Sparse Stabilization and Control of Consensus Models

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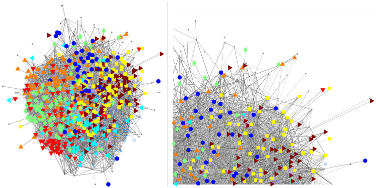
Sparse Control of Large Groups
Rutgers University
March 15, 2013

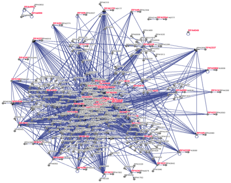
Introduction

High dimensional particle systems arise in many modern applications:

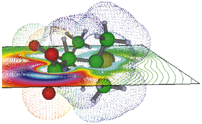
Image halftoning via variational dithering.



Large Facebook "friendship" network



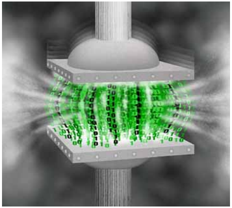
Dynamical data analysis: R. palustris protein-protein interaction network.



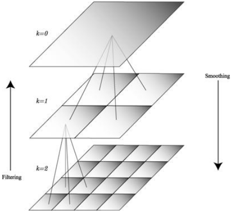
Computational chemistry: molecule simulation.

Relevant techniques

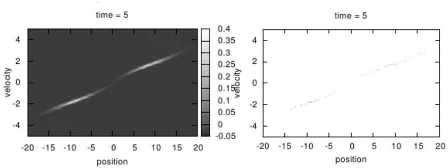
These tasks are addressed by common tools and concepts:



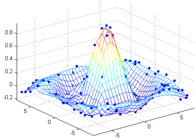
Compression



Multiscale



Mean field limit



High dimensional approximation

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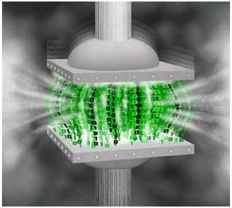

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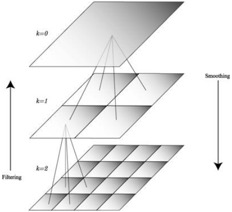

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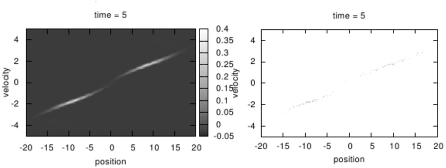

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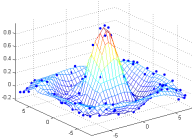


High dimensional approximation

MODELING, SIMULATION, LEARNING, CONTROL

Compression

"–Compression can be mathematically expressed as - numerically - approximating a certain function, sometimes explicitely given or, as more often, only implicitly given as a solution of a certain equation or variational problem, by using the minimal/optimal amount of degrees of freedom.–"

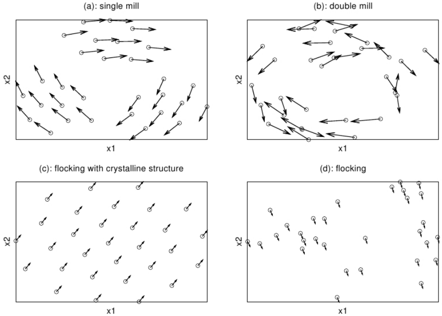
Social dynamics

We consider Dynamical Systems of mutual distances $\mathcal{D}x = (\|x_i - x_j\|)_{ij}$:

$$\dot{x}_i(t) = f_i(\mathcal{D}x(t)) + \sum_{j=1}^{N} f_{ij}(\mathcal{D}x(t)) x_j(t).$$

Several "social forces" encoded in f_i and f_{ij}:

- Repulsion-attraction
- Self-drive
- Noise/uncertainty
- ...



Patterns related to different balance of social forces.

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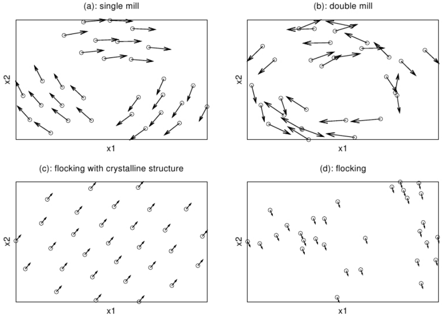

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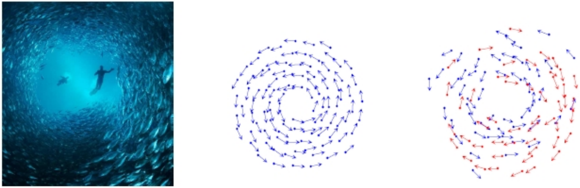
Understanding how superposition of re-iterated binary "social forces" yields global self-organization.

An example inspired by nature



Mills in nature and in our simulations.

J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, *Particle, kinetic, hydrodynamic models of swarming*, within the book "Mathematical modeling of collective behavior in socio-economic and life-sciences", Birkhäuser (Eds. Lorenzo Pareschi, Giovanni Naldi, and Giuseppe Toscani), 2010.

Consensus emergence

The Cucker-Smale model:

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) \in \mathbb{R}^d \end{cases},$$

where $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$ governs the rate of communication.

[1] The *Laplacian* L of A is given by $L = D - A$, with $D = \mathrm{diag}(d_1, \dots, d_N)$ and $d_k = \sum_{j=1}^{N} a_{kj}$

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$$\begin{cases} \dot{x} = v \\ \dot{v} = -L_x v \end{cases}$$

where L_x is the Laplacian of the matrix[1] $(a(\|x_j - x_i\|)/N)_{i,j=1}^N$ and depends on x.

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- Mean-velocity conservation: $\frac{d}{dt}\bar{v}(t) = \frac{1}{N}\sum_{i=1}^N \dot{v}_i(t) = \frac{1}{N^2}\sum_{i=1}^N \sum_{j=1}^N \frac{v_j - v_i}{(1+\|x_j - x_i\|^2)^\beta} \equiv 0.$

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Unconditional consensus emergence

Without loss of generality $\bar{v} = 0$ and $\bar{x}(t) = \bar{x}(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$.

