Fractional Calculus and Some Problems

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Background of fractional derivatives

- Its origins go back more to the year 1695 and the conversation of L'Hopital and Leibniz (throught the letters):
  Leibniz: **Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?**
  De L'Hospital: **What if n=1/2?**
  Leibniz: **It will lead to a paradox, from which one day useful consequences will be drawn.**

- After that, many famous mathematicians, like J. Fourier, N. H. Abel, J. Liouville, B. Riemann and others, contributed to the development of the Fractional Calculus.
- The theory of derivatives and integrals of arbitrary order took more or less finished form by the end of the XIX century.
- Fractional differentiation may be introduced in several different ways – fractional derivatives of Riemann-Liouville; Grünwald-Letnikov; Caputo; Miller-Ross.

- For three centuries the theory of fractional derivatives was developed as a pure theoretical field of mathematics, useful only for mathematicians.
- In the last few decades, fractional differentiation turned out to be very useful in various fields: physics (classic and quantum mechanics, thermodynamics, etc.), chemistry, biology, economics, engineering, signal and image processing, and control theory.

The web site of Podlubny:

http://people.tuke.sk/igor.podlubny/fc_resources.html

How to get fractional derivatives of exponential function

We know already:

$$D^1 e^{\lambda x} = \lambda e^{\lambda x}\,, \quad D^2 e^{\lambda x} = \lambda^2 e^{\lambda x}\,, \ldots \quad D^n e^{\lambda x} = \lambda^n e^{\lambda x}\,,$$

when n is an integer. Why not to replace n by $1/2$ and write

$$D^{1/2} e^{\lambda x} = \lambda^{1/2} e^{\lambda x} \ ?$$

Why not to go further and put $\sqrt{2}$ instead of n? Let us write

$$D^{\alpha} e^{\lambda x} = \lambda^{\alpha} e^{\lambda x} \qquad \text{(exp)}$$

for any value of α, integer, rational, irrational, or complex.

We naturally want

$$e^{\lambda x} = D^1\left(D^{-1}\left(e^{\lambda x}\right)\right).$$

Since $e^{\lambda x} = D^1\left(\frac{1}{\lambda}e^{\lambda x}\right)$, we have

$$D^{-1}\left(e^{\lambda x}\right) = \frac{1}{\lambda}e^{\lambda x} = \int e^{\lambda x}dx.$$

Similarly,

$$D^{-2}\left(e^{\lambda x}\right) = \int\int e^{\lambda x}dxdx.$$

So it is reasonable to interpret D^α when α is a negative integer $-n$ as the nth iterated integral.

D^α represents a derivative if α is a positive real number, and an integral if α is a negative real number.

Examples

- Let $f(x) = e^{2x}$, $0 < \alpha < 1$. Then $D^{\alpha}(e^{2x}) = 2^{\alpha} e^{2x}$. We see that

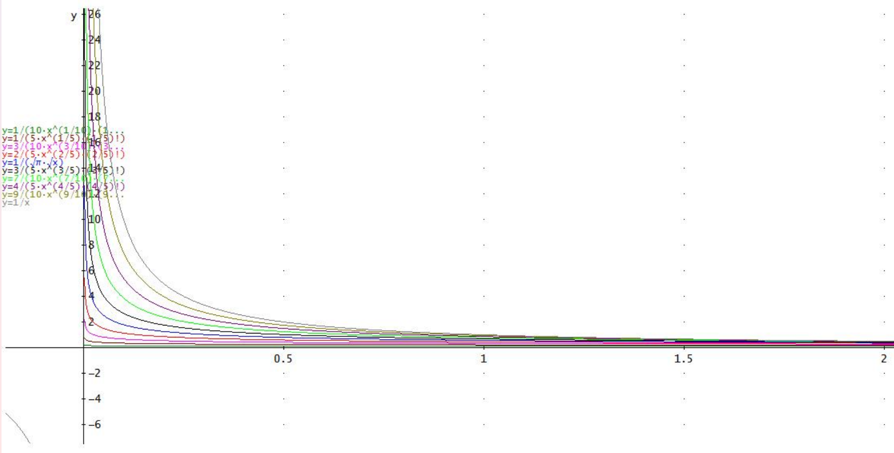
$$D^0 (f(x)) = f(x) \leq D^{\alpha} (f(x)) \leq D^1 (f(x))$$

- Let $f(x) = e^{\frac{1}{3}x}$, $0 < \alpha < 1$. Then $D^{\alpha}(\frac{1}{3}x) = \left(\frac{1}{3}\right)^{\alpha} e^{\frac{1}{3}x}$. We see that

$$D^1 (f(x)) \leq D^{\alpha} (f(x)) \leq D^0 (f(x))$$

$$\frac{d^{\alpha}}{dx^{\alpha}}1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$$

where we replace $n! = 1 \cdot 2 \cdot 3 \cdot \ldots (n-1)n$ by more general Gamma function $\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt\,, \quad t > 0.$



The fractional derivative of a constant is not equal to zero for $0 < \alpha < 1$. However, there is an agreement with the classical calculus:

$$\alpha \to 1 \Rightarrow \Gamma(1-\alpha) \to +\infty \Rightarrow D^1(1) = 0$$

First trial to define fractional derivative

We can extend the idea of a fractional derivative to a large number of functions. Given any function that can be expanded in a Taylor series in powers of x,

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n ,$$

and assuming we can differentiate term by term we get

$$D^\alpha f(x) = \sum_{n=0}^{+\infty} a_n D^\alpha x^n = \sum_{n=0}^{+\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} . \qquad \text{(Liouville)}$$

Do we get a contradiction?

