Notes and Solutions to Problems

in

Arnold's

Mathematical Methods of Classical Mechanics[1]

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[1]Second Edition, Springer New York, 1989.

A few years ago I read through Arnold's classic, made extensive notes for myself and gave solutions to many of the problems in the text. Recently I came across my old notes, and thought it would be beneficial to some students and interested readers to share these. In my notes below "text" refers to Arnold's book; references are given as x/y, where x stands for the line number(s) (with $x < 0$ counting from the bottom of the page excluding footnotes and captions of illustrations) and y stands for the page number(s). For longer notes x/y signifies the start of the respective passage.
As always with first drafts, typos and errors are nearly unavoidable. Comments, notes, additions are alays welcome; please do not hesitate to send them to gtoth camden.rutgers.edu.

Part I: Newtonian Mechanics

-3/12 PROBLEM. We assume that the stone has unit mass. (See the footnote on p. 11 of the text.) Let r be the distance of the stone from the center of the earth, r_0 the radius of the earth, and M_e the mass of the earth. r is a function of time t with initial conditions $r(0) = r_0$ and $\dot{r}(0) = v_2$. The equation of motion is

$$\ddot{r} = -\frac{GM_e}{r^2}.$$

Multiplying through by $\dot{r}$, we have

$$\ddot{r} \cdot \dot{r} = \left(\frac{\dot{r}^2}{2}\right)^{\cdot} = -GM_e\frac{\dot{r}}{r^2} = -gr_0^2\frac{\dot{r}}{r^2},$$

where $GM_e/r_0^2 = g$ is the magnitude of the gravitational acceleration vector $\mathbf{g}$ on the surface of the earth, $g = |\mathbf{g}|$. Since we are looking for the second cosmic velocity,[1] we have $\lim_{t\to\infty} r(t) = \infty$, and we can integrate on $[0, \infty)$ as

$$\frac{1}{2}\int_0^\infty \left(\dot{r}^2\right)^{\cdot} dt = -\frac{1}{2}\dot{r}(0)^2 = -\frac{1}{2}v_2^2 = -gr_0^2\int_0^\infty \frac{\dot{r}}{r^2}\,dt = -gr_0^2\int_{r_0}^\infty \frac{dr}{r^2} = -gr_0^2\frac{1}{r_0} = -gr_0,$$

where minimality of v_2 is used in $\lim_{t\to\infty}\dot{r}(t) = 0$. Hence, we obtain

$$v_2 = \sqrt{2gr_0} \approx \sqrt{2\cdot 9.80665\cdot 6,378,000}\,m/sec \approx 11,184.52625\,m/sec.$$

-3/18 PROBLEM. The total energy is

$$\frac{1}{2}\dot{x}^2 + U(x) = E.$$

[1] "The **first cosmic velocity** v_1 is the velocity of motion on a circular orbit of radius close to the radius of the earth." (See 7/41 in the text and also below.) The **second cosmic velocity** v_2 is the minimum velocity of motion on a straight line that extends infinitely far from the earth.

Hence
$$\frac{dx}{dt} = \dot{x} = +\sqrt{2(E - U(x))},$$
where the positive sign indicates motion in one direction with x increasing in t. Inverting, we obtain
$$\frac{dt}{dx} = \frac{1}{\sqrt{2(E - U(x))}}.$$
Integrating, we arrive at
$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}.$$

-3/19 Problem. Expanding the potential energy U at the maximum ξ, we get
$$U(x) = U(\xi) + U'(\xi)(x - \xi) + \frac{U''(\xi)}{2}(x - \xi)^2 + \cdots$$
Now, $U(\xi) = E$, $U'(\xi) = 0$ (as ξ is a critical point), and $U''(\xi) \le 0$ (since at ξ the potential energy U attains maximum). With these, we calculate
$$\dot{x}^2 = 2(E - U(x)) = -U''(\xi)(x - \xi)^2 + \cdots$$
(See -3/18 abve.) Hence
$$\dot{x} = \pm\sqrt{-U''(\xi)}(x - \xi).$$
But $\dot{x} = y$, so that the equations of the tangent lines are
$$y = \pm\sqrt{-U''(\xi)}(x - \xi).$$

1/20 Problem. Let $x_1 < x_2$ be the (consecutive) intersection points of the closed phase curve with the x-axis. As the region enclosed by the phase curve is symmetric with repect to the first axis, by the definition of the integral, we have
$$\begin{aligned} \frac{1}{2}S &= \int_{x_1}^{x_2} y\,dx = \int_0^{T/2} y\,\dot{x}\,dt = \int_0^{T/2} y\sqrt{2(E - (U(x))}\,dt \\ &= \int_0^{T/2} \dot{x}\sqrt{2(E - (U(x))}\,dt = \int_{x_1}^{x_2} \sqrt{2(E - U(x))}\,dx, \end{aligned}$$
where we used the fact that the direction of the motion is positive. Differentiating under the integral sign **with respect to** E, we obtain
$$\frac{dS}{dE} = \sqrt{2}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}.$$

On the other hand, by the problem at -3/18 above, we have

$$\frac{T}{2} = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}.$$

Combining these, we arrive at

$$\frac{dS}{dE} = T.$$

5/20 PROBLEM. Using the expansion of U in problem at -3/19, we have $U'(x) = U''(\xi)(x - \xi) + \cdots$, where $U''(\xi) \geq 0$ since U attains minimum at ξ. Linearizing the differential equation governing the motion at $(\xi, 0)$ we thus obtain

$$\ddot{x} = -U''(\xi)(x - \xi).$$

The (shifted) solutions $x-\xi$ are linear combinations of $\cos(\sqrt{U''(\xi)}t)$ and $\sin(\sqrt{U''(\xi)}t)$ with period

$$T = \frac{2\pi}{\sqrt{U''(\xi)}}.$$

7/24 PROBLEM. The solution of the system in Example 1 at -5/23 can be written as

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \cos t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} + \sin t \begin{pmatrix} c_2 \\ -c_1 \\ c_4 \\ -c_3, \end{pmatrix}$$

where conservation of energy gives $2E_0 = x_1^2 + x_2^2 + y_1^2 + y_2^2 = c_1^2 + c_2^2 + c_3^2 + c_4^2$. This level surface π_{E_0} is the 4-sphere in $\mathbb{R}^4$ with radius $\sqrt{2E_0}$ and center at the origin. The vectors in the linear combination on the left-hand side of the equation above are orhonormal. Hence the phase curve is a circle with center at the origin. Scaling, we may asume $2E_0 = 1$.

14/24 PROBLEM. Introducing complex arithmetic, we have

$$z_1 = x_1 + iy_1 = (c_1 + ic_2)\cos t + (c_2 - ic_1)\sin t = (c_1 + ic_2)(\cos t + i\sin t) = e^{it}(c_1 + ic_2),$$

and similarly

$$z_2 = x_2 + iy_2 = (c_3 + ic_4)(\cos t + i\sin t) = e^{it}(c_3 + ic_4).$$

With the identification $\mathbb{R}^4 = \mathbb{C}^2$, $(c_1, c_2, c_3, c_4) \mapsto (z_1, z_2)$, we have $2E_0 = |z_1|^2 + |z_2|^2$, and for the "Hopf map" we obtain

$$w = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{c_1 + ic_2}{c_3 + ic_4} \in \mathbb{C}.$$

This shows that, under the Hopf map, every phase curve is mapped to a point. In addition, the pre-image of a point is a single phase curve. Indeed, $z_1/z_2 = z_1'/z_2'$ implies $z_1/z_1' = z_2/z_2' = \lambda \in \mathbb{C}$, and $|z_1|^2 + |z_2|^2 = |z_1'|^2 + |z_2'|^2$ $(z_1, z_2, z_1', z_2' \neq 0, \infty)$, gives $|\lambda| = 1$, that is $\lambda = e^{it}$ for some $t \in \mathbb{R}$.
Note that $\mathbb{C}$ plus the point at infinity ∞ can be identified with the 2-sphere S^2 via the stereographic projection $h : S^2 \to \mathbb{C}$ (from the North pole $(0, 0, 1)$) whose inverse is given by[2]

$$h^{-1}(z) = \left(\frac{2z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right), \quad z \in \mathbb{C}.$$

Substituting $z = z_1/z_2$, we obtain

$$\begin{aligned} h^{-1}\left(\frac{z_1}{z_2}\right) &= \left(\frac{2z_1/z_2}{|z_1/z_2|^2 + 1}, \frac{|z_1/z_2|^2 - 1}{|z_1/z_2|^2 + 1} \right) = \left(\frac{2z_1\bar{z}_2}{|z_1|^2 + |z_2|^2}, \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} \right) \\ &= \left(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2\right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3, \end{aligned}$$

since we set $2E_0 = 1$ (7/24). This gives the usual representation of the Hopf map $S^3 \to S^2$ associating to $(z_1, z_2) \in S^3 \subset \mathbb{C}$ the point $(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2) \in S^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$.

12/30 We have

$$(\mathbf{F}, d\mathbf{S}) = \mathbf{\Phi}(r)(\mathbf{e}_r, d\mathbf{r}) = \mathbf{\Phi}(r)\left(\frac{\mathbf{r}}{r}, d\mathbf{r}\right) = \mathbf{\Phi}(r)\frac{(\mathbf{r}, d\mathbf{r})}{r} = \frac{1}{2}\mathbf{\Phi}(r)\frac{d(r^2)}{r} = \mathbf{\Phi}(r)dr.$$

-6/31 In coordinates $\mathbf{e}_r = (\cos\varphi, \sin\varphi)$ and $\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi)$.

2/32 Indeed, by the above

$$\begin{aligned} \dot{\mathbf{e}}_r &= \dot{\varphi}(-\sin\varphi, \cos\varphi) = \dot{\varphi}\mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\varphi &= -\dot{\varphi}(\cos\varphi, \sin\varphi) = -\dot{\varphi}\mathbf{e}_r. \end{aligned}$$

12/34 Here we use the lemma at the bottom of page 31.

[2]See Toth, G., *Glimpses of Algebra and Geometry*, Second Edition, Springer NY (2000), Section 7, p. 88.

-10/34 Strictly speaking, we assume here that r increases with t so that the square root is positive.

8/37 PROBLEM 1. Under the substitution $x = M/r$, the effective potential energy $V(r) = U(r) + M^2/(2r^2)$ becomes

$$W(x) = V\left(\frac{M}{x}\right) = U\left(\frac{M}{x}\right) + \frac{x^2}{2}.$$

With the differential $dx = -M dr/r^2$, the apsidal angle is

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{M/r^2}{\sqrt{2(E - V(r))}}\, dr = -\int_{x_{\max}}^{x_{\min}} \frac{dx}{\sqrt{2(E - W(x))}} = \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2(E - W(x))}}.$$

13/37 PROBLEM 2. The derivative of the effective potential energy with respect to $' = d/dr$ is $V'(r) = U'(r) - M^2/r^3$, and therefore the equation of motion is

$$\ddot{r} = -V'(r) = -U'(r) + \frac{M^2}{r^3}.$$

Letting $x = r - r_0$, we expand the right-hand side at $x = 0$ and obtain

$$\begin{aligned}
\ddot{x} &= -V'(x + r_0) = -V'(r_0) - V''(r_0)x + \cdots = -V''(r_0)x^2 + \cdots \\
&= -U'(x + r_0) + \frac{M^2}{(x + r_0)^3} = -U'(r_0) - U''(r_0)\,x + \frac{M^2}{r_0^3} - \frac{3M^3}{r_0^4}\,x + \cdots \\
&= -\left(U''(r_0) + \frac{3M^2}{r_0^4}\right)\,x + \cdots = -\left(U''(r_0) + \frac{3U'(r_0)}{r_0}\right)\,x + \cdots
\end{aligned}$$

where $V'(r_0) = U'(r_0) - M^2/r_0^3 = 0$ as $r = r_0$ is a circular orbit.
The linearized equation is

$$\ddot{x} = -V''(r_0)x = -\left(U''(r_0) + \frac{3U'(r_0)}{r_0}\right)\,x = -\frac{r_0 U''(r_0) + 3U'(r_0)}{r_0}\,x.$$

The circular orbit $r = r_0$ is stable if $V''(r_0) > 0$, or equivalently, if $r_0 U''(r_0) + 3U'(r_0) > 0$. Assuming this, the period of oscillation for x and hence for $r = r_0 + x$ is

$$T = \frac{2\pi}{\sqrt{V''(r_0)}} = 2\pi\sqrt{\frac{r_0}{r_0 U''(r_0) + 3U'(r_0)}}.$$

The apsidal angle Φ is the amount by which φ increases from a maximum to the consecutive minimum. This happens in $T/2$ amount of time. Moreover $\dot{\varphi} = M/r^2 \approx$

$M/r_0^2 = \sqrt{U'(r_0)/r_0}$, so that

$$\begin{aligned}\Phi &= \dot{\varphi}\frac{T}{2} \approx \Phi_{\text{cir}} = \frac{M}{r_0^2}\frac{\pi}{\sqrt{V''(r_0)}} \\ &= \sqrt{\frac{U'(r_0)}{r_0}}\,\pi\sqrt{\frac{r_0}{r_0U''(r_0)+3U'(r_0)}} \\ &= \pi\sqrt{\frac{U'(r_0)}{3U'(r_0)+r_0U''(r_0)}}.\end{aligned}$$

15/37 Problem 3. Clearly, Φ_{cir} is independent of r_0 if and only if $rU''(r)/U'(r)$ is indepedent of r, or equivalently, $r(\ln|U'(r)|)' = \beta$ for some constant $\beta \geq -3$ (our condition of stability is $>$ and $U'(r) > 0$ for r near r_0). Hence $(\ln|U'(r)|)' = \beta/r$, and therefore $\ln|U'(r)| = \beta\ln r + B$. Adjusting the last constant, this gives $U'(r) = br^\beta$ ($b = \pm e^B$). Hence, for $\beta \neq -1$, $U(r) = br^{\beta+1}/(\beta+1) = ar^\alpha$, $\alpha \geq -2, \alpha \neq 0$ ($\alpha = \beta+1$, and $a = b/(\beta+1)$), and for $\beta = -1$, $U(r) = b\ln(r)$ (adjusting U by a constant). In both cases, we obtain $\Phi_{\text{cir}} = \pi/\sqrt{\beta+3}$ which is equal to $\pi/\sqrt{\alpha+2}$ with $\alpha = 0$ in the second case. In particular, the first two values of α for which all bounded orbits are closed are for $\alpha = -1$, $U(r) = a/r$, and we have $\Phi_{\text{cir}} = \pi$, and, for $\alpha = 2$, $U(r) = a\,r^2$, and we have $\Phi_{\text{cir}} = \pi/2$.

