Errata and Notes

for

Glimpses of Algebra and Geometry (2nd Edition)

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page(s)/line(s)

57/9
T_n and U_n are called the Chebyshev polynomials; they have extended literature.

64/-5
Conversely, every complex number $z \in \mathbb{C}$ is the root of a quadraic polynomial in real coefficients. Indeed, $a + bi$, $a, b \in \mathbb{R}$, is a root of the quadratic polynomial $x^2 - 2ax + a^2 + b^2 = 0$.

97/-6
Recall a few facts from the metric geometry of the Euclidean plane $\mathbb{R}^2$. The equation for the line containing two points $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$ can be written as

$$(y_1 - y_0)x - (x_1 - x_0)y = x_0 y_1 - x_1 y_0,$$

or equivalently

$$(x_0 - x)(y - y_1) - (x - x_1)(y_0 - y) = 0.$$

(Note that, in the second equation with the variable point $q = (x, y)$, the left-hand side is the (signed) area of the parallelogram with vertices at the origin, $p_0 - q$, $q - p_1$ and $p_0 - p_1$. It expresses the fact that q is on the line containing the points p_0 and p_1 if and only if the parallelogram is degenerate (has area zero), that is, its four vertices are collinear.)
Assume that the line ℓ contains two distinct points $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$. For $t \in \mathbb{R}$, we define the point

$$p_t = (1 - t)p_0 + tp_1 = ((1 - t)x_0 + tx_1, (1 - t)y_0 + ty_1).$$

We claim that $\ell = \{p_t \,|\, t \in \mathbb{R}\}$. The indeterminate $t \in \mathbb{R}$ is called and affine parameter of the line ℓ.
First, for $t \in \mathbb{R}$, we have $p_t \in \ell$ since

$$\begin{aligned}(y_1 - y_0)\left((1 - t)x_0 + tx_1\right) &- (x_1 - x_0)\left((1 - t)y_0 + ty_1\right) \\ &= (1 - t)\left((y_1 - y_0)x_0 - (x_1 - x_0)y_0\right) + t\left((y_1 - y_0)x_1 - (x_1 - x_0)y_1\right) \\ &= (1 - t)(x_0 y_1 - x_1 y_0) + t(x_0 y_1 - x_1 y_0) = x_0 y_1 - x_1 y_0.\end{aligned}$$

We need to show the converse. If $x_0 \neq x_1$ then we let $t = (x - x_0)/(x_1 - x_0)$, or equivalently, $x = (1 - t)x_0 + tx_1$. Substituting this into the equation of the line, a simple computation gives $y = (1 - t)y_0 + ty_1$. If $y_0 \neq y_1$ then we let $t = (y - y_0)/(y_1 - y_0)$, or equivalently, $y = (1 - t)y_0 + ty_1$. Substituting this into the

equation of the line again, we obtain $x = (1-t)x_0 + tx_1$. The converse follows.
The line segment with end-points p_0 and p_1 is given by

$$[p_0, p_1] = \{p_t \,|\, 0 \le t \le 1\}.$$

97/-2
The Euclidean distance d satistfies the following properties:
1. Non-negativity: $d(p_0, p_1) \ge 0$ for all $p_0, p_1 \in \mathbb{R}^2$, and $d(p_0, p_1) = 0$ if and only if $p_0 = p_1$.
2. Symmetry: $d(p_0, p_1) = d(p_1, p_0)$ for all $p_0, p_1 \in \mathbb{R}^2$.
3. (Strict) Triangle Inequality: $d(p_0, p_1) \le d(p_0, q) + d(q, p_1)$ for all $p_0, p_1, q \in \mathbb{R}^2$. The triangle inequality is strict in the sense that equality holds if and only if $q \in [p_0, p_1]$.
Non-negativity and symmetry are clear. We only need to show 3.
Letting $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$, $q = (x_2, y_2)$, we denote $a = x_0 - x_2$, $b = x_2 - x_1$, and $c = y_0 - y_2$, $d = y_2 - y_1$, so that, we have $a + b = x_0 - x_1$ and $c + d = y_0 - y_1$.
For the triangle inequality, we need to show

$$\sqrt{(a+b)^2 + (c+d)^2} \le \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2}.$$