We wrote

$$D^\alpha e^x = e^x \,. \tag{1}$$

Let us now compare this with our definition (Liouville) to see if they agree. From the Taylor series,

$$e^x = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n$$

and our definition (Liouville) gives

$$D^\alpha e^x = \sum_{n=0}^{+\infty} \frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)} \tag{2}$$

- The expressions (1) and (2) do not match!
- We have discovered a contradiction that historically has caused great problems (controversy between 1835 and 1850).

Iterated Integrals

We could write $D^{-1}f(x) = \int f(x)dx$, but the right-hand side is indefinite. Instead, we will write

$$D^{-1}f(x) = \int_0^x f(t)dt\,.$$

The second integral will then be

$$D^{-2}f(x) = \int_0^x \int_0^{t_2} f(t_1)dt_1 dt_2\,.$$

The region of integration is a triangle, and if we interchange the order of integration, we can write the iterated integral as a single integral (method of Dirichlet, 1908):

$$D^{-2}f(x) = \int_0^x \int_{t_1}^x f(t_1)dt_2 dt_1 = \int_0^x f(t_1) \int_{t_1}^x dt_2 dt_1 = \int_0^x f(t_1)(x-t_1)dt_1$$

Dirichlet: writing iterated integrals as a single integral

Using the same procedure we can show that

$$D^{-3}f(x) = \frac{1}{2}\int_0^x f(t)(x-t)^2 dt\,,$$

$$D^{-4}f(x) = \frac{1}{2\cdot 3}\int_0^x f(t)(x-t)^3 dt$$

and, in general,

$$D^{-n}f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1} dt\,.$$

Now, as we have previously done, let us replace the n with an arbitrary α and the factorial with the gamma function...

Second trial to define fractional derivative

... replacing n by α, factorial by gamma function:

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1}dt\,. \qquad \text{(Riemann)}$$

Remark

- As $t \to x$, $x - t \to 0$. The integral diverges for every $\alpha \leq 0$; when $0 < \alpha < 1$ the improper integral converges.
- Since (Riemann) converges only for positive α, it is truly a *fractional integral*.

- The choice of zero for the lower limit was arbitrary. It could have been a.
- In general, people who work in the field use the notation ${}_aD_x^{\alpha} f(x)$ indicating limits of integration going from a to x. With this notation we have:

Riemann-Liouville integral

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \, . $$

- We are not surprised that fractional integrals involve limits, because integrals involve limits.
- Since ordinary derivatives do not involve limits of integration, we were not expecting fractional derivatives to involve such limits!
- We think of derivatives as local properties of functions.
- The fractional derivative symbol D^{α} incorporates both derivatives (positive α) and integrals (negative α).

Integrals are between limits, it turns out that fractional derivatives are between limits too.

What was the reason of contradiction?

- The reason for the contradiction is that we used two different limits of integration!
- We have

$$ {}_aD_x^{-1}e^{\lambda x} = \int_a^x e^{\lambda t}dt = \frac{1}{\lambda}e^{\lambda x} - \frac{1}{\lambda}e^{\lambda a} $$

- We get the form we want when $\frac{1}{\lambda}e^{\lambda a} = 0$, i.e., when $\lambda a = -\infty$.
- If λ is positive, then $a = -\infty$:

Weyl fractional derivative

$$ {}_{-\infty}D_x^{\alpha}e^{\lambda x} = \lambda^{\alpha}e^{\lambda x} $$

We have

$$ {}_aD_x^{-1}x^p = \int_a^x t^p dt = \frac{x^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}. $$

Again, we want $\frac{a^{p+1}}{p+1} = 0$. This will be the case when $a = 0$. We conclude that

$$ {}_0D_x^{\alpha}x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}. $$

Explanation

First we calculated ${}_{-\infty}D_x^{\alpha}e^{\lambda x}$; the second time we calculated ${}_0D_x^{\alpha}e^{\lambda x}$.

Riemann-Liouville: Fractional Integrals and Derivatives

- Fractional integral of f of order α:

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \,, \quad \alpha > 0 \,. $$

- Let $\alpha > 0$ and let m be the smallest integer exceeding α. Then we define the *fractional derivative of f* of order α as

$$ \begin{aligned} {}_aD_x^{\alpha} f(x) &= \frac{d^m}{dx^m} \left[{}_aD_x^{-(m-\alpha)} f(x) \right] \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x f(t)(x-t)^{m-\alpha-1} dt \,. \end{aligned} $$

FD of integer order is the ordinary derivative

If $\alpha = n \in \mathbb{N}$, then $m = n + 1$ and ${}_aD_x^n f(x) = \frac{d^n}{dx^n} f(x)$.