-7/37 Problem 4. The original setup for the effective potential is

$$V(r) = U(r) + \frac{M^2}{2r^2} \leq E, \quad r_{\min} \leq r \leq r_{\max}, \quad V(r_{\min}) = V(r_{\max}) = E.$$

In Problem 1 above we used the substitution $x = M/r$. With this we have

$$W(x) = V\left(\frac{M}{x}\right) = U\left(\frac{M}{x}\right) + \frac{x^2}{2} \leq E,\ x_{\min} \leq x \leq x_{\max},\ W(x_{\min}) = W(x_{\max}) = E,$$

where $x_{\min} = M/r_{\max}$ and $x_{\max} = M/r_{\min}$.
We now introduce yet another new variable y via $x = y\,x_{\max}$. With this $x_{\min} = y_{\min}x_{\max}$, so that $y_{\min} = x_{\min}/x_{\max}$ and $y_{\max} = 1$. With these, we define $W^*(y) = W(y\,x_{\max})/x_{\max}^2$, and obtain

$$W^*(y) = \frac{y^2}{2} + \frac{1}{x_{\max}^2}U\left(\frac{M}{yx_{\max}}\right) \leq \frac{E}{x_{\max}^2}, \quad y_{\min} \leq y \leq 1, \quad W^*(y_{\min}) = W^*(1) = \frac{E}{x_{\max}^2}.$$

From Problem 1, the apsidal angle becomes

$$\begin{aligned}\Phi &= \int_{y_{\min}}^{1} \frac{x_{\max}\, dy}{\sqrt{2(E - W(y\, x_{\max}))}} = \int_{y_{\min}}^{1} \frac{dy}{\sqrt{2\left(\frac{E}{x_{\max}^2} - \frac{1}{x_{\max}^2} W(y\, x_{\max})\right)}} \\ &= \int_{y_{\min}}^{1} \frac{dy}{\sqrt{2(W^*(1) - W^*(y))}},\end{aligned}$$

Note hat the integral is improper at both end-points $y_{\min}$ and 1.
Assume $\lim_{r\to\infty} U(r) = \infty$. In the setting of Problem 3 above this gives $U(r) = a\, r^{\alpha}$, $a, \alpha > 0$, or $U(r) = b \log(r)$, $b > 0$.
Assume now that $E \to \infty$ so that $x_{\max} \to \infty$.
In the two cases above we have

$$\begin{aligned}\frac{1}{x_{\max}^2} U\left(\frac{M}{x_{\max}}\right) &= \frac{a\, M^{\alpha}}{x_{\max}^{2+\alpha}}, \quad a > 0\ \alpha > 0, \\ \frac{1}{x_{\max}^2} U\left(\frac{M}{x_{\max}}\right) &= = \frac{b \log M}{x_{\max}^2} - b\frac{\log x_{\max}}{x_{\max}^2}, \quad b > 0.\end{aligned}$$

Both of these converge to zero as $x_{\max} \to \infty$. Hence, we obtain

$$\lim_{E\to\infty} W^*(1) = \frac{1}{2}.$$

Thus, since $W^*(y) \le W^*(1)$, $y_{\min} \le y \le 1$, the left-hand side is bounded as $E \to \infty$. On the other hand, $W^*(1) = E/x_{\max}^2$, so that $\lim_{E\to\infty} E/x_{\max}^2 = 1/2$, and hence $\lim_{E\to\infty} W^*(y_{\min}) = 1/2$ follow. Now, as $E \to \infty$, we also have $y_{\min} \to 0$.
For fixed $y > 0$ in

$$\begin{aligned}\frac{1}{x_{\max}^2} U\left(\frac{M}{y x_{\max}}\right) &= \frac{a\, M^{\alpha}}{y^{\alpha} x_{\max}^{2+\alpha}}, \quad a > 0,\ \alpha > 0 \\ \frac{1}{x_{\max}^2} U\left(\frac{M}{y x_{\max}}\right) &= = \frac{b \log(M/y)}{x_{\max}^2} - b\frac{\log x_{\max}}{x_{\max}^2}, \quad b > 0.\end{aligned}$$

As before, both coverge to zero as $x_{\max} \to \infty$, so that we have

$$\lim_{E\to\infty} W^*(y) = \frac{y^2}{2}, \quad y > 0.$$

Finally, for the integral, we have

$$\begin{aligned}\lim_{x_{\max}\to\infty} \int_{y_{\min}}^{1} \frac{dy}{\sqrt{2(W^*(1) - W^*(y))}} &= \int_0^1 \frac{dy}{\sqrt{2\,(1/2 - y^2/2)}} \\ = \int_0^1 \frac{dy}{\sqrt{1 - y^2}} &= \int_0^{\pi/2} dt = \frac{\pi}{2},\end{aligned}$$

where we used the substitution $y = \sin t$ and $dy = \cos t\, dt$. We finally arrive at

$$\lim_{E\to\infty} \Phi(E, M) = \int_0^1 \frac{dy}{\sqrt{2(W^*(1) - W^*(y))}} = \frac{\pi}{2}.$$

1/38 PROBLEM 5. The graph of the effective potential is as in Figure 34 on p. 38. As $E \to 0^-$, we have $r_{\rm min}$ the r-intercept of the graph, and $r_{\rm max} = \infty$. For the first, we have $k/r_{\rm min}^{\beta} = M^2/(2r_{\rm min}^2)$ which gives $r_{\rm min}^{2-\beta} = M^2/(2k)$. These give $x_{\rm min} = 0$ and $x_{\rm max}^{2-\beta} = 2k/M^{\beta}$. Note that the latter also follows from

$$W^*(y) = \frac{y^2}{2} - \frac{k}{x_{\rm max}^2}\left(\frac{yx_{\rm max}}{M}\right)^{\beta} = \frac{y^2}{2} - \frac{k}{M^{\beta}}\frac{y^{\beta}}{x_{\rm max}^{2-\beta}}$$

by setting $y_{\rm max} = 1$ as

$$W^*(1) = \frac{1}{2} - \frac{k}{M^{\beta}}\frac{1}{x_{\rm max}^{2-\beta}} = \frac{E}{x_{\rm max}^2},$$

and letting $E \to 0^-$. With this, we have

$$W^*(y) = \frac{y^2}{2} - \frac{y^{\beta}}{2}$$

We thus obtain

$$\Phi_0 = \lim_{E\to 0^-} \Phi = \int_0^1 \frac{dy}{\sqrt{y^{\beta} - y^2}} = \frac{\pi}{2-\beta}.$$

As for the computation of the last integral, we substitute

$$y^{2-\beta} = u^2 \quad \text{and} \quad dy = 2/(2-\beta) \cdot u^{2/(2-\beta)-1} du$$

(and follow up with the second substiution $u = \sin t$, $du = \cos\, dt$). We obtain

$$\int_0^1 \frac{dy}{\sqrt{y^{\beta} - y^2}} = \frac{2}{2-\beta}\int_0^1 \frac{u^{-\frac{\beta}{2-\beta}} u^{\frac{2}{2-\beta}-1}}{\sqrt{1-u^2}}\, du = \frac{2}{2-\beta}\int_0^1 \frac{du}{\sqrt{1-u^2}} = \frac{\pi}{2-\beta}.$$

-2/38 We use the substitution $x = M/r$ and $dx = -M dr/r^2$, and integrate

$$\begin{aligned}
\varphi &= \int \frac{(M/r^2)\, dr}{\sqrt{2(E + k/r - M^2/2r^2}} = \int \frac{dx}{\sqrt{2E + 2kx/M - x^2}} \\
&= -\int \frac{dx}{\sqrt{(2E + k^2/M^2) - (x - k/M)^2}} = -\int \frac{dy}{\sqrt{a^2 - y^2}} \\
&= \arccos\left(\frac{y}{a}\right) = \arccos\left(\frac{M/r - k/M}{\sqrt{2E + k^2/M^2}}\right)
\end{aligned}$$

where $a = \sqrt{2E + k^2/M^2}$ and $y = x - k/M$.

16/40 Proof. We provide some computational details in the proof of Kepler's third law: Let $p = M^2/k$ and $e = \sqrt{1 + 2EM^2/k^2}$ (7/39). Then, noting that $E < 0$, we calculate

$$a = \frac{p}{1-e^2} = \frac{M^2/k}{1-(1+2EM^2/k^2)} = \frac{k}{2|E|}.$$

(-3/39). Since $e = \sqrt{a^2 - b^2}/a$, we obtain

$$b = a\sqrt{1-e^2} = \frac{k}{2|E|}\sqrt{\frac{2|E|/M^2}{k^2}} = \frac{M}{\sqrt{2|E|}}.$$

Thus, since $M/2$ is the sectorial velocity, we arrive at

$$T = \frac{2\pi ab}{M} = \frac{2\pi}{M}\frac{k}{2|E|}\frac{M}{\sqrt{2|E|}} = 2\pi\frac{k}{\sqrt{2|E|}^3} = 2\pi\frac{a^{3/2}}{\sqrt{k}}.$$

7/41 Problem. We use the notations and setup in -6/12: $r = r_0$ is a circular orbit, $GM_e = gr_0^2$, and the angular momenum is $M = r_0 v_1$. The equation of motion on p. 34 is

$$\ddot{r} = -\frac{GM_e}{r^2} + \frac{M^2}{r^3}$$

(5/34). For the circular orbit $r = r_0$, we therefore have

$$\frac{GM_e}{r_0^2} = \frac{M^2}{r_0^3},$$

or equivalently $\sqrt{r_0 GM_e} = M = r_0 v_1$. These give $v_1 = \sqrt{GM_e/r_0} = \sqrt{gr_0}$. Combining this with the earlier formula $v_2 = \sqrt{2gr_0}$ (-3/12 above), we arrive at $v_2 = \sqrt{2}v_1$.

11/41 Problem. We let $r = r_0 + r_1$, $r_0 = 1$, $r_1 << 1$, $\varphi = \varphi_0 + \varphi_1$, $\varphi_0 = t$, $\varphi_1 << 1$. Substituting $r = 1 + r_1$ and $\varphi = t + \varphi_1$ into the equations of motion (-1/33)

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{1}{r^2} \quad \text{and} \quad 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$$

we obtain

$$\ddot{r}_1 - (1 + r_1)(1 + \dot{\varphi}_1)^2 = -\frac{1}{(1+r_1)^2} \quad \text{and} \quad 2\dot{r}_1(1 + \dot{\varphi}_1) + (1 + r_1)\ddot{\varphi}_1 = 0.$$

Linearizing

$$\ddot{r}_1 - 1 - r_1 - 2\dot{\varphi}_1 = -1 + 2r_1 \quad \text{and} \quad 2\dot{r}_1 + \ddot{\varphi}_1 = 0,$$

we obtain

$$\ddot{r}_1 = 3r_1 + 2\dot{\varphi}_1 \quad \text{and} \quad \ddot{\varphi}_1 = -2\dot{r}_1.$$

The scaling factor due to our choices $r_0 = 1$ and $\varphi_0 = t$ is $v_1/10 = \sqrt{gr_0}/10 = \sqrt{9.80665 \cdot 6378000}/10 = 790.8654355\ldots \approx 800$, where v_1 is the first cosmic velocity (7/41). With this, the initial conditions are $r_1(0) = \varphi_1(0) = \dot{\varphi}_1(0) = 0$ and $\dot{r}_1(0) = -1/800$. Letting $x = r_1$, $y = \dot{r}_1$, $z = \dot{\varphi}_1$, we obtain the linear system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 3x + 2z \\ \dot{z} &= -2y, \end{aligned}$$

with the initial conditions $x(0) = 0$, $y(0) = -1/800$, $z(0) = 0$.
This can be easily resolved by observing that $(2x + z)^{\cdot} = 0$ so that $2x + z = C = 0$. The system reduces to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x, \end{aligned}$$

with solution $x = A \sin t$, $y = A \cos t$ and $z = -2A \sin t$, $A = -1/800$. Playing these back to r_1 and φ_1, we get

$$\begin{aligned} r_1(t) &= -\frac{1}{800} \sin t \\ \dot{r}_1(t) &= -\frac{1}{800} \cos t \\ \dot{\varphi}_1(t) &= \frac{1}{400} \sin t \\ \varphi_1(t) &= -\frac{1}{400} \cos t, \end{aligned}$$

and finally

$$\begin{aligned} r(t) &= 1 - \frac{1}{800} \sin t \\ \varphi(t) &= t - \frac{1}{400} \cos t, \end{aligned}$$

with perigee $1 - 1/800$ and apogee $1 + 1/800$.

3/49 Assume we have a two point system $\mathbf{r} = (\mathbf{r}_i, \mathbf{r}_j)$ with $\mathbf{F} = (\mathbf{F}_{ij}, \mathbf{F}_{ji}) =$

$(\mathbf{F}_{ij}, -\mathbf{F}_{ij})$. We calculate

$$\begin{aligned} U_{ij}(\mathbf{r}) &= \int_{\mathbf{r}_0}^{\mathbf{r}} (\mathbf{F}, d\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} ((\mathbf{F}_{ij}, d\mathbf{r}_i) + (\mathbf{F}_{ji}, d\mathbf{r}_j)) \\ &= \int_{\mathbf{r}_0}^{\mathbf{r}} (\mathbf{F}_{ij}, d(\mathbf{r}_i - \mathbf{r}_j)) = \int_{\mathbf{r}_0}^{\mathbf{r}} (f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)\, \mathbf{e}_{ij}, d(|\mathbf{r}_i - \mathbf{r}_j|\, \mathbf{e}_{ij})) \\ &= \int_{\mathbf{r}_0}^{\mathbf{r}} (f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|), d|\mathbf{r}_i - \mathbf{r}_j|) = \int_{\mathbf{r}_0}^{\mathbf{r}} f_{ij}(\rho)\, d\rho. \end{aligned}$$

Moreover

$$-\frac{\partial U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} = -f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)\frac{\partial |\mathbf{r}_i - \mathbf{r}_j|}{\mathbf{r}_i} = f_{ij}\mathbf{e}_{ij} = \mathbf{F}_{ij},$$

where he last but one equality is because

$$\frac{\partial |\mathbf{r}_i - \mathbf{r}_j|}{\partial \mathbf{r}_i} = \frac{\partial \sqrt{(\mathbf{r}_i - \mathbf{r}_j)(\mathbf{r}_i - \mathbf{r}_j)}}{\partial \mathbf{r}_i} = \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|} = -\mathbf{e}_{ij}.$$

9/50 The equations of motion are $m_1\ddot{\mathbf{r}}_1 = -\partial U/\partial \mathbf{r}_1$ and $m_2\ddot{\mathbf{r}}_2 = -\partial U/\partial \mathbf{r}_2$, where $U = U(|\mathbf{r}_1 - \mathbf{r}_2|)$. For $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the computation in 3/49 above gives

$$\frac{\partial U}{\partial \mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}_1} = -\frac{\partial U}{\partial \mathbf{r}_2}.$$

Note also that

$$\left(\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\right)^{\cdot\cdot} = -\frac{1}{m_2 + m_2}\left(\frac{\partial U}{\partial \mathbf{r}_1} + \frac{\partial U}{\partial \mathbf{r}_2}\right) = \mathbf{0}$$

which means that the center of mass does a uniform linear motion. Finally, we have

$$m_1 m_2 \ddot{\mathbf{r}} = m_1 m_2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = -m_2\frac{\partial U}{\partial \mathbf{r}_1} + m_1\frac{\partial U}{\partial \mathbf{r}_2} = -(m_1 + m_2)\frac{\partial U}{\partial \mathbf{r}},$$

or equivalently

$$\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}},$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

Part II: Lagrangian Mechanics

6/59: The change from Cartesian to polar coordiates gives $\dot{x}_1^2 + \dot{x}_2^2 = \dot{r}^2 + r^2\dot{\varphi}^2$, so that, we have $L_{\text{pol}} = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}$, and hence

$$\Phi_{\text{pol}} = \int_{t_0}^{t_1} \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}\, dt.$$

Here we have $r = f(t)$ and $\varphi = \varphi(t)$. The Euler-Lagrange equations in polar coordinates are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} &= 0, \end{aligned}$$

where $L = L_{\text{pol}}$. Upon substitution, the first equation is

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}}\right) = \frac{r\dot{\varphi}^2}{\sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}}.$$

For the second equation, we have

$$\frac{d}{dt}\left(\frac{r^2\dot{\varphi}}{\sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}}\right) = 0,$$

which gives

$$\frac{1}{\sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}} = \frac{C}{r^2\dot{\varphi}}, \quad C \in \mathbb{R}.$$

We now use this to eliminate the radicals in the first equation above, and obtain

$$\frac{d}{dt}\left(\frac{\dot{r}}{r^2\dot{\varphi}^2}\right) = \frac{\dot{\varphi}}{r}.$$

We seek the solution for this equation in polar form $r = f(\varphi)$ with derivative $\dot{r} = f'(\varphi)\dot{\varphi}$, where $' = d/d\varphi$. Upon substitution, we obtain

$$\frac{d}{dt}\left(\frac{f'(\varphi)}{f^2(\varphi)}\right) = \frac{\dot{\varphi}}{f(\varphi)}.$$

Rewriting this in terms of differential forms

$$d\left(\frac{f'(\varphi)}{f^2(\varphi)}\right) = -\left(\frac{1}{f(\varphi)}\right)'' \cdot d\varphi = \frac{d\varphi}{f(\varphi)},$$

we arrive at

$$\left(\frac{1}{f(\varphi)}\right)'' + \frac{1}{f(\varphi)} = 0.$$

The solution is given by

$$\frac{1}{f(\varphi)} = a\cos\varphi + b\sin(\varphi), \quad a, b \in \mathbb{R},$$

or equivalently

$$r = f(\varphi) = \frac{1}{a\cos\varphi + b\sin(\varphi)}.$$

This is the equation of a line in polar cooordinates.