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Theorem (Cucker-Smale, Ha-Tadmor, Carrillo-F.-Rosado-Toscani)

Let $(x(t), v(t)) \in C^1([0, +\infty), \mathbb{R}^{2d\times N})$ be the solution of the Cucker-Smale system. We denote

$$\begin{cases} \mathcal{V}(t) = \max\limits_{i=1,\dots N} \|v_i(t)\|, \quad \mathcal{V}_0 = \mathcal{V}(0), \\ \mathcal{X}(t) = \max\limits_{i=1,\dots N} \|x_i(t) - x_i(0)\|, \quad \mathcal{X}_0 = \mathcal{X}(0). \end{cases}$$

If $0 < \beta < \frac{1}{2}$ then

$$\mathcal{V}(t) \leq \mathcal{V}_0 e^{-a(2\bar{\mathcal{X}})t} \to 0, t \to \infty, \quad \exists \bar{\mathcal{X}} > 0.$$

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Actually one has $\mathcal{V}(t) \to 0$ also for $\beta = 1/2$.

Conditional consensus emergence for a generic communication rate $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

and

$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

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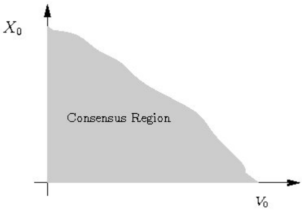

Theorem (Ha-Ha-Kim)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\sqrt{N}\int_{\sqrt{NX_0}}^{\infty} a(\sqrt{2}r)dr > \sqrt{V_0}\,.$$

Then the solution with initial data (x_0, v_0) tends to consensus.

Non-consensus events

If $\beta > 1/2$ then for $a(\cdot) = a_\beta(\cdot)$ the consensus condition is not satisfied by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$.

There are counterexamples to consensus emergence (Caponigro-F.-Piccoli-Trélat).

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Consider $\beta = 1$ and $x(t) = x_1(t) - x_2(t)$, $v(t) = v_1(t) - v_2(t)$ relative pos. and vel. of two agents on the line:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\dfrac{v}{1+x^2} \end{cases}$$

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$.

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By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$

Hence, if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$

Self-organization Vs organization by intervention

We introduce the notion of *organization via intervention*.

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Admissible controls: measurable functions $u = (u_1, \dots, u_N) : [0, +\infty) \to \mathbb{R}^N$ such that $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant M:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$

for $i = 1, \dots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$.

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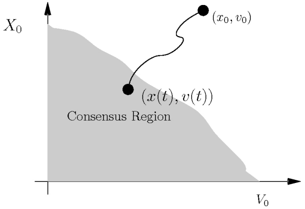

Our aim is then to find admissible controls steering the system to the consensus region.

Total control

Proposition (Caponigro-F.-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exist* $T > 0$ *and* $u : [0, T] \to (\mathbb{R}^d)^N$*, with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated solution tends to consensus.*

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Proof.

Consider a solution of system with initial data (x_0, v_0) associated with a feedback control $u = -\alpha(v - \bar{v})$, with $0 < \alpha \leq M/(N\sqrt{B(v_0, v_0)})$.

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\end{aligned}$$

Therefore $V(t) \leq e^{-2\alpha t} V(0)$ and $V(t)$ tends to 0 exponentially fast as $t \to \infty$. Moreover $\sum_{i=1}^N \|u_i\| \leq M$. $\square$

More economical choices?

We wish to make

$$\begin{aligned}\frac{d}{dt}V(t) &= \frac{d}{dt}B(v(t), v(t)) \\ &= -2B(L_x v(t), v(t)) + 2B(u(t), v(t))\end{aligned}$$

the smallest possible and use the minimal amount of intervention:

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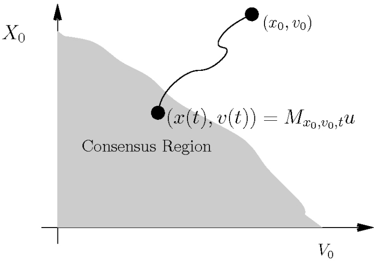
the smallest possible and use the minimal amount of intervention: minimize $B(u(t), v(t))$ with additional sparsity constraints.

Linear dynamical systems



Were the dependence of the trajectory $(x(t), v(t))$ at the time t on the control $u := \{u(s) : s \in [0, t]\}$ linear

$$(x(t), v(t)) = M_{x_0,v_0,t}u,$$

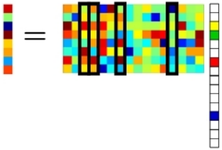
then a rather general theory of linear compression would apply.

Compressed sensing enters the picture

Theorem

Given a matrix $M \in \mathbb{R}^{k \times d}$, $k \ll d$, with incoherency properties $M^T M \approx I$, and

$$x = Mu + e \in \mathbb{R}^k, \quad \|e\| \leq \varepsilon,$$



the vector $\hat{u}$ computed by

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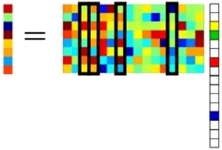

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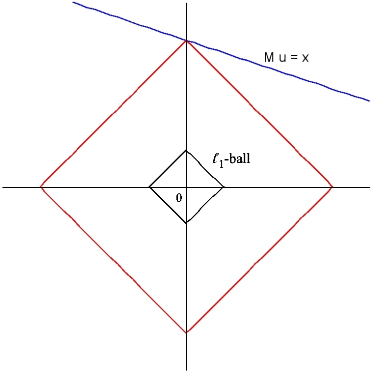
$$\hat{u} = \arg \min_{\|Mv - x\| \leq \varepsilon} \|v\|_{\ell_1} := \sum_{i=1}^{d} |v_i|, \quad (\ell_1)$$

has the approximation property

$$\|\hat{u} - u\| \leq C_1 \frac{\sigma_K(u)_1}{\sqrt{K}} + C_2 \varepsilon,$$

where $\sigma_K(v)_1 = \|v - v_{[K]}\|_{\ell_1}$, best-$K$-term approx. error. If u is sparse then $\sigma_K(u)_1 = 0$.

