Gravitational field equations on Fefferman space-times

Elisabetta Barletta[1]
Sorin Dragomir[1]
Howard Jacobowitz[2]

Abstract. The total space $\mathfrak{M} \approx \mathbb{H}_1 \times S^1$ of the canonical circle bundle over the 3-dimensional Heisenberg group $\mathbb{H}_1$ is a space-time with the Lorentzian metric F_{θ_0} (Fefferman's metric) associated to the canonical Tanaka-Webster flat contact form θ_0 on $\mathbb{H}_1$. The matter and energy content of $\mathfrak{M}$ is described by the energy-momentum tensor $\mathring{T}_{\mu\nu}$ (the trace-less Ricci tensor of F_{θ_0}) as an effect of the non flat nature of Feferman's metric F_{θ_0}. We study the gravitational field equations $R_{\mu\nu} - (1/2)\, R\, g_{\mu\nu} = \mathring{T}_{\mu\nu}$ on $\mathfrak{M}$. We consider the first order perturbation $g = F_{\theta_0} + \epsilon\, h$, $\epsilon << 1$, and linearize the field equations about F_{θ_0}. We determine a Lorentzian metric g on $\mathfrak{M}$ which solves the linearized field equations corresponding to a diagonal perturbation h.

1. Introduction

The *Fefferman metric* (cf. [14]) F_θ is a Lorentzian metric on the total space $C(M)$ of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ over a strictly pseudoconvex CR manifold M endowed with a positively oriented contact form θ. It has been discovered by C. Fefferman (cf. [19]) in relation to the study of the boundary behavior of the Bergman kernel (cf. [18]) of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ ($n \geq 2$), to start with as a Lorentzian metric on the product manifold $\partial\Omega \times S^1$. Complex analysis in several complex variables thus exhibits a nowadays largely exploited (cf. [5], [14]) yet not fully understood relationship between Lorentzian geometry on one hand, and CR geometry and subelliptic theory (cf. [4], [22]) on the other. The principal bundle $S^1 \to C(M) \to M$ carries *Graham's connection* i.e. the natural connection 1-form $\sigma \in C^\infty(T^*(C(M)))$ discovered in [21].

[1]Università degli Studi della Basilicata, Dipartmento di Matematica, Informatica ed Economia, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy, e-mail `elisabetta.barletta@unibas.it`, `sorin.dragomir@unibas.it`

[2]Department of Mathematical Sciences, Rutgers University at Camden, Armitage Hall, 311 N. 5th Street, Camden, NJ 08102, USA, `jacobowi@crab.rutgers.edu`

Let $X^\uparrow \in C^\infty(T(C(M)))$ denote the horizontal lift of the tangent vector field $X \in C^\infty(T(M))$ with respect to Graham's connection σ. If $S \in C^\infty(T(C(M)))$ and $\xi \in C^\infty(T(M))$ are respectively the tangent to the S^1-action and the *Reeb vector* field associated to the contact form θ then $\xi^\uparrow - S$ is a globally defined time-like vector field on $C(M)$, thus giving a time orientation of the Lorentzian manifold $(C(M), F_\theta)$. The synthetic object $(C(M),\, F_\theta\,,\, \xi^\uparrow - S)$ is then a space-time (cf. [11]). Our purpose through the present paper is to continue the investigation (cf. [2], [6], [10], [15]) of the relationship between CR and pseudohermitian geometry (and the underlying subelliptic theory, cf. [3]) and space-time physics, as prompted by the occurrence of Fefferman's metric associated to a pseudohermitian manifold (M, θ) and, viceversa, by the occurrence of strictly pseudoconvex CR structures associated to shear free null geodesic congruences on a Lorentzian manifold (cf. [24], [27], [30], [32]). By a result of J.M. Lee (cf. [28]) F_θ is never an Einstein metric i.e. there is no $\Lambda \in \mathbb{R}$ such that $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$ where $R_{\mu\nu}$ is the Ricci tensor field of the Lorentzian manifold $(C(M), F_\theta)$. In the spirit of space-time physics, to produce a gravitation theory on $C(M)$ one should look for solutions to Einstein field equations. In the present paper we formulate a number of fundamental questions, the first of which is **I) what is the convenient form of Einstein's equations, to be considered on $C(M)$?** A moment's thought shows that

$$\mathrm{Ric}(g)_{\mu\nu} = 0 \tag{1}$$

[Einstein's equations for empty space, where $\mathrm{Ric}(g)$ is the Ricci tensor of $g \in \mathrm{Lor}(C(M))$, the dependent variable in equations (1)] is not an appropriate choice [e.g. by the aforementioned result in [28], Fefferman's metric F_θ is never a solution to (1)]. It is one of our purposes to answer the question posed above, although confined to the case where $M = \mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ is the 3-dimensional *Heisenberg group*. This may be organized as a strictly pseudoconvex CR manifold CR isomorphic to the boundary of the Siegel domain $\Omega = \{(z, w) \in \mathbb{C}^2 : \mathrm{Im}(w) > z\overline{z}\}$ (cf. e.g. [14], p. 14). Let $\mathfrak{M} = C(\mathbb{H}_1)$ be the total space of the canonical circle bundle over $\mathbb{H}_1$ (cf. [14], p. 119). Let

$$\theta_0 = dt + i\,\left(z\, d\overline{z} - \overline{z}\, dz\right)$$

be the *canonical* contact form on $\mathbb{H}_1$. Then θ_0 is positively oriented and Tanaka-Webster flat. Let $F_{\theta_0} \in \mathrm{Lor}(\mathfrak{M})$ be the Fefferman metric of the pseudohermitian manifold $(\mathbb{H}_1, \theta_0)$ (cf. [14], p. 128). The metric F_{θ_0} is not flat and its curvature may be thought of as the effect of a matter distribution on $\mathfrak{M}$. The matter, or energy, content of $\mathfrak{M}$ is then described by the energy-momentum tensor $\mathring{T}_{\mu\nu}$ which is the traceless

Ricci tensor of F_{θ_0}. $\mathring{T}_{\mu\nu}$ is further discussed in § 5, showing that $\mathring{T}_{\mu\nu}$ describes an incoherent matter distribution (*incoherent dust*) on $\mathfrak{M}$. $\mathfrak{M}$ is a 4-dimensional manifold diffeomorphic to $\mathbb{H}_1 \times S^1$ so that it may not be covered by a globally defined coordinate system. However one may adopt the globally defined nonholonomic frame

$$X_0 = 2\,S, \quad X_1 = L^{\uparrow}, \quad X_2 = \overline{L}^{\uparrow}, \quad X_3 = \xi_0^{\uparrow}, \tag{2}$$

where $S \in \mathfrak{X}(\mathfrak{M})$ is the tangent to the S^1 action on $\mathfrak{M}$, $\xi_0 \in \mathfrak{X}(\mathfrak{M})$ is the Reeb vector field of $(\mathbb{H}_1\,,\,\theta_0)$, and $\overline{L} = \partial/\partial\overline{z} - i\,z\,\partial/\partial t$ is the *Lewy operator* (cf. [14], p. 12). Components of tensor fields on $\mathfrak{M}$ are then intended with respect to $\{X_\mu\,:\,0 \leq \mu \leq 3\}$ and are globally defined smooth functions on $\mathfrak{M}$, perhaps complex valued. Through the present paper we shall study Einstein's gravitational field equations

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} = \mathring{T}_{\mu\nu} \tag{3}$$

on $\mathfrak{M}$ in the presence of the matter distribution described by $\mathring{T}_{\mu\nu}$. Here $R_{\mu\nu} = \mathrm{Ric}(g)_{\mu\nu}$ and R are respectively the Ricci and scalar curvature of the Lorentzian metric $g \in \mathrm{Lor}(\mathfrak{M})$ [whose components $g_{\mu\nu}$ are the unknown functions in the PDEs system (3)]. Any physics discussion of gravity on $\mathfrak{M}$ requires a solution to (3). While the search for exact solutions to (3) is deferred to further work, we wish (in the spirit of the classical approach by A. Einstein, [17]) to linearize the field equations (3) with the manifest purpose of producing at least a solution to the linearized equations. Another fundamental question posed in the present paper is then **II) what is a convenient base point** $g_0 \in \mathrm{Lor}(\mathfrak{M})$ [about which one ought to linearize (3)] **and what is an appropriate choice of perturbation matrix** h**?** We consider first order perturbations

$$g = F_{\theta_0} + \epsilon\,h, \quad \epsilon << 1, \tag{4}$$

and linearize equations (3) about $g_0 = F_{\theta_0}$. Our computational approach, and in particular the choice of frame (2), draws inspiration from the recent work [24] (relating solutions g of Einstein equations $R_{\mu\nu} = \Lambda\,g_{\mu\nu}$ on $M \times \mathbb{R}$ to the local embeddability of the CR structure on M associated to g). In a previous paper (cf. [7]) we exploited the formal similarity between $\mathbb{R}^4$ with the (flat) Minkowski metric and $\mathbb{H}_1$ with the canonical (Tanaka-Webster flat) contact form θ_0 in order to produce pseudohermitian analogs of classical results in [17]. The postulated field equation (for empty space) in [7] was

$$R_{1\overline{1}} = 0 \tag{5}$$

[where $R_{1\bar{1}}$ is the pseudohermitian Ricci tensor associated to an arbitrary contact form on $\mathbb{H}_1$, cf. [14] and § 2 of this paper] and one linearized (5) by considering perturbations of the form $\theta_0 + \epsilon\,\theta$ ($\epsilon <<$ 1). Any positively oriented contact form θ on $\mathbb{H}_1$ is related to θ_0 by $\theta = e^{2u}\theta_0$ for some $u \in C^\infty(\mathbb{H}_1\,,\,\mathbb{R})$. By a result of J.M. Lee (cf. [28]) this yields the relationship

$$F_\theta = e^{2\,u\circ\pi}\,F_{\theta_0}$$

between the corresponding Fefferman metrics and then the perturbation $\theta_0 + \epsilon\,\theta$ induces a perturbation of the form (4) with

$$h = e^{2\,u\circ\pi} F_{\theta_0}\,. \tag{6}$$

However, as shown by our discussion in § 2, (6) is not an appropriate choice of perturbation matrix, for (6) exhibits $h_{00} = 0$, and the h_{00} component of the perturbation matrix is classically responsible for the potential ϕ (in whose central force field $-\nabla\phi$ the geodesic motion of a particle should occur, in the classical limit of weak fields and low velocities). One then answers question (II) by choosing the perturbation (4) where h is a $(0,2)$-tensor field on $\mathfrak{M}$ with $h_{00} \neq 0$, allowing one to mimic the classical limit of the gravitational field equations in nonempty space (cf. e.g. [1], p. 277-280). The resulting linearized field equations (66)-(72) have a formidable aspect leaving little hope in the search for an explicit solution. We therefore confine ourselves to the case of a diagonal perturbation matrix h for which the linearized field equations are found to be

$$\Delta_b\, h_{00} = 0, \quad \xi_0(L\, h_{00}) = 0, \tag{7}$$

$$L^2(h_{22}) + \overline{L}^2(h_{11}) - 2h_{33} = 0, \quad \xi_0(h_{22}) = 0, \tag{8}$$

$$\xi_0^2(h_{00}) + 2h_{33} = 0, \quad L\xi_0(h_{22}) - 2i\,\overline{L}(h_{33}) = 0, \quad \Delta_b\, h_{33} = 0, \tag{9}$$

where Δ_b is the sublaplacian of $(\mathbb{H}_1\,,\,\theta_0)$, cf. [14], p. 111 [the Hörmander operator associated to the Hörmander system of vector fields $\{X, Y\}$ on $\mathbb{H}_1$, where $L = \frac{1}{2}(X - iY)$, cf. [26]]. Δ_b is a subelliptic operator of order 1/2 (in the sense of [20]) thus bringing subelliptic theory (cf. e.g. [16]) into the picture. It is one of our main points (here and in [7]) that the appropriate mathematical analysis entering gravity theory on $\mathfrak{M}$ should rely on the relationship between hyperbolic and subelliptic theory (as following from $\pi_*\Box = \Delta_b$, according to a result in [28]). A finding in the present paper is an explicit nontrivial solution h to (7)-(9). The solution is discussed in section § 5.