Squaring both sides we have

$$(a+b)^2 + (c+d)^2 \le a^2 + b^2 + c^2 + d^2 + 2\sqrt{a^2 + c^2}\sqrt{b^2 + d^2}.$$

Expanding and simplifying, we obtain

$$ab + cd \le \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}.$$

Squaring both sides again, we arrive at the Cauchy-Schwarz inequality:

$$(ab + cd)^2 \le (a^2 + c^2)(b^2 + d^2).$$

Since the steps that we made are reversible, we obtain that the triangle inequality is equivalent to the Cauchy-Schwarz inequality above.
The latter, however, is a direct consequence of the identity

$$(ab + cd)^2 + (ad - bc)^2 = (a^2 + c^2)(b^2 + d^2)$$

which can be verified by expanding all parentheses. Thus, the Cauchy-Schwarz inequality, and thereby the triangle inequality follow.
For the strictness of the triangle inequality: For the "if" part, assuming that equality holds in the triangle inequality, and thereby in the Cauchy-Schwarz inequality, the identity above implies

$$ad - bc = (x_0 - x_2)(y_2 - y_1) - (x_2 - x_1)(y_0 - y_2) = 0.$$

By the above, this means that q is on the line ℓ containing the points p_0, p_1. Since $q \in \ell$, we have $q = p_t$ for some $t \in \mathbb{R}$. With this, we have $d(p_0, q) = d(p_0, p_t) = |t|d(p_0, p_1)$ and $d(q, p_1) = d(p_t, p_1) = |1 - t|d(p_0, p_1)$. Hence $|t| + |1 - t| = 1$ holds. This means that $0 \leq t \leq 1$ so that $q = p_t \in [p_0, p_1]$. The claim follows.
The "only if" part is obvious since $q \in [p_0, p_1]$ implies $q = p_t$ for some $0 \leq t \leq 1$, and thus $d(p_0, q) + d(q, p_1) = d(p_0, p_t) + d(p_t, p_1) = td(p_0, p_1) + (1 - t)d(p_0, p_1) = d(p_0, p_1)$.

98/7
We claim that every isometry $S \in Iso\,(\mathbb{R}^2)$ is invertible: Since S is clearly injective, we need to show that it is also surjecive. A quick proof of this is the following: Let $v \in \mathbb{R}^2$ be the vector terminating at $S(0)$, and consider the composition $U = T_{-v} \circ S$. Then, U is an isometry which fixes the origin; that is, $U(0) = 0$. By the parallelogram rule of addition of vectors, $U : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear (isometric) transformation. Since U is injective, linear algebra tells us that it must also be surjective. Hence, $S = T_v \circ U$ must also be surjective. Thus S^{-1} exist, and, clearly, it is also an isometry.
A somewhat longer proof, without the recourse to linear algebra, is the following: We first claim that S maps lines onto lines. Indeed, let $\ell \in \mathbb{R}^2$ be a line and $p_0, p_1 \in \ell$ two distinct points. For $0 \leq t \leq 1$, the point $p_t = (1 - t)p_0 + tp_1 \in [p_0, p_1]$ satisfies

$$d(S(p_0), S(p_t)) = d(p_0, p_t) = td(p_0, p_1) = td(S(p_0), S(p_1)).$$

Similarly

$$d(S(p_t), S(p_1)) = d(p_t, p_1) = (1 - t)d(p_0, p_1) = (1 - t)d(S(p_0), S(p_1)).$$

Adding, we obtain

$$d(S(p_0), S(p_t)) + d(S(p_t), S(p_1)) = d(S(p_0), S(p_1)).$$

By strictness of the triangle inequality, we get $S(p_t) \in [S(p_0), S(p_1)]$, and hence, by the above

$$S(p_t) = (1 - t)S(p_0) + tS(p_1), \quad 0 \leq t \leq 1.$$

We obtain that S maps the line segment $[p_0, p_1]$ onto the line seqment $[S(p_0), S(p_1)]$. A simple induction in the use of an equidistant subdivision of ℓ now shows that $S(\ell)$ is a line.
Given a triangle with vertices $p, q, r \in \mathbb{R}^2$, by what we have just shown, the isometry S maps the sides of the triangle onto the sides of the triangle with vertices $S(p), S(q), S(r)$. Applying the same argument to line sements with end-points as one of the vertices and a (varying) point of the opposite side, we obtain that S maps triangles onto (congruent) triangles. Finally, again by induction in the use of a subdivision of $\mathbb{R}^2$ into congruent triangles, it follows that S is onto.

