &= Y_{\mathcal{E}^o}(Y_W(u, \Psi(g)^{-1}(x)), \phi(g)x_0)Y_W(v, \Psi(g)^{-1}(x)) \\ &= L_{\Psi(g)}Y_{\mathcal{E}^o}(Y_W(u, x), x_0)Y_W(v, x) \\ &= L_{\Psi(g)}Y_W(Y(u, x_0)v, x) \\ &= Y_W(Y(u, x_0)v, \Psi(g)^{-1}(x)). \end{aligned}$$

As U generates V as a vertex algebra, it follows from (2.31) that

$$Y_W(L(g)v, x) = L_{\Psi(g)}(Y_W(v, x)) \quad \text{for all } g \in \Gamma, v \in V.$$

That is, Y_W is a homomorphism of vertex Γ -algebras. By Proposition 2.16, (W, Y_W) is a quasi V -module at infinity. \square

3. LIE ALGEBRA $\hat{\mathfrak{g}}(\infty)[\Gamma]$ AND VERTEX ALGEBRA $V_{\hat{\mathfrak{g}}}(\ell, 0)$

In this section, we recall from [Li6] the Lie algebra $\hat{\mathfrak{g}}(\infty)[\Gamma]$ and the main results on the relation between $\hat{\mathfrak{g}}(\infty)[\Gamma]$ and vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$, including Theorem 5.14, and as our main result we establish the converse of this theorem.

Let \mathfrak{g} be a (possibly infinite-dimensional) Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Associated to the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, one has an (untwisted) affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k},$$

where \mathbf{k} is central and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\delta_{m+n,0}\langle a, b \rangle \mathbf{k}$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Defining $\deg(\mathfrak{g} \otimes t^m) = -m$ for $m \in \mathbb{Z}$ and $\deg \mathbf{k} = 0$ makes $\hat{\mathfrak{g}}$ a \mathbb{Z} -graded Lie algebra. For $a \in \mathfrak{g}$, form a generating function

$$a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) x^{-n-1}.$$

Let ℓ be a complex number. Denote by \mathbb{C}_ℓ the 1-dimensional $(\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbf{Ck})$ -module with $\mathfrak{g} \otimes \mathbb{C}[t]$ acting trivially and with \mathbf{k} acting as scalar ℓ . Form the induced $\hat{\mathfrak{g}}$ -module

$$(3.1) \quad V_{\hat{\mathfrak{g}}}(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbf{Ck})} \mathbb{C}_\ell.$$

Set $\mathbf{1} = 1 \otimes \mathbf{1}$ and then identify \mathfrak{g} as a subspace of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ through the linear map $a \rightarrow a(-1)\mathbf{1}$. It is well known (cf. [FZ]) that there exists a unique vertex-algebra structure on $V_{\hat{\mathfrak{g}}}(\ell, 0)$ with $\mathbf{1}$ as the vacuum vector and with $Y(a, x) = a(x)$ for $a \in \mathfrak{g}$. Defining $\deg \mathbf{1} = 0$ makes $V_{\hat{\mathfrak{g}}}(\ell, 0)$ a \mathbb{Z} -graded $\hat{\mathfrak{g}}$ -module and the vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ equipped with this \mathbb{Z} -grading is a \mathbb{Z} -graded vertex algebra.

Let Γ be a subgroup of $\text{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, consisting of automorphisms of \mathfrak{g} that preserve $\langle \cdot, \cdot \rangle$. Each $g \in \Gamma$ lifts canonically to an automorphism of the \mathbb{Z} -graded Lie algebra $\hat{\mathfrak{g}}$, and then to an automorphism of the \mathbb{Z} -graded vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$. In this way, Γ acts on vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ by automorphisms that preserve the \mathbb{Z} -grading. Let $\phi : \Gamma \rightarrow \mathbb{C}^\times$ be any group homomorphism. For $g \in \Gamma$, set

$$L(g) = \phi(g)^{L(0)} g \in GL(V_{\hat{\mathfrak{g}}}(\ell, 0)),$$

where $L(0)$ denotes the linear operator on $V_{\hat{\mathfrak{g}}}(\ell, 0)$ defined by $L(0)v = nv$ for $v \in V_{\hat{\mathfrak{g}}}(\ell, 0)_{(n)}$ with $n \in \mathbb{Z}$. This defines a vertex Γ -algebra structure on $V_{\hat{\mathfrak{g}}}(\ell, 0)$.