-2/60 EXAMPLE 2. In polar coordinates $q_1 = r$ and $q_2 = \varphi$ we have $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi$, where $\mathbf{e}_r = (\cos\varphi, \sin\varphi, 0)$ and $\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$ (-6/31 and 2/32 above). The kinetic energy $T = (m/2)\dot{\mathbf{r}}^2 = (m/2)(\dot{r}^2 + r^2\dot{\varphi}^2)$.
In a **central field** $U(\mathbf{q}) = U(q_1) = U(r)$, and the Lagrangian is

$$L = (\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

The generalized momenta $\mathbf{p} = \partial L/\partial\dot{\mathbf{q}}$ in polar coordinates are

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_2 = \frac{\partial L}{\partial \dot{q}_2} = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}.$$

The Lagrange equations $\dot{\mathbf{p}} = \partial L/\partial\mathbf{q}$ are

$$\begin{aligned}(m\dot{r})^{\cdot} &= mr\dot{\varphi}^2 - \frac{\partial U}{\partial r}\\ (mr^2\dot{\varphi})^{\cdot} &= 0.\end{aligned}$$

The second equation gives the law of conservation of the angular momentum

$$p_2 = mr^2\dot{\varphi} = C.$$

In a **non-central field** $U = U(r, \varphi)$ the first Lagrange equation holds and the second is

$$(mr^2\dot{\varphi})^{\cdot} = \dot{p}_2 = -\frac{\partial U}{\partial \varphi}.$$

We rewrite this as follows. First, we calculate the angular momentum

$$\mathbf{M} = m[\mathbf{r}, \dot{\mathbf{r}}] = [r\mathbf{e}_r, \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi] = mr^2\dot{\varphi}[\mathbf{e}_r, \mathbf{e}_\varphi] = mr^2\dot{\varphi}\mathbf{e}_z = p_2\mathbf{e}_z,$$

where $\mathbf{e}_z = [\mathbf{e}_r, \mathbf{e}_\varphi] = (0,0,1)$. This gives

$$(\mathbf{M}, \mathbf{e}_z)^{\cdot} = \dot{p}_2 = -\frac{\partial U}{\partial \varphi}.$$

On the other hand

$$\frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \varphi}d\varphi = dU = -(\mathbf{F}, d\mathbf{r}) = -(\mathbf{F}, \mathbf{e}_r)dr - r(\mathbf{F}, \mathbf{e}_\varphi)d\varphi$$

where we used $d\mathbf{r} = \mathbf{e}_r dr + r\mathbf{e}_\varphi d\varphi$. Hence

$$-\frac{\partial U}{\partial \varphi} = r(\mathbf{F}, \mathbf{e}_\varphi) = r(\mathbf{F}, [\mathbf{e}_z, \mathbf{e}_r]) = r(\mathbf{e}_z, [\mathbf{e}_r, \mathbf{F}]) = r([\mathbf{e}_r, \mathbf{F}], \mathbf{e}_z) = ([r\mathbf{e}_r, \mathbf{F}], \mathbf{e}_z) = ([\mathbf{r}, \mathbf{F}], \mathbf{e}_z),$$

where $[\mathbf{e}_z, \mathbf{e}_r] = \mathbf{e}_\varphi$. Combining these, we arrive at the following

$$(\mathbf{M}, \mathbf{e}_z)^{\cdot} = ([\mathbf{r}, \mathbf{F}], \mathbf{e}_z).$$

The left-hand side is the rate of change of the angular momentum relative to the z-axis, and the right-hand side is the moment of force relative to the z-axis.

-5/61 Let $y = f(x)$ and assume convexity $f''(x) > 0$. By definition, $g(p) = F(p, x(p)) = p \cdot x(p) - f(x(p))$, where, for given p, the equation $f'(x(p)) = p$ defines $x(p)$. Since

$$\frac{dg}{dp} = x(p) + (p - f'(x(p)))\frac{dx}{dp}, \quad ' = \frac{d}{dx},$$

we obtain[3]

$$\frac{dg}{dp}(p) = x(p) \quad \Leftrightarrow \quad \frac{df}{dx}(x(p)) = p.$$

13/62 Example 3. For $f(x) = x^\alpha/\alpha$, we have $f'(x) = x^{\alpha-1}$ so that $x(p)^{\alpha-1} = p$ gives $x(p) = p^{1/(\alpha-1)}$. Using this, we calculate

$$g(p) = p \cdot x(p) - f(x(p)) = p \cdot p^{1/(\alpha-1)} - \frac{p^{\alpha/(\alpha-1)}}{\alpha} = \left(1 - \frac{1}{\alpha}\right) p^{\frac{1}{1-1/\alpha}} = \frac{p^\beta}{\beta},$$

where $1/\alpha + 1/\beta = 1$.

3/63 Convexity of g. Differentiating $(df/dx)(x(p) = p$ (-5/61), we obtain

$$\frac{d}{dp}f'(x(p)) = f''(x(p))\frac{dx}{dp} = 1.$$

[3] Assume that the strict inequality is relaxed to $f''(x) \geq 0$. If, for example, $f(x) = p_0 x$ for some $p_0 > 0$ then g has the domain $\{p_0\}$ since $f'(x) = p_0$ and $f'(x(p)) = p_0 = p$.

This gives

$$\frac{dx}{dp} = \frac{1}{f''(x(p))}.$$

Using this, have

$$\frac{d^2g}{dp^2} = \frac{dx}{dp} = \frac{1}{f''(x(p))} > 0.$$

Convexity of g follows.

13/63 Geometric interpretation of $G(x,p) = x \cdot p - g(p)$. For fixed p, the function G is linear in x and $\partial G/\partial x = p$ so that

$$G(x(p),p) = x(p) \cdot p - g(p) = f(x(p)).$$

Now fix $x = x_0$ and vary p. Then, the values of $G(x_0,p)$ for varying p will be the ordinates of the intersection of points of the vertical line $x = x_0$ and the lines tangent to $y = f(x)$ with slopes p. By convexity, these tangent lines are below the graph of f. Hence $\max_p G(x_0,p) = f(x_0)$ and the maximum is attained at $p = p(x_0) = f'(x_0)$. Hence, we obtain $G(x_0, p(x_0)) = f(x_0)$.

1/65: We write in a matrix form $f(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$, where $A = (f_{ij})_{i,j=1}^n$ is a symmetric positive definite matrix, $A > 0$, so that A^{-1} is also symmetric and positive definite. Since $\partial f/\partial \mathbf{x} = 2A\mathbf{x}$, by definition $2A\mathbf{x}(\mathbf{p}) = \mathbf{p}$, so that $\mathbf{x}(\mathbf{p}) = (1/2)A^{-1}\mathbf{p}$. We now calculate

$$f(\mathbf{x}(\mathbf{p})) = (A\mathbf{x}(\mathbf{p}), \mathbf{x}(\mathbf{p})) = \frac{1}{4}(AA^{-1}\mathbf{p}, A^{-1}\mathbf{p}) = \frac{1}{4}(A^{-1}\mathbf{p}, \mathbf{p}),$$

and hence

$$\begin{aligned} g(\mathbf{p}) &= F(\mathbf{p}, \mathbf{x}(\mathbf{p})) = (\mathbf{p}, \mathbf{x}(\mathbf{p})) - f(\mathbf{x}(\mathbf{p})) \\ &= \frac{1}{2}(\mathbf{p}, A^{-1}\mathbf{p}) - \frac{1}{4}(\mathbf{p}, A^{-1}\mathbf{p}) = \frac{1}{4}(A^{-1}\mathbf{p}, \mathbf{p}) = f(\mathbf{x}(\mathbf{p})). \end{aligned}$$

By duality, we also have

$$g(\mathbf{p}(\mathbf{x})) = f(\mathbf{x}).$$

(See also the lemma on p. 66 of the text.)

-8/66 To the proof of the theorem:

$$\mathbf{p}\dot{\mathbf{q}} = \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} = \frac{\partial T}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} = 2T,$$

where the last but one equality is because U does not depend on $\dot{q}$, and the last equality is because T is a homogeneous function of degree 2; that is, we have $T(\lambda\dot{\mathbf{q}}) = \lambda^2 T(\dot{\mathbf{q}})$, $\lambda \in \mathbb{R}$.

11/68: To the proof of Corollary 3:

$$\frac{d}{dt}H(p',q') = \frac{\partial H}{\partial p'}\dot{p}' + \frac{\partial H}{\partial q'}\dot{q}' = \dot{q}'\dot{p}' - \dot{p}'\dot{q}' = 0,$$

so that $H(p',q') = c$.

-4/74 Example 4. The first digit of 2^n is $m \in 1,2,\ldots,9$ if and only if $m \cdot 10^\ell \le 2^n < (m+1)\cdot 10^\ell$, for some ℓ. This condition is equivalent to $\log_{10} m + \ell \le n\log_{10} 2 < \log_{10}(m+1) + \ell$, or what is the same

$$\log_{10} m \le (n\log_{10} 2)\,(\mathrm{mod}\ 1) < \log_{10}(m+1).$$

Let $I = [\log_{10} m, \log_{10}(m+1)] \subset [0,1)$. Since $\log_{10} 2$ is irrational,[4] by the Equidistribution Theorem,[5] we have

$$\lim_{n\to\infty} \frac{1}{n}\,|\{0 \le j < n \,|\, (j\log_{10} 2)(\mathrm{mod}\ 1) \in I\}| = \log_{10}(m+1) - \log_{10}(m) = \log_{10}\left(\frac{m+1}{m}\right).$$

Thus, m as the first digit of 2^n occurs with frequency $\log_{10}((m+1)/m)$. In particular, 7 occurs with frequency $\log_{10}(8/7) \approx 0.05799\ldots$ and 8 occurs with frequency $\log_{10}(9/8) \approx 0.05115\ldots$.

-8/77 If $\varphi : U \to \varphi(U)$ is a chart then the inverse map $\varphi^{-1} : \varphi(U) \to U$ will be denoted by $\mathbf{q} = (q_1,\ldots,q_n)$ with component functions $q_i : \varphi(U) \to \mathbb{R}$, $i = 1,\ldots,n$. This provides coordinates for points in the range $\varphi(U)$ of the chart.

-5/80: This is simply the chain rule applied to $f_j(\boldsymbol{\varphi}(t)) = 0$, giving $(\mathrm{grad}\, f_j)(\boldsymbol{\varphi}(t)) \perp \dot{\boldsymbol{\varphi}}(t)$, $j = 1,\ldots,n-k$, so that $\dot{\boldsymbol{x}} = \dot{\boldsymbol{\varphi}}(0)$ is tangent to M at $\boldsymbol{x} = \boldsymbol{\varphi}(0)$.

15/81 The definition of a tangent vector as an equivalence classe of curves in 3/81 is equivalent to the customary definition of a tangent vector at $\boldsymbol{x} \in M$ as a linear differential operator $\boldsymbol{\xi}$ (satisfying the Leibniz property) acting on real functions locally defined on M near $\boldsymbol{x}$. The equivalence is given by a representative curve of the class $\boldsymbol{\varphi}(t)$ in M (with $\boldsymbol{\varphi}(0) = \boldsymbol{x}$) acting on a function μ on M by $\xi \cdot \mu = (d/dt)\mu(\boldsymbol{\varphi}(t))|_{t=0}$. This, applied to the component functions q_i of the inverse of a chart $\boldsymbol{\varphi}$ covering $\boldsymbol{x}$

[4] $\log_{10} 2 = a/b$, $a,b \in \mathbb{Z}$, would imply $2^{b-a} = 5^a$.

[5] See, for example, Toth, G., *Elements of Mathematics – History and Foundations*, Springer, New York, 2021; p. 130.

gives $\boldsymbol{\xi} \cdot q_i = (d/dt)q_i(\boldsymbol{\varphi}(t))|_{t=0} = (dq_i)(\boldsymbol{\xi}) = \xi_i$. Hence the local components of the tangent vector $\boldsymbol{\xi}$ are ξ_i, $i = 1, \ldots, n$, and $\boldsymbol{\xi} \in T_{\boldsymbol{x}}(M)$ corresponds to the vector $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

-6/85: We already derived the change from Cartesian to polar coordinates formula $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$. With $r = r(z)$ we have $\dot{r} = r_z\dot{z}$ so that

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (1 + r_z)\dot{z}^2 + r(z)^2\dot{\varphi}^2.$$

Moreover, for the inverse $z = z(r)$, we have $\dot{z} = z_r\dot{r}$, and hence

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\varphi}^2 + z_r^2\dot{r}^2 = (1 + z_r^2)\dot{r}^2 + r^2\dot{\varphi}^2.$$

10/87 We calculate

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \;&=\; (r\cos q\cos(\boldsymbol{\omega}t)\dot{q} - r\sin q\sin(\boldsymbol{\omega}t)\boldsymbol{\omega})^2 \\ &\quad +(r\cos q\sin(\boldsymbol{\omega}t)\,\dot{q} + r\sin q\cos(\boldsymbol{\omega}t)\,\boldsymbol{\omega})^2 + (-r\sin q\,\dot{q})^2 \\ &\quad = r^2\boldsymbol{\omega}^2\sin^2 q + r^2\dot{q}^2.\end{aligned}$$

-3/87 The potential energy

$$V(q) = A\cos q - B\sin^2 q = B\cos^2 q + A\cos q - B = B\left(\cos q + \frac{A}{2B}\right)^2 - B\left(1 + \left(\frac{A}{2B}\right)^2\right)$$

has zeros at

$$\cos q = \frac{-A \pm \sqrt{A^2 + 4B^2}}{2B} = -\frac{A}{2B} \pm \sqrt{\left(\frac{A}{2B}\right)^2 + 1},$$

and $\pm$ in front of the last radical sign can only be positive because of the range of the cosine. With this, we obtain

$$\cos q = \sqrt{\left(\frac{A}{2B}\right)^2 + 1} - \frac{A}{2B} = \frac{1}{\sqrt{\left(\frac{A}{2B}\right)^2 + 1} + \frac{A}{2B}} > 0.$$

The citical points of V are given by

$$\frac{\partial V}{\partial q} = -A\sin q - B\sin q\cos q = -\sin q\,(A + 2B\cos q) = 0.$$

The crtical points are $q = 0, \pi, 2\pi$ and, for $2B > A$, that is, for $A/(2B) < 1$, these are the only ones. For $2B < A$, there are two additional critical points given by $\cos q = -A/(2B)\,(> -1)$.

-9/88 Proof of Noether's Theorem. We have

$$0 = \frac{\partial L(\Phi, \dot{\Phi})}{\partial s} = \frac{\partial L}{\partial q}\frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial t}\frac{\Phi}{\partial s},$$

where in the last term on the right-hand side we interchanged the partial derivatives $\partial/\partial s$ and $\partial/\partial t$. (As noted in the text, the partial derivatives are taken at $\Phi(s,t)$ and $\dot{\Phi}(s,t)$.) We now freeze s and replace $\partial L/\partial q$ in the first term of the right-hand side using Lagrange's equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q},$$

and obtain

$$0 = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)\frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial t}\frac{\Phi}{\partial s} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial \Phi}{\partial s}\right).$$

Noether's theorem follows.

Note. The lines 6-7/89 are superfluos. The box at the end of line 9/89 should be moved to the end of line 16/89; and the word "Remark" should be suppressed.

1/91 By definition, we have $\int L\,dt = \int L_1\,d\tau$. The first integral is

$$I_1 = I_1\left(q, t, \frac{dq}{d\tau}, \frac{dt}{d\tau}\right) \quad \text{with} \quad I(q, \dot{q}, t) = I_1(q, t, \dot{q}, 1).$$

Since $\partial h^s(q,t)/\partial s = (0,1)$, we have

$$\begin{aligned} I_1 &= I_1\left(q, t, \frac{dq}{d\tau}, \frac{dt}{d\tau}\right) = \frac{\partial L_1}{\partial(dq/d\tau)} \cdot 0 + \frac{\partial L_1}{\partial(dt/d\tau)} \cdot 1 \\ &= \frac{\partial L}{\partial \dot{q}}\left(\frac{dq}{d\tau}\right)\left(-\frac{1}{(dt/d\tau)^2}\right)\frac{dt}{d\tau} + L(q, \dot{q}) \\ &= -\frac{\partial L}{\partial \dot{q}}\,\dot{q} + L = -\dot{q}^2 + \frac{1}{2}\dot{q}^2 - U = -\left(\frac{1}{2}\dot{q}^2 + U\right) = -E. \end{aligned}$$

-1/91 The Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - U(\mathbf{x}) - NU_1(\mathbf{x}),$$

so that $\mathbf{R} = m\ddot{\mathbf{x}} + \partial U/\partial \mathbf{x} = -NU_1/\partial \mathbf{x} = \mathbf{F}(\mathbf{x})$.

-1/93 The variation calculates as

$$\begin{aligned}\delta\Phi = \delta\Phi(\boldsymbol{\xi}) &= \frac{\partial}{\partial s}\Phi(\mathbf{x}+s\boldsymbol{\xi})\bigg|_{s=0} = \frac{\partial}{\partial s}\bigg|_{s=0}\int_{t_0}^{t_1}\left(\frac{1}{2}\dot{(\mathbf{x}+s\boldsymbol{\xi})}^2 - U(\mathbf{x}+s\boldsymbol{\xi})\right)dt \\ &= \frac{\partial}{\partial s}\bigg|_{s=0}\int_{t_0}^{t_1}\left(\frac{1}{2}(\dot{\mathbf{x}}+s\dot{\boldsymbol{\xi}})^2 - U(\mathbf{x}+s\boldsymbol{\xi})\right)dt \\ &= \int_{t_0}^{t_1}\left(\dot{\mathbf{x}}\cdot\dot{\boldsymbol{\xi}} - \frac{\partial U}{\partial \dot{\mathbf{x}}}\cdot\boldsymbol{\xi}\right)dt = -\int_{t_0}^{t_1}\left(\ddot{\mathbf{x}} + \frac{\partial U}{\partial \dot{\mathbf{x}}}\right)\cdot\boldsymbol{\xi}\,dt\end{aligned}$$

14/101 Instead, we can write

$$\dot{\mathbf{q}}_0 = \mathbf{0} \quad \text{and} \quad \mathbf{p}_0 = \frac{\partial L}{\partial \dot{\mathbf{q}}}\bigg|_{\mathbf{q}_0} = \frac{\partial T}{\partial \dot{\mathbf{q}}}\bigg|_{\mathbf{q}_0} = \mathbf{0}.$$

7/102 We have

$$T_2 = \frac{a}{2}\dot{q}^2 \quad \text{and} \quad U_2 = \frac{b}{2}q^2$$

so that

$$L_2 = T_2 - U_2 = \frac{a}{2}\dot{q}^2 - \frac{b}{2}q^2.$$

Hence the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

reduces to $a\ddot{q} = -bq$; that is, $\ddot{q} = -(b/a)\,q = -\omega_0^2$.