Grünwald-Letnikov: other approach fractional derivative

We know the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

as well as we can get:

$$f''(x) = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

and more general:

$$D^n f(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - jh)$$

Grünwald-Letnikov: other approach fractional derivative

How to generalized the latter?? By a generalization of binomial:

$$\binom{\alpha}{j} = \begin{cases} \frac{n(n-1)\cdots(n-j+1)}{j!} & j > 0 \\ 1 & j = 0 \end{cases}$$

or:

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha-j+1)}.$$

Then:

$${}_a d_t^{\alpha} f(x) := \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{n}{j} f(x-jh)$$

Caputo definition from 1967

$$ {}_{a}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} \frac{d^{m} f(t)}{dt^{m}} (x-t)^{m-\alpha-1} dt \,. $$

Love and Fractional Models

- When one wants to include memory effects, i.e., the influence of the past on the behavior of the system at present time, one may use fractional derivatives to describe such effect.
- Fractional models have been shown by many researchers to adequately describe the operation of a variety of physical, engineering, and biological processes and systems.
- Such models are represented by differential equations of non-integer order.

W. M. Ahmad and R. El-Khazali, Fractional-order dynamical models of love, Chaos Solitons Fractals **33** (2007), no. 4, 1367–1375.

What was the first application?

- The 1st application of FC belongs to Niels Henrik Abel in 1823. Abel applied the FC in the solution of an integral equation which arises in the formulation of the Tautochrone Problem.
- This problem, sometimes called the *isochrone problem*, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead placed on the wire slides to the lowest point of the wire in the *same time* regardless of where the bead is placed.

Nagumo Theorem for fractional system

If somebody wants to speak about viability he should start from the very beginning, namely *Nagumo Theorem, 1942*.

Our contribution

- We formulated viability results for nonlinear fractional differential equations with the Caputo derivative and fractional differential inclusions, namely, sufficient conditions that in both cases guarantee existence of viable solutions. A fractional differential inclusion that we considered was defined via the single-valued control problem.
  E. Girejko, D. Mozyrska, and M. Wyrwas. A sufficient condition of viability for fractional differential equations with the Caputo derivative. Journal of Mathematical Analysis and Applications, 381, 2011..

Nagumo Theorem for fractional system

Our contribution

- We gave necessary conditions of viability for nonlinear fractional differential equations with the Riemann-Liouville derivative.
  D. Mozyrska, E. Girejko, and M. Wyrwas. A necessary condition of viability for fractional differential equations with initialization. Computers and Mathematics with Applications, 62(9):36423647,2011.

Positive and cone solutions of fractional systems

Working with fractional systems one can pose problems of existence and uniqueness of solutions staying in a certain set of constraints... so again something around viability theory.

Our contribution

- We formulated conditions for existing cone and positive solutions (kind of viability) of fractional system. Nonlinear fractional cone systems involving the Caputo fractional derivative were considered. We established sufficient conditions for the existence of at least one cone solution of such systems. We gave sufficient conditions for the unique existence of the cone solution of a nonlinear fractional cone system.

  D. Mozyrska, E. Girejko, and M. Wyrwas. Nonlinear fractional cone systems with the Caputo derivative. Applied Mathematics Letters, 25(4), 2012.

Discrete fractional systems

Our contribution

- We did some basic stability work, namely, using the Lyapunov's direct method, the stability of discrete non-autonomous systems with the nabla Caputo fractional difference was studied. We gave conditions for different kind of stability.

  M. Wyrwas, E. Girejko, and D. Mozyrska. Stability of discrete fractional- order nonlinear systems with the nabla caputo difference. In IFAC Joint Conference: 5th Symposium on System, Structure and Control: SSSC2013, 11th Workshop on Time-delay Systems, 6th Workshop on Fractional Differentiation and its Applications, pages 167171, Grenoble, February 2013. International Federation of Automatic Control.

Discrete fractional systems

Our contribution

- We examined some fractional difference operators and discussed their properties. We gave a characterization of three operators that we call Grünwald-Letnikov, Riemann-Liouville and Caputo like difference operators.
  D. Mozyrska and E. Girejko. Advances in Harmonic Analysis and Operator Theory: The Stefan Samko Anniversary Volume, volume 229, chapter Overview of the fractional h-difference operators, pages 253267. Springer,2013.

http://katmat.pb.bialystok.pl/pcm10/prezentacje/TorresDelfimFM0.PDF

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Agnieszka B. Malinowska and Delfim F. M. Torres, Introduction to the Fractional Calculus of Variations, Imperial College Press, London and World Scientific Publishing, Singapore, 2012.

A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, "Theory and applications of fractional differential equations", Elsevier, Amsterdam, 2006.

K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974.

I. Podlubny, "Fractional differential equations", Academic Press, Inc., San Diego, CA, 1999.