Acknowledgements Part of this paper has been written while H. Jacobowitz visited (June and October 2014) at the Department of Mathematics, Informatics and Economy (DiMIE) of the University of Basilicata. He expresses his gratitude for support from INdAM (Rome, Italy) and DiMIE (Potenza, Italy). E. Barletta and S. Dragomir acknowledge support from P.R.I.N. 2012.

Contents

2. Fefferman space-times

Notations, conventions and fundamental results (in CR and pseudohermitian geometry) adopted through this paper are those in [14]. Some of the basic material is recalled in this section, for the needs of the more physics oriented reader. Let $\mathbb{H}_1$ be the 3-dimensional Heisenberg group i.e. the Lie group $\mathbb{C} \times \mathbb{R}$ with the product

$$(z, t) \cdot (w, s) = (z + w, \, t + s + 2\,\mathrm{Im}\,(z\overline{w}))\,. \tag{10}$$

$\mathbb{H}_1$ is a strictly pseudoconvex CR manifold, of CR dimension 1, with the CR structure $T_{0,1}(\mathbb{H}_1)$ spanned by the *Lewy operator*

$$\overline{L} = \partial/\partial\overline{z} - iz\,\partial/\partial t.$$

One sets as customary $T_{1,0}(\mathbb{H}_1) = \overline{T_{0,1}(\mathbb{H}_1)}$ (overbars denote complex conjugates). A relevant second order differential operator (occurring in the linearized field equations (7)-(9)) is the *sublaplacian* (cf. [14])

$$\Delta_b u = -\left(L\overline{L}u + \overline{L}Lu\right), \quad u \in C^2(\mathbb{H}_1),$$

coinciding with the *Hörmander operator* $-\frac{1}{2}\left(X^2 + Y^2\right)$ (cf. [26]) associated to the Hörmander system of vector fields $\{X, Y\}$ on $\mathbb{H}_1$, where $L = \frac{1}{2}(X - iY)$. The sublaplacian Δ_b is formally similar to the Laplace-Beltrami operator of a Riemannian manifold, yet it isn't elliptic. Nevertheless Δ_b is a positive, formally self-adjoint, degenerate elliptic (in the sense of [13]) operator which is subelliptic of order 1/2 (cf. [20])

and hence hypoelliptic (cf. [25], a feature that Δ_b shares with elliptic operators). Also for every $u \in C^2(\mathbb{H}_1)$

$$\Delta_b u = \frac{1}{2}\,\Delta_0 u + 2i\,\frac{\partial}{\partial t}\left(z\,\frac{\partial u}{\partial z} - \overline{z}\,\frac{\partial u}{\partial \overline{z}}\right) - 2|z|^2\,\frac{\partial^2 u}{\partial t^2} \tag{11}$$

where Δ_0 is the ordinary Laplacian $-(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on $\mathbb{R}^2$ (with $z = x + iy$).

A complex p-form ω is a $(p,0)$-*form* if $T_{0,1}(\mathbb{H}_1)\,\rfloor\,\omega = 0$. A top degree $(p,0)$-form is a $(2,0)$-form. The *canonical bundle* $\mathbb{C} \to K(\mathbb{H}_1) \to \mathbb{H}_1$ is the bundle of all $(2,0)$-forms (a complex line bundle). There is a natural action of $\mathbb{R}^+$ (the multiplicative positive reals) on $K^0(\mathbb{H}_1) = K(\mathbb{H}_1) \setminus \{\text{zero section}\}$ and the quotient space $\mathfrak{M} = K^0(\mathbb{H}_1)/\mathbb{R}^+$ is the total space of a principal S^1-bundle (the *canonical circle bundle* over $\mathbb{H}_1$). Let $\pi : \mathfrak{M} \to \mathbb{H}_1$ be the projection. Let θ be a positively oriented contact form on $\mathbb{H}_1$ i.e. a real 1-form such that

$$\operatorname{Ker}(\theta) = H(\mathbb{H}_1) = \operatorname{Re}\{T_{1,0}(\mathbb{H}_1) \oplus T_{0,1}(\mathbb{H}_1)\}$$

(the Levi, or maximally complex, distribution on $\mathbb{H}_1$), $\theta \wedge d\theta$ is a volume form on $\mathbb{H}_1$, and the Levi form associated to θ

$$G_\theta(Z, \overline{Z}) = -i(d\theta)(Z, \overline{Z}), \quad Z \in T_{1,0}(\mathbb{H}_1),$$

$(i = \sqrt{-1})$ is positive definite. If $\theta^1 = dz$ then each class $[\omega]$ mod $\mathbb{R}^+$ of $\omega \in K^0(\mathbb{H}_1)_x$ may be represented as

$$[\omega] = \left[\lambda\,\left(\theta \wedge \theta^1\right)_x\right] \in \mathfrak{M}_x\,, \quad \lambda \in \mathbb{C}^*\,, \quad x \in \mathbb{H}_1\,.$$

In particular the canonical circle bundle is trivial i.e.

$$\mathfrak{M} \approx S^1 \times \mathbb{H}_1\,, \quad \Phi : [\omega] \longmapsto \left(\frac{\lambda}{|\lambda|}\,,\, x\right), \tag{12}$$

is a C^∞ diffeomorphism. The Heisenberg group $\mathbb{H}_1$ also carries a Riemannian metric g_θ [the *Webster metric* of $(\mathbb{H}_1\,,\,\theta)$] given by

$$g_\theta(V, W) = G_\theta(V, W), \quad g_\theta(V, \xi) = 0, \quad g_\theta(\xi, \xi) = 1,$$

for any $V, W \in H(\mathbb{H}_1)$. The pair $(H(\mathbb{H}_1)\,,\,G_\theta)$ is a sub-Riemannian structure on $\mathbb{H}_1$ (cf. [9]) and the Webster metric g_θ is a contraction of G_θ (cf. [31]). For each $u \in C^1(\mathbb{H}_1)$ the gradient of u is given by $g_\theta(\nabla u, V) = V(u)$ for any $V \in \mathfrak{X}(\mathbb{H}_1)$. The *horizontal gradient* of u is

$$\nabla^H u = \Pi_H \nabla u$$

where $\Pi_H : T(\mathbb{H}_1) \to H(\mathbb{H}_1)$ is the projection associated to the decomposition $T(\mathbb{H}_1) = H(\mathbb{H}_1) \oplus \mathbb{R}\xi$.

Given a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ ($n \geq 2$) a Lorentzian metric F on $\partial\Omega \times S^1$, known nowadays as *Fefferman's metric* (cf. [14]), was built by C. Fefferman (cf. [19]). A key feature of F was that its *restricted* conformal class

$$\{e^{u \circ \pi} F : u \in C^\infty(\partial\Omega, \mathbb{R})\}$$

is a biholomorphic invariant of Ω. Here $\pi : \partial\Omega \times S^1 \to \partial\Omega$ is the projection. This prompted the question (also due to C. Fefferman, [19]) whether F admits an intrinsic construction for each strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^{n+1}$, such that the restricted conformal class is a CR invariant of M. The question was settled by J.M. Lee (cf. [28]) who built a Lorentzian metric F_θ on the total space $C(M)$ of the canonical circle bundle over a strictly pseudoconvex CR manifold M, not necessarily embedded, in correspondence to any fixed positively oriented contact form θ on M. The metric F_θ is computed in terms of pseudohermitian invariants [of the pseudohermitian manifold (M, θ)] and the construction of F_θ is such that, whenever M is the boundary of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the Lorentzian manifold $(C(\partial\Omega), F_\theta)$ is conformally diffeomorphic to $(\partial\Omega \times S^1, F)$. Through the present paper we only need J.M. Lee's result for $M = \mathbb{H}_1$ i.e.

$$F_\theta = \pi^* \tilde{G}_\theta + 2\,(\pi^*\theta) \odot \sigma\,, \quad \tilde{G}_\theta = 2 g_{1\bar{1}}\, \theta^1 \odot \theta^{\bar{1}}\,,$$

$$\sigma = \frac{1}{3}\left\{ d\gamma + \pi^* \left(i\, \omega_1{}^1 - \frac{i}{2}\, g^{1\bar{1}}\, dg_{1\bar{1}} - \frac{\rho}{8}\, \theta \right) \right\},$$

where γ is a local fibre coordinate on $\mathfrak{M}$ and $\omega_1{}^1$ is the connection 1-form of the Tanaka-Webster connection ∇ of $(\mathbb{H}_1\,, \theta)$ i.e. $\nabla L = \omega_1{}^1 \otimes L$. Let $\xi = \xi_\theta \in \mathfrak{X}(\mathbb{H}_1)$ be the *Reeb vector* field of $(\mathbb{H}_1, \theta)$ i.e. the unique nowhere zero, globally defined tangent vector field on $\mathbb{H}_1$, transverse to the Levi distribution $H(\mathbb{H}_1)$, determined by

$$\theta(\xi) = 1, \quad \xi \rfloor d\theta = 0.$$

For simplicity we set $\xi_0 = \xi_{\theta_0}$. Also if R^∇ is the curvature tensor field of ∇ then we set $T_1 = L$, $T_{\bar{1}} = \overline{L}$, $T_0 = \xi$ and

$$R_A{}^D{}_{BC} T_D = R^\nabla(T_B\,, T_C) T_A\,, \quad A, B, C, \dots \in \{1,\ \bar{1},\ 0\},$$

$$g_{1\bar{1}} = G_\theta(L, \overline{L}), \quad g^{1\bar{1}} = 1/g_{1\bar{1}}\,, \quad R_{1\bar{1}} = R_1{}^1{}_{1\bar{1}}\,, \quad \rho = g^{1\bar{1}}\, R_{1\bar{1}}\,.$$

Here $g_{1\bar{1}}$, $R_{1\bar{1}}$ and ρ are respectively the *Levi invariant*, the *pseudohermitian Ricci tensor*, and the *pseudohermitian scalar curvature* of $(\mathbb{H}_1\,, \theta)$. Let us set $\mathfrak{M} = C(\mathbb{H}_1)$ for simplicity. By a result of R.C.