-9/102 Problem. For the arc length, we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{\partial U}{\partial x}\right)^2}.$$

Hence

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{ds}{dx}\right)^2\left(\frac{dx}{dt}\right)^2 = \left(1 + \left(\frac{\partial U}{\partial x}\right)^2\right)\dot{x}^2.$$

3/103 Problem. The arc length along the wire is

$$q(x) = \int_{x_0}^{x}\sqrt{1 + \left(\frac{\partial U}{\partial z}\right)^2}\,dz,$$

or equivalently

$$\left(\frac{dq}{dx}\right)^2 = 1 + \left(\frac{\partial U}{\partial x}\right)^2.$$

With this, we calculate

$$2T = \dot{q}^2 = \left(\frac{dq}{dt}\right)^2 = \left(\frac{dq}{dx}\right)^2 \left(\frac{dx}{dt}\right)^2 = \left(1 + \left(\frac{dU}{dx}\right)^2\right) \dot{x}^2.$$

In differential forms, we write this as

$$dq = \sqrt{1 + \left(\frac{dU}{dx}\right)^2}\, dx.$$

In terms of the variable x, the Lagrangian is

$$L(x, \dot{x}) = T - U(x) = \frac{1}{2}\left(1 + \left(\frac{dU}{dx}\right)^2\right) \dot{x}^2 - U(x).$$

With q this writes as

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - V(q).$$

Here $U(x) = U(x(q)) = V(q)$ (with $x = x(q)$, the inverse $q = q(x)$), so that

$$V(q) = V\left(\int_{x_0}^{x} \sqrt{1 + \left(\frac{dU}{dz}\right)^2}\, dz\right).$$

-9/103 Problem. We have two quadratic forms $(A\mathbf{q}, \mathbf{q})$ and $(B\mathbf{q}, \mathbf{q})$, $A^t = A$ and $B^t = B$, such that $A > 0$, positive definite. For the simultaneous diagonalization of A and B, we first consider the Cholesky decomposition $A = L \cdot L^t$, where L is a lower triangular matrix with positive diagonal entries $\mu_i > 0$, $i = 1, \ldots, n$. Note that, since $\det A = \det L \cdot \det(L^t) = (\det L)^2 = \det(L^2)$, the eigenvalues of A are μ_i^2, $i = 1, \ldots, n$. Since $A > 0$ and $A^t = A$, L is unique.[6] With this, consider $L^{-1}B(L^{-1})^t$. This matrix is clearly symmetric and hence diagonalizable: $L^{-1}B(L^{-1})^t = UDU^t$, where $U \in O(n)$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal with diagonal entries $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$. Finally, we set $C = (LU)^t = U^t L^t$. With these, we calculate

$$A = L \cdot L^t = LU \cdot U^t L^t = (LU) \cdot (LU)^t = C^t \cdot C,$$

[6] One can also use $A^{1/2}$ instead of L; we define $A^{1/2} = UD^{1/2}U^t$, where $A = UDU^t$, $U \in O(n)$, and D is a diagonal matrix.

and

$$B = LU \cdot D \cdot U^t L^t = (LU) \cdot D \cdot (LU)^t = C^t \cdot D \cdot C.$$

Hence

$$(A\mathbf{q}, \mathbf{q}) = (C^t C\mathbf{q}, \mathbf{q}) = (C\mathbf{q}, C\mathbf{q}) = (\mathbf{Q}, \mathbf{Q}) = \sum_{i=1}^{n} Q_i^2,$$

and

$$(B\mathbf{q}, \mathbf{q}) = (C^t DC\mathbf{q}, \mathbf{q}) = (DC\mathbf{q}, C\mathbf{q}) = (D\mathbf{Q}, \mathbf{Q}) = \sum_{i=1}^{n} \lambda_i Q_i^2.$$

Finally, we have

$$\begin{aligned} \det(B - \lambda A) &= \det(C^t DC - \lambda C^t) = \det(C^t(D - \lambda I)C) \\ &= \det(C^t)\det(D - \lambda I)\det(C) \\ &= \det(C)^2 \det(D - \lambda I) \\ &= \prod_{i=1}^{n} \mu_i^2 \prod_{i=1}^{n} (\lambda_i - \lambda). \end{aligned}$$

-4/105 EXAMPLE 1. The potential energy of the ith pendulum, $i = 1, 2$, is

$$1 - \cos q_i \approx 1 - \left(1 - \frac{q_i^2}{2}\right) = \frac{q_i^2}{2}.$$

Letting $q_1 = (Q_1 + Q_2)/\sqrt{2}$ and $q_2 = (Q_1 - Q_2)/\sqrt{2}$, we have $q_1 - q_2 = \sqrt{2}Q_2$. We calculate

$$\begin{aligned} U &= \frac{1}{2}\left(q_1^2 + q_2^2 + \alpha(q_1 - q_2)^2\right) \\ &= \frac{1}{2}\left(\frac{(Q_1 + Q_2)^2}{2} + \frac{(Q_1 - Q_2)^2}{2} + 2\alpha Q_2^2\right) \\ &= \frac{1}{2}\left(Q_1^2 + (1 + 2\alpha)Q_2^2\right) = \frac{1}{2}\left(\boldsymbol{\omega}_1^2 Q_1^2 + \boldsymbol{\omega}_2^2 Q_2^2\right), \quad \boldsymbol{\omega}_1 = 1,\ \boldsymbol{\omega}_2 = \sqrt{1 + 2\alpha}. \end{aligned}$$

4/109 PROBLEM. CHARACTERISTIC FREQUENCIES OF THE DOUBLE PLANAR PENDULUM (FIGURE 88). Using θ_i, $i = 1, 2$, the angles of inclination of the point masses m_i, $i = 1, 2$, for the Cartesian coordinates, we have $x_1 = \ell_1 \sin\theta_1$, $y_1 = \ell_1 \cos\theta_1$, $x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$, and $y_2 = \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2$. Differentating, we obtain

$$\begin{aligned} \dot{x}_1 &= \ell_1 \cos\theta_1 \cdot \dot{\theta}_1 \\ \dot{y}_1 &= -\ell_1 \sin\theta_1 \cdot \dot{\theta}_1 \\ \dot{x}_2 &= \ell_1 \cos\theta_1 \cdot \dot{\theta}_1 + \ell_2 \cos\theta_2 \cdot \dot{\theta}_2 \\ \dot{y}_2 &= -\ell_1 \sin\theta_1 \cdot \dot{\theta}_1 - \ell_2 \sin\theta_2 \cdot \dot{\theta}_2. \end{aligned}$$

Hence

$$\begin{aligned}\dot{x}_1^2+\dot{y}_1^2 &= \ell_1^2\dot{\theta}_1^2\\ \dot{x}_2^2+\dot{y}_2^2 &= \ell_1^2\dot{\theta}_1^2+\ell_2^2\dot{\theta}_2^2+2\ell_1\ell_2\cos(\theta_1-\theta_2)\,\dot{\theta}_1\,\dot{\theta}_2.\end{aligned}$$

With these the kinetic and potential energies are

$$\begin{aligned}T &= \frac{m_1}{2}(\dot{x}_1^2+\dot{y}_1^2)+\frac{m_2}{2}(\dot{x}_2^2+\dot{y}_2^2)=\frac{m_1+m_2}{2}\ell_1^2\dot{\theta}_1^2+\frac{m_2}{2}\ell_2^2\dot{\theta}_2^2+m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\,\dot{\theta}_1\,\dot{\theta}_2\\ U &= -m_1gy_1-m_2gy_2=-(m_1+m_2)g\ell_1\cos\theta_1-m_2g\ell_2\cos\theta_2.\end{aligned}$$

For the Lagrangian equations ($L=T-U$ and $q_i=\theta_i$, $i=1,2$):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_i}\right)=\frac{\partial L}{\partial\theta_i},\quad i=1,2$$

we calculate

$$\begin{aligned}\frac{\partial L}{\partial\dot{\theta}_1} &= \frac{\partial T}{\partial\dot{\theta}_1}=(m_1+m_2)\ell_1^2\dot{\theta}_1+m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\dot{\theta}_2\\ \frac{\partial L}{\partial\dot{\theta}_2} &= \frac{\partial T}{\partial\dot{\theta}_2}=m_2\ell_2^2\dot{\theta}_2+m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\dot{\theta}_1\\ \frac{\partial L}{\partial\theta_1} &= \frac{\partial U}{\partial\theta_1}=-m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2-(m_1+m_2)g\ell_1\sin\theta_1\\ \frac{\partial L}{\partial\theta_1} &= \frac{\partial U}{\partial\theta_1}=m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2-m_2g\ell_2\sin\theta_2.\end{aligned}$$

Substituting these into the Lagrange equations, and simplifying, we obtain

$$\begin{aligned}(m_1+m_2)\ell_1\,\ddot{\theta}_1+m_2\ell_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_2+m_2\ell_2\sin(\theta_1-\theta_2)\,\dot{\theta}_2^2 &= -(m_1+m_2)g\sin\theta_1\\ m_2\ell_2\,\ddot{\theta}_2+m_2\ell_1\cos(\theta_1-\theta_2)\,\ddot{\theta}_1-m_2\ell_1\sin(\theta_1-\theta_2)\,\dot{\theta}_1^2 &= -m_2g\sin\theta_2.\end{aligned}$$

Linearizing, we get

$$\begin{aligned}T_2 &= \frac{m_1+m_2}{2}\ell_1^2\dot{\theta}_1^2+\frac{m_2}{2}\ell_2^2\dot{\theta}_2^2+m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2=\frac{1}{2}(A\dot{\boldsymbol{\theta}},\dot{\boldsymbol{\theta}})\\ U_2 &= \frac{m_1+m_2}{2}g\ell_1\theta_1^2+\frac{m_2}{2}g\ell_2\theta_2^2=\frac{1}{2}(B\dot{\boldsymbol{\theta}},\dot{\boldsymbol{\theta}})\end{aligned}$$

(since $\partial^2U/\partial\theta_1^2=(m_1+m_2)g\ell_1$, $\partial^2U/\partial\theta_2^2=m_2g\ell_2$, $\partial^2U/\partial\theta_1\partial\theta_2=0$), where $\boldsymbol{\theta}=(\theta_1,\theta_2)$, and

$$A=\begin{pmatrix}(m_1+m_2)\ell_1^2 & m_2\ell_1\ell_2\\ m_2\ell_1\ell_2 & m_2\ell_2^2\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}(m_1+m_2)g\ell_1 & 0\\ 0 & m_2g\ell_2\end{pmatrix}$$

With these, we calculate

$$\begin{aligned}\det(B-\lambda A) &= \det\begin{pmatrix} g(m_1+m_2)\ell_1 - \lambda(m_1+m_2)\ell_1^2 & -m_2\ell_1\ell_2 \\ -m_2\ell_1\ell_2 & gm_2\ell_2 - \lambda m_2\ell_2^2 \end{pmatrix} \\ &= \left(g(m_1+m_2)\ell_1 - \lambda(m_1+m_2)\ell_1^2\right)\left(gm_2\ell_2 - \lambda m_2\ell_2^2\right) - m_2^2\ell_1^2\ell_2^2 \\ &= (m_1+m_2)\ell_1\ell_2\,\lambda^2 - (m_1+m_2)g(\ell_1+\ell_2)\,\lambda + (m_1+m_2)g^2 - m_2\ell_1\ell_2 = 0.\end{aligned}$$

Rearranging, we obtain

$$\lambda^2 - g\left(\frac{1}{\ell_1}+\frac{1}{\ell_2}\right)\lambda + g^2\frac{1}{\ell_1\ell_2} - \frac{m_2}{m_1+m_2} = 0$$

Letting $g/\ell_i = \rho_1$, $i = 1, 2$, and $\mu = m_1/m_2$, this rewrites as

$$\lambda^2 - (\rho_1+\rho_2)\,\lambda + \rho_1\rho_2 - \frac{1}{1+\mu} = 0,$$

or equivalently

$$(\lambda-\rho_1)(\lambda-\rho_2) = \left(\lambda - \frac{g}{\ell_1}\right)\left(\lambda - \frac{g}{\ell_1}\right) = \frac{1}{1+\mu}.$$

Solving, and returning to our original variables, we get

$$2\lambda_{1,2} = 1 \pm \sqrt{(\rho_1-\rho_2)^2 + \frac{1}{1+\mu}} = 1 \pm \sqrt{\left(\frac{g}{\ell_1}-\frac{g}{\ell_2}\right)^2 + \frac{1}{1+\mu}}$$

-4/114 Problem 1. We have $(d/dt)\,g^t(x) = f(g^t(x), t)$. Assume $g^t g^s = g^{t+s}$. We have

$$\begin{aligned}f(g^t(g^s(x)), t) &= \frac{d}{dt}g^t(g^s(x)) = \frac{d}{dt}g^{t+s}(x) = \frac{d}{du}g^u(x)\big|_{u=t+s} \\ &= f(g^u(x), u)\big|_{u=t+s} = f(g^{t+s}(x), t+s).\end{aligned}$$

At $t = 0$ and $y = g^{-s}(y)$, this gives

$$f(y, 0) = f(y, s).$$

-2/114 Problem 2. Assume f is periodic with period T. We have

$$\begin{aligned}\frac{d}{ds}(g^s(g^T(x))) &= f(g^s(g^T(x)), s) \\ \frac{d}{ds}(g^{s+T}(x)) &= f(g^{s+T}(x), s+T) = f(g^{s+T}(x), s).\end{aligned}$$

Thus, $g^s(g^T(x))$ and $g^{s+T}(x)$ satisfy the same differential equation with the same initial condition at $s = 0$. By unicity, they must be equal: $g^{s+T}(x) = g^s(g^T(x))$.