Consider the following completion of affine Lie algebra $\hat{\mathfrak{g}}$:

$$(3.2) \quad \hat{\mathfrak{g}}(\infty) = \mathfrak{g} \otimes \mathbb{C}((t^{-1})) \oplus \mathbf{Ck},$$

where

$$(3.3) \quad [a \otimes p(t), b \otimes q(t)] = [a, b] \otimes p(t)q(t) + \text{Res}_t p'(t)q(t) \langle a, b \rangle \mathbf{k}$$

for $a, b \in \mathfrak{g}$, $p(t), q(t) \in \mathbb{C}((t^{-1}))$.

The following is a construction of a family of new Lie algebras by using Lie algebra $\hat{\mathfrak{g}}(\infty)$ (cf. [Li6], Proposition 5.12; [GKK])⁵:

Proposition 3.1. *Let \mathfrak{g} be a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ and let Γ be an automorphism group of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, satisfying the condition that for any $u, v \in \mathfrak{g}$,*

$$[gu, v] = 0 \quad \text{and} \quad \langle gu, v \rangle = 0 \quad \text{for all but finitely many } g \in \Gamma.$$

Let $\Psi : \Gamma \rightarrow G$ be a group homomorphism and set $g(x) = \Psi(g)(x) \in G$ for $g \in \Gamma$. Define a new bilinear multiplicative operation $[\cdot, \cdot]_\Gamma$ on vector space $\hat{\mathfrak{g}}(\infty) = \mathfrak{g} \otimes \mathbb{C}((t^{-1})) \oplus \mathbf{Ck}$ by

$$\begin{aligned} [a \otimes p(t), \mathbf{k}]_\Gamma &= 0 = [\mathbf{k}, a \otimes p(t)]_\Gamma, \\ [a \otimes p(t), b \otimes q(t)]_\Gamma &= \sum_{g \in \Gamma} [ga, b] \otimes p(g^{-1}(t))q(t) + \text{Res}_t q(t) \frac{d}{dt} p(g^{-1}(t)) \langle ga, b \rangle \mathbf{k} \end{aligned}$$

⁵This is the corrected version of Proposition 5.12 in [Li6]. In the original proof, an action of Γ on $\hat{\mathfrak{g}}(\infty)$ was defined by

$$g(a \otimes p(t) + \lambda \mathbf{k}) = ga \otimes p(g(t)) + \lambda \mathbf{k}$$

for $g \in \Gamma$, $a \in \mathfrak{g}$, $p(t) \in \mathbb{C}((t^{-1}))$, $\lambda \in \mathbb{C}$, which is not a left action if $\Psi(\Gamma)$ is not abelian.

for $a, b \in \mathfrak{g}$, $p(t), q(t) \in \mathbb{C}((t^{-1}))$. Then the subspace, linearly spanned by vectors

$$ga \otimes p(t) - a \otimes p(g(t))$$

for $g \in \Gamma$, $a \in \mathfrak{g}$, $p(t) \in \mathbb{C}((t^{-1}))$, is a two-sided ideal of the non-associative algebra, and the quotient algebra, which we denote by $\hat{\mathfrak{g}}(\infty)[\Gamma]$, is a Lie algebra.

Proof. For $g \in \Gamma$, $a \in \mathfrak{g}$, $p(t) \in \mathbb{C}((t^{-1}))$, $\lambda \in \mathbb{C}$, define

$$g(a \otimes p(t) + \lambda \mathbf{k}) = ga \otimes p(g^{-1}(t)) + \lambda \mathbf{k}.$$

It is straightforward to show that Γ acts on $\hat{\mathfrak{g}}(\infty)$ by automorphisms, satisfying the condition that for any $u, v \in \hat{\mathfrak{g}}(\infty)$,

$$[gu, v] = 0 \quad \text{for all but finitely many } g \in \Gamma.$$

Then it follows immediately from [Li5] (Lemma 4.1). □

Let

$$\pi : \hat{\mathfrak{g}}(\infty) \rightarrow \hat{\mathfrak{g}}(\infty)[\Gamma]$$

denote the natural linear map. For $a \in \mathfrak{g}$, set

$$(3.4) \quad a_{\Gamma}(x) = \sum_{n \in \mathbb{Z}} \pi(a \otimes t^n) x^{-n-1} \in (\hat{\mathfrak{g}}(\infty)[\Gamma])[[x, x^{-1}]].$$