Graham (cf. [21]) σ is a connection 1-form on the principal bundle $S^1 \to \mathfrak{M} \to \mathbb{H}_1$. In particular for $\theta = \theta_0$

$$\tilde{G}_{\theta_0} = 2\,\theta^1 \odot \theta^{\bar{1}}\,, \quad \sigma = \frac{1}{3}\,d\gamma,$$

so that σ is flat. Let $S \in \mathfrak{X}(\mathfrak{M})$ be the tangent to the S^1 action and choose the local fibre coordinate γ such that $S = (3/2)\,\partial/\partial\gamma$ on $\mathcal{U}(\varphi_0) = \Phi^{-1}\,[\omega(\varphi_0) \times \mathbb{H}_1]$, where $\omega(\varphi_0) = \{e^{i\varphi} : |\varphi - \varphi_0| < \pi\}$ with $\varphi_0 \in \mathbb{R}$. If $V \in \mathfrak{X}(\mathbb{H}_1)$ then let $V^\uparrow \in \mathfrak{X}(\mathfrak{M})$ be the horizontal lift of V with respect to σ i.e.

$$V^\uparrow_p \in \mathrm{Ker}(\sigma)_p\,, \quad (d_p\pi)V^\uparrow_p = V_x\,, \quad p \in \mathfrak{M}, \quad x = \pi(p) \in \mathbb{H}_1\,.$$

Then $X_\theta = \xi^\uparrow - S$ is a globally defined time-like vector field on the Lorentzian manifold $(\mathfrak{M},\, F_\theta)$ i.e. X_θ is a time orientation, so that $(\mathfrak{M},\, F_\theta\,,\, X_\theta)$ is a space-time. If $z = x + iy$ and t are canonical coordinates on $\mathbb{H}_1$, we endow $\mathfrak{M}$ with the local coordinates

$$(x^\alpha) \equiv (x^0\,,\, x^j) \equiv (\gamma,\ x \circ \pi,\ y \circ \pi,\ t \circ \pi),$$

where $0 \le \alpha \le 3$, $1 \le j \le 3$. With respect to (x^α) the Fefferman metric F_{θ_0} reads

$$F_{\theta_0} = 2\left[\left(dx^1\right)^2 + \left(dx^2\right)^2\right] + \frac{2}{3}\left[dx^3 + 2\left(x^1\,dx^2 - x^2\,dx^1\right)\right] \odot dx^0\,.$$

We shall write the geodesics equations for the Lorentzian manifold $(\mathfrak{M}, F_{\theta_0})$ as the Euler-Lagrange equations of the variational principle

$$\delta \int F_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu\, ds = 0 \tag{13}$$

i.e.

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}\,, \quad L(x, \dot{x}) \equiv F_{\mu\nu}(x)\,\dot{x}^\mu\,\dot{x}^\nu\,.$$

Since

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & -\frac{2}{3}\,y & \frac{2}{3}\,x & \frac{1}{3} \\ -\frac{2}{3}\,y & 2 & 0 & 0 \\ \frac{2}{3}\,x & 0 & 2 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$$

the Euler-Lagrange equations of (13) are

$$\frac{d^2 x^0}{ds^2} = 0, \tag{14}$$

$$\frac{d^2 x^1}{ds^2} - \frac{2}{3}\frac{dx^0}{ds}\frac{dx^2}{ds} = 0, \tag{15}$$

$$\frac{d^2 x^2}{ds^2} + \frac{2}{3}\frac{dx^0}{ds}\frac{dx^1}{ds} = 0, \tag{16}$$

$$\frac{d^2 x^3}{ds^2} - \frac{4}{3}\,x^1\,\frac{dx^0}{ds}\,\frac{dx^1}{ds} - \frac{4}{3}\,x^2\,\frac{dx^0}{ds}\,\frac{dx^2}{ds} = 0. \tag{17}$$

Straightforward integration of the ODE system (14)-(17) leads to the two families of geodesics of $(\mathfrak{M},\, F_{\theta_0})$

$$\gamma(s) = B_0\,, \quad z(s) = x(s) + i\,y(s) = \alpha\,s + \beta, \quad t(s) = a\,s + b, \tag{18}$$
$$\alpha,\, \beta \in \mathbb{C}, \quad a,\, b,\, B_0 \in \mathbb{R},$$

and

$$\gamma(s) = A_0\,s + B_0\,, \quad A_0\,, B_0 \in \mathbb{R}, \quad A_0 \neq 0, \tag{19}$$

$$(z(s),\, t(s)) = \left(-\frac{1}{\lambda}\,i\,\alpha\,,\, k\right)\cdot\left(\beta\,e^{-i\lambda\,s}\,,\, \left(\frac{1}{\lambda}\,|\alpha|^2 + \lambda\,|\beta|^2\right)\,s\right), \tag{20}$$
$$\alpha,\, \beta \in \mathbb{C}, \quad k \in \mathbb{R}, \quad \lambda = \frac{2}{3}\,A_0\,.$$

The dot product in (20) is given by (10). A parallel of (14)-(17) to

$$\frac{d\dot{x}^\mu}{ds} + \left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\} \dot{x}^\alpha\,\dot{x}^\beta = 0, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{ds}\,,$$

furnishes the list of all nonzero Christoffel symbols of F_{θ_0} i.e.

$$\left\{ \begin{array}{c} 1 \\ 02 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 20 \end{array} \right\} = -\frac{1}{3}\,, \quad \left\{ \begin{array}{c} 2 \\ 01 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ 10 \end{array} \right\} = \frac{1}{3}\,, \tag{21}$$

$$\left\{ \begin{array}{c} 3 \\ 01 \end{array} \right\} = \left\{ \begin{array}{c} 3 \\ 10 \end{array} \right\} = -\frac{2}{3}\,x, \quad \left\{ \begin{array}{c} 3 \\ 02 \end{array} \right\} = \left\{ \begin{array}{c} 3 \\ 20 \end{array} \right\} = -\frac{2}{3}\,y. \tag{22}$$

Consequently if $G = \det\,[F_{\mu\nu}]$ the Ricci curvature

$$r_{\beta\nu} = \left\{ \begin{array}{c} \alpha \\ \nu\beta \end{array} \right\}_{|\alpha} - \left(\log\sqrt{-G}\right)_{|\beta|\nu} +$$
$$+ \left\{ \begin{array}{c} \sigma \\ \nu\beta \end{array} \right\} \left(\log\sqrt{-G}\right)_{|\sigma} - \left\{ \begin{array}{c} \sigma \\ \alpha\beta \end{array} \right\} \left\{ \begin{array}{c} \alpha \\ \nu\sigma \end{array} \right\}$$

of $(\mathfrak{M}, F_{\theta_0})$ is

$$r_{00} = \frac{2}{9}\,, \quad r_{i0} = 0, \quad r_{ij} = 0, \quad 1 \leq i, j \leq 3. \tag{23}$$

Indeed $G = -4/9$ hence

$$r_{\beta\nu} = \left\{ \begin{array}{c} \alpha \\ \nu\beta \end{array} \right\}_{|\alpha} - \left\{ \begin{array}{c} \sigma \\ \alpha\beta \end{array} \right\} \left\{ \begin{array}{c} \alpha \\ \nu\sigma \end{array} \right\}.$$

Next [by (21)-(22)] $\left\{ \begin{array}{c} \alpha \\ 00 \end{array} \right\} = 0$ and for instance

$$r_{00} = -\left\{ \begin{array}{c} \sigma \\ \alpha 0 \end{array} \right\} \left\{ \begin{array}{c} \alpha \\ 0\sigma \end{array} \right\} = -2 \left\{ \begin{array}{c} 1 \\ 20 \end{array} \right\} \left\{ \begin{array}{c} 2 \\ 01 \end{array} \right\} = \frac{2}{9}\,.$$

The remaining components of $r_{\mu\nu}$ may be computed in a similar manner. The Lorentzian manifold $(\mathfrak{M}, F_{\theta_0})$ has nonzero curvature, which may be thought of as the result of a distribution of matter on $\mathfrak{M}$. Therefore the matter, or energy, content of $\mathfrak{M}$ is described by the energy-momentum tensor

$$T^{\mu\nu} = r^{\mu\nu} - \frac{1}{2}\, r\, F^{\mu\nu}\,, \quad r^{\mu\nu} = F^{\mu\alpha} F^{\nu\beta} r_{\alpha\beta}\,, \quad r = F^{\mu\nu} r_{\mu\nu}\,.$$

Yet [by (23)] $r = F^{00} r_{00} = 0$ hence

$$[T^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{24}$$

It is our purpose, as explained in § 1, to consider first order perturbations $g = F_{\theta_0} + \epsilon\, h$ (with $\epsilon << 1$) and i) look at geodesic motion in the field g, at the Newtonian limit of velocities, and ii) linearize Einstein field equations on $\mathfrak{M}$ in the presence of the matter distribution described by $T^{\mu\nu}$ above. In a previous paper (cf. [7]) we started with first order perturbations $\theta_\epsilon = \theta_0 + \epsilon\, \theta$ of the canonical contact form θ_0. The space of all positively oriented contact forms on $\mathbb{H}_1$ is parametrized by $C^\infty(\mathbb{H}_1, \mathbb{R})$ hence $\theta = e^{2u}\theta_0$ for some smooth function $u : \mathbb{H}_1 \to \mathbb{R}$. Therefore

$$\theta_\epsilon = e^{2u_\epsilon}\, \theta_0\,, \quad u_\epsilon = \log \sqrt{1 + \epsilon\, e^{2u}}\,. \tag{25}$$

By a result of J.M. Lee (25) yields (cf. [28])

$$F_{\theta_\epsilon} = \left(1 + \epsilon\, e^{2u\circ\pi}\right)\, F_{\theta_0}$$

i.e. $F_{\theta_\epsilon} = F_{\theta_0} + \epsilon\, h$ where $h = F_\theta$. It is then tempting to use $h_{\mu\nu} = (F_\theta)_{\mu\nu}$ as a perturbation matrix, a case in which the linearized field equations will have θ, and then the scalar field u, as the unknown function. Yet for this choice of h one has $h_{00} = F_\theta(\partial/\partial\gamma\, , \, \partial/\partial\gamma) = 0$. Consequently, should one set $x^0 = \gamma = c\,\tau$ and interpret τ as a "time" coordinate, the geodesic motion equations on $(\mathfrak{M},\, F_{\theta_\epsilon})$ will not reduce (for $\epsilon << 1$ and $\|\mathbf{v}\|/c << 1$) to any reasonable pseudohermitian analog to Newton's law of motion in classical mechanics (as we shall show in § 3). This is intuitively clear from the classical argument (cf. e.g. [1], p. 124) that geodesic motion in a weak gravitational field modeled by $g_0 + \epsilon h$ where $g_0 = -c^2\, d\tau^2 + dx^2 + dy^2 + dt^2$ is the ordinary Minkowski metric on $\mathbb{R}^4$ leads, for $\|\mathbf{v}\|/c << 1$, to Newton's law of motion $d^2\mathbf{r}/d\tau^2 = -\nabla\phi$ in a central force field whose potential ϕ is essentially the h_{00} entry of the perturbation matrix [i.e. $\phi = (c^2\epsilon/2)\, h_{00}$].