Geometrical interpretation



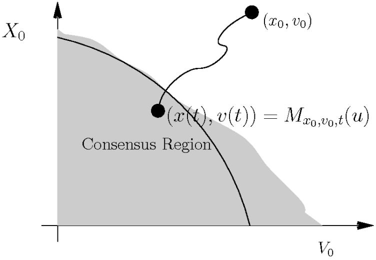
Minimal ℓ_1-norm solution.

Assume $d = 2$ and $k = 1$. Hence $\mathcal{F}(x) = \{z : Mu = x\}$ is just a line in $\mathbb{R}^2$. If we exclude that there exists $\eta \in \ker M$ such that $|\eta_1| = |\eta_2|$ or, equivalently,

$$|\eta_i| < |\eta_{\{1,2\}\setminus\{i\}}|$$

for all $\eta \in \ker M$ and for one $i = 1, 2$, then the solution to (ℓ_1) is a sparse solution.

Nonlinear dynamical systems



What to do if the dependence of the trajectory $(x(t), v(t))$ at the time t on the control $u := \{u(s) : s \in [0, t]\}$ is nonlinear

$$(x(t), v(t)) = M_{x_0,v_0,t}(u)?$$

Can we again use ℓ_1-minimization as a criterion for sparsifying the control?

Greedy sparse control

Theorem (Caponigro-F.-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exist* $T > 0$ *and a sparse control* $u : [0, T] \to (\mathbb{R}^d)^N$*, with* $\sum_{i=1}^N \|u_i(t)\| \le M$ *for every* $t \in [0, T]$ *such that the associated AC solution tends to consensus.*

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$$\min B(v, u) + \frac{\gamma(x)}{N} \sum_{i=1}^{N} \|u_i\| \quad \textit{subject to} \ \sum_{i=1}^{N} \|u_i\| \leq M \, ,$$

where

$$\gamma(x) = \sqrt{N} \int_{\sqrt{NB(x,x)}}^{\infty} a(\sqrt{2}r)dr.$$

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The control $u(t)$ *is a sparse vector with at most one nonzero coordinate, i.e.,* $u_i(t) \neq 0$ *for a unique* $i \in \{1, \ldots, N\}$ *and* $u_j(t) = 0$ *for* $j \neq i$ *for almost every* $t \in [0, T]$*.*

Explicit sparse control

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Otherwise there exists a "best index" $i \in \{1, \ldots, N\}$ such that

$$\|v_{\perp_i}\| > \gamma(x) \quad \text{and} \quad \|v_{\perp_i}\| \geq \|v_{\perp_j}\| \quad \text{for every } j = 1, \ldots, N.$$

Therefore we can choose $i \in \{1, \ldots, N\}$ satisfying it, and a control law

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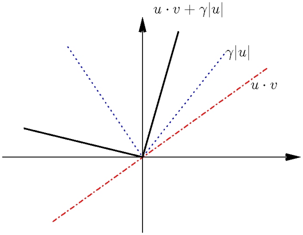
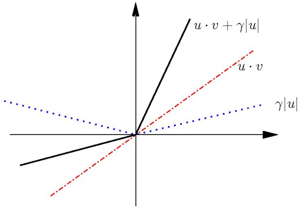
Hence the control acts on the most "stubborn". We may call this control the "shepherd dog strategy". This choice of the control makes $V(t) = B(v(t), v(t))$ vanishing in finite time, hence there exists T such that $B(v(t), v(t)) \leq \gamma(x)^2$, $t \geq T$.

Geometrical interpretation in the scalar case



For $|v| \leq \gamma$ the minimal solution $u \in [-M, M]$ is zero.



For $|v| > \gamma$ the minimal solution $u \in [-M, M]$ is $|u| = M$.

Instantaneous optimality of the greedy strategy

Consider generic control u (solution of the variation problem) of components

$$u_i(x,v) = \begin{cases} 0 & \text{if } v_{\perp_i} = 0 \\ -\alpha_i \dfrac{v_{\perp_i}}{\|v_{\perp_i}\|} & \text{if } v_{\perp_i} \neq 0 \end{cases}$$

where $\alpha_i \geq 0$ such that $\sum_{i=1}^N \alpha_i \leq M$.

Theorem (Caponigro-F.-Piccoli-Trélat)

The 1-sparse control is the minimizer of

$$\mathcal{R}(t,u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

among all the control of the previous form.

A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.

Time-sparse control: sampling-and-hold

Definition (Sampling solution)

Let $U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \mapsto f(x,u)$ be continuous and locally Lipschitz in x uniformly on compact subset of $\mathbb{R}^n \times U$. Given a feedback $u : \mathbb{R}^n \to U$, $\tau > 0$, and $x_0 \in \mathbb{R}^n$ we define the *sampling solution* of

$$\dot{x} = f(x, u(x)), \quad x(0) = x_0$$

as the continuous (actually piecewise C^1) function $x : [0, T] \to \mathbb{R}^n$ solving recursively for $k \geq 0$

$$\dot{x}(t) = f(x(t), u(x(k\tau))), \quad t \in [k\tau, (k+1)\tau]$$

using as initial value $x(k\tau)$ the endpoint of the solution on the preceding interval and starting with $x(0) = x_0$.

Time-sparse control: sampling-and-hold

We define $u = u(x, v)$ via the following criterion. If $B(v, v) \geq \gamma(B(x, x))^2$ then let $i \in \{1, \dots, N\}$ be the smallest index such that

$$\|v_{\perp_i}\| \geq \gamma(B(x, x)) \quad \text{and} \quad \|v_{\perp_i}\| \geq \|v_{\perp_j}\| \quad \text{for every } j = 1, \dots, N.$$

and set

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Theorem (Caponigro-F.-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *consider the control* u *given above. There exists* $\tau_0 = \tau_0(M, N, x_0, v_0) > 0$ *small enough, such that for all* $0 < \tau \leq \tau_0$ *the sampling solution associated with the control* u*, the sampling time* τ*, and initial datum* (x_0, v_0) *tends to consensus in time* $T \leq \frac{N}{2M}(\sqrt{V(0)} - \gamma(\bar{X}))$, $\bar{X} = 2B(x_0, x_0) + \frac{2N^4}{M^2} B(v_0, v_0)^2$.