99/18

If an isometry fixes two distinct points $p_0, p_1 \in \mathbb{R}^2$ then it fixes every point on the line ℓ through p_0 and p_1. This is because

$$S(p_t) = (1-t)S(p_0) + tS(p_1) = (1-t)p_0 + tp_1 = p_t, \quad t \in \mathbb{R}.$$

If an isometry fixes two distict points $p_0, p_1 \in \mathbb{R}^2$ then it is either the identity, or a reflection in the line ℓ passing through p_0, p_1. Indeed, let $q \in \mathbb{R}^2$ be a point not in ℓ. Since the triangles with vertices p_0, p_1, q and $p_0, p_1, S(q)$ are congruent then either $S(q) = q$ or $S(q) = R_\ell(q)$. In the first case S is the identity, in the second, $R_\ell \circ S$ is the identity. Since R_ℓ^2 is the identity, the stamement follows.

As an immediate byproduct we see that if an isometry fixes three non-collinear points in $\mathbb{R}^2$ then it fixes every point in $\mathbb{R}^2$; that is, the isometry is the identity.

99/-14

Given three points $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, we let

$$\omega(p_0, p_1, p_2) = (x_2 - x_0)(y_2 - y_1) - (x_2 - x_1)(y_2 - y_0).$$

For example $\omega((0,0),(1,0),(0,1)) = 1$. As above, $\omega(p_0, p_1, p_2)$ is the signed area of the paralelogram with vertices $0, p_0 - p_2, p_2 - p_1, p_0 - p_1$ or (applying a translation) that of the parallelogram with vertices $p_1, p_0 + p_1 - p_2, p_2, p_0$. In particular, $p_0, p_1, p_2 \in \mathbb{R}^2$ are not collinear if and only if $\omega(p_0, p_1, p_2) \neq 0$.

Clearly, ω is unchanged by a simultaneous translation or a rotation of the arguments, and it changes sign by a reflection of the arguments in a line.

We call a triplet (p_0, p_1, p_2) positively oriented if $\omega(p_0, p_1, p_2) > 0$. Otherwise, (p_0, p_1, p_2) is negatively oriented. Given $p_0, p_1 \in \mathbb{R}^2$ distinct, the points $p_2 \in \mathbb{R}^2$ such that $\omega(p_0, p_1, p_2) > 0$ fill an open half-plane whose boundary line passes through p_0 and p_1. The other half-plane corresponds to negatively oriented triplets.

Any two positively oriented triplets (p_0, p_1, p_2) and (q_0, q_1, q_2) can be connected by a continuous (in fact, piecewise linear) one-parameter family of positively oriented triplets.

An isometry S is direct if it does not change the sign of the orientation of triplets; that is, if S maps positively oriented triples to positively oriented triplets, and the same holds for negatively oriented triplets. More concisely, S is direct if

$$\omega(p_0, p_1, p_2)\omega(S(p_0), S(p_1), S(p_2)) > 0$$

for any triplet of non-collinear points $p_0, p_1, p_2 \in \mathbb{R}^2$. Otherwise, S is opposite.

S is direct if and only if the determinant $\det U > 0$ (and hence $= 1$), where $S = T_v \circ U$

as above. Indeed, if the matrix of U is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we have

$$\begin{aligned} &\omega((0,0),(1,0),(0,1))\omega(S(0,0),S(1,0),S(0,1)) \\ &\quad = \omega((0,0),(1,0),(0,1))\omega(U(0,0),U(1,0),U(0,1)) \\ &\quad = ad - bc = \det U. \end{aligned}$$

Similarly, S is opposite if and only if $\det U = -1$.
Clearly, translations and rotations are direct, and reflections (in lines) and glides are opposite. The composition of two isometries is direct if they have the same orientation; otherwise it is opposite.