-7/119 PROBLEM. The equation of motion is $\ddot{x} = -f^2(t)\,x$, where

$$f(t) = \begin{cases} \boldsymbol{\omega} + \epsilon & \text{if } 0 < t < \pi \\ \boldsymbol{\omega} - \epsilon & \text{if } \pi < t < 2\pi. \end{cases}$$

and $f(t + 2\pi) = f(t)$, $\epsilon << 1$. Then $A = A_2 \cdot A_1$, $k = 1, 2$, where

$$A_k = \begin{pmatrix} c_k & \frac{1}{\boldsymbol{\omega}_k} s_k \\ -\boldsymbol{\omega}_k s_k & c_k \end{pmatrix}$$

and $c_k = \cos \pi\boldsymbol{\omega}_k$, $s_k = \sin \pi\boldsymbol{\omega}_k$, $\boldsymbol{\omega}_{1,2} = \boldsymbol{\omega} \pm \epsilon$. The boundary of the zone of stability is given by

$$\begin{aligned} |\mathrm{tr}\, A| &= \left|\mathrm{tr}\left(\begin{pmatrix} c_2 & \frac{1}{\boldsymbol{\omega}_2} s_2 \\ -\boldsymbol{\omega}_2 s_k & c_2 \end{pmatrix}\begin{pmatrix} c_1 & \frac{1}{\boldsymbol{\omega}_1} s_1 \\ -\boldsymbol{\omega}_1 s_1 & c_1 \end{pmatrix}\right)\right| = \left|c_1c_2 - \frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1}\, s_1 s_2 - \frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2}\, s_1 s_2 + c_1 c_2\right| \\ &= \left|c_1 c_2 - \left(\frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1} + \frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2}\right) s_1 s_2 + c_1 c_2\right| = |2c_1c_2 - 2(1+\Delta)\, s_1 s_2| = 2, \end{aligned}$$

where Δ is defined by

$$\frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1} + \frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2} = 2(1+\Delta).$$

We calculate

$$\frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2} = \frac{\boldsymbol{\omega}+\epsilon}{\boldsymbol{\omega}-\epsilon} = 1 + 2\frac{\epsilon}{\boldsymbol{\omega}-\epsilon} = 1 + 2\frac{\frac{\epsilon}{\boldsymbol{\omega}}}{1-\frac{\epsilon}{\boldsymbol{\omega}}} = 1 + 2\left(\frac{\epsilon}{\boldsymbol{\omega}} + \frac{\epsilon^2}{\boldsymbol{\omega}^2} + \cdots\right)$$

Similarly, we have

$$\frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1} = \frac{\boldsymbol{\omega}-\epsilon}{\boldsymbol{\omega}+\epsilon} = 1 - 2\frac{\epsilon}{\boldsymbol{\omega}+\epsilon} = 1 - 2\frac{\frac{\epsilon}{\boldsymbol{\omega}}}{1+\frac{\epsilon}{\boldsymbol{\omega}}} = 1 - 2\left(\frac{\epsilon}{\boldsymbol{\omega}} - \frac{\epsilon^2}{\boldsymbol{\omega}^2} + \cdots\right)$$

These give

$$\frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1} + \frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2} = 2\left(1 + 2\left(\frac{\epsilon^2}{\boldsymbol{\omega}^2} + \frac{\epsilon^4}{\boldsymbol{\omega}^4} + \cdots\right)\right) = 2(1+\Delta)$$

so that

$$\Delta = 2\frac{\epsilon^2}{\boldsymbol{\omega}^2} + 2\frac{\epsilon^4}{\boldsymbol{\omega}^4} + \cdots = 2\frac{\epsilon^2}{\boldsymbol{\omega}^2} + O\left(\epsilon^4\right) << 1.$$

Moreover, we have

$$\begin{aligned} 2c_1c_2 &= 2\cos(\pi\boldsymbol{\omega}_1)\cos(\pi\boldsymbol{\omega}_2) = \cos(2\pi\epsilon) + \cos(2\pi\boldsymbol{\omega}) \\ 2s_1s_2 &= 2\sin(\pi\boldsymbol{\omega}_1)\sin(\pi\boldsymbol{\omega}_2) = \cos(2\pi\epsilon) - \cos(2\pi\boldsymbol{\omega}). \end{aligned}$$

Substituing these back to the trace formula above, we obtain that the boundary of the zone of stability is given by

$$|\cos(2\pi\epsilon) + \cos(2\pi\boldsymbol{\omega}) - (1+\Delta)(\cos(2\pi\epsilon) - \cos(2\pi\boldsymbol{\omega}))| = 2.$$

This gives

$$-\cos(2\pi\epsilon)\Delta + \cos(2\pi\boldsymbol{\omega})(2+\Delta) = \pm 2,$$

or equivalently

$$\cos(2\pi\boldsymbol{\omega}) = \frac{\pm 2 + \Delta\cos(2\pi\epsilon)}{2+\Delta} = \pm 1 - \frac{\Delta}{2+\Delta}(\pm 1 - \cos(2\pi\epsilon)).$$

We give details in the first case

$$\cos(2\pi\boldsymbol{\omega}) = 1 - \frac{\Delta}{2+\Delta}(1 - \cos(2\pi\epsilon))\ (\approx 1).$$

We write $\boldsymbol{\omega} = k + a$, where k is a positive integer and $a << 1$. We have

$$\cos(2\pi\boldsymbol{\omega}) = \cos(2\pi a) = 1 - 2\pi^2 a^2 + O(a^4),$$

and

$$\frac{\Delta}{2+\Delta} = \frac{\Delta/2}{1+\Delta/2} = \frac{\Delta}{2} - \frac{\Delta^2}{4} + \cdots = \frac{\epsilon^2}{\boldsymbol{\omega}^2} + O(\epsilon^4),$$

where we used the previous estimate on Δ. Substituting these into the formula for $\cos(2\pi\boldsymbol{\omega})$ above, we obtain

$$\cos(2\pi a) = 1 - \left(\frac{\epsilon^2}{\boldsymbol{\omega}^2} + O(\epsilon^4)\right)\left(2\pi^2\epsilon^2 + O(\epsilon^4)\right) = 1 - \frac{1}{2}\left(2\pi\frac{\epsilon^2}{\boldsymbol{\omega}}\right)^2 + O(\epsilon^6).$$

Simplifying, we get

$$a^2 + O(a^4) = \frac{\epsilon^4}{\boldsymbol{\omega}^2} + O(\epsilon^6).$$

Hence[7]

$$a = \pm\frac{\epsilon^2}{\boldsymbol{\omega}} + o(\epsilon^2),$$

or equivalently

$$\boldsymbol{\omega} = k + a = k \pm \frac{\epsilon^2}{\boldsymbol{\omega}} + o(\epsilon^2) = k \pm \frac{\epsilon^2}{k} + o(\epsilon^2),$$

where we used

$$\frac{1}{\boldsymbol{\omega}} = \frac{1}{k+a} = \frac{1}{k} - \frac{a}{k^2} + O(a^2).$$

[7] Note the typos in the text.

Similar computations in the second case give

$$\boldsymbol{\omega} = k + \frac{1}{2} \pm \frac{\epsilon}{\pi(k+1/2)} + o(\epsilon).$$

3/121 PROBLEM. The equation of motion is $\ddot{x} = (\boldsymbol{\omega}^2 \pm d^2)\, x$, $d^2 > \boldsymbol{\omega}^2$, where

$$\boldsymbol{\omega}^2 = \frac{g}{\ell} \quad \text{and} \quad d^2 = \frac{c}{\ell},$$

and $\pm c$ is the constant acceleration of the point of suspension over half of the period $\tau << 1$. The equation of the corresponding parabola is

$$y = -\frac{4a}{\tau^2}\, t(t-\tau) = -\frac{4a}{\tau^2}\,(t^2 - \tau t),$$

where $a << \ell$ is the amplitude of the oscillation of the point of suspension (at $t = \tau/2$, we have $y = a$). We have $\ddot{y} = -8a/\tau^2 = \mp c$, and hence $d^2 = 8a/(\ell\tau^2)$.
The equation of motion can easily be solved, and we obtain[8]

$$A_1 = \begin{pmatrix} \cosh k\tau & \frac{1}{k}\sinh k\tau \\ k\sinh k\tau & \cosh k\tau \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \cos \Omega\tau & \frac{1}{\Omega}\sin \Omega\tau \\ -\Omega \sin \Omega\tau & \cos \Omega\tau \end{pmatrix},$$

where $k^2 = d^2 + \boldsymbol{\omega}^2$ and $\Omega^2 = d^2 - \boldsymbol{\omega}^2$.
Letting $A = A_2 \cdot A_1$, the condition of stability is

$$\begin{aligned} |\mathrm{tr}\, A| &= \left| \mathrm{tr} \left(\begin{pmatrix} \cos \Omega\tau & \frac{1}{\Omega}\sin \Omega\tau \\ -\Omega \sin \Omega\tau & \cos \Omega\tau \end{pmatrix} \begin{pmatrix} \cosh k\tau & \frac{1}{k}\sinh k\tau \\ k\sinh k\tau & \cosh k\tau \end{pmatrix} \right) \right| \\ &= \left| 2\cosh k\tau \cos \Omega\tau + \left(\frac{k}{\Omega} - \frac{\Omega}{k} \right) \sinh k\tau \sin \Omega\tau \right| < 2. \end{aligned}$$

We now introduce the new parameters

$$\epsilon^2 = \frac{a}{\ell} << 1 \quad \text{and} \quad \mu^2 = \frac{g}{c} << 1.$$

With these, we have

$$k\tau = 2\sqrt{2}\epsilon\sqrt{1+\mu^2} \quad \text{and} \quad \Omega\tau = 2\sqrt{2}\epsilon\sqrt{1-\mu^2}.$$

Indeed, we calculate

$$k\tau = \tau\sqrt{d^2 + \boldsymbol{\omega}^2} = \tau d\sqrt{1 + \frac{\boldsymbol{\omega}^2}{d^2}} = \tau d\sqrt{1 + \frac{g}{c}} = 2\sqrt{2}\sqrt{\frac{a}{\ell}}\sqrt{1+\mu^2} = 2\sqrt{2}\epsilon\sqrt{1+\mu^2}$$

[8] The text uses ch = cosh and sh = sinh.

and

$$\Omega\tau = \tau\sqrt{d^2 - \boldsymbol{\omega}^2} = \tau\sqrt{\frac{c}{\ell} - \frac{g}{\ell}} = \tau\sqrt{\frac{c}{\ell}}\sqrt{1 - \frac{g}{c}} = \tau d\sqrt{1-\mu^2} = 2\sqrt{2}\epsilon\sqrt{1-\mu^2}.$$

Moreover

$$\frac{k}{\Omega} - \frac{\Omega}{k} = \frac{k\tau}{\Omega\tau} - \frac{\Omega\tau}{k\tau} = \sqrt{\frac{1+\mu^2}{1-\mu^2}} - \sqrt{\frac{1-\mu^2}{1+\mu^2}} = 2\mu^2 + O(\mu^4)$$

since

$$\frac{d}{dx}\left(\sqrt{\frac{1+x^2}{1-x^2}} - \sqrt{\frac{1-x^2}{1+x^2}}\right)_{x=0} = 2.$$

Continuing the computations in the use of the expansions of the hyperbolic functions, we have[9]

$$\begin{aligned} \cosh k\tau &= 1 + \frac{1}{2}k^2\tau^2 + \frac{1}{24}k^4\tau^4 + \cdots = 1 + 4\epsilon^2(1+\mu^2) + \frac{8}{3}\epsilon^4(1+\mu^2)^2 + \cdots \\ &= 1 + 4\epsilon^2(1+\mu^2) + \frac{8}{3}\epsilon^4 + o(\epsilon^4 + \mu^4). \end{aligned}$$

Similarly

$$\cos\Omega\tau = 1 - 4\epsilon^2(1-\mu^2) + \frac{8}{3}\epsilon^4 + o(\epsilon^4 + \mu^4).$$

Finally

$$\begin{aligned} &\left(\frac{k\tau}{\Omega\tau} - \frac{\Omega\tau}{k\tau}\right)\sinh k\tau \sin\Omega\tau = \big(2\mu^2 + O(\mu^4)\big)\cdot 2\sqrt{2}\epsilon\sqrt{1+\mu^2}\cdot 2\sqrt{2}\epsilon\sqrt{1-\mu^2} \\ &\quad = 16\epsilon^2\mu^2\sqrt{1-\mu^4} + o(\epsilon^4+\mu^4) = 16\epsilon^2\mu^2 + o(\epsilon^4+\mu^4). \end{aligned}$$

Putting everything together and discarding the terms of magnitude $o(\epsilon^4 + \mu^4)$, the condition of stability is

$$2\left(1 + 4\epsilon^2(1+\mu^2) + \frac{8}{3}\epsilon^4\right)\left(1 - 4\epsilon^2(1-\mu^2) + \frac{8}{3}\epsilon^4\right) + 16\epsilon^2\mu^2 < 2.$$

This gives

$$2\left(1 - 16\epsilon^4 + \frac{16}{3}\epsilon^4 + 8\epsilon^2\mu^2\right) + 16\epsilon^2\mu^2 < 2,$$

[9]Here and below, for the final estimates we can use the AM-GM inequality in different settings.

where, once again, we discarded the the terms of magnitude $o(\epsilon^4+\mu^4)$. We rearrange and simplify to obtain

$$3\mu^2 < 2\epsilon^2.$$

Playing this back to our original parameters, we finally arrive at

$$\frac{g}{c} < \frac{2a}{3\ell}.$$

The rest of the example follows.

3/126: We have $\mathbf{r}=\mathbf{0}$ and $\dot{\mathbf{Q}}=\mathbf{0}$. With respect to an orthonormal basis in k where the third axis is $\mathbb{R}\boldsymbol{\omega}$, we have

$$U(t) = \begin{pmatrix} \cos|\boldsymbol{\omega}|t & -\sin|\boldsymbol{\omega}|t & 0 \\ \sin|\boldsymbol{\omega}|t & \cos|\boldsymbol{\omega}|t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $B(t) = U(t)B(0)$, we have $\mathbf{q} = B(t)\mathbf{Q} = U(t)B(0)\mathbf{q}$ so that $\dot{\mathbf{q}} = \dot{B}\mathbf{Q} = \dot{U}(t)B(0)\mathbf{Q}$. Since $U(t)^{-1}\mathbf{q} = B(0)\mathbf{Q}$ and $U(t)^{-1} = U(t)^t$, we obtain $\dot{\mathbf{q}} = \dot{U}(t)U(t)^t\mathbf{q}$. We now calculate

$$\begin{aligned} \dot{U}(t)U(t)^t &= |\boldsymbol{\omega}| \begin{pmatrix} -\sin|\boldsymbol{\omega}|t & -\cos|\boldsymbol{\omega}|t & 0 \\ \cos|\boldsymbol{\omega}|t & -\sin|\boldsymbol{\omega}|t & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos|\boldsymbol{\omega}|t & \sin|\boldsymbol{\omega}|t & 0 \\ -\sin|\boldsymbol{\omega}|t & \cos|\boldsymbol{\omega}|t & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -|\boldsymbol{\omega}| & 0 \\ |\boldsymbol{\omega}| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Using this, we continue

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} 0 & -|\boldsymbol{\omega}| & 0 \\ |\boldsymbol{\omega}| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -|\boldsymbol{\omega}|q_2 \\ |\boldsymbol{\omega}|q_1 \\ 0 \end{pmatrix}.$$

On the other hand

$$[\boldsymbol{\omega}, \mathbf{q}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & |\boldsymbol{\omega}| \\ q_1 & q_2 & q_3 \end{vmatrix} = (-|\boldsymbol{\omega}|q_2, |\boldsymbol{\omega}|q_1, 0).$$

Thus, we have $\dot{\mathbf{q}} = [\boldsymbol{\omega}, \mathbf{q}]$.

8/127 With respect to an orthonormal basis $\{e_1, e_2, e_3\} \subset k$, we have

$$A\mathbf{q} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -q_2\omega_3 + q_3\omega_2 \\ q_1\omega_3 - q_3\omega_1 \\ -q_1\omega_2 + q_2\omega_1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = [\boldsymbol{\omega}, \mathbf{q}].$$

12/127 Indeed, we have

$$(\mathbf{p}, [\boldsymbol{\omega}, \mathbf{q}]) = ([\mathbf{p}, \boldsymbol{\omega}], \mathbf{q}) = -([\boldsymbol{\omega}, \mathbf{p}], \mathbf{q}), \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^3.$$

-6/130 Since $\boldsymbol{\Omega}$ is orthogonal to $[\boldsymbol{\Omega}, \mathbf{Q}]$, we have

$$|[\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \mathbf{Q}]]| = |\boldsymbol{\Omega}||[\boldsymbol{\Omega}, \mathbf{Q}]| = |\boldsymbol{\Omega}|^2|\mathbf{Q}|\sin\theta = |\boldsymbol{\Omega}|^2 r,$$

where θ is the angle between $\boldsymbol{\Omega}$ and $\mathbf{Q}$.

2/131 Here, as usual, $\mathbf{x} = B\mathbf{X}$.

9/131 Note that

$$m\ddot{\mathbf{q}} = f(\mathbf{q}, \dot{\mathbf{q}}) = f(B\mathbf{Q}, (B\mathbf{Q})^{\cdot}) = BF(\mathbf{Q}, \dot{\mathbf{Q}}).$$

-8/131 Example 1. From the equation of motion we get $\ddot{\mathbf{Q}}_1 = \mathbf{g}$, $\dot{\mathbf{Q}}_1(0) = \mathbf{0}$, and so $\mathbf{Q}_1(t) = \mathbf{Q}_1(0) + \mathbf{g}t^2/2$. Moreover

$$\ddot{\mathbf{Q}}_2 = 2[\dot{\mathbf{Q}}_1, \boldsymbol{\Omega}] + 2[\dot{\mathbf{Q}}_2, \boldsymbol{\Omega}] = 2[\mathbf{g}t, \boldsymbol{\Omega}] + 2[\dot{\mathbf{Q}}_2, \boldsymbol{\Omega}] = 2[\mathbf{g}t, \boldsymbol{\Omega}] + O(|\boldsymbol{\Omega}|^2), \quad |\boldsymbol{\Omega}| << 1,$$

where the estimate is because $|\dot{\mathbf{Q}}_2| = O(|\boldsymbol{\Omega}|)$. Hence

$$\mathbf{Q}_2(t) \approx \frac{t^3}{3}[\mathbf{g}, \boldsymbol{\Omega}] = \frac{2t}{3}[\mathbf{h}, \boldsymbol{\Omega}], \quad \mathbf{h} = \mathbf{g}\frac{t^2}{2}.$$

Now

$$|[\mathbf{h}, \boldsymbol{\Omega}]| = |\mathbf{h}|\,|\boldsymbol{\Omega}|\,\sin(\pi/2 + \lambda) = |\mathbf{h}|\,|\boldsymbol{\Omega}|\,\cos\lambda,$$

and $gt^2/2 = 250$ gives $t^2 \approx 500/g \approx 50$, and so $t \approx \sqrt{50} \approx 7$. With this

$$\frac{2t}{3}[\mathbf{h}, \boldsymbol{\Omega}] = \frac{14}{3}|\mathbf{h}|\,|\boldsymbol{\Omega}|\cos\lambda \approx \frac{14}{3} \cdot 250 \cdot 7 \cdot 10^{-5} \cdot \frac{1}{2}\,m \approx 4\,cm,$$

where we used $|\boldsymbol{\Omega}| \approx 7.3 \cdot 10^{-5}$ (-5/126) and $\cos\lambda \approx 1/2$.