Complexty of consensus

Given a stuitable compact set $\mathcal{K} \subset (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ of initial conditions, control bound $M > 0$, number of agents $N \in \mathbb{N}$, and arrival time $T > 0$, we define

$$\begin{aligned} n &:= n(M, N, \mathcal{K}, T) \\ &= \inf \sup_{(x_0, v_0) \in \mathcal{K}} \left\{ \sum_{\ell=0}^{k-1} \# \operatorname{supp}(u(t_\ell)) : \right. \\ &\qquad\qquad \left. (x(T; u), v(T, u)) \text{ is in the consensus region } \right\} \end{aligned}$$

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$$\bar{T} = \bar{T}(M, N, x_0, v_0) = \frac{N}{2M}(\sqrt{V(0)} - \gamma(\bar{X})),$$

$$n(M, N, \mathcal{K}, T) \leq \begin{cases} \infty, & T < \bar{T} \\ \frac{\sup_{(x_0,v_0)\in\mathcal{K}} \bar{T}(M,N,x_0,v_0)}{\inf_{(x_0,v_0)\in\mathcal{K}} \tau_0(M,N,x_0,v_0)}, & T \geq \bar{T} \end{cases}$$

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Presently lower bounds are not yet given.

Sparse controllability near the consensus manifold

Consensus manifold is $(\mathbb{R}^d)^N \times \mathcal{V}_f$, where
$\mathcal{V}_f = \{(v_1, \dots, v_N) \in (\mathbb{R}^d)^N \mid v_1 = \cdots = v_N \in \mathbb{R}^d\}$.

Theorem (Caponigro-F.-Piccoli-Trélat)

For every $M > 0$, for almost every $\tilde{x} \in (\mathbb{R}^d)^N$ and for every $\tilde{v} \in \mathcal{V}_f$, for every time $T > 0$, there exists a neighborhood W of $(\tilde{x}, \tilde{v})$ in $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ such that, for all points (x_0, v_0) and (x_1, v_1) of W, for every index $i \in \{1, \dots, N\}$, there exists an admissible componentwise and time sparse sparse control u, every component of which is zero except the i^{th} (that is, $u_j(t) = 0$ for every $j \neq i$ and every $t \in [0, T]$), steering the control system from (x_0, v_0) to (x_1, v_1) in time T.

Global sparse controllability

Corollary (Caponigro-F.-Piccoli-Trélat)

For every $M > 0$, for every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for almost every $(x_1, v_1) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$, there exist $T > 0$ and an admissible componentwise and time sparse control $u : [0, T] \to (\mathbb{R}^d)^N$, such that the corresponding solution starting at (x_0, v_0) arrives at the consensus point (x_1, v_1) within time T.

Sparse optimal control

The problem is to minimize, for a given $\gamma > 0$

$$\mathcal{J}(u) := \int_0^T \sum_{i=1}^N \left(\left(v_i(t) - \frac{1}{N} \sum_{j=1}^N v_j(t) \right)^2 \quad + \quad \gamma \sum_{i=1}^N \|u_i(t)\| \right) dt,$$

$$\text{s.t.} \ \sum \|u_i\| \leq M$$

where the state is a trajectory of the control system

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$

with initial constraint

$$(x(0), v(0)) = (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N.$$

Beyond a greedy approach: sparse optimal control

Theorem (Caponigro-F.-Piccoli-Trélat)

For every (x_0, v_0) in $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for every $M > 0$, and for every $\gamma > 0$ the optimal control problem has an optimal solution. The optimal control $u(t)$ is "usually" instantaneously a vector with at most one nonzero coordinate.

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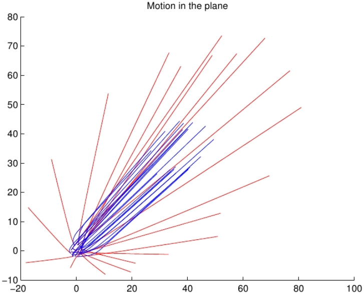
The PMP ensures the existence of $\lambda \geq 0$ and of a nontrivial covector $(p_x, p_v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ satisfying the adjoint equations, for $i = 1, \ldots, N$,

$$\begin{cases} \dot{p}_{x_i} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \dfrac{a(\|x_j - x_i\|)}{\|x_j - x_i\|} \langle x_j - x_i, v_j - v_i \rangle (p_{v_j} - p_{v_i}) \\[2ex] \dot{p}_{v_i} = -p_{x_i} - \dfrac{1}{N} \displaystyle\sum_{j \neq i} a(\|x_j - x_i\|)(p_{v_j} - p_{v_i}) - 2\lambda v_i + \dfrac{2\lambda}{N} \displaystyle\sum_{j=1}^{N} v_j. \end{cases}$$

The application of the PMP leads to minimize

$$\min \sum_{i=1}^{N} \langle p_{v_i}, u_i \rangle + \lambda\gamma \sum_{i=1}^{N} \|u_i\|, \qquad \text{subject to } \sum_{i=1}^{N} \|u_i\| \leq M.$$

Conclusion



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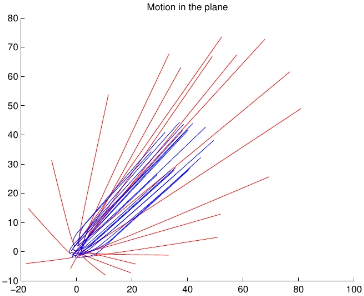

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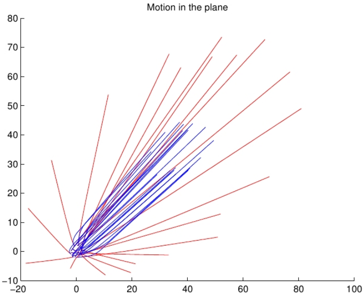

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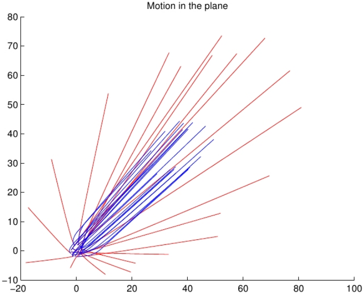

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- In case pattern formation cannot be ensured, we introduced the concept of *organization by external intervention*.
- We proved that the most effective *greedy strategy* to achieve consensus emergence is by instantaneous 1-sparse controls.
- We showed that *maximally sparse* optimal control are also expected when considering ℓ_1-norm constraints.

