The Existence of CR Structures

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The most important example of a CR manifold:

$$M^{2n+1} \subset \mathbf{C}^{n+1}$$

$$\begin{aligned} B &= T^{0,1} \cap (C \otimes T(M)) \\ &= \{L = \sum_{1}^{n+1} \alpha_j \frac{\partial}{\partial \overline{z}_j} \mid L = X + iY,\ X \in TM,\ Y \in TM\} \end{aligned}$$

Properties

- rank $B = n$
- $B \cap \overline{B} = \{0\}$
- $[B, B] \subset B$

Abstract CR structure

$B \subset C \otimes T(M)$ is a CR structure of codimension k on M^{2n+k} if

- rank $B = n$
- $B \cap \overline{B} = \{0\}$
- $[B, B] \subset B$

Example If $M^{2n+k} \subset \mathbf{C}^{n+k}$ is "generic" at $p \in M$, then near p

$$B = T^{0,1} \cap (C \otimes T(M))$$

is a CR structure of codimension k.

Digression Not every abstract CR structure is realizable.

When does M^{2n+k} admit a CR structure of codimension k?
Need that M admits an almost complex structure:

- rank $B = n$
- $B \cap \overline{B} = \{0\}$
- ~~$[B, B] \subset B$~~

The question becomes
When can B be deformed to a CR structure?
Seek $B_t \subset C \otimes T(M)$, for $0 \leq t \leq 1$ with

$$\mathrm{rank} B_t = n$$

$B_0 = B$
$B_t \cap \overline{B_t} = \{0\}$

Conjecture

Let M be an open and orientable manifold. If the cohopmology groups $H^q(M, Z)$ vanish for q large (depending on k), then every almost CR structure of codimension k may be deformed to a CR structure.

Known results

$k = 0$ $\quad q \geq \frac{\dim M}{2}$

$k = \dim M - 2$ $\quad q \geq \dim M + 1$ (That is, there is no cohomological restriction.)

Related questions

- Can every CR structure on M be approximated by a C^ω structure?
- Can every CR structure on an open subset of M be extended to a CR structure on all of M?
- Can a non-degenerate CR structure on an open subset of M be extended to a CR structure of the same signature on all of M?

Heuristic motivation

Observe

If M^{2n+k} has a map into $\mathbf{C}^{n+k}$ that is generic at all points of M, then

$$B = f_*(C \otimes T(M)) \cap T^{0,1}$$

determines a CR structure of codimension k.

Given some $B^n \subset C \otimes T(M)$ would like to find some complex manifold X^{n+k} and a generic map

$$f : M \to X.$$

Actually, X will be of real dimension $4n + 3k$ and foliated by complex manifolds of complex dimension $n + k$.
X is used to provide half of the proof of the conjecture:

Theorem

Let B be a continuous almost CR structure of codimension k on M^{2n+k}. If $(C \otimes T(M))/B$ is isomorphic to the normal bundle of a Haefliger CR structure then B is homotopic through almost CR structures to a C^{ω} CR structure.

Approach modeled on classical foliation theory

Analysts A foliation means the Frobenius Theorem: A subbundle $K \subset TM$ defines a foliation if and only if $[K, K] \subset K$.

Topologists A foliation of codimension q is a collection of open sets

$$M = \bigcup \mathcal{O}_i,$$

submersions onto open subsets of $\mathbf{R}^q$

$$f_i : \mathcal{O}_i \to U_i,$$

and compatibility conditions.

Haefliger: To show that any $K \subset TM$ may be deformed to the tangent bundle of a foliation

- Step 1 If TM/K is isomorphic to the normal bundle of a Haefliger structure then the Gromov-Phillips Theorem may be used to find the foliation.
- Step 2 Find topological conditions on M, depending on $\mathrm{rank} K$, such that for all K, TM/K is isomorphic to such a normal bundle.

Gromov-Phillips Theorem

Simplest case

$$\begin{aligned} f: &\quad M \to X \\ g: &\quad TM \to TX \end{aligned}$$

with

$$g : TM_p \to TX_{f(p)}$$

surjective at each $p \in M$.

Theorem

There exists a submersion $F : M \to X$.

Haefliger Structures

An example and then the definition.

If M^{2n+k} has a C^ω CR structure of co-dimension k then there exists an open covering

$$M = \cup \mathcal{O}_j$$

and C^ω CR embeddings

$$f_j : \mathcal{O}_j \to \mathbf{C}^{n+k}.$$

Further, for each pair i, j with $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ there exist open sets U_{ij} containing $f_i(\mathcal{O}_i \cap \mathcal{O}_j)$ and biholomorphism $\gamma_{ij} : U_{ji} \to U_{ij}$ with

$$f_i = \gamma_{ij} \circ f_j \text{ on } \mathcal{O}_i \cap \mathcal{O}_j.$$

and

$$\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}.$$

Definition

A Haefliger CR structure of codimension k on M^{2n+k} consists of

- An open covering $M = \cup \mathcal{O}_j$,
- continuous maps $f_j : \mathcal{O}_j \to \mathbf{C}^{n+k}$,
- local biholomorphisms γ_{ij} of $\mathbf{C}^{n+k}$ defined for each pair (i, j) such that $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ satisfying
  1. $$\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$$
     at all points where both sides are defined
  2. and
     $$f_i = \gamma_{ij} \circ f_j \text{ on } \mathcal{O}_i \cap \mathcal{O}_j.$$

The normal bundle

The normal bundle ν of a Haefliger CR structure is the $\mathbf{C}^{n+k}$ bundle over M with transitions functions $d\gamma_{ij}$.

Lemma

The normal bundle ν of a Haefliger CR structure admits in a neighborhood of the zero section a foliation of dimension $2n + k$ transverse to the bundle fibers.

Micro-foliation

Step 1 for CR Structures

We assume $C \otimes T(M)/B$ is isomorphic to the normal bundle of a Haefliger structure and find a CR structure.
Write X in place of ν.

- $\dim X = 4n + 3k$.
- X has transverse foliations $\mathcal{F}^{2n+2k}$ and $\mathcal{F}^{2n+k}$.
- The leaves of $\mathcal{F}^{2n+2k}$ are complex manifolds V^{n+k}.

1- Use $\mathcal{F}^{2n+k}$ to define

$$p : C \otimes T(X) \to T^{1,0}V.$$

2- Use p and $C \otimes T(M)/B \cong \nu$ to construct a surjective map

$$C \otimes T(M) \to T^{1,0}V|_M$$

with kernel B.

3- Use the h-principle to find a map

$$F : M \to X$$

with

$$p \circ F_* : C \otimes T(M) \to T^{1,0}V$$

surjective.

4- Observe that $B = ker\ p \circ F_*$ satisfies

- rank $B = n$
- $B \cap \overline{B} = \{0\}$
- $[B, B] \subset B$.

Step 2 for CR Structures

$$
\begin{array}{ccc}
 & & \mathcal{B} \oplus GL(n) \\
 & & \downarrow \Gamma(\nu_{\mathcal{B}}) \oplus \mathrm{id} \\
M & \xrightarrow{\Gamma(\nu) \oplus \Gamma(B)} & BGL(n+k) \oplus BGL(n)
\end{array}
$$

The obstructions to lifting

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ M & \rightarrow & X \end{array}$$

lie in

$$H^{i+1}(M, \pi_i(F)).$$

So if

$$\pi_j(F) = 0$$

for $0 \leq j \leq \dim M$, and if

$$H^j(M, Z) = 0$$

for $N + 2 \leq j \leq \dim M$, then all maps $M \rightarrow X$ lift.