The next section is then devoted to the study of geodesic motion on $\mathfrak{M}$, at the Newtonian limit of velocities, in the presence of a perturbation $F_{\theta_0} + \epsilon\, h$ of the Fefferman metric F_{θ_0}, where the perturbation matrix h is allowed to have a nontrivial component h_{00} corresponding to our choice of a "time" coordinate γ, though independent of γ (so that h is a *stationary* perturbation).

3. Geodesic motion in the classical limit

To build a gravity theory on $\mathfrak{M}$ starting from small perturbations of $(\mathfrak{M},\, F_{\theta_0})$ one is led to a third fundamental question as to **III) what is an appropriate choice of time coordinate on $\mathfrak{M}$?** There is no moral distinction between the Cartesian coordinates $(x^0,\, x,\, y,\, z)$ on $\mathbb{R}^4$ and one may rather arbitrarily fix a coordinate, say x^0, and set $x^0 = c\,t$. This symmetry is lost when looking at the local coordinate system $(\gamma,\, x\circ\pi,\, y\circ\pi,\, t\circ\pi)$ induced on $\mathfrak{M}$ by the Cartesian coordinates $(x,\, y,\, t)$ on $\mathbb{H}_1 \approx \mathbb{R}^3$. It is natural to regard $\mathbb{H}_1$ as ordinary *space* hence our choice is to think of (the vertical lift of) $(x,\, y,\, z)$ as *space* coordinates and of the additional local fibre coordinate $x^0 = \gamma$ as the *time* coordinate: this is however intimately tied to $(x,\, y,\, z)$ through the manifold structure of $\mathfrak{M}$ (in the spirit of general - as opposed to special - relativity theory, formulated on an arbitrary 4-dimensional manifold of vanishing Euler-Poincaré characteristic). Let h be a symmetric $(0,2)$-tensor field on $\mathfrak{M}$ and let us set

$$g = F_{\theta_0} + \epsilon\, h, \quad \epsilon << 1. \tag{26}$$

Let $C : (-\delta, \delta) \to \mathfrak{M}$ be a time-like curve in $(\mathfrak{M},\, g)$ locally given by

$$C(\tau) = \left(c\,\tau,\ C^1(\tau),\ C^2(\tau),\ C^3(\tau)\right), \quad |\tau| < \delta, \tag{27}$$

where c is the velocity of light in vacuum. By the implicit function theorem any smooth curve C in $\mathfrak{M}$ may be locally represented as in (27) i.e. such that time in $\mathfrak{M}$ is a parameter along C. However τ is not the proper time i.e. time as perceived by an observer attached to C [and a change of parameter as given by (28) is needed]. Let $v^j = dC^j/d\tau$, $1 \leq j \leq 3$, and $\beta = \|\mathbf{v}\|/c$ where $\mathbf{v} = (v^1, v^2, v^3)$ and $\|\mathbf{v}\| = \left(\sum_{j=1}^{3}(v^j)^2\right)^{1/2}$. If

$$s = \phi(\tau) = \int_0^\tau \left[-g_{C(u)}\left(\dot{C}(u)\,,\,\dot{C}(u)\right)\right]^{\frac{1}{2}} du \tag{28}$$

and $x^\mu(s) = C^\mu(\phi^{-1}(s))$ then the geodesic equations are

$$\frac{d^2x^\alpha}{ds^2} + \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{29}$$

Here

$$\left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} = g^{\alpha\sigma}\ \{\mu\nu, \sigma\}\,, \quad \{\mu\nu, \sigma\} = \frac{1}{2}\left(g_{\mu\sigma|\nu} + g_{\nu\sigma|\mu} - g_{\mu\nu|\sigma}\right),$$

$$f_{|\mu} = \partial f/\partial x^\mu\,, \quad f \in C^1(\mathfrak{M}).$$

We wish to derive the geodesic equations of motion (29) in the gravitational field g, in the weak field ($\epsilon << 1$) and low velocity ($\beta << 1$) limit. In the subsequent elementary asymptotic analysis we drop terms of order $O(\epsilon^2)$, $O(\beta^2)$, $O(\epsilon\,\beta)$ and higher. Throughout we assume that the perturbation matrix $[h_{\mu\nu}]$ in (26) doesn't depend on the fibre coordinate γ i.e.

$$h_{\mu\nu|0} = 0. \tag{30}$$

As $x^0 = \gamma$ is thought of as a time coordinate the assumption (30) corresponds to a static (i.e. time independent) perturbation matrix. Let us set $\mathbf{r} = \pi \circ C$. The tangent vector along C is

$$\dot{C}(\tau) = c\left(\frac{\partial}{\partial\gamma}\right)_{C(\tau)} + v^i(\tau)\left(\frac{\partial}{\partial x^i}\right)_{C(\tau)}$$

hence

$$\tilde{G}_{\theta_0\,,\,\mathbf{r}(\tau)}(\dot{\mathbf{r}}(\tau), \dot{\mathbf{r}}(\tau)) = 2\left[\left(v^1\right)^2 + \left(v^2\right)^2\right],$$

$$\theta_{0,\mathbf{r}(\tau)}(\dot{\mathbf{r}}(\tau)) = v^3 + 2\left[C^1(\tau)\,v^2 - C^2(\tau)\,v^1\right],$$

$$F_{\theta_0\,,\,C(\tau)}(\dot{C}(\tau), \dot{C}(\tau)) = -c^2\,H(\tau) + 2\left[\left(v^1\right)^2 + \left(v^2\right)^2\right],$$

so that

$$g_{C(\tau)}(\dot{C}(\tau), \dot{C}(\tau)) = -c^2\,H(\tau) + 2\left[\left(v^1\right)^2 + \left(v^2\right)^2\right] + \tag{31}$$

$$+\epsilon\,h_{\mu\nu}(C(\tau))\frac{dC^\mu}{d\tau}\frac{dC^\nu}{d\tau}$$

where

$$H(\tau) = \frac{2}{3}\left[2\,C^2(\tau)\,\frac{v^1}{c} - 2\,C^1(\tau)\,\frac{v^2}{c} - \frac{v^3}{c}\right] = O(\beta).$$

On the other hand

$$h_{\mu\nu}(C(\tau))\frac{dC^\mu}{d\tau}\frac{dC^\nu}{d\tau} =$$

$$= c^2\left[h_{00}(C(\tau)) + 2\,h_{0j}(C(\tau))\,\frac{v^j}{c} + h_{jk}(C(\tau))\frac{v^j}{c}\frac{v^k}{c}\right] \approx$$

(by dropping $O(\beta^2)$)

$$\approx c^2\left[h_{00}(C(\tau)) + 2\,h_{0j}(C(\tau))\,\frac{v^j}{c}\right]$$

and (31) becomes

$$g_{C(\tau)}\left(\dot{C}(\tau), \dot{C}(\tau)\right) = c^2\left[-H(\tau) + \epsilon\,h_{00}(C(\tau))\right]. \tag{32}$$

As a consequence of (32)

$$\phi'(\tau) \approx c\left[H(\tau) - \epsilon\,h_{00}(C(\tau))\right]^{\frac{1}{2}}$$

so that

$$\frac{dx^\lambda}{ds}(\phi(\tau)) = \frac{1}{c}\,\left[H(\tau) - \epsilon\,h_{00}(C(\tau))\right]^{-\frac{1}{2}}\,\frac{dC^\lambda}{d\tau}. \tag{33}$$

Differentiation with respect to τ in (33) together with

$$\left[H(\tau) - \epsilon\,h_{00}(C(\tau))\right]^{-1} = \frac{1}{H(\tau)} + \frac{\epsilon}{H(\tau)^2}\,h_{00}(C(\tau)) + O(\epsilon^2)$$

yields

$$c^2\left[H(\tau) - \epsilon\,h_{00}(C(\tau))\right]\,\frac{d^2x^\lambda}{ds^2} = \frac{d^2C^\lambda}{d\tau^2} - \tag{34}$$

$$-\frac{1}{2}\left[H(\tau) - \epsilon\,h_{00}(C(\tau))\right]^{-1}\left[\frac{2}{3c}\,G(\tau) - \epsilon\,h_{00|j}(C(\tau))\,\frac{dC^j}{d\tau}\right]\frac{dC^\lambda}{d\tau}$$

where

$$G(\tau) = 2C^2(\tau)\,\frac{d^2C^1}{d\tau^2} - 2C^1(\tau)\,\frac{d^2C^2}{d\tau^2} - \frac{d^2C^3}{d\tau^2}.$$

Yet

$$\epsilon\,h_{00|j}(C(\tau))\,\frac{v^j}{c} = O(\epsilon\beta)$$

and (34) reads

$$c^2\left(H - \epsilon\,h_{00}\right)\frac{d^2x^\lambda}{ds^2} = \frac{d^2C^\lambda}{d\tau^2} - \frac{1}{3c}\left(\frac{1}{H} + \frac{\epsilon}{H^2}\,h_{00}\right)\,G(\tau)\,\frac{dC^\lambda}{d\tau}. \tag{35}$$

Next (by (33))

$$\left\{ \begin{array}{c} \mu \\ \lambda\sigma \end{array} \right\} (C(\tau)) \frac{dx^\lambda}{ds}(\phi(\tau)) \frac{dx^\sigma}{ds}(\phi(\tau)) =$$

$$= \frac{1}{c^2\,(H - c\,h_{00})} \left\{ \begin{array}{c} \mu \\ \lambda\sigma \end{array} \right\} (C(\tau)) \frac{dC^\lambda}{d\tau} \frac{dC^\sigma}{d\tau}$$

or (by dropping $O(\beta^2)$)

(36) $$\left\{ \begin{array}{c} \mu \\ \lambda\sigma \end{array} \right\} \frac{dx^\lambda}{ds} \frac{dx^\sigma}{ds} = \frac{1}{H - \epsilon\, h_{00}} \left\{ \begin{array}{c} \mu \\ 00 \end{array} \right\}.$$

Moreover

$$\left\{ \begin{array}{c} \mu \\ 00 \end{array} \right\} = g^{\mu\nu}\, \{00, \nu\} = -\frac{1}{2}\, g^{\mu\nu}\, g_{00|\nu}$$

so that

(37) $$\left\{ \begin{array}{c} \mu \\ 00 \end{array} \right\} = -\frac{\epsilon}{2}\, g^{\mu\nu}\, h_{00|\nu}\,.$$

Next one may observe that

(38) $$g^{\mu\nu} = F^{\mu\nu} - \epsilon\, h^{\mu\nu} + O(\epsilon^2)$$

where (cf. e.g. [4])