99/-4
Using $S = T_v \circ R_\theta(p)$, another proof is the following. Write $T_v = R_{\ell'} \circ R_\ell$, $\ell \| \ell'$, and then $R_\theta(p) = R_\ell \circ R_{\ell''}$, we obtain $S = R_{\ell'} \circ R_{\ell''}$. Now $\ell' \| \ell''$ since S has no fixed point. Thus, S is a translation.

99/-3
More precisely, $a = |v|/(2\tan(\theta/2))$ is the height of the isosceles triangle.

127/-1
As evey reflecion is the inverse of itself, it follows that any Möbius transformation has an inverse.

134/-6
Actually, at most three reflections as $z \mapsto p\bar{z} + q$, $|p| = 1$, is an isometry.

134/-1
Clearly, $c \neq 0$ is equivalent to $g(\infty) \neq \infty$. If this holds the isometric circle exists, and the computations show that $g \circ R_{S_g}$ is an isometry h of $\mathbb{R}^2$. This gives $g = h \circ R_{S_g}$, where $h \in \mathit{Iso}\,(\mathbb{R}^2)$. In particular, every Möbius transformation is the composition of at most four reflections (first, in three lines, and the last in the isometric circle).
For completeness: If $c = 0$, that is, $g(\infty) = \infty$, then $g(z) = (a/d)z + (b/d) = a^2 z + ab$ since $ad = 1$. If $z_0 \in \mathbb{C}$ is a fixed point of g then $(a^2 - 1)z_0 = -ab$.
We distingish two cases. 1. If $a^2 = 1$ then, for $b = 0$, g is the identity, and, for $b \neq 0$, there is no fixed point and $g(z) = z + ab$, so that g is a translation. 2. If $a^2 \neq 1$ then $z_0 = -ab/(a^2 - 1)$ is the unique fixed point of g in $\mathbb{C}$. We then have $g(z) - z_0 = a^2 z + ab + ab/(a^2 - 1) = a^2(z - z_0)$, $z \in \mathbb{C}$. Hence g is a central dilatation with center z_0 and ratio a^2.

148/-5
We use the substitution $x = \cos u$, $dx = -\sin u du$.

204/1
Let $O(\mathbb{R}^3)$ denote the group of linear isometries of $\mathbb{R}^3$, and $SO(\mathbb{R}^3) \subset O(\mathbb{R}^3)$ the subgroup of all direct linear isomeries. Theorem 10 implies that $SO(\mathbb{R}^3)$ is the group of all spatial rotations of $\mathbb{R}^3$ which are linear, that is their rotation axes pass through the origin.

256/1
Cayley's theorem and the subsequent remark assert that $SO(\mathbb{R}^3)$, the group of all spatial rotations of $\mathbb{R}^3$, is isomorphic with the group $SU(2)/\{\pm I\}$.

320/-11
By Theorem 16, we have an isomorphism $S^3/\{\pm 1\} \to SO(\mathbb{R}^3)$. If we use the natural identification $S^3 = SU(2)$ given by $(\lambda, \mu) \in \mathbb{C}^2$, $|\lambda|^2 + |\mu|^2 = 1$, associated to $\begin{pmatrix} \lambda & -\bar{\mu} \\ \mu & \bar{\lambda} \end{pmatrix}$ (as in Cayley's theorem) then we obtain the isomorphism $SU(2)/\{\pm 1\} \to SO(\mathbb{R}^3)$. Note that this is not quite the inverse of the isomorphism given by Cayley's theorem, but would become the inverse if we used the reverse basis k, j, i. (This remark is due to Claus Diem.)

334/-4
Delete the sentence "Using more modern ...a well-defined function on $\mathbb{C}P^1 = \hat{\mathbb{C}}$."

376/7

$$= \frac{(48u^2a - 12ub - c)^3}{64a^2(12u(ac - b^2) - bc)}$$