-5/138 We have

$$|\mathbf{v}_i| = |\mathbf{V}_i| = |[\boldsymbol{\Omega}, \mathbf{Q}_i]| = |\boldsymbol{\Omega}|\,|\mathbf{Q}_i|\sin\theta_i = \boldsymbol{\Omega} \cdot \mathbf{r}_i$$

(Figure 115).

1/141 Example. The total mass

$$m = \int_{-a}^{a}\int_{-b}^{b} \rho\, dx\, dy = 4ab\rho,$$

so that $\rho = m/(4ab)$. The moment of inertia

$$I_y = \frac{m}{4ab}\int_{-a}^{a}\int_{-b}^{b} x^2\, dx\, dy = \frac{m}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{ma^2}{3}.$$

Similarly

$$I_x = \frac{mb^2}{3}.$$

Finally, we arrive at

$$I_z = \int_{-a}^{a}\int_{-b}^{b} r^2\rho\, dx\, dy = \int_{-a}^{a}\int_{-b}^{b} (x^2+y^2)\,\rho\, dx\, dy = I_x + I_y.$$

9/141 Problem. We have

$$\begin{aligned} I_z &= \iiint_{\mathcal{B}} (x^2+y^2)\rho\, dV \le \iiint_{\mathcal{B}} (x^2+y^2)\rho\, dV + 2\iiint_{\mathcal{B}} z^2\rho\, dV \\ &= \iiint_{\mathcal{B}} (y^2+z^2)\rho\, dV + \iiint_{\mathcal{B}} (x^2+z^2)\rho\, dV = I_x + I_y, \end{aligned}$$

with equality if and only if $\iiint_{\mathcal{B}} z^2\rho\, dV = 0$ if and only if $z = 0$; that is, $\mathcal{B}$ is planar.

12/141 Problem. First, consider the ball $\mathcal{B}$ of radius R and mass density ρ. The total mass

$$m = \iiint_{\mathcal{B}} \rho\, dV = \rho\frac{4\pi}{3}R^3,$$

so that $\rho = 3m/(4\pi R^3)$. We now use spherical coordinates $x = r_0\cos\theta$, $y = r_0\sin\theta$, $r_0 = r\sin\varphi$, $z = r\cos\varphi$, $0 \le \theta < 2\pi$, $0 \le \varphi \le \pi$, with $dV = r^2\sin^2\varphi\, dr\, d\varphi\, d\theta$, and calculate

$$\begin{aligned} I_z &= \frac{3m}{4\pi R^3}\iiint_{\mathcal{B}} (x^2+y^2)\rho\, dV = \frac{3m}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin^2\varphi\cdot r^2\sin^2\varphi\, dr\, d\varphi\; d\theta \\ &= \frac{3m}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^4\sin^4\varphi\, dr\, d\varphi\; d\theta = \frac{3m}{10}R^2\int_0^{\pi}\sin^4\varphi\, d\varphi \\ &= \frac{3m}{10}R^2\left[\frac{1}{4}\frac{\cos(3\varphi)}{3} - \frac{3}{4}\cos\varphi\right]_0^{\pi} = m\frac{2R^2}{5}. \end{aligned}$$

For the ellipsoid $\mathcal{E}$ given by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$
we change the variables $u = x/a$, $v = y/b$, $w = z/c$ with $u^2 + v^2 + w^2 \leq 1$ and $du\,dv\,dw = dx\,dy\,dz/(abc)$, and obtain

$$m = \iiint_{\mathcal{E}} \rho\,dV = \rho\frac{4\pi}{3}abc$$

so that $\rho = 3m/(4\pi abc)$. Moreover

$$I_z = \iiint_{\mathcal{E}} (x^2 + y^2)\rho\,dx\,dy\,dz = abc \iiint_{\mathcal{B}_0} (a^2u^2 + b^2v^2)\,\rho\,du\,dv\,dw$$

where $\mathcal{B}_0$ is the unit ball ($R = 1$). Using spherical coordinates again, we calculate

$$\begin{aligned}
\iiint_{\mathcal{B}_0} (a^2u^2)\,\rho\,du\,dv\,dw &= a^2\rho \iiint_{\mathcal{B}_0} u^2\,du\,dv\,dw \\
&= \rho \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 \sin^2\varphi \cos^2\theta \cdot r^2 \sin^2\varphi\,dr\,d\varphi\,d\theta \\
&= a^2\rho \int_0^{2\pi} \cos^2\theta\,d\theta \cdot \int_0^{\pi} \int_0^1 r^4 \sin^4\varphi\,d\varphi\,d\theta = \rho\frac{4\pi}{15}a^2.
\end{aligned}$$

Similarly
$$\iiint_{\mathcal{B}_0} (b^2v^2)\,\rho\,du\,dv\,dw = \rho\frac{4\pi}{15}b^2.$$
Putting these together, we obtain

$$I_z = abc\rho\frac{4\pi}{15}(a^2 + b^2) = \frac{3m}{4\pi abc}abc\frac{4\pi}{15}(a^2 + b^2) = m\frac{a^2 + b^2}{5}.$$

Permuting the coordinates, we also obtain

$$I_x = m\frac{b^2 + c^2}{5} \quad \text{and} \quad I_y = m\frac{a^2 + c^2}{5}.$$

15/141 Problem. Steiner's Theorem. We let ρ be the mass density, 0 the center of mass of $\mathcal{B}$, also the origin of the coordinate system. Set up the coordinate system such that the axis through 0 is the z-axis. We let d_0 (and not r) denote the distance betwen the two parallel axes, and assume that the second axis is through

$(d_0, 0, 0)$ parallel to the z-axis. We will use cylindrical coordinates. We let r denote the distance from the z-axis. We then have

$$I_0 = I_z = \iiint_{\mathcal{B}} r^2 \rho \, dV,$$

and hence (by the law of cosine) the moment of inertia with respect to the second axis is

$$I = \iiint_{\mathcal{B}} (r^2 + d_0^2 - 2rd_0 \cos\theta)\, \rho \, dV = I_0 + d_0^2 m - 2d_0 \iiint_{\mathcal{B}} r \cos\theta \, dV.$$

The last integral is the z-coordinate of the center of mass of $\mathcal{B}$ since $x = r\cos\theta$, and hence it vanishes.

-1/141 FIGURE 119. We have

$$\mathbf{grad}\,(A\mathbf{\Omega}, \mathbf{\Omega}) = 2A\mathbf{\Omega} = 2\mathbf{M},$$

where $\mathcal{E} = \{\mathbf{\Omega} \,|\, (A\mathbf{\Omega}, \mathbf{\Omega}) = 1\}$ (see p. 146 of the text).

5/142 PROBLEM. The center of mass is displaced by $\epsilon/(m\epsilon)\mathbf{Q}$, where $\mathbf{Q} = (x_1, x_2, x_3)$. Since

$$\frac{\epsilon}{m+\epsilon} = \frac{\epsilon}{m}\left(1 - \frac{\epsilon}{m} + \frac{\epsilon^2}{m^2} - \cdots\right) = \frac{\epsilon}{m} + O(\epsilon^2),$$

by Steiner's theorem, moving the axis to a parallel axis, the moments of inertia differ by $O(\epsilon^2)$. Keeping the center of mass in the original position we thus incur an error of $O(\epsilon^2)$. The change in the kinetic energy (proof of Lemma and its Corollary on p. 137 of the text) is

$$T = T_0 + \frac{1}{2}(A\mathbf{\Omega}, \mathbf{\Omega}) = T_0 + \frac{\epsilon}{2}|[\mathbf{\Omega}, \mathbf{Q}]|^2 + O(\epsilon^2),$$

where

$$T_0 = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

is the original kinetic energy. We now calculate the inertia operator as

$$\begin{aligned}
2T &= 2T_0 + \epsilon\left[(\Omega_2 x_3 - \Omega_3 x_2)^2 + (\Omega_1 x_3 - \Omega_3 x_1)^2 + (\Omega_1 x_2 - \Omega_2 x_1)^2\right] + O(\epsilon^2) \\
&= \left(I_1 + \epsilon(x_2^2 + x_3^2)\right)\Omega_1^2 + \left(I_2 + \epsilon(x_1^2 + x_3^2)\right)\Omega_2^2 + \left(I_3 + \epsilon(x_1^2 + x_2^2)\right)\Omega_3^2 \\
&\quad -2\epsilon x_2 x_3 \Omega_2\Omega_3 - 2\epsilon x_1 x_3 \Omega_1\Omega_3 - 2\epsilon x_1 x_2 \Omega_1\Omega_2 + O(\epsilon^2) \\
&= \left(\begin{pmatrix} I_1 + \epsilon(x_2^2 + x_3^2) & -\epsilon x_1 x_2 & -\epsilon x_1 x_3 \\ -\epsilon x_1 x_2 & I_2 + \epsilon(x_1^2 + x_3^2) & -\epsilon x_2 x_3 \\ -\epsilon x_1 x_3 & -\epsilon x_2 x_3 & I_3 + \epsilon(x_1^2 + x_2^2) \end{pmatrix}\begin{pmatrix}\Omega_1 \\ \Omega_2 \\ \Omega_3\end{pmatrix}, \begin{pmatrix}\Omega_1 \\ \Omega_2 \\ \Omega_3\end{pmatrix}\right) + O(\epsilon^2).
\end{aligned}$$

We denote by $A(\epsilon)$ the matrix above. This is the inertia operator up to $O(\epsilon^2)$. The eigenvalues $I_i(\epsilon)$ and corresponding eigenvectors $\mathbf{e}_i(\epsilon)$, $i = 1, 2, 3$, of $A(\epsilon)$ are easily determined as follows:

$$
\begin{aligned}
I_1(\epsilon) &= I_1 + \epsilon(x_2^2 + x_3^2) + O(\epsilon^2) \qquad \mathbf{e}_1(\epsilon) = \left(1, \epsilon\frac{x_1x_2}{I_2 - I_1}, \epsilon\frac{x_1x_3}{I_3 - I_1}\right) + O(\epsilon^2) \\
I_2(\epsilon) &= I_2 + \epsilon(x_1^2 + x_3^2) + O(\epsilon^2) \qquad \mathbf{e}_2(\epsilon) = \left(-\epsilon\frac{x_1x_2}{I_2 - I_1}, 1, \epsilon\frac{x_2x_3}{I_3 - I_2}\right) + O(\epsilon^2) \\
I_3(\epsilon) &= I_3 + \epsilon(x_1^2 + x_2^2) + O(\epsilon^2) \qquad \mathbf{e}_3(\epsilon) = \left(-\epsilon\frac{x_1x_3}{I_3 - I_1}, -\epsilon\frac{x_2x_3}{I_3 - I_2}, 1\right) + O(\epsilon^2).
\end{aligned}
$$

1/146 Since $M_i = I_i\Omega_1$, $i = 1, 2, 3$, we have

$$
\begin{aligned}
2T &= 2E = (A\mathbf{\Omega}, \mathbf{\Omega}) = (\mathbf{M}, \mathbf{\Omega}) = M_1\Omega_1 + M_2\Omega_2 + M_3\Omega_3 \\
&= I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}.
\end{aligned}
$$

Note that the angular momentum $\mathbf{m} \in k$ is conserved (independent of t). Since $\mathbf{m} = B_t\mathbf{M}$, we have $|\mathbf{M}|^2 = M^2 = M_1^2 + M_2^2 + M_3^2 = |\mathbf{m}|^2$, where $\mathbf{M} = (M_1, M_2, M_3)$. The inertia ellipsoid $\mathcal{E} = \{\mathbf{\Omega} \mid | (A\mathbf{\Omega}, \mathbf{\Omega}) = 1\} \subset K$, and $B_t\mathcal{E} \subset k$. Hence $\mathbf{\Omega}/\sqrt{2T} \in \mathcal{E}$ so that $\boldsymbol{\xi} = \boldsymbol{\omega}/\sqrt{2T} \in B_t\mathcal{E}$. Now, $\mathbf{grad}\,(A\mathbf{\Omega}, \mathbf{\Omega}) = 2A\mathbf{\Omega} = 2\mathbf{M}$ so that the normal to $B_t\mathcal{E}$ at $\boldsymbol{\xi}$ is parallel to $\mathbf{m}$.
Finally

$$
(\boldsymbol{\xi}, \mathbf{m}) = \frac{1}{\sqrt{2T}}(\boldsymbol{\omega}, \mathbf{m}) = \frac{1}{\sqrt{2T}}(\mathbf{\Omega}, \mathbf{M}) = \sqrt{2T}
$$

is constant.

4/147 Since $I_1 \neq I_2 = I_3$, we have Ω_1 constant, and

$$
\begin{aligned}
\frac{d\Omega_2}{dt} &= \frac{I_2 - I_1}{I_2}\Omega_1\Omega_3 = \lambda\Omega_3 \\
\frac{d\Omega_3}{dt} &= \frac{I_1 - I_2}{I_2}\Omega_1\Omega_2 = -\lambda\Omega_2,
\end{aligned}
$$

where

$$
\lambda = \frac{I_2 - I_1}{I_2}\Omega_1.
$$

Solving, we obtain

$$
\mathbf{\Omega} = (\Omega_1, A\sin(\lambda t), A\cos(\lambda t)) = \Omega_1\mathbf{e}_1 + A(\sin(\lambda t)\mathbf{e}_2 + \cos(\lambda t)\mathbf{e}_3).
$$

This gives

$$\mathbf{M} = I_1\Omega_1\mathbf{e}_1 + I_2A(\sin(\lambda t)\mathbf{e}_2 + \cos(\lambda t)\mathbf{e}_3).$$

In particular, $\mathbf{e}_1, \mathbf{\Omega}, \mathbf{M}$ are coplanar in K, and hence $B_t\mathbf{e}_1, \boldsymbol{\omega}, \mathbf{m}$ are coplanar in k. Moreover, $|\boldsymbol{\omega}| = |\mathbf{\Omega}| = \Omega_1^2 + A^2$ is constant and so are each of the three angles between $\mathbf{e}_1, \mathbf{\Omega}, \mathbf{M}$ with cosines

$$\begin{aligned}
\frac{(\mathbf{e}_1, \mathbf{\Omega})}{|\mathbf{\Omega}|} &= \frac{\Omega_1}{\sqrt{\Omega_1^2 + A^2}} \\
\frac{(\mathbf{e}_1, \mathbf{M})}{|\mathbf{M}|} &= \frac{I_1\Omega_1}{\sqrt{I_1^2\Omega_1^2 + I_2^2A^2}} \\
\frac{(\mathbf{\Omega}, \mathbf{M})}{|\mathbf{\Omega}|\,|\mathbf{M}|} &= \frac{I_1\Omega_1^2 + I_2A^2}{\sqrt{\Omega_1^2 + A^2}\sqrt{I_1^2\Omega_1^2 + I_2^2A^2}}.
\end{aligned}$$

By the equations for $\mathbf{\Omega}$ and $\mathbf{M}$ above, we have

$$\mathbf{M} = I_1\Omega_1\mathbf{e}_1 + I_2(\mathbf{\Omega} - \Omega_1\mathbf{e}_1) = (I_1 - I_2)\Omega_1\mathbf{e}_1 + I_2\mathbf{\Omega} = I_2(\mathbf{\Omega} - \lambda\mathbf{e}_1),$$

since $(I_1 - I_2)\Omega_1 = -\lambda I_2$. We obtain

$$\mathbf{\Omega} = \frac{\mathbf{M}}{I_2} + \lambda\mathbf{e}_1.$$

In k this gives

$$\boldsymbol{\omega} = \frac{\mathbf{m}}{I_2} + \lambda B_t\mathbf{e}_1.$$

Hence

$$\left[\frac{\mathbf{m}}{I_2}, B_t\mathbf{e}_1\right] = [\boldsymbol{\omega}, B_t\mathbf{e}_1] = (B_t\mathbf{e}_1)^{\cdot}$$

as $\boldsymbol{\omega}$ is the instantaneous angular velocity of $B_t\mathbf{e}_1$ (see p. 126 of the text). This shows that $B_t\mathbf{e}_1$ rotates around the fixed angular momentum vector $\mathbf{m}$ with angle $|\mathbf{m}|t/I_2$. Finally

$$\left[\frac{\mathbf{m}}{I_2}, \boldsymbol{\omega}\right] = \left[\frac{\mathbf{m}}{I_2}, \lambda B_t\mathbf{e}_1\right] = \lambda\left[\frac{\mathbf{m}}{I_2}, B_t\mathbf{e}_1\right] = \lambda(B_t\mathbf{e}_1)^{\cdot} = (\lambda B_t\mathbf{e}_1)^{\cdot} = \dot{\boldsymbol{\omega}}$$

since $\dot{\mathbf{m}} = \mathbf{0}$. We obtain that $\boldsymbol{\omega}$ also rotates around the fixed angular momentum vector through the angle $|\mathbf{m}|t/I_2$. The angular velocity of the procession is $\boldsymbol{\omega}_{\mathbf{pr}} = \mathbf{m}/I_2$.