(39) $$[F^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & \frac{1}{2} & 0 & y \\ 0 & 0 & \frac{1}{2} & -x \\ 3 & y & -x & 2|z|^2 \end{pmatrix}$$

[the inverse of $F_{\mu\nu} = F_{\theta_0}(\partial/\partial x^\mu\,,\, \partial/\partial x^\nu)$] and $h^{\mu\nu} = F^{\alpha\mu} F^{\sigma\nu} h_{\alpha\sigma}$. One may conclude (by (37)-(38))

(40) $$\left\{ \begin{array}{c} \mu \\ 00 \end{array} \right\} = -\frac{\epsilon}{2}\, F^{\mu\nu}\, h_{00|\nu}\,.$$

Equations (35)-(36) and (40) imply geodesic motion (29) is governed by

(41) $$\frac{d^2C^\mu}{d\tau^2} - \frac{1}{3c}\left(1 + \frac{\epsilon}{H}\, h_{00}\right) \frac{G}{H} \frac{dC^\mu}{d\tau} = \frac{c^2\epsilon}{2}\, F^{\mu k}\, h_{00|k}$$

i.e. for $\mu = j$ (respectively for $\mu = 0$)

(42) $$\frac{d^2C^j}{d\tau^2} - \frac{1}{3c} \frac{G}{H}\, v^j = \frac{c^2\epsilon}{2}\, F^{jk}\, h_{00|k}\,,$$

$$-\frac{1}{3}\left(1+\frac{\epsilon}{H}\,h_{00}\right)\frac{G}{H}=\frac{c^2\epsilon}{2}\,F^{0k}\,h_{00|k}\,. \tag{43}$$

By (39) and $[1+(\epsilon/H)h_{00}]^{-1}\approx 1-(\epsilon/H)\,h_{00}$ equation (43) simplifies to

$$-\frac{1}{3}\frac{G}{H}=\frac{3c^2\epsilon}{2}\,\frac{\partial h_{00}}{\partial t} \tag{44}$$

and substitution from (44) into (42) leads to

$$\frac{d^2C^j}{d\tau^2}=\frac{c^2\epsilon}{2}F^{jk}\,h_{00|k}\,. \tag{45}$$

Next (by taking into account (39))

$$F^{1k}\,h_{00|k}=\frac{1}{2}\,X(h_{00})\,,\quad F^{2k}\,h_{00|k}=\frac{1}{2}\,Y(h_{00}),\quad F^{3k}\,h_{00|k}=V(h_{00}),$$

where

$$X=\frac{\partial}{\partial x}+2y\,\frac{\partial}{\partial t}\,,\quad Y=\frac{\partial}{\partial y}-2x\,\frac{\partial}{\partial t}\,,$$

$$V=y\,\frac{\partial}{\partial x}-x\,\frac{\partial}{\partial y}+2|z|^2\,\frac{\partial}{\partial t}=y\,X-x\,Y,$$

(so that $L=\frac{1}{2}(X-iY)$). Summing up, equations (44)-(45) are

$$G(\tau)=-\frac{9c^2\epsilon}{2}\,H(\tau)\,\frac{\partial h_{00}}{\partial t}(C(\tau)), \tag{46}$$

$$\frac{d^2C^1}{d\tau^2}=\frac{c^2\epsilon}{4}\,X(h_{00})_{C(\tau)}\,,\quad \frac{d^2C^2}{d\tau^2}=\frac{c^2\epsilon}{4}\,Y(h_{00})_{C(\tau)}\,, \tag{47}$$

$$\frac{d^2C^3}{d\tau^2}=\frac{c^2\epsilon}{2}\,V(h_{00})_{C(\tau)}\,. \tag{48}$$

An inspection reveals (46) as a linear combination of (47)-(48). Finally if

$$\frac{d^2\mathbf{r}}{d\tau^2}=\frac{d^2C^j}{d\tau^2}\left(\frac{\partial}{\partial x^j}\right)_{\mathbf{r}(\tau)}$$

then (47)-(48) become

$$\frac{d^2\mathbf{r}}{d\tau^2}=\frac{c^2\epsilon}{4}\,\{X(h_{00})X+Y(h_{00})Y\}_{\mathbf{r}(\tau)}\,. \tag{49}$$

For each $u\in C^1(\mathbb{H}_1)$ let $\nabla^H u$ be the horizontal gradient of u with respect to the canonical contact form θ_0. If $E_1=(1/\sqrt{2})X$ and $E_2=(1/\sqrt{2})Y$ then $\nabla^H u=\sum_{a=1}^2 E_a(u)\,E_a$ and (49) becomes

$$\frac{d^2\mathbf{r}}{d\tau^2}=-\left(\nabla^H\phi\right)_{\mathbf{r}(\tau)} \tag{50}$$

where

$$\phi = -\frac{c^2 \epsilon}{2} h_{00} . \tag{51}$$

Viceversa, given the classical potential ϕ, the motion of a particle will follow a geodesic of $(\mathfrak{M}, g)$ if the g_{00} term of the metric has the form $g_{00} = -(2/c^2)\,\phi$. The other components of g enter our asymptotic scheme only through the assumptions that they are time independent and nearly Feferman (i.e. close to the components of F_{θ_0}). As shown in the successive § 4 if (26) is a solution to Einstein's field equations [in a nonempty space, whose matter content is described by the energy-momentum tensor (24)] to order $O(\epsilon)$ then $\Delta_b \phi = 0$.

4. Gravitational field equations

By recent work of C.D. Hill & J. Lewandowski & P. Nurowski (cf. [24]) local embeddability of 3-dimensional CR manifolds M is closely tied to the existence of Lorentzian metrics on $M \times \mathbb{R}$ satisfying Einstein's equations $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$. The knowledge that CR structures may be associated, in a natural manner, to certain classes of Lorentzian metrics is certainly older and goes back to the work by I. Robinson & A. Trautman (cf. [30]) and A. Trautman (cf. [32]). Cf. also [27] and [10]. Fefferman metrics belong to this class, and they may actually be characterized (cf. [21]) as the Lorentzian metrics g on the 4-dimensional manifold $\mathfrak{M}$ admitting a null Killing vector field $K \in \mathfrak{X}(\mathfrak{M})$ such that $K \rfloor W = K \rfloor C = 0$ and $\mathrm{Ric}(K, K) > 0$, where W, C and Ric are the Weyl, Cotton and Ricci tensors of g. Given such a Lorentzian metric g, the leaf space $\mathfrak{M}/K$ may be organized, at least locally, as a C^∞ manifold M carrying a strictly pseudoconvex CR structure and a positively oriented contact form θ such that $(\mathfrak{M}, g)$ be locally isometric to $(C(M), F_\theta)$. To (locally) embed a 3-dimensional CR manifold M one needs (cf. e.g. [12]) to determine two functionally independent (local) CR functions f_a, $a \in \{1, 2\}$, on M, so that $(f_1, f_2) : M \to \mathbb{C}^2$ is (locally) a CR immersion. The subtle approach to the problem by C.D. Hill et al. (cf. *op. cit.*) is to write the Cartan structure equations (for the Einstein metric g at hand)

$$d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\alpha \wedge \Gamma^\alpha{}_\nu = \frac{1}{2} R^\mu{}_{\nu\alpha\beta}\, \Theta^\alpha \wedge \Theta^\beta$$

with respect to a special (local) frame $\{\Theta^\mu : 0 \le \mu \le 3\}$ [with Θ^a, $a \in \{1, 2\}$, complex 1-forms and Θ^b, $b \in \{3, 4\}$ real forms such that g admits a particularly simple local representation i.e. $g_{12} = g_{21} = 1$, $g_{03} = g_{30} = 1$, and $g_{\mu\nu} = 0$ otherwise] and find indices $(\mu_0\,, \nu_0)$ such that $\Gamma_{\mu_0\nu_0} = 0$ is (as a *consequence* of Einstein's equations) an involutive complex

Pfaffian system on (an open subset of) $\mathfrak{M}$. Then, by a combination of the classical Frobenius theorem (for real Pfaffian systems) and existence of isothermal coordinates (on a Riemann surface) one may represent $\Gamma_{\mu_0\nu_0}$ as $\Gamma_{\mu_0\nu_0} = h\, d\zeta$ for some complex function ζ with $d\zeta \wedge d\overline{\zeta} \neq 0$, whose projection on M gives the desired (local) CR function. The existence of a solution $g_{\mu\nu}$ to the field equations (3) [rather than $R_{\mu\nu} = \Lambda g_{\mu\nu}$] written on an open set of $M \times \mathbb{R}$ is expected [in view of Hill-Lewandowski-Nurowski's scheme, producing CR functions] to require peculiar properties of the base CR structure. These properties are so far unknown and will be addressed in further work. It should be mentioned that we neither continue nor explain the results in [24] but merely conduct our calculations with respect to a special (globally defined) coframe chosen as in [24] i.e.

$$\Theta^1 = \pi^*\theta^1\,,\ \ \Theta^2 = \pi^*\theta^{\overline{1}}\,,\ \ \Theta^3 = 2\, F_{\theta_0}(S\,,\,\cdot\,),\ \ \Theta^0 = \sigma,$$

so that $\Theta^3 = \pi^*\theta_0$. Then the Fefferman metric F_{θ_0} admits the simple representation

$$F_{\theta_0} = 2\left\{\Theta^1 \odot \Theta^2 + \Theta^3 \odot \Theta^0\right\}.$$

Let $\{X_\mu : 0 \leq \mu \leq 3\}$ be the dual frame i.e. $\Theta^\mu(X_\nu) = \delta^\mu_\nu$. Then

$$X_1 = L^\uparrow\,, \quad X_2 = \overline{L}^\uparrow\,, \quad X_3 = \xi_0^\uparrow\,, \quad X_0 = 2S, \tag{52}$$

where horizontal lifting is meant with respect to Graham's connection 1-form $\sigma_0 = (1/3)\, d\gamma$. If $F_{\mu\nu} = F_{\theta_0}(X_\mu\,,\, X_\nu)$ and $[F^{\mu\nu}] = [F_{\mu\nu}]^{-1}$ then

$$F_{12} = F_{21} = 1, \quad F_{03} = F_{30} = 1,$$

$$F^{12} = F^{21} = 1, \quad F^{03} = F^{30} = 1,$$

$$(\mu\nu) \notin \{(12),\ (21),\ (03),\ (30)\} \Longrightarrow F_{\mu\nu} = 0, \quad F^{\mu\nu} = 0.$$

Let D be the Levi-Civita connection of $(\mathfrak{M}, g)$ and R^D its curvature tensor field. We adopt the conventions

$$D_{X_\mu} X_\nu = \Gamma^\alpha_{\mu\nu} X_\alpha\,,$$

$$R^\alpha{}_{\mu\beta\nu} X_\alpha = R^D(X_\beta\,,\, X_\nu) X_\mu\,, \quad R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}\,.$$