Define a linear character $\phi : \Gamma \rightarrow \mathbb{C}^{\times}$ by

$$\phi(g) = \frac{d}{dx} \Psi(g)(x) \quad \text{for } g \in \Gamma.$$

We say that a $\hat{\mathfrak{g}}(\infty)[\Gamma]$ -module W is of level $\ell \in \mathbb{C}$ if \mathbf{k} acts on W as scalar ℓ . The following is a modification of Lemma 5.13 in [Li6]:

Lemma 3.2. *Let W be a vector space and let $\ell \in \mathbb{C}$. Then a $\hat{\mathfrak{g}}(\infty)[\Gamma]$ -module structure of level ℓ on W amounts to a linear map*

$$\theta : \mathfrak{g} \rightarrow \mathcal{E}^o(W); \quad a \mapsto a_W(x),$$

satisfying the conditions that

$$(3.5) \quad (ga)_W(x) = \phi(g)^{-1} a_W(g^{-1}(x))$$

for $g \in \Gamma$, $a \in \mathfrak{g}$ and that

$$(3.6) \quad [a_W(x_1), b_W(x_2)] \\ = \sum_{g \in \Gamma} [ga, b]_W(x_2) x_1^{-1} \delta \left(\frac{g^{-1}(x_2)}{x_1} \right) + \ell \langle ga, b \rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{g^{-1}(x_2)}{x_1} \right)$$

for $a, b \in \mathfrak{g}$.

Proof. Let $g(x) = \alpha x + \beta \in G$ with $\alpha \in \mathbb{C}^{\times}$, $\beta \in \mathbb{C}$. Then $g^{-1}(x) = \alpha^{-1}(x - \beta)$ and

$$x^{-1} \delta \left(\frac{g(t)}{x} \right) = x^{-1} \delta \left(\frac{\alpha t + \beta}{x} \right) = \alpha^{-1} t^{-1} \delta \left(\frac{x - \beta}{\alpha t} \right) = \alpha^{-1} t^{-1} \delta \left(\frac{g^{-1}(x)}{t} \right).$$

Using this we get

$$\begin{aligned}
(ga)_\Gamma(x) &= \sum_{n \in \mathbb{Z}} \pi(ga \otimes t^n) x^{-n-1} = \sum_{n \in \mathbb{Z}} \pi(a \otimes g(t)^n) x^{-n-1} \\
&= \pi \left(a \otimes x^{-1} \delta \left(\frac{g(t)}{x} \right) \right) \\
&= \phi(g)^{-1} \pi \left(a \otimes t^{-1} \delta \left(\frac{g^{-1}(x)}{t} \right) \right) \\
&= \phi(g)^{-1} a_\Gamma(g^{-1}(x)),
\end{aligned}$$

proving (3.5). As for (3.6), notice that

$$\begin{aligned}
\sum_{m, n \in \mathbb{Z}} g^{-1}(t)^m t^n x_1^{-m-1} x_2^{-n-1} &= x_1^{-1} \delta \left(\frac{g^{-1}(t)}{x_1} \right) x_2^{-1} \delta \left(\frac{t}{x_2} \right) \\
&= x_1^{-1} \delta \left(\frac{g^{-1}(x_2)}{x_1} \right) x_2^{-1} \delta \left(\frac{t}{x_2} \right)
\end{aligned}$$

and

$$\begin{aligned}
&\text{Res}_t \sum_{m, n \in \mathbb{Z}} t^n \frac{d}{dt} g^{-1}(t)^m x_1^{-m-1} x_2^{-n-1} \\
&= -\text{Res}_t \sum_{m, n \in \mathbb{Z}} n t^{n-1} g^{-1}(t)^m x_1^{-m-1} x_2^{-n-1} \\
&= \text{Res}_t \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{g^{-1}(t)}{x_1} \right) t^{-1} \delta \left(\frac{x_2}{t} \right) \\
&= \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{g^{-1}(x_2)}{x_1} \right).
\end{aligned}$$

Then (3.6) follows from the construction of $\hat{\mathfrak{g}}(\infty)[\Gamma]$. \square

We have (cf. [Li6], Theorem 5.14)⁶:

Theorem 3.3. *Let $\ell \in \mathbb{C}$ and let W be any $\hat{\mathfrak{g}}(\infty)[\Gamma]$ -module of level ℓ such that $a_\Gamma(x) \in \mathcal{E}^\circ(W)$ for $a \in \mathfrak{g}$. Then there exists a unique structure of a quasi module at infinity on W for the vertex Γ -algebra $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ with $Y_W(a, x) = a_\Gamma(x)$ for $a \in \mathfrak{g}$, where \mathfrak{g}° denotes the opposite Lie algebra of \mathfrak{g} .*