High-dimensional dynamical systems: the general model

First, some notation:

- $d \in \mathbb{N}$ - dimension (very large!!),
- $N \in \mathbb{N}$ - number of agents, typically $N = d^\alpha$, $\alpha > 0$;
- $x = \{x_1, \ldots, x_N\} \in \mathbb{R}^{N \times d}$, where $x_i \in \mathbb{R}^d$, $i = 1, \ldots, N$,
- $\mathcal{D} : \mathbb{R}^{N \times d} \to \mathbb{R}^{N \times N}$, $\mathcal{D}x = (|x_i - x_j|)_{i,j=1}^N$ is the adjacency matrix of x;
- $f_i : \mathbb{R}^{N \times N} \to \mathbb{R}^d, \quad i = 1, \ldots, N,$;
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We are interested in the

- dimensionality reduction and numerical simulation

of dynamical systems of the type

$$\dot{x}_i(t) = f_i(\mathcal{D}x(t)) + \sum_{j=1}^N f_{ij}(\mathcal{D}x(t))x_j(t), \quad x(0) = x^0 \in \mathbb{R}^{N \times d},$$

describing the dynamics of multiple complex agents, interacting on the basis of their mutual “social” distance.

The application framework

With the development of communication technology and *Internet*, larger and larger groups of people will access

- information (interactive database access, trends in scientific literature and in newspapers ...)
- services (Google, the financial market ...)
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We are facing very difficult challenges due to the "curse of dimensionality", as our individuals are not physical particles and need a large number d of degrees of freedom to be described.

Relevant assumptions

We assume the following Lipschitz and boundedness properties of f_i and f_{ij}, namely

$$\|f_i(a) - f_i(b)\| \le L\|a - b\|_\infty,$$

$$\max_i \sum_j |f_{ij}(a)| \le L',$$

$$\max_i \sum_j |f_{ij}(a) - f_{ij}(b)| \le L''\|a - b\|_\infty,$$

for every $a, b \in \mathbb{R}^{N \times N}$. Here, $\|a - b\|_\infty := \max_{i,j} |a_{ij} - b_{ij}|$.

A classical result

Theorem (Convergence of the Euler scheme)

Assume $f_{ij} = 0$*. Fix* $x^0 \in \mathbb{R}^{N \times d}$ *and let* $x(t)$ *be the unique solution of the ODE system*

$$\dot{x}(t) = f(\mathcal{D}x(t)), \qquad x(0) = x^0,$$

on the interval $[0, T]$*,* $T > 0$*.*

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Fix $h > 0$ *and* $t_n := nh$:

$$\tilde{x}_{n+1} = \tilde{x}_n + hf(\mathcal{D}\tilde{x}_n)\,, \qquad \tilde{x}_0 = \tilde{x}^0\,,$$

for $n = 1, 2, \dots$.

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for $n = 1, 2, \dots$.

Then, we have the estimate for $e_n = \|x(t_n) - \tilde{x}_n\|$,

$$e_n \leq \exp(Lt_n) \left(e_0 + ht_n \frac{\|f(\mathcal{D}\tilde{x}^0)\|}{2} \right) .$$

Exponential complexity reduction in d

The complexity of this algorithm stems from the evaluation of $f(\mathcal{D}x)$ which can be (generically) estimated by

$$\mathcal{O}(d \times N^2).$$

Our first aim is to find an appropriate *model* of appropriate reduction of the dynamical system to $\log(d)$ dimensions and consequently the complexity to

$$\mathcal{O}(\log(d) \times N^2).$$

Dimensionality reduction via Johnson-Lindenstrauss embeddings

Again some notation

- $\varepsilon > 0$ - a distortion parameter from J-L Lemma, see below,
- $n_0 \in \mathbb{N}$ - number of iterations,
- $\mathcal{N} = n_0 N$ - number of iterations times number of agents
- $k = \mathcal{O}(\varepsilon^{-2} \log(\mathcal{N}))$, new lower dimension - see below,
- $M \in \mathbb{R}^{k \times d}$ - randomly generated matrix, see below,
- $\mathcal{D} : \mathbb{R}^{N \times d} \to \mathbb{R}^{N \times N}$, $\mathcal{D}x = (\|x_i - x_j\|)_{i,j=1}^N$ is the adjacency matrix in high-dimension and similarly defined $\mathcal{D}' : \mathbb{R}^{N \times k} \to \mathbb{R}^{N \times N}$, $\mathcal{D}'y = (\|y_i - y_j\|)_{i,j=1}^N$, the one in low-dimension.

Dimensionality reduction via Johnson-Lindenstrauss embeddings

Lemma (Johnson and Lindenstrauss)

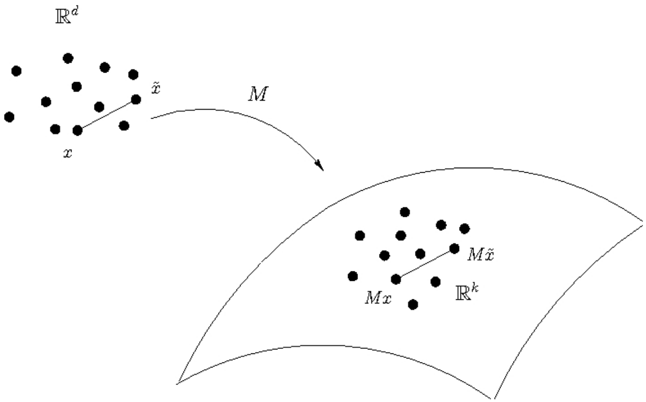
Let $\mathcal{P}$ be an arbitrary set of $\mathcal{N}$ points in $\mathbb{R}^d$. Given $\varepsilon > 0$, there exists

$$k_0 = \mathcal{O}(\varepsilon^{-2} \log(\mathcal{N})),$$

such that for all integers $k \geq k_0$, there exists a $k \times d$ random matrix M for which with high probability, for all $x, \tilde{x} \in \mathcal{P}$

$$(1 - \varepsilon)\|x - \tilde{x}\|^2 \leq \|Mx - M\tilde{x}\|^2 \leq (1 + \varepsilon)\|x - \tilde{x}\|^2.$$