As D is torsion-free $[X_\mu\,,\, X_\nu] = \left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\right) X_\alpha$. Hence

$$R^\alpha{}_{\mu\beta\nu} = X_\beta\left(\Gamma^\alpha_{\nu\mu}\right) - X_\nu\left(\Gamma^\alpha_{\beta\mu}\right) + \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\beta\sigma} - \Gamma^\sigma_{\beta\mu}\Gamma^\alpha_{\nu\sigma} - \Gamma^\sigma_{\beta\nu}\Gamma^\alpha_{\sigma\mu} + \Gamma^\sigma_{\nu\beta}\Gamma^\alpha_{\sigma\mu}$$

and the Ricci curvature is

$$R_{\mu\nu} = X_\alpha\left(\Gamma^\alpha_{\nu\mu}\right) - X_\nu\left(\Gamma^\alpha_{\alpha\mu}\right) + \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\alpha\sigma} - \Gamma^\sigma_{\alpha\nu}\Gamma^\alpha_{\sigma\mu}\,. \tag{53}$$

From now on we represent the perturbation matrix as

$$[h_{\mu\nu}] = \begin{pmatrix} a & \overline{v} & v & u \\ \overline{v} & \overline{\omega} & b & \overline{\beta} \\ v & b & \omega & \beta \\ u & \overline{\beta} & \beta & \alpha \end{pmatrix}, \quad h_{\mu\nu} = h\left(X_\mu\, ,\, X_\nu\right),$$

with $a,\, b,\, u,\, \alpha \in C^\infty(\mathbb{H}_1\, ,\, \mathbb{R})$ and $v,\, \beta,\, \omega \in C^\infty(\mathbb{H}_1\, ,\, \mathbb{C})$. Indeed if $b = h_{12} = h_{21}$ then $\overline{b} = \overline{h(X_1, X_2)} = h(X_2, X_1) = b$ so b is real valued.

Lemma 1. *The Ricci tensor of* $(\mathfrak{M}, g)$ *is given by*

$$R_{00} = 2 + \frac{\epsilon}{2}\,\Delta_b a + 4\epsilon(u-b) - 2i\epsilon\left[L(v) - \overline{L}(\overline{v})\right], \tag{54}$$

$$\begin{aligned} R_{10} = R_{01} = \frac{\epsilon}{2}\,&\left[L^2(v) - L\overline{L}(\overline{v}) + \xi_0 L(a)\right] + \\ &+ i\,\epsilon\,\left[L(2u-b) - \xi_0(\overline{v})\right] + 2\,\epsilon\,\overline{\beta}\,, \end{aligned} \tag{55}$$

$$\begin{aligned} R_{20} = R_{02} = \frac{\epsilon}{2}\,&\left\{\overline{L}^2(\overline{v}) - L\overline{L}(v) + \xi_0\overline{L}(a)\right\} - \\ &- i\,\epsilon\,\left[\overline{L}(2u-b) - \xi_0(v)\right] + 2\,\epsilon\,\beta, \end{aligned} \tag{56}$$

$$\begin{aligned} R_{30} = R_{03} = \frac{\epsilon}{2}\,&\left\{\Delta_b u + L\xi_0(v) + \overline{L}\xi_0(\overline{v}) + \xi_0^2(a)\right\} + \\ &- i\,\epsilon\,\left[L(\beta) - \overline{L}(\overline{\beta})\right] + 2\,\epsilon\,\alpha, \end{aligned} \tag{57}$$

$$R_{11} = \epsilon\,\left[\xi_0 L(\overline{v}) - L^2(u)\right] + i\,\epsilon\,\xi_0(\overline{\omega}), \tag{58}$$

$$R_{22} = \epsilon\,\left[\xi_0\overline{L}(v) - \overline{L}^2(u)\right] - i\,\epsilon\,\xi_0(\omega), \tag{59}$$

$$R_{33} = \frac{\epsilon}{2}\,\Delta_b\alpha + \epsilon\,\left[-\xi_0^2(b) + L\xi_0(\beta) + \overline{L}\xi_0(\overline{\beta})\right], \tag{60}$$

$$\begin{aligned} R_{21} = \frac{\epsilon}{2}\,&\left[-2\,L\overline{L}\,(u+b) + \xi_0 L(v) + \xi_0\overline{L}(\overline{v}) + L^2(\omega) + \overline{L}^2(\overline{\omega})\right] + \\ &+ 2\,i\,\epsilon\,\left[L(\beta) - \overline{L}(\overline{\beta})\right] - i\,\epsilon\,\xi_0(u+b) - 2\epsilon\alpha, \end{aligned} \tag{61}$$

$$\begin{aligned} R_{12} = \frac{\epsilon}{2}\,&\left[-2\,\overline{L}L(u+b) + \xi_0 L(v) + \xi_0\overline{L}(\overline{v}) + L^2(\omega) + \overline{L}^2(\overline{\omega})\right] + \\ &+ 2\,i\,\epsilon\,\left[L(\beta) - \overline{L}(\overline{\beta})\right] + i\,\epsilon\,\xi_0(u+b) - 2\epsilon\alpha\,, \end{aligned} \tag{62}$$

$$\begin{aligned} R_{31} = R_{13} = i\,\epsilon\,&\left[L(\alpha) - \xi_0(\overline{\beta})\right] + \\ + \frac{\epsilon}{2}\,&\left[-L\xi_0\,(u+b) + \xi_0^2(\overline{v}) + L^2(\beta) - L\overline{L}(\overline{\beta}) + \overline{L}\xi_0(\overline{\omega})\right], \end{aligned} \tag{63}$$

$$R_{32} = R_{23} = i\,\epsilon\,\left[\xi_0(\beta) - \overline{L}(\alpha)\right] + \tag{64}$$

$$+\frac{\epsilon}{2}\left[-\overline{L}\xi_0(u+b)+\xi_0^2(v)+\overline{L}^2(\overline{\beta})-\overline{L}L(\beta)+\xi_0 L(\omega)\right],$$

to order $O(\epsilon)$. One has

$$R_{20}=\overline{R_{10}}\,,\quad R_{22}=\overline{R_{11}}\,,\quad R_{21}=\overline{R_{12}}\,,\quad R_{32}=\overline{R_{31}}\,.$$

In particular the scalar curvature of $(\mathfrak{M}, g)$ is given by

$$R=\epsilon\left[\Delta_b\,(2u+b)+\xi_0^2(a)\right]+2\epsilon\left[L\xi_0(v)+\overline{L}\xi_0(\overline{v})\right]+ \tag{65}$$

$$+\epsilon\left[L^2(\omega)+\overline{L}^2(\overline{\omega})\right]+2\,i\,\epsilon\left[L(\beta)-\overline{L}(\overline{\beta})\right]-2\,\epsilon\,\alpha$$

to order $O(\epsilon)$.

The computational details leading to Lemma 1 are relegated to Appendix A. Let

$$T^{\mu\nu}\,\partial/\partial x^\mu\odot\partial/\partial x^\nu=2\,\partial/\partial x^3\otimes\partial/\partial x^3=\mathring{T}^{\mu\nu}\,X_\mu\odot X_\nu$$

be the energy-momentum tensor as considered in § 2. Its components with respect to the frame (52) are $\mathring{T}^{\mu\nu}=2\,\delta_3^\mu\delta_3^\nu$ hence $\mathring{T}_{\mu\nu}=g_{\mu\alpha}g_{\nu\beta}\mathring{T}^{\alpha\beta}$ is given by

$$\left[\mathring{T}_{\mu\nu}\right]=\begin{pmatrix}2(1+2\epsilon u) & 2\epsilon\overline{\beta} & 2\epsilon\beta & 2\epsilon\alpha\\ 2\epsilon\overline{\beta} & 0 & 0 & 0\\ 2\epsilon\beta & 0 & 0 & 0\\ 2\epsilon\alpha & 0 & 0 & 0\end{pmatrix}+O(\epsilon^2).$$

Also the trace-less Ricci tensor of $(\mathfrak{M}, g)$ is

$$\left[R_{\mu\nu}-\frac{1}{2}\,R\,g_{\mu\nu}\right]=$$

$$=\begin{pmatrix}R_{00} & R_{01} & R_{02} & R_{03}-\frac{1}{2}\,R\\ R_{10} & R_{11} & R_{12}-\frac{1}{2}\,R & R_{13}\\ R_{20} & R_{21}-\frac{1}{2}\,R & R_{22} & R_{23}\\ R_{30}-\frac{1}{2}\,R & R_{31} & R_{32} & R_{33}\end{pmatrix}+O(\epsilon^2)$$

hence (by Lemma 1 and $\left[L,\overline{L}\right]=-2i\xi_0$) Einstein's equations $R_{\mu\nu}-\frac{1}{2}\,R\,g_{\mu\nu}=\mathring{T}_{\mu\nu}$ are [to order $O(\epsilon)$]

$$\Delta_b a-4\left\{2b+i\left[L(v)-\overline{L}(\overline{v})\right]\right\}=0, \tag{66}$$

$$\xi_0(La)+2i\,L(2u-b)+L^2(v)-\overline{L}L(\overline{v})=0, \tag{67}$$

$$\Delta_b\,(u+b)+L\xi_0(v)+\overline{L}\xi_0(\overline{v})+L^2(\omega)+\overline{L}^2(\overline{\omega})+ \tag{68}$$

$$+4i\left[L(\beta)-\overline{L}(\overline{\beta})\right]-2\alpha=0$$

$$L^2(u)-\xi_0 L(\overline{v})-i\,\xi_0(\overline{\omega})=0, \tag{69}$$

$$\Delta_b(2u+b)+\xi_0^2(a)+2\,L\overline{L}(u+b)+L\xi_0(v)+\overline{L}\xi_0(\overline{v})- \\ -2i\left[L(\beta)-\overline{L}(\overline{\beta})\right]+2i\,\xi_0(u+b)+2\alpha=0 \tag{70}$$

$$-L\xi_0\,(u+b)+\xi_0^2(\overline{v})+L^2(\beta)-L\overline{L}(\overline{\beta})+\overline{L}\xi_0(\overline{\omega})+ \\ +2i\,\left[L(\alpha)-\xi_0(\overline{\beta})\right]=0, \tag{71}$$

$$\Delta_b\alpha-2\,\xi_0^2(b)+2\left[L\xi_0(\beta)+\overline{L}\xi_0(\overline{\beta})\right]=0, \tag{72}$$

(merely conjugate equations were omitted). Equations (66)-(72) are Einstein's equations on $\mathfrak{M}$ linearized about F_{θ_0}. To provide a peudohermitian analog to Einstein's results (producing a spherically symmetric solution to the gravitational field equations, cf. [17]) one should look for solutions with Heisenberg spherical symmetry i.e. $h_{\mu\nu}(x)=f_{\mu\nu}(r)$ with $r=|x|$ for some functions $f_{\mu\nu}$ to be determined from (66)-(72). Here $|x|=\left(|z|^4+t^2\right)^{1/4}$ is the Heisenberg norm of $x=(z,t)\in\mathbb{H}_1$. While this is left as an open problem, a nontrivial solution to (66)-(72) furnishing a diagonal perturbation matrix $[h_{\mu\nu}]=\operatorname{diag}(a,\,\overline{\omega},\,\omega,\,\alpha)$ may be determined as follows. If $b=u=0$ and $\beta=v=0$ then the system (66)-(72) becomes

$$\Delta_b a=0, \tag{73}$$

$$\xi_0(La)=0, \tag{74}$$

$$L^2(\omega)+\overline{L}^2(\overline{\omega})-2\alpha=0, \tag{75}$$

$$\xi_0(\omega)=0, \tag{76}$$

$$\xi_0^2(a)+2\alpha=0, \tag{77}$$

$$L\xi_0(\omega)-2i\,\overline{L}(\alpha)=0, \tag{78}$$

$$\Delta_b\alpha=0. \tag{79}$$

We need the following

Lemma 2. *Any real valued CR function on $\mathbb{H}_1$ is a constant.*

Proof. Let f be a $\mathbb{R}$-valued CR function i.e. f is a C^1 solution to the tangential Cauchy-Riemann equations $\overline{L}f=0$. By complex conjugation $Lf=0$ hence

$$0=\left[L\,,\,\overline{L}\right]f=-2i\,\xi_0(f)$$

implying that f is a constant. Q.e.d.