-2/151 For a rotation of a rigid body fixed at O with angular velocity $\boldsymbol{\omega} = \omega\mathbf{e}$ around the $\mathbf{e}$ axis, the kinetic energy is $K = (1/2)I_{\mathbf{e}}\omega^2$ (15/138). The angular momentum vector is $\mathbf{m} = I_{\mathbf{e}}\boldsymbol{\omega} = I_{\mathbf{e}}\omega\,\mathbf{e}$, and the angular momentum with respect to the $\mathbf{e}$ axis is $m_{\mathbf{e}} = (\mathbf{m}, \mathbf{e}) = I_{\mathbf{e}}\omega$. Hence $\partial L/\partial\omega = \partial T/\partial\omega = I_{\mathbf{e}}\omega = m_{\mathbf{e}}$ (even in the presence of potential energy independent of ω). In our case at hand, $\mathbf{e} = \mathbf{e}_z$, $I_{\mathbf{e}} = I_z$, etc. and $\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{e}_z$ ($\varphi = \psi = 0$), so that we have[10] $\partial L/\partial\dot{\varphi} = \partial T/\partial\dot{\varphi} = m_z$. Note, finally, that the Lagrange equation $(d/dt)(\partial L/\partial\dot{\varphi}) = \partial L/\partial\varphi = 0$ (along with φ being cylic) directly implies that $\partial L/\partial\dot{\varphi} = m_z$ is preserved. The discussion is similar for m_3.

8/152 Eliminating $\dot{\psi}$ from the two first integrals (-2/151), we obtain

$$M_z = \dot{\varphi}(I_1\sin^2\theta + I_3\cos^2\theta) + \left(\frac{M_3}{I_3} - \dot{\varphi}\cos\theta\right) I_3\cos\theta.$$

Simplifying, we get

$$M_z = \dot{\varphi} I_1 \sin^2\theta + M_3\cos\theta,$$

or equivalently

$$\dot{\varphi} = \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta}.$$

Using this, we also obtain

$$\dot{\psi} = \frac{M_3}{I_3} - \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta}\cos\theta.$$

Using these in the formula for the kinetic energy (-5/151), the total energy is

$$E = \frac{I_1}{2}\left(\dot{\theta}^2 + \frac{(M_z - M_3\cos\theta)^2}{I_1^2\sin^2\theta}\right) + \frac{M_3^2}{2I_3} + mg\ell\cos\theta$$

or

$$E' = \frac{I_1}{2}\dot{\theta}^2 + U_{\text{eff}}(\theta),$$

where

$$U_{\text{eff}}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + mg\ell\cos\theta.$$

Using the subsitutions in -13/152

$$\alpha = \dot{\theta}^2 + \frac{(a - b\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta$$

and $u = \cos\theta$, $\dot{u} = -\sin\theta\cdot\dot{\theta}$, $\dot{u}^2 = \sin^2\theta\cdot\dot{\theta}^2$, we obtain

$$\dot{u}^2 \quad = \quad \alpha\sin^2\theta - (a - b\cos\theta)^2 - \beta\cos\theta\sin^2\theta.$$

[10] In the text $m_z = M_z$.

Equivalently

$$\dot{u}^2 = \alpha(1-u^2) - (a-bu)^2 - \beta u(1-u^2) = (\alpha - \beta u)(1-u^2) - (a-bu)^2 = f(u).$$

2/159 We have $\theta = \theta_0 + x$, $x << 1$, where $M_z = M_3 \cos\theta_0$. Thus

$$\begin{aligned}\cos\theta &= \cos(\theta_0 + x) = \cos\theta_0 \cos x - \sin\theta_0 \sin x = \cos\theta_0 - x\sin\theta_0 + \cdots \\ \sin\theta &= \sin(\theta_0 + x) = \sin\theta_0 \cos x + \cos\theta_0 \sin x = \sin\theta_0 + x\cos\theta_0 + \cdots\end{aligned}$$

Using these, we expand the effective potential energy

$$\begin{aligned}U_{\text{eff}}\Big|_{g=0} &= \frac{(M_z - M_3\cos(\theta_0+x))^2}{2I_1\sin^2(\theta_0+x)} = \frac{M_3^2}{2I_1}\frac{(\cos\theta_0 - \cos(\theta_0+x))^2}{\sin^2(\theta_0+x)} \\ &= \frac{I_3^2\omega_3^2}{2I_1}\left(\frac{x\sin\theta_0 + \cdots}{\sin\theta_0 + x\cos\theta_0 + \cdots}\right)^2 = \frac{I_3^2\omega_3^2}{I_1}\cdot\frac{x^2}{2} + \cdots\end{aligned}$$

and the potential energy

$$mg\ell\cos\theta = mg\ell\cos(\theta_0 + x) = mg\ell\cos\theta_0 - xmg\ell\sin\theta_0 + \cdots$$

We now apply the lemma (-8/157) to $f = U_{\text{eff}}|_{g=0}$, $g = \epsilon$, $h(x) = m\ell\cos(\theta_0 + x) = m\ell\cos\theta_0 - xm\ell\sin\theta_0 + \cdots$, with

$$A = \frac{I_3^2\omega_3^2}{I_1}, \quad B = m\ell\cos\theta_0, \quad C = -m\ell\sin\theta_0.$$

According to the lemma, the function

$$f_\epsilon(x) = f(x) + \epsilon h(x) = \frac{I_3^2\omega_3^2}{I_1}\cdot\frac{x^2}{2} + \epsilon m\ell(\cos\theta_0 - x\sin\theta_0) + \cdots,$$

has a minimum at

$$x_g = \frac{I_1 m\ell\sin\theta_0}{I_3^2\omega_3^2} g + O(g^2), \quad g = \epsilon.$$

Now, by the previous computation

$$\dot{\varphi} = \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta} = \frac{M_3}{I_1\sin\theta_0}\cdot x + \cdots$$

Part III: Hamiltonian Mechanics

Chapter 7: "Differential forms" is a short introduction to the differential calculus of forms on manifolds. This material can also be found in several introductory texts on differential geometry.

-8/202 Using the notations in the text, if, in local coordinates, we have $\mathbf{p} = \sum_{i=1}^n p_i\, dq_i$ then $\omega^1(\boldsymbol{\xi}) = f^*\mathbf{p}(\boldsymbol{\xi}) = \mathbf{p}(f_*\boldsymbol{\xi}) = \sum_{i=1}^n p_i \circ f \cdot f^*(dq_i) = \sum_{i=1}^n p_i \circ f \cdot d(q_i \circ f) = \sum_{i=1}^n p_i\, dq_i = \mathbf{p} \wedge d\mathbf{q}$, where, in the last but one equality we identified p_i with $p_i \circ f$ and q_i with $q_i \circ f$, $i = 1, \ldots, n$.

1/204 Example. As in the text, we have the isomorphism $TM_{\mathbf{x}} \to T^*M_{\mathbf{x}}$ given by $\boldsymbol{\xi} \mapsto \omega^1_{\boldsymbol{\xi}}$, where $\omega^1_{\boldsymbol{\xi}}(\boldsymbol{\eta}) = \omega^2(\boldsymbol{\xi}, \boldsymbol{\eta})$. In particular, we have

$$\frac{\partial}{\partial \mathbf{p}} \mapsto -d\mathbf{q} \quad \text{and} \quad \frac{\partial}{\partial \mathbf{q}} \mapsto d\mathbf{p}.$$

Indeed, we calculate

$$\omega^1_{\partial/\partial \mathbf{p}}(\boldsymbol{\eta}) = \omega^2\left(\boldsymbol{\eta}, \frac{\partial}{\partial \mathbf{p}}\right) = d\mathbf{p} \wedge d\mathbf{q}\left(\boldsymbol{\eta}, \frac{\partial}{\partial \mathbf{p}}\right) = -d\mathbf{p} \wedge d\mathbf{q}\left(\frac{\partial}{\partial \mathbf{p}}, \boldsymbol{\eta}\right) = -d\mathbf{q}(\boldsymbol{\eta}).$$

The proof for the second correspondence is analogous.
Hence, for the differential of the Hamiltonian

$$dH = \frac{\partial H}{\partial \mathbf{p}}\, d\mathbf{p} + \frac{\partial H}{\partial \mathbf{q}}\, d\mathbf{q} = \frac{\partial H}{\partial \mathbf{p}}\, \omega^1_{\partial/\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\, \omega^1_{\partial/\partial \mathbf{p}} = \omega^1_{\frac{\partial H}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{p}}}$$

we obtain

$$I\, dH = \frac{\partial H}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{p}}.$$

Thus, for $\mathbf{x} = (\mathbf{q}, \mathbf{p})$, the differential equation $\dot{\mathbf{x}} = I\, dH(\mathbf{x})$ is equivalent to

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \quad \text{and} \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}.$$

16/205 With the notations in the text, for $f' : [0,1] \times [0,\tau] \to M^{2n}$, we have

$$\begin{aligned}\int_{J\gamma} \omega^2 &= \int_{[0,1]\times[0,\tau]} f'^*\omega^2 = \int_0^1 \int_0^\tau (f'^*\omega^2)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) dt\, ds \\ &= \int_0^1 \int_0^\tau \omega^2\left(f'_*\left(\frac{\partial}{\partial s}\right), f'_*\left(\frac{\partial}{\partial t}\right)\right) dt\, ds = \int_0^1 \int_0^\tau \omega^2\left(\boldsymbol{\xi}, \boldsymbol{\eta}\right) dt\, ds.\end{aligned}$$

-8/221 Problem. The fact that $Sp(2)$ is isomorphic with $SL(2, \mathbb{R})$, the Lie group of 2×2 matrices with real entries and determinant 1, is clear since a 2×2-matrix

A with real entries is an element in $Sp(2)$ if and only if $A^tIA = \Omega$, where I is the operator of the symplectic basis in $\mathbb{R}^2$. In this basis we have $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Setting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we obtain $A^tIA = (ad - bc)I$, and the first statement follows.
We now turn to the second statement that $SL(2, \mathbb{R})$ is homeomorphic with the interior of a solid 3-dimensional torus.
We will use the fact that the projective special linear group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm E\}$, where E is the denity, is isomorphic with the group of linear fractional transformations

$$\left\{ g \,\middle|\, g(z) = \frac{az + b}{cz + d}, z \in \mathbb{C}, \ ad - bc = 1, \ a, b, c, d \in \mathbb{R} \right\}$$

with real coefficients, which, in turn, is the group $Iso^+(H^2)$ of (orientation preserving) isometries of the hyperbolic plane $H^2 = \{z \in \mathbb{C} \,|\, \Im(z) > 0\}$, the upper half plane model of hyperbolic geometry.[11] The isomorphism is given by

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad - bc = 1, \ a, b, c, d \in \mathbb{R} \quad \leftrightarrow \quad z \mapsto \frac{az + b}{cz + d}, \quad z \in H^2.$$

With this, we first show that any $g \in Iso^+(H^2)$ which fixes $i \in H^2$, $g(i) = i$, must be of the form $g = k_\theta$, $0 \leq \theta < 2\pi$, where[12]

$$k_\theta(z) = \frac{\cos(\theta/2)z - \sin(\theta/2)}{\sin(\theta/2)z + \cos(\theta/2)}, \quad z \in H^2.$$

Indeed, with the linear fractional representation of g, the condition $g(i) = i$ gives $(ai + b)/(ci + d) = i$, that is, $ai + b = -c + di$. Hence $a = d$ and $b = -c$, and the determinant condition $ad - bc = 1$ reduces to $a^2 + b^2 = 1$. Hence $a = \cos(\theta/2)$ and $b = -\sin(\theta/2)$, for a unique $\theta \in [0, 2\pi]$.
Second, we claim that any $g \in Iso^+(H^2)$ can be uniquely decomposed as

$$g = k_\theta \circ a_\lambda \circ n_\mu, \quad 0 \leq \theta \leq 2\pi, \quad \lambda > 0, \quad \mu \in \mathbb{R},$$

where

$$a_\lambda(z) = \lambda^2 \cdot z \quad \text{and} \quad n_\mu(z) = z + \mu, \quad z \in H^2,$$

are central dilatations and horizontal translations, respectively, in $Iso\,(H^2)$.
To show the claim, we let $g^{-1}(i) = -\mu + i/\lambda^2 \in H^2$ with $\lambda > 0$, $\mu \in \mathbb{R}$ uniquely

[11] See, for example, Toth, G., *Glimpses of Algebra and Geometry*, Second Edition, Springer NY (2000), Section 13, p. 139.
[12] The parametrization includes half angles since $g_0 = g_{2\pi} = I$.

determined by g. Then, letting $h = g \circ n_{-\mu} \circ a_{1/\lambda}$, we have $h(i) = g(-\mu + i/\lambda^2) = g(g^{-1}(i)) = i$. Since h fixes i, by the above, we have $h = k_\theta$, for a unique $0 \leq \theta < 2\pi$. Hence $g = h \circ a_{1/\lambda}^{-1} \circ n_{-\mu}^{-1} = k_\theta \circ a_\lambda \circ n_\mu$ as claimed. Unicity is also clear.
As a consequence, we also obtain the Iwasawa decomposition for $SL(2, \mathbb{R})$ as follows. We define three subgroups of $SL(2, \mathbb{R})$ by

$$\begin{aligned} K &= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| 0 \leq \theta < 2\pi \right\} \\ A &= \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \middle| \lambda > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \middle| \mu \in \mathbb{R} \right\}. \end{aligned}$$

Then any element $A \in SL(2\mathbb{R})$ can be unquely decomposed as $A = k \cdot a \cdot n$, where $k \in K$, $a \in A$ and $n \in N$.
For the proof, we note that, under the isomorphism $PSL(2, \mathbb{R}) \cong Iso^+(H^2)$, the set of matrices in K parametrized by $0 \leq \theta < 2\pi$ splits into two subsets, one parametrized by $0 \leq \theta < \pi$ and the other by $\pi \leq \theta < 2\pi$. Since $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$, these correpond to $\pm$ the respective set for $\theta/2$ paramerized by $0 \leq \theta < 2\pi$. Moreover, the typical matrices in A and N correspond to a_λ and n_μ, respectively.
Finally, we have the homeomorphisms $K \approx S^1$, $A \approx \mathbb{R}_+$, and $N \approx \mathbb{R}$, so that $SL(2, \mathbb{R}) \approx S^1 \times \mathbb{R} \times \mathbb{R}_+$, the interior of a solid torus.

7/234 We are in $\mathbb{R}^3$. By 8/187 EXAMPLE 2, to a vector field $\mathbf{v}$ there corresponds the 1-form $\omega^1_{\mathbf{v}}$ such that $\omega^1_{\mathbf{v}}(\boldsymbol{\xi}) = (\mathbf{v}, \boldsymbol{\xi})$. The **circulation** of $\mathbf{v}$ over a curve l is given by the integral

$$\int_{c_1} \omega^1_{\mathbf{v}} = \int_l (\mathbf{v}, dl)$$

of the 1-form $\omega^1_{\mathbf{v}}$ on a chain c_1 representing l.
To a vector field $\mathbf{v}$ there also corresponds the 2-form $\omega^2_{\mathbf{v}}$ such that $\omega^2_{\mathbf{v}}(\boldsymbol{\xi}, \boldsymbol{\eta}) = (\mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\eta})$. The **flux** of $\mathbf{v}$ through an oriented surface S is given by the integral

$$\int_{c_2} \omega^2_{\mathbf{v}} = \int_S (\mathbf{v}, d\mathbf{n}),$$

where the 2-chain c_2 is represented by S.
Now, we have

$$d\omega^1_{\mathbf{v}} = \omega^2_{\mathbf{curl\,v}} = \omega^2_{\mathbf{r}}$$

and, if $l = \partial S$, then, Stokes' theorem asserts that

$$\int_l (\mathbf{v}, dl) = \int_S (\mathbf{curl}\,\mathbf{v}, d\mathbf{n}) = \int_S (\mathbf{r}, d\mathbf{n}),$$

or equivalently

$$\int_{c_1} \omega^1_{\mathbf{v}} = \int_{c_2} \omega^2_{\mathbf{r}}, \quad c_1 = \partial c_2$$

(-13/194).