Dimensionality reduction via Johnson-Lindenstrauss embeddings



Restricted Isometry Property

Definition

A $k \times d$ matrix $\tilde{M}$ is said to have the Restricted Isometry Property of order $K \leq d$ and level $\delta \in (0, 1)$ if

$$(1 - \delta)\|x\|^2 \leq \|\tilde{M}x\|^2 \leq (1 + \delta)\|x\|^2$$

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Theorem (Krahmer, Ward)

Fix $\eta > 0$ and $\varepsilon > 0$, and consider a finite set $\mathcal{P} \subset \mathbb{R}^d$ of cardinality $|\mathcal{P}| = \mathcal{N}$. Set $K \geq 40 \log \frac{4\mathcal{N}}{\eta}$, and suppose that the $k \times d$ matrix $\tilde{M}$ satisfies the Restricted Isometry Property of order K and level $\delta \leq \varepsilon/4$. Let $\xi \in \mathbb{R}^d$ be a Rademacher sequence, i.e., uniformly distributed on $\{-1, 1\}^d$. Then with probability exceeding $1 - \eta$,

$$(1 - \varepsilon)\|x\|^2 \leq \|Mx\|^2 \leq (1 + \varepsilon)\|x\|^2.$$

uniformly for all $x \in \mathcal{P}$, where $M := \tilde{M}\operatorname{diag}(\xi)$.

Some stochastic constructions of RIP$\rightarrow$JL matrices

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More matrices: by random sampling of bounded basis (e.g., Fourier basis) or random circulant matrices.

Projection of the dynamical system

We consider the system of ordinary differential equations in the fixed form with the initial condition

$$x_i(0) = x_i^0 \,, \qquad i = 1, \ldots, N \,.$$

The Euler method for this system is given by this initial condition and

$$x_i^{n+1} := x_i^n + h \left[f_i(\mathcal{D}x^n) + \sum_{j=1}^{N} f_{ij}(\mathcal{D}x^n) x_j^n \right] , \quad n = 0, \ldots, n_0 - 1.$$

where $h > 0$ is the time step and $n_0 := T/h$ is the number of iterations.

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If $M \in \mathbb{R}^{k \times d}$ is a matrix, we may consider the associated Euler method in $\mathbb{R}^k$, namely

$$y_i^0 := M x_i^0,$$

$$y_i^{n+1} := y_i^n + h \left[M f_i(\mathcal{D}' y^n) + \sum_{j=1}^{N} f_{ij}(\mathcal{D}' y^n) y_j^n \right] , \quad n = 0, \ldots, n_0 - 1.$$

A first surprising result

Theorem (Fornasier, Haškovec, Vybíral)

Given a matrix $M \in \mathbb{R}^{k\times d}$ such that

$$\left\| Mf_i(\mathcal{D}' y^n) - Mf_i(\mathcal{D} x^n) \right\| \le (1+\varepsilon) \left\| f_i(\mathcal{D}' y^n) - f_i(\mathcal{D} x^n) \right\|,$$
$$\|Mx_j^n\| \le (1+\varepsilon)\|x_j^n\|,$$
$$(1-\varepsilon)\|x_i^n - x_j^n\| \le \|Mx_i^n - Mx_j^n\| \le (1+\varepsilon)\|x_i^n - x_j^n\|$$

for all $i, j = 1, \dots, N$ and all $n = 0, \dots, n_0$. Let us also assume, that $\alpha \ge \max_j \|x_j^n\|$ for all $n = 0, \dots, n_0$, $j = 1, \dots, N$. Let

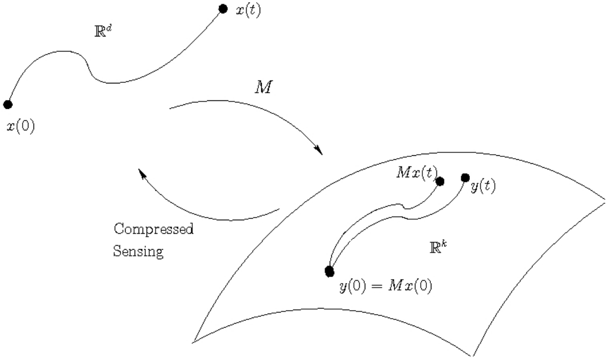
$$e_i^n := \|y_i^n - Mx_i^n\|, \; i = 1, \dots, N \text{ and } n = 0, \dots, n_0$$

and put $\mathcal{E}^n := \max_i e_i^n$. Then

$$\mathcal{E}^n \le \varepsilon hnB \exp(hnA),$$

where $A := L' + 2(1+\varepsilon)(L + \alpha L'')$ and $B := 2\alpha(1+\varepsilon)(L+\alpha L'')$.

Visual explanation



A continuous Johnson-Lindenstrauss Lemma

Theorem (Fornasier, Haškovec, Vybíral)

Let $\varphi : [0,1] \to \mathbb{R}^d$ be a $\mathcal{C}^1$ curve. Let $0 < \varepsilon < \varepsilon' < 1$,

$$\gamma := \max_{\xi \in [0,1]} \frac{\|\dot{\varphi}(\xi)\|}{\|\varphi(\xi)\|} < \infty \quad \textit{and} \quad \mathcal{N} \geq (\sqrt{d}+1) \cdot \frac{\gamma}{\varepsilon' - \varepsilon}.$$

Let k be such a dimension, that a randomly chosen (and properly normalized) projector M satisfies the statement of the Johnson-Lindenstrauss Lemma with ε, d, k and $\mathcal{N}$ arbitrary points. Then

$$(1-\varepsilon')\|\varphi(t)\| \leq \|M\varphi(t)\| \leq (1+\varepsilon')\|\varphi(t)\|, \qquad t \in [0,1]$$

holds with the same probability.