Let $(a,\,\omega,\,\alpha)$ be a solution to (73)-(79). As $[L\,,\,\xi_0]=0$ equation (74) shows that $\xi_0(a)$ is a real valued CR function on $\mathbb{H}_1$ hence (by Lemma

2) $\xi_0(a) = C$ for some $C \in \mathbb{R}$. Then [by (77)] $\alpha = 0$ and the system (73)-(79) reduces to

$$\Delta_b a = 0, \tag{80}$$

$$L^2(\omega) + \overline{L}^2(\overline{\omega}) = 0, \tag{81}$$

$$\xi_0(\omega) = 0. \tag{82}$$

Equation (82) shows that $\omega(z,t) = f(z) + i\, g(z)$ for some real valued functions f, g of one complex variable. Then (80)-(82) reads [cf. also (11)]

$$a_{xx} + a_{yy} = 0, \tag{83}$$

$$f_{xx} - f_{yy} + 2\, g_{xy} = 0, \tag{84}$$

with $z = x + iy$. Moreover $\xi_0(a) = C$ yields $a(z,t) = Ct + F(z)$ with $\Delta_0 F = 0$ [by (83)]. A solution with spherical symmetry $F(z) = \psi(|z|)$ is $\psi(\rho) = \frac{1}{2\pi} \log \rho$ (the fundamental solution to Laplace equation in the plane) hence

$$a(z,t) = C\, t + \frac{1}{2\pi} \log |z|. \tag{85}$$

As to equation (84) one looks for solutions of the form $f(z) = A(\eta)$ and $g(z) = B(\eta)$ with $\eta = x/|y|$ so that

$$\left(1 - \eta^2\right) A''(\eta) - 2\eta\, A'(\eta) = \pm \left[\eta\, B''(\eta) + B'(\eta)\right]$$

according to whether $\pm y > 0$. In particular if

$$\left(1 - \eta^2\right) A'(\eta) = k, \quad \eta\, B'(\eta) = m,$$

for some constants $k, m \in \mathbb{R}$ then

$$A(\eta) = k \log \left| \frac{1-\eta}{1+\eta} \right| + \ell, \quad B(\eta) = m \log |\eta| + p,$$

with k, ℓ, m, $p \in \mathbb{R}$. Finally

$$\omega = k \log \left| \frac{x - |y|}{x + |y|} \right| + \ell + i \left[m \log \left| \frac{x}{y} \right| + p \right]. \tag{86}$$

Summing up [by (85)-(86) and $\alpha = 0$]

$$g = \epsilon \left[t + \frac{1}{2\pi} \log |z| \right] \left(\Theta^0\right)^2 + \tag{87}$$

$$+ \epsilon \left[\log \left| \frac{x - |y|}{x + |y|} \right| - i \log \left| \frac{x}{y} \right| \right] \left(\Theta^1\right)^2 +$$

$$+\epsilon \left[\log \left| \frac{x - |y|}{x + |y|} \right| + i \, \log \left| \frac{x}{y} \right| \right] \left(\Theta^2\right)^2 +$$

$$+2 \left(\Theta^0 \odot \Theta^3 + \Theta^1 \odot \Theta^2\right)$$

is a solution to Einstein's equations (3) to order $O(\epsilon)$.

5. Conclusions and final comments

The total space $\mathfrak{M}$ of the principal circle bundle over the 3-dimensional Heisenberg group $\mathbb{H}_1$, endowed with the Fefferman metric F_{θ_0} and the time orientation $\xi_0^{\uparrow} - S$, is a space-time. The metric F_{θ_0} is not flat and one may identify, as foundational for general relativity theory, geometry to matter content of space described by the energy-momentum tensor $\mathring{T}^{\mu\nu} = \rho_0 \, u^\mu u^\nu$ where $\rho_0 = 2$ and the four-velocity flow is $u^\mu = \delta_3^\mu$.

The formal similarity between $\mathring{T}^{\mu\nu}$ (the trace-less Ricci tensor of F_{θ_0}) and (9.7) in [1], p. 263, allows for the physical interpretation of $\mathring{T}^{\mu\nu}$ as a field of non-interacting incoherent matter described by a four-vector field of flow u^μ and a scalar proper density field ρ_0, moving at low [by also comparing $\mathring{T}^{\mu\nu}$ with (9.84) in [1], p. 277] velocity.

Linearization of Einstein's equations $R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \mathring{T}_{\mu\nu}$ about F_{θ_0} is expected to produce linear PDEs whose principal part is the wave operator $\Box$ (the Laplace-Beltrami operator of the Lorentzian metric F_{θ_0}). Indeed the linearized field equations (73)-(78) involve the sublaplacian Δ_b of $(\mathbb{H}_1 \, , \, \theta_0)$, which is the same as $\Box$ on functions not depending on the "time" coordinate γ [for, by a result of J.M. Lee, [28], $\pi_* \Box = \Delta_b$].

Geodesic motion on $(\mathfrak{M}, \, F_{\theta_0} + \epsilon \, h)$ in the classical limit of velocities $\|\mathbf{v}\|/c << 1$ was shown to be motion of a particle in a force field $\mathbf{F}$ which is the horizontal gradient $\mathbf{F} = -\nabla^H \phi$ rather than the full gradient (with respect to the Webster metric g_{θ_0}) of the potential $\phi = -(c^2 \epsilon/2) \, h_{00}$ which, as a consequence of linearized field equations, satisfies $\Delta_b \phi = 0$. We derive a particular nontrivial solution h to (73)-(78) given by

$$\begin{pmatrix} t + \frac{1}{2\pi} \log|z| & 0 & 0 & 0 \\ 0 & \log \left| \frac{x-|y|}{x+|y|} \right| - i \, \log \left| \frac{x}{y} \right| & 0 & 0 \\ 0 & 0 & \log \left| \frac{x-|y|}{x+|y|} \right| + i \, \log \left| \frac{x}{y} \right| & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The problem of existence of solutions to (66)-(71) which aren't necessary diagonal is expected to depend on the further progress of linear subelliptic theory and its applications to CR geometry (cf. [16] for the state-of-the-art).

Appendix A. Linearized Ricci curvature

Our purpose in Appendix A is to give a proof of Lemma 1. The methods consist of a mix of tensor calculus (within pseudohermitian geometry on $\mathbb{H}_1$) and principal bundle techniques (cf. [23]).

Let $\mathring{D}$ be the Levi-Civita connection of $(\mathfrak{M}, F_{\theta_0})$ and let us set $\mathring{D}_{X_\mu} X_\nu = \mathring{\Gamma}^\alpha_{\mu\nu} X_\alpha$. Using

$$[X_\mu\,,\, X_\nu] = \left(\mathring{\Gamma}^\alpha_{\mu\nu} - \mathring{\Gamma}^\alpha_{\nu\mu}\right) X_\alpha\,,$$

$$2\,g(D_X Y\,,\, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) +$$
$$+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X),$$

for any $X, Y, Z \in \mathfrak{X}(\mathfrak{M})$, one derives

$$\Gamma^\lambda_{\alpha\beta} = g^{\mu\lambda}\Gamma_{\alpha\beta\mu} + B^\lambda_{\alpha\beta} \tag{88}$$

where

$$B^\lambda_{\alpha\beta} = \frac{1}{2}\Big\{\mathring{\Gamma}^\lambda_{\alpha\beta} - \mathring{\Gamma}^\lambda_{\beta\alpha} +$$
$$+\left(\mathring{\Gamma}^\sigma_{\mu\alpha} - \mathring{\Gamma}^\sigma_{\alpha\mu}\right) g_{\sigma\beta} g^{\mu\lambda} + \left(\mathring{\Gamma}^\sigma_{\mu\beta} - \mathring{\Gamma}^\sigma_{\beta\mu}\right) g_{\sigma\alpha} g^{\mu\lambda}\Big\}\,,$$

$$\Gamma_{\alpha\beta\mu} = \frac{1}{2}\left\{X_\alpha(g_{\beta\mu}) + X_\beta(g_{\alpha\mu}) - X_\mu(g_{\alpha\beta})\right\}.$$

We shall use (88) to compute $\Gamma^\lambda_{\alpha\beta}$ to order $O(\epsilon)$. To this end we determine $\mathring{\Gamma}^\lambda_{\alpha\beta}$ from (cf. [2] for $n = 1$ and $\theta = \theta_0$)

$$\mathring{D}_{X^\uparrow} Y^\uparrow = (\mathring{\nabla}_X Y)^\uparrow - (d\theta_0)(X,Y)\,\xi^\uparrow + \sigma_0\left(\left[X^\uparrow\,,\, Y^\uparrow\right]\right) S, \tag{89}$$

$$\mathring{D}_{X^\uparrow}\xi^\uparrow = 0, \tag{90}$$

$$\mathring{D}_{\xi^\uparrow} X^\uparrow = (\mathring{\nabla}_\xi X)^\uparrow\,, \tag{91}$$

$$\mathring{D}_{X^\uparrow} S = \mathring{D}_S X^\uparrow = \frac{1}{2}(JX)^\uparrow\,, \tag{92}$$

$$\mathring{D}_S S = \mathring{D}_S \xi^\uparrow = \mathring{D}_{\xi^\uparrow} S = \mathring{D}_{\xi^\uparrow}\xi^\uparrow = 0, \tag{93}$$

for any $X, Y \in H(\mathbb{H}_1)$. Here $\mathring{\nabla}$ is the Tanaka-Webster connection of $(\mathbb{H}_1\,,\,\theta_0)$. Also ξ is short for ξ_0. Using (89)-(93) one profits from the progress in [2], relating the Lorentzian geometry on $(\mathfrak{M}, F_{\theta_0})$ to pseudohermitian geometry on $(\mathbb{H}_1\,,\,\theta_0)$, and bearing a strong analogy to the techniques of B. O'Neill, [29]. However the results in [29] are derived for Riemannian submersions while $\pi : (\mathfrak{M},\, F_{\theta_0}) \to (\mathbb{H}_1\,,\, g_{\theta_0})$