10/242 PROBLEM. By the definition of S, and with the differential on the extended phase space, we have

$$dS = \mathbf{p}\,d\mathbf{q} - \mathbf{P}\,d\mathbf{Q} + \frac{\partial S}{\partial t}\,dt$$

Hence

$$\begin{aligned} \mathbf{p}\,d\mathbf{q} - H\,dt &= \mathbf{P}\,d\mathbf{Q} - H\,dt - \frac{\partial S}{\partial t}dt + dS \\ &= \mathbf{P}\,d\mathbf{Q} - \left(H + \frac{\partial S}{\partial t}\right)dt + dS \\ &= \mathbf{P}\,d\mathbf{Q} - K\,dt + dS \end{aligned}$$

7/243 ... an integral curve γ ...

7/244 We have

$$\dot{\mathbf{q}}\,\delta\mathbf{p} + \mathbf{p}\,\delta\dot{\mathbf{q}} = \dot{\mathbf{q}}\,\delta\mathbf{p} + \mathbf{p}\,\delta\dot{\mathbf{q}} + \dot{\mathbf{p}}\,\delta\mathbf{q} - \dot{\mathbf{p}}\,\delta\mathbf{q} = (\mathbf{p}\,\delta\mathbf{q})^{\cdot} + \dot{\mathbf{q}}\,\delta\mathbf{p} - \dot{\mathbf{p}}\,\delta\mathbf{q}.$$

-8/246 We have $\dot{\mathbf{q}} = ds/d\tau$.

-10/247 The metric $d\rho$ is called **conformal** to ds, a term coined by Gauss.

10/254 See the remark at -6/244.

11/255 We have

$$\mathbf{p}\dot{\mathbf{q}} = \frac{\partial S_{\mathbf{q}_0}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \frac{d}{dt}S_{\mathbf{q}_0}(\mathbf{q}(t)) = 1.$$

-6/256 THE INITIAL CONDITION: $t = t_0$ gives $\mathbf{p} = \partial S_0/\partial\mathbf{q}$ and $H dt = 0$, so that the integral in -9/256 is

$$\begin{aligned} \int_{(\mathbf{q}_0,t_0)}^{(\mathbf{q},t_0)} L(\mathbf{q},\dot{\mathbf{q}},t)\,dt &= \int_{(\mathbf{q}_0,t_0)}^{(\mathbf{q},t_0)} (\mathbf{p}\cdot\dot{\mathbf{q}} - H)\,dt = \int_{(\mathbf{q}_0,t_0)}^{(\mathbf{q},t_0)} \mathbf{p}\,d\mathbf{q} \\ &= \int_{\mathbf{q}_0}^{\mathbf{q}} \frac{\partial S_0}{\partial \mathbf{q}}\,d\mathbf{q} = S_0(\mathbf{q}) - S_0(\mathbf{q}_0) = S(\mathbf{q},t_0) - S_0(\mathbf{q}_0). \end{aligned}$$

-6/258 PROBLEM. Since $g(\mathbf{p},\mathbf{q}) = (\mathbf{P}(\mathbf{p},\mathbf{q}),\mathbf{Q}(\mathbf{p},\mathbf{q}))$, we have $\mathbf{P}\wedge\mathbf{Q} = \det\partial(\mathbf{P},\mathbf{Q})/\partial(\mathbf{p},\mathbf{q})\cdot \mathbf{p}\wedge\mathbf{q} = g^*(d\mathbf{p}\wedge d\mathbf{q})$. Therefore, taking the differential of both sides of (1), we obtain $d\mathbf{p}\wedge d\mathbf{q} - d\mathbf{P}\wedge d\mathbf{Q} = ddS(\mathbf{p},\mathbf{q}) = 0$, and the statement follows.

-1/258 We have

$$\det\frac{\partial(\mathbf{Q},\mathbf{q})}{\partial(\mathbf{p},\mathbf{q})} = \det\begin{pmatrix} \partial\mathbf{Q}/\partial\mathbf{p} & 0 \\ 0 & E\end{pmatrix} = \det\frac{\partial\mathbf{Q}}{\partial\mathbf{p}}.$$

-7/262 Thus

$$T = a^2\frac{\dot\xi^2}{2} + b^2\frac{\dot\eta^2}{2},\quad p_\xi = a^2\dot\xi,\quad p_\eta = b^2\dot\eta$$

and so

$$\begin{aligned} H &= \frac{p_\xi^2}{2a^2} + \frac{p_\eta^2}{2b^2} - \frac{k}{r_1} - \frac{k}{r_2} = 2\sin^2\alpha\cdot p_\xi^2 + 2\cos^2\alpha\cdot p_\eta^2 - \frac{k(r_1+r_2)}{r_1 r_2} \\ &= 2p_\xi^2\frac{\xi^2-4c^2}{\xi^2-\eta^2} + 2p_\eta^2\frac{4c^2-\eta^2}{\xi^2-\eta^2} - \frac{4k\xi}{\xi^2-\eta^2}. \end{aligned}$$

-7/263 Setting $\partial S/\partial\xi = p_\xi$ and $\partial S/\partial\eta = p_\eta$ as in (2) in p. 259, we obtain the Hamilton-Jacobi equation

$$2\left(\frac{\partial S}{\partial\xi}\right)^2\frac{\xi^2-4c^2}{\xi^2-\eta^2} + 2\left(\frac{\partial S}{\partial\eta}\right)\frac{4c^2-\eta^2}{\xi^2-\eta^2} - \frac{4k\xi}{\xi^2-\eta^2} = K,$$

or equivalently

$$2\left(\frac{\partial S}{\partial\xi}\right)^2(\xi^2-4c^2) + 2\left(\frac{\partial S}{\partial\eta}\right)^2(4c^2-\eta^2) = K(\xi^2-\eta^2) + 4k\xi$$

(note the missing factor 2 in the text which does not effect the final form of S). Note finally that here $\mathbf{q} = (\xi,\eta)$ and $\mathbf{Q} = (c_1, c_2)$.

-9/270 Using the setup in the text,[13] and the proof of the lemma at -11/205 and -7/207, we have

$$\begin{aligned} \iint_{\sigma(\epsilon)}\omega^2 &= \int_{J\gamma}\omega^2 = \int_0^\epsilon\left(\int_{g_\tau\gamma} dH\right)d\tau = \int_0^\epsilon\left(\int_\gamma g_\tau^* dH\right)d\tau \\ &= \int_0^\epsilon\left(\int_\gamma d(H\circ g_\tau)\right)d\tau = \int_0^\epsilon\left(\int_\gamma dH\right)d\tau \\ &= \epsilon\int_\gamma dH = H(y) - H(x). \end{aligned}$$

[13]The text changes the notation from p. 205 in the use of the double integral sign as well as changes g^t to g_t.

18/272 For the definition of conditionally periodic motion, see -5/285.

-9/272 That is dF and dH are linearly independent 1-forms pointwise.

11/273 For the proof, see Problems 2-3 on pp. 215-216.

-13/274 Note that g_i^t is defined for all $t \in \mathbb{R}$ since M^n is compact.

-4/275 PROBLEM. $s \in t + V$ implies $s - t \in V$. Hence $s - t = 0$ giving $s = t$.

-13/280 By Hooke's law, $F(q) = -q$, so that $F(q) = -U'(q)$ gives $U(q) = q^2/2$.

-4/280 Since $p = r\cos\varphi$ and $q = r\sin\varphi$, we have $dp = \cos\varphi\, dr - r\sin\varphi\, d\varphi$ and $dq = \sin\varphi\, dr + r\cos\varphi\, dp$. These give

$$dp \wedge dq = r\cos^2\varphi\, dr \wedge d\varphi - r\sin^2\varphi\, d\varphi \wedge dr = r\, dr \wedge d\varphi = d(r^2/2) \wedge d\varphi.$$

Thus, $I(p,q) = I(r) = r^2/2$.

-10/281 In fact, $\varphi = \partial S/\partial I$, so that

$$d\varphi\big|_{I=\text{const}} = \frac{\partial^2 S}{\partial I \partial q}\, dq = \frac{\partial p}{\partial I}\, dq.$$

Integrating, we obtain

$$\oint_{M_h} d\varphi = \oint_{M_h} \frac{\partial p}{\partial I}\, dq = \frac{d}{dI}\oint_{M_h} p\, dq = \frac{d\,\Delta S(I)}{dI}.$$

5/282 PROBLEM. The ellipse M_h is given by $a^2p^2 + b^2q^2 = 2h$, so that $\Pi(h) = 2\pi h/(ab)$. Therefore, we have $I(h) = h/(ab)$ and $h(I) = ab\, I$. The third equation in (4) gives

$$H\left(\frac{\partial S(I,q)}{\partial q}, q\right) = h(I) = ab\, I,$$

or equivalently

$$a^2\left(\frac{\partial S(I,q)}{\partial q}\right)^2 + b^2q^2 = 2ab\, I.$$

Thus

$$\frac{\partial S(I,q)}{\partial q} = \sqrt{\frac{2ab\, I - b^2q^2}{a^2}} = \frac{b}{a}\sqrt{2\frac{a}{b}I - q^2} = p.$$

Integrating, up to an additive constant, we get

$$S(I,q) = \frac{b}{a}\int \sqrt{2\frac{a}{b}I - q^2}\, dq = \frac{b}{2a}\left(q\sqrt{2\frac{a}{b}I - q^2} + 2\frac{a}{b}I\arctan\left(\frac{q}{\sqrt{2\frac{a}{b}I - q^2}}\right)\right)$$

or, in terms of p above

$$S(I,q) = \frac{pq}{2} + I \arctan\left(\frac{bq}{ap}\right), \quad p = \frac{b}{a}\sqrt{2\frac{a}{b}I - q^2}.$$

Finally, differentiating under the integral sign, we obtain

$$\varphi = \frac{\partial S}{\partial I} = \int \frac{dq}{\sqrt{2\frac{a}{b}I - q^2}}\, dq = \arctan\left(\frac{q}{\sqrt{2\frac{a}{b}I - q^2}}\right) = \arctan\left(\frac{bq}{ap}\right).$$

Note that, the original ellipse $a^2p^2+b^2q^2 = 2h$ is parametrized by $\sqrt{2h}(\cos(\varphi)/a, \sin(\varphi)/b)$.

-13/287 See also the note at -4/74.

5/289 Problem. Using Example 2 at -9/24, we have $U = x^2/2+\omega^2y^2/2$, where $\omega^2 = 2$, so that $\omega = \sqrt{2}$. Since $|x|, |y| \leq 1$, we have $x = \sin(t + \varphi_0)$ and $y = \sin(\omega t + \psi_0)$. The kinetic energy is

$$\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}\left(\cos^2(t + \varphi_0) + \omega^2 \cos^2(\omega t + \psi_0)\right).$$

Using the calculus formula

$$\int_0^T \cos^2(\omega t + c)\, dt = \frac{T}{2} + \frac{1}{4\omega} \sin(2\omega T + 2c), \quad c \in \mathbb{R},$$

we obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_0^T \left(\dot{x}^2 + \dot{y}^2\right)\, dt = \frac{1}{4} + \frac{\omega^2}{4} = \frac{3}{4},$$

since

$$\lim_{T\to\infty} \frac{\sin(2T + 2\varphi_0)}{T} = \lim_{T\to\infty} \frac{\sin(2\omega T + 2\psi_0)}{T} = 0.$$

-9/290 Problem. For fixed $0 \neq \mathbf{k} \in \mathbb{Z}^n$, the set $\mathbf{k}^\perp = \{\boldsymbol{\omega} \in \mathbb{R}^n \,|\, (\boldsymbol{\omega}, \mathbf{k}) = 0\}$ is a hyperplane in $\mathbb{R}^n$, and hence of Lebesgue measure zero in $\mathbb{R}^n$. The set

$$\{\boldsymbol{\omega} \in \mathbb{R}^n \,|\, \exists\, 0 \neq \mathbf{k} \in \mathbb{Z}^n, (\boldsymbol{\omega}, \mathbf{k}) = 0\} = \bigcup_{0\neq\mathbf{k}\in\mathbb{Z}^n} \mathbf{k}^\perp$$

is the union of countably many sets of Lebesgue measure zero, therefore, it is itself of Lebesgue measure zero.

-7/292 Problem. We have

$$\varphi = \varphi_0 + \omega t, \quad \varphi_0 = \varphi(0).$$

Since $\tilde{g}(\varphi) = g(\varphi) - \bar{g}$, where $\bar{g} = \int_0^{2\pi} g(\varphi)\, d\varphi$, we get $\int_0^{2\pi} \tilde{g}(\varphi)\, d\varphi = 0$. Since g is 2π-periodic, it follows that h defined by $h(\varphi) = \int_0^{\varphi} \tilde{g}(\varphi)\, d\varphi$ is also 2π-periodic. Using $\dot{I} = \epsilon g(\varphi)$, we now calculate

$$\begin{aligned} I(t) - I(0) &= \epsilon \int_0^t g(\varphi_0 + \omega t)\, dt = \epsilon \bar{g}\, t + \epsilon \int_0^t \tilde{g}(\varphi_0 + \omega t)\, dt \\ &= \epsilon \bar{g}\, t + \frac{\epsilon}{\omega} \int_{\varphi_0}^{\varphi_0 + \omega t} \tilde{g}(\varphi)\, d\varphi = \epsilon \bar{g}\, t + h(\varphi_0 + \omega t) - h(\varphi_0), \end{aligned}$$

where, in the last but one equality, we performed he substitution $\varphi = \varphi_0 + \omega t$, $d\varphi = \omega\, dt$. (Note the slight discrepancy with the text.)

9/293 Here, we have $\dot{\mathbf{I}} = \epsilon g(\mathbf{I}, \boldsymbol{\varphi})$, where $g(\mathbf{I}, \boldsymbol{\varphi}) = \partial H_1(\mathbf{I}, \boldsymbol{\varphi})/\partial \boldsymbol{\varphi}$ since $H_0(\mathbf{I})$ does not depend on $\boldsymbol{\varphi}$.

-14/295 In fact, we have

$$\int_0^{2\pi} \frac{\partial k}{\partial \varphi}\, d\varphi = k(2\pi) - k(0) = 0$$

since k is 2π-periodic.

18/297 If q is the angle of inclination, then, assuming unit mass $m = 1$, the kinetic and potential energies are

$$T = \frac{1}{2}\dot{q}^2 = \frac{p^2}{2l^2} \quad \text{and} \quad U = g(l - l\cos q) \approx gl(1 - (1 - q^2/2)) = gl\frac{q^2}{2},$$

where the momentum $p = l\dot{q}$ and the potential energy is zero at $q = 0$. Due to the approximation, the Hamiltonian is for small oscillations only.

-9/299 We have

$$\begin{aligned} \dot{\varphi} &= \frac{\partial K}{\partial I} = \frac{\partial H_0}{\partial I} + \epsilon \frac{\partial^2 S}{\partial I \partial \lambda} = \omega(I, \lambda) + \epsilon\, f(I, \varphi; \lambda) \\ \dot{I} &= -\frac{\partial K}{\partial \varphi} = -\epsilon \frac{\partial^2 S}{\partial \varphi \partial \lambda} = \epsilon\, g(I, \varphi; \lambda), \end{aligned}$$

since H_0 does not depend on φ.

1/300 We have $I(t) - J(t) = I(t) - J(0) = I(t) - I(0)$.

4/300 Here, as ususal, $H = h$, and, using the area formula for the ellipse

$$\frac{a^2}{2h}p^2 + \frac{b^2}{2h}q^2 = 1$$

we obtain

$$I = \frac{1}{2\pi}\pi\frac{\sqrt{2h}}{a}\frac{\sqrt{2h}}{b} = \frac{h}{ab} = \frac{h}{\omega}.$$

-10/300 Using

$$H = \frac{p^2}{2l^2} + lg\frac{q^2}{2} = h$$

at 18/297, we obtain (as above)

$$I = \frac{1}{2\pi}\pi\sqrt{2hl^2}\sqrt{\frac{2h}{lg}} = \sqrt{\frac{l}{g}}h = \sqrt{\frac{l}{g}}\left(\frac{p^2}{2l^2} + lg\frac{q^2}{2}\right) = \frac{1}{2}l^{3/2}g^{1/2}q_{\max}^2,$$

since $p = 0$ for $q = q_{\max}$. Since I is constant on the phase trajectory (ellipse), dividing, we obtain

$$\frac{q_{\max}(t)}{q_{\max}(0)} = \left(\frac{l(0)}{l(t)}\right)^{3/4}.$$