A continuous Johnson-Lindenstrauss Lemma

The condition

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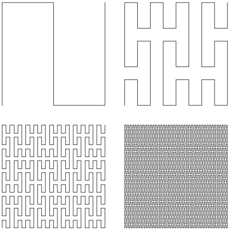
is necessary.y



Peano's space-filling curve

By lifting a suitable parametrization a Peano's space-filling curve on the unit sphere $\mathbb{S}^{d-1}$, one generates a curve with infinite speed (i.e., the condition does not hold), and at the same time it generates any possible vector including those in the kernel of M, hence

$$(1-\varepsilon')\|\varphi(t)\| \leq \|M\varphi(t)\|$$

cannot hold!

Projecting the continuous system

Theorem (Fornasier, Haškovec, Vybíral)

Let $x(t) \in \mathbb{R}^{d \times N}$, $t \in [0, T]$, *be the solution of the given ODE system, such that* $\max_{t \in [0,T]} \max_{i,j} \|x_i(t) - x_j(t)\| \leq \alpha$. *Let us fix* $k \in \mathbb{N}$, $k \leq d$, *and a matrix* $M \in \mathbb{R}^{k \times d}$ *such that*

$$(1 - \varepsilon)\|x_i(t) - x_j(t)\| \leq \|Mx_i(t) - Mx_j(t)\| \leq (1 + \varepsilon)\|x_i(t) - x_j(t)\|,$$

for all $t \in [0, T]$ *and* $i, j = 1, \ldots, N$. *Let* $y(t) \in \mathbb{R}^{k \times N}$, $t \in [0, T]$ *be the solution of the projected (continuous) system such that for a suitable* $\beta > 0$, $\max_{t \in [0,T]} \max_i \|y_i(t)\| \leq \beta$. *Let us define the columnwise* ℓ_2*-error* $e_i(t) := \|y_i(t) - Mx_i(t)\|$ *for* $i = 1, \ldots, N$ *and*

$$\mathcal{E}(t) := \max_{i=1,\ldots,N} e_i(t).$$

Then we have the estimate

$$\mathcal{E}(t) \leq \varepsilon \alpha t (L\|M\| + L''\beta) \exp\left[(2L\|M\| + 2\beta L'' + L')t\right].$$

Verifying the crucial condition

According to our continuous Johnson-Lindenstrauss Lemma

$$(1-\varepsilon)\|x_i(t)-x_j(t)\| \leq \|Mx_i(t)-Mx_j(t)\| \leq (1+\varepsilon)\|x_i(t)-x_j(t)\|\,,$$

for all $t \in [0, T]$ and i, $j = 1, \ldots, N$, is verified if the necessary condition

$$\sup_{t\in[0,T]} \max_{i,j} \frac{\|\dot{x}_i(t)-\dot{x}_j(t)\|}{\|x_i(t)-x_j(t)\|} \leq \gamma < \infty\,,$$

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holds. It is, for instance, trivially satisfied when the right hand sides f_i, f_{ij} have the following Lipschitz continuity:

$$\begin{aligned} \|f_i(\mathcal{D}x)-f_j(\mathcal{D}x)\| &\leq L'''\|x_i-x_j\| \qquad \text{for all } i,j=1,\ldots,N, \\ \|f_{i\ell}(\mathcal{D}x)-f_{j\ell}(\mathcal{D}x)\| &\leq L''''\|x_i-x_j\| \qquad \text{for all } i,j,\ell=1,\ldots,N. \end{aligned}$$

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We will show relevant examples below for which the condition is verified.

Optimal information recovery?

We would like to address the following two fundamental questions:

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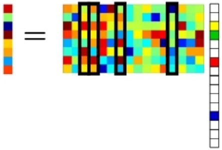
The first question was implicitly addressed already in the 70's by Kashin and later by Garnaev and Gluskin, as one can put in relationship the optimal recovery from (random) linear projections with Gelfand width of ℓ_p-balls. It was only with the development of the theory of *compressed sensing* that an answer to the second question was provided, showing that ℓ_1-minimization actually performs an *optimal recovery* of vectors in high dimension from random linear projections to low dimension.

Compressed sensing enters the picture

Theorem

Given a matrix $M \in \mathbb{R}^{k\times d}$ with the RIP of order $2K$ and level $\delta < 0.4$, and



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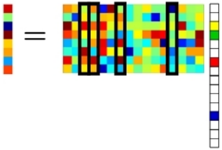

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The vector $\hat{x}$ computed by $\hat{x} = \arg\min_{\|Mz-y\|\leq\varepsilon} \|z\|_1 := \sum_{i=1}^d |z_i|$, has the approximation property

$$\|\hat{x} - x\| \leq C_1 \frac{\sigma_K(x)_1}{\sqrt{K}} + C_2\varepsilon,$$

where $\sigma_K(z)_1 = \|z - z_{[K]}\|_1$, best-$K$-term approx. error.

A second surprising algorithmic result

As a consequence of this theorem, by projecting and simulating in parallel the dynamical system d_k-times, $d_k \leq \frac{d}{k}$ in lower dimension

$$\dot{y}_i^\ell = M^\ell f_i(\mathcal{D}' y^\ell) + \sum_{j=1}^{N} f_{ij}(\mathcal{D}' y^\ell) y_j^\ell \,, \qquad y_i^\ell(0) = M^\ell x_i^0 \,, \quad j = 1, \ldots, d_k,$$

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we can assemble the following system

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Therefore we can compute $\hat{x}_i$ such that

$$\|\hat{x}_i - x_i\| \leq C_1 \frac{\sigma_{K'}(x_i)_1}{\sqrt{K'}} + C_2 \varepsilon,$$

with $K' = \mathcal{O}\left(\frac{d_k k}{1 + \log(d/(d_k k))}\right)$.