Proof. Set

$$U = \{a_\Gamma(x) \mid a \in \mathfrak{g}\} \subset \mathcal{E}^\circ(W).$$

For $a, b \in \mathfrak{g}$, there exist $g_1, \dots, g_r \in \Gamma$ such that $[ga, b] = 0$ and $\langle ga, b \rangle = 0$ for $g \notin \{g_1, \dots, g_r\}$. It follows from (3.6) that

$$(x_1 - g_1^{-1}(x_2))^2 \cdots (x_1 - g_r^{-1}(x_2))^2 [a_\Gamma(x_1), b_\Gamma(x_2)] = 0.$$

Thus U is a Γ -local subspace of $\mathcal{E}^\circ(W)$. From (3.5), $\Gamma \cdot U = U$. By Theorem 2.15, U generates a vertex Γ -algebra $\langle U \rangle$ with W as a quasi module-at-infinity where $Y_W(\alpha(x), x_0) = \alpha(x_0)$ for $\alpha(x) \in \langle U \rangle$. Combining (3.6) with Lemma 2.8 we get

$$a_\Gamma(x)_0 b_\Gamma(x) = -[a, b]_\Gamma(x), \quad a_\Gamma(x)_1 b_\Gamma(x) = -\ell \langle a, b \rangle 1_W, \quad \text{and} \quad a_\Gamma(x)_n b_\Gamma(x) = 0$$

for $n \geq 2$. In view of the universal property of $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ (cf. [Li5]), there exists a vertex-algebra homomorphism from $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ to $\langle U \rangle$, sending a to $a_\Gamma(x)$ for $a \in \mathfrak{g}$.

⁶This is a corrected version of Theorem 5.14 in [Li6] with a complete proof.

Consequently, W is a quasi module-at-infinity for $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ viewed as a vertex algebra. Furthermore, for $g \in \Gamma$, $a \in \mathfrak{g}$ we have

$$Y_W(L(g)a, x) = Y_W(\phi(g)^{L(0)}ga, x) = \phi(g)(ga)_\Gamma(x) = a_\Gamma(g^{-1}(x)) = Y_W(a, g^{-1}(x)).$$

As \mathfrak{g} generates $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ as a vertex algebra, it follows from Lemma 2.18 that W is a quasi module-at-infinity for $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ viewed as a vertex Γ -algebra. \square

Furthermore, we have the following converse of Theorem 3.3:

Theorem 3.4. *Let (W, Y_W) be any quasi module at infinity for the vertex Γ -algebra $V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$ such that the associated homomorphism $\Psi : \Gamma \rightarrow G$ is one-to-one. Then W is a module for Lie algebra $\hat{\mathfrak{g}}(\infty)[\Gamma]$ of level ℓ with $a_\Gamma(x) = Y_W(a, x)$ for $a \in \mathfrak{g}$.*

Proof. Note that for $u, v \in \mathfrak{g} \subset V_{\hat{\mathfrak{g}}^\circ}(-\ell, 0)$, we have

$$u_0v = -[u, v], \quad u_1v = -\ell\langle u, v \rangle \mathbf{1}, \quad \text{and} \quad u_i v = 0 \quad \text{for } i \geq 2.$$

Let $a, b \in \mathfrak{g}$. Using Theorem 2.17 and the facts above we get

$$\begin{aligned} & [Y_W(a, x_1), Y_W(b, x_2)] \\ &= - \sum_{g \in \Gamma} Y_W([ga, b], x_2) x_1^{-1} \delta \left(\frac{g^{-1}(x_2)}{x_1} \right) + \ell \langle ga, b \rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{g^{-1}(x_2)}{x_1} \right), \end{aligned}$$

noticing that $L(g) = \phi(g)^{L(0)}g$ and $L(0)a = a$. On the other hand, we have

$$Y_W(ga, x) = \phi(g)^{-1}Y_W(L(g)a, x) = \phi(g)^{-1}Y_W(a, g^{-1}(x)) \quad \text{for } g \in \Gamma, a \in \mathfrak{g}.$$

It then follows from Lemma 3.2 that W is a module for $\hat{\mathfrak{g}}(\infty)[\Gamma]$ of level ℓ with $a_\Gamma(x) = Y_W(a, x)$ for $a \in \mathfrak{g}$. \square

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