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with $K' = \mathcal{O}\left(\frac{d_k k}{1 + \log(d/(d_k k))}\right)$. The computation of $\hat{x}_i$ can be parallelized!

M. Fornasier, *Domain decomposition methods for linear inverse problems with sparsity constraints*, Inverse Problems, Vol. 23, 2007, pp. 2505-2526.

Interesting examples

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$$\dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i).$$

The function $g : \mathbb{R} \to \mathbb{R}$ is given by $a(t) = \frac{G}{(1+t^2)^{\beta}}$, $t > 0$ and bounded by $a(0) = G > 0$.

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- the D'Orsogna-Chuang-Bertozzi-Chayes model, which is given by

$$\dot{x}_i = v_i \in \mathbb{R}^d,$$
$$\dot{v}_i = (a - b\|v_i\|^2)v_i - \frac{1}{N}\sum_{j\neq i} \nabla U(\|x_i - x_j\|),$$

  where a and b are positive constants and $U : \mathbb{R} \to \mathbb{R}$ is a smooth potential.

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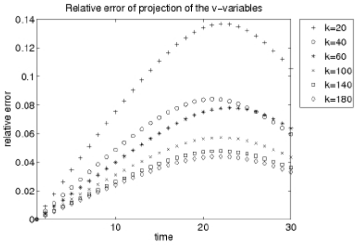
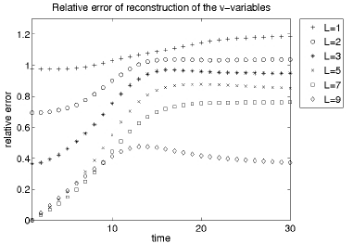
In this case, though, the matrix M should be better a partial orthogonal random matrix (for instance a random partial Fourier matrix), as $MB_i(t)$, $i = 1, \ldots, N$ are mutually independent k-dimensional Brownian motions!

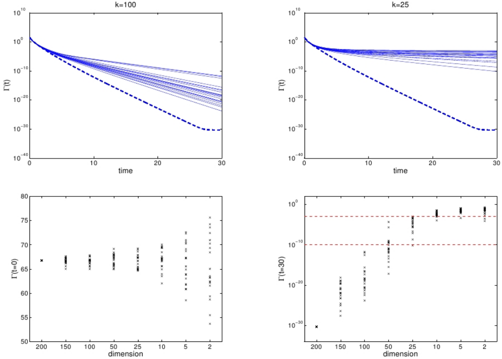
Numerical results





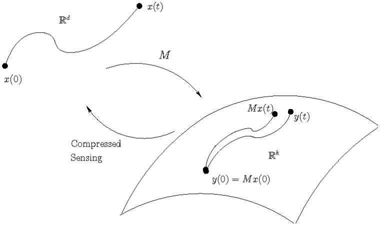
Numerical results showing the time evolution of the relative error of projection (left panel) and relative error of recovery via ℓ_1-minimization (right panel) of the v-variables for the Cucker-Smale model.

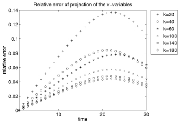
Numerical results: stability of consensus after random projection

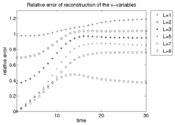


Numerical results for $\beta = 1.62$: First row shows the evolution of $\Gamma(t) = V(t)$ of the CS-system projected to dimension $k = 100$ (left) and $k = 25$ (right) in the twenty realizations, compared to the original system (bold dashed line). Second row shows the initial values $V(t = 0)$ and final values $V(t = 30)$ in all the performed simulations.

Conclusion

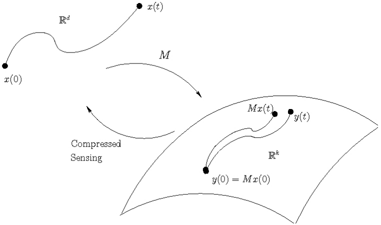


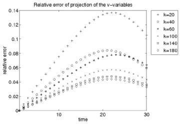


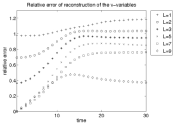


- We defined a general class of dynamical systems modeling social interactions

Conclusion







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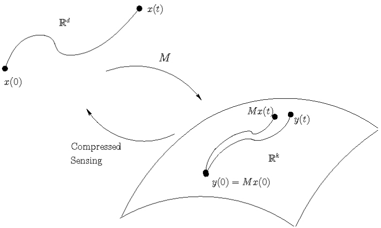

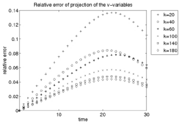

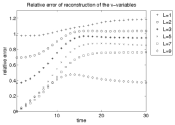

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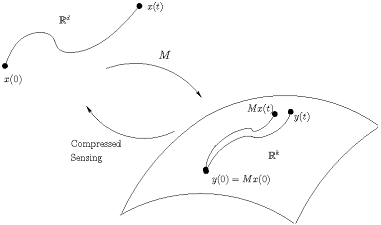

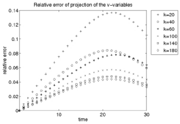

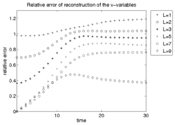

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A few info

- **WWW:** http://www-m15.ma.tum.de/

- **References**:

  - M. Caponigro, M. Fornasier, B. Piccoli, and E. Trélat, *Sparse stabilization and optimal control of the Cucker-Smale model*, submitted to SIAM Review, 2012.
  - J. A. Carrillo, M. Fornasier, J. Rosado, and G. Toscani, *Asymptotic flocking dynamics for the kinetic Cucker-Smale model*, SIAM. J. Math. Anal., Vol. 42, no. 1, 2010, pp. 218-236.
  - M. Fornasier, J. Haškovec and J. Vybíral, *Particle systems and kinetic equations modeling interacting agents in high dimension*, to appear in Multiscale Modeling and Simulation, 2012.
  - M. Fornasier and F. Solombrino, *Mean field optimal control*, in preparation