isn't even semi-Riemannian (the fibres of π are degenerate). Formula (89) is useful together with

$$\left[X^{\uparrow}\,,\,Y^{\uparrow}\right] = [X,Y]^{\uparrow} \tag{94}$$

(cf. (35) in [2] for $n=1$) showing that $\sigma_0([X^{\uparrow},Y^{\uparrow}])=0$. Formulas (89)-(93) yield

$$\mathring{\Gamma}^{\alpha}_{01} = \mathring{\Gamma}^{\alpha}_{10} = i\,\delta^{\alpha}_{1}\,,\quad \mathring{\Gamma}^{\alpha}_{02} = \mathring{\Gamma}^{\alpha}_{20} = -i\,\delta^{\alpha}_{2}\,,\quad \mathring{\Gamma}^{\alpha}_{12} = -\mathring{\Gamma}^{\alpha}_{21} = -i\,\delta^{\alpha}_{3}\,, \tag{95}$$

and the remaining connection coefficients are zero. Similar to (38)

$$g^{\mu\nu} = F^{\mu\nu} - \epsilon\, h^{\mu\nu} + O(\epsilon^2)$$

with the new meaning of components of tensors involved, as related to the nonholonomic frame $\{X_\mu : 0 \le \mu \le 3\}$ rather than $\{\partial/\partial x^\mu : 0 \le \mu \le 3\}$. Hence

$$g^{\mu\lambda}\Gamma_{\alpha\beta\mu} = \frac{\epsilon}{2}\,F^{\mu\lambda}\,\{X_\alpha(h_{\beta\mu}) + X_\beta(h_{\alpha\mu}) - X_\mu(h_{\alpha\beta})\} \tag{96}$$

with the corresponding modification of (88). Next (95) implies

$$\left[B^{\lambda}_{\alpha\beta}\right]_{0\le\alpha,\beta\le 3} = \tag{97}$$

$$= \begin{pmatrix} 0 & i\,g_{03}g^{2\lambda} & -i\,g_{03}g^{1\lambda} & 0 \\ i\,g_{03}g^{2\lambda} & 2i\,g_{13}g^{2\lambda} & B^{\lambda}_{12} & i\,g_{33}g^{2\lambda} \\ -i\,g_{03}g^{1\lambda} & B^{\lambda}_{21} & -2i\,g_{23}g^{1\lambda} & -i\,g_{33}g^{1\lambda} \\ 0 & i\,g_{33}g^{2\lambda} & -i\,g_{33}g^{1\lambda} & 0 \end{pmatrix}$$

where

$$B^{\lambda}_{12} = i\left(-\delta^{\lambda}_{3} + g_{23}g^{2\lambda} - g_{13}g^{1\lambda}\right), \quad B^{\lambda}_{21} = i\left(\delta^{\lambda}_{3} + g_{23}g^{2\lambda} - g_{13}g^{1\lambda}\right).$$

Note that $h^{\mu\nu} = F^{\mu\beta}F^{\nu\gamma}h_{\beta\gamma}$ are

$$[h^{\mu\nu}] = \begin{pmatrix} \alpha & \beta & \overline{\beta} & u \\ \beta & \omega & b & v \\ \overline{\beta} & b & \overline{\omega} & \overline{v} \\ u & v & \overline{v} & a \end{pmatrix}$$

hence $B^{\lambda}_{\mu\nu}$ simplifies to

$$\left[B^{0}_{\mu\nu}\right]_{0\le\mu,\nu\le 3} = \begin{pmatrix} 0 & -i\,\epsilon\,\overline{\beta} & i\,\epsilon\,\beta & 0 \\ -i\,\epsilon\,\overline{\beta} & 0 & 0 & 0 \\ i\,\epsilon\,\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O(\epsilon^2), \tag{98}$$

$$\left[B^{1}_{\mu\nu}\right]_{0\le\mu,\nu\le 3} = \tag{99}$$

$$= \begin{pmatrix} 0 & i\,[1+\epsilon(u-b)] & i\,\epsilon\,\omega & 0 \\ i\,[1+\epsilon(u-b)] & 2i\,\epsilon\,\overline{\beta} & i\,\epsilon\,\beta & i\,\epsilon\,\alpha \\ i\,\epsilon\,\omega & i\,\epsilon\,\beta & 0 & 0 \\ 0 & i\,\epsilon\,\alpha & 0 & 0 \end{pmatrix} + O(\epsilon^2),$$

(100) $$\left[B^2_{\mu\nu}\right]_{0\le\mu,\nu\le 3} =$$

$$= \begin{pmatrix} 0 & -i\,\epsilon\,\overline{\omega} & -i\,[1+\epsilon(u-b)] & 0 \\ -i\,\epsilon\,\overline{\omega} & 0 & -i\,\epsilon\,\overline{\beta} & 0 \\ -i\,[1+\epsilon(u-b)] & -i\,\epsilon\,\overline{\beta} & -2i\,\epsilon\,\beta & -i\,\epsilon\,\alpha \\ 0 & 0 & -i\,\epsilon\,\alpha & 0 \end{pmatrix} + O(\epsilon^2),$$

(101) $$\left[B^3_{\mu\nu}\right]_{0\le\mu,\nu\le 3} =$$

$$= \begin{pmatrix} 0 & -i\,\epsilon\,\overline{v} & i\,\epsilon\,v & 0 \\ -i\,\epsilon\,\overline{v} & 0 & -i & 0 \\ i\,\epsilon\,v & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O(\epsilon^2).$$

Then (88) yields

(102) $$\Gamma^0_{00} = -\frac{\epsilon}{2}\,\xi(a), \;\; \Gamma^1_{00} = -\frac{\epsilon}{2}\,\overline{L}(a), \;\; \Gamma^2_{00} = -\frac{\epsilon}{2}\,L(a), \;\; \Gamma^3_{00} = 0,$$

(103) $$\Gamma^0_{10} = \Gamma^0_{01} = -i\,\epsilon\,\overline{\beta} + \frac{\epsilon}{2}\,[L(u) - \xi(\overline{v})],$$

$$\Gamma^1_{10} = \Gamma^1_{01} = i\,[1+\epsilon(u-b)] + \frac{\epsilon}{2}\,\left[L(v) - \overline{L}(\overline{v})\right],$$

$$\Gamma^2_{10} = \Gamma^2_{01} = -i\,\epsilon\,\overline{\omega}, \quad \Gamma^3_{10} = \Gamma^3_{01} = -i\,\epsilon\,\overline{v} + \frac{\epsilon}{2}\,L(a),$$

(104) $$\Gamma^0_{20} = \Gamma^0_{02} = i\,\epsilon\,\beta + \frac{\epsilon}{2}\,\left[\overline{L}(u) - \xi(v)\right], \;\; \Gamma^1_{20} = \Gamma^1_{02} = i\,\epsilon\,\omega,$$

$$\Gamma^2_{20} = \Gamma^2_{02} = -i[1+\epsilon(u-b)] + \frac{\epsilon}{2}\,\left[\overline{L}(\overline{v}) - L(v)\right],$$

$$\Gamma^3_{20} = \Gamma^3_{02} = i\,\epsilon\,v + \frac{\epsilon}{2}\,\overline{L}(a),$$

(105) $$\Gamma^0_{30} = \Gamma^0_{03} = 0, \;\; \Gamma^1_{30} = \Gamma^1_{03} = \frac{\epsilon}{2}\,\left[\xi(v) - \overline{L}(u)\right],$$

$$\Gamma^2_{30} = \Gamma^2_{03} = \frac{\epsilon}{2}\,[\xi(\overline{v}) - L(u)], \;\; \Gamma^3_{30} = \Gamma^3_{03} = \frac{\epsilon}{2}\,\xi(a),$$

and in particular

(106) $$\Gamma^\lambda_{\lambda 0} = 0.$$

Moreover

(107) $$\Gamma^0_{11} = \frac{\epsilon}{2}\left[2\,L(\overline{\beta}) - \xi(\overline{\omega})\right], \;\; \Gamma^1_{11} = 2i\epsilon\,\overline{\beta} + \frac{\epsilon}{2}\,\left[2\,L(b) - \overline{L}(\overline{\omega})\right],$$

$$\Gamma^2_{11} = \frac{\epsilon}{2}\, L(\overline{\omega}), \quad \Gamma^3_{11} = \epsilon\, L(\overline{v}),$$

(108)
$$\Gamma^0_{21} = \frac{\epsilon}{2}\, \left[\overline{L}(\overline{\beta}) + L(\beta) - \xi\,(b)\right],$$
$$\Gamma^1_{21} = i\epsilon\beta + \frac{\epsilon}{2}\, L(\omega), \quad \Gamma^2_{21} = -i\epsilon\overline{\beta} + \frac{\epsilon}{2}\, \overline{L}(\overline{\omega}),$$
$$\Gamma^3_{21} = i + \frac{\epsilon}{2}\left[\overline{L}(\overline{v}) + L(v)\right],$$

(109)
$$\Gamma^0_{12} = \frac{\epsilon}{2}\, \left[\overline{L}(\overline{\beta}) + L(\beta) - \xi\,(b)\right],$$
$$\Gamma^1_{12} = i\epsilon\beta + \frac{\epsilon}{2}\, L(\omega), \quad \Gamma^2_{12} = -i\epsilon\overline{\beta} + \frac{\epsilon}{2}\, \overline{L}(\overline{\omega}),$$
$$\Gamma^3_{12} = -i + \frac{\epsilon}{2}\left[\overline{L}(\overline{v}) + L(v)\right],$$

(110)
$$\Gamma^0_{31} = \Gamma^0_{13} = \frac{\epsilon}{2}\, L(\alpha),$$
$$\Gamma^1_{31} = \Gamma^1_{13} = i\,\epsilon\,\alpha + \frac{\epsilon}{2}\, \left[\xi(b) + L(\beta) - \overline{L}(\overline{\beta})\right],$$
$$\Gamma^2_{31} = \Gamma^2_{13} = \frac{\epsilon}{2}\,\xi(\overline{\omega}), \quad \Gamma^3_{31} = \Gamma^3_{13} = \frac{\epsilon}{2}\, [\xi(\overline{v}) + L(u)],$$

and in particular

(111)
$$\Gamma^\lambda_{\lambda 1} = \epsilon\, L(u+b).$$

Moreover

(112)
$$\Gamma^0_{22} = \frac{\epsilon}{2}\, \left[2\,\overline{L}(\beta) - \xi(\omega)\right], \quad \Gamma^1_{22} = \frac{\epsilon}{2}\,\overline{L}(\omega),$$
$$\Gamma^2_{22} = -2i\epsilon\beta + \frac{\epsilon}{2}\, \left[2\,\overline{L}(b) - L(\omega)\right], \quad \Gamma^3_{22} = \epsilon\,\overline{L}(v),$$

(113)
$$\Gamma^0_{32} = \Gamma^0_{23} = \frac{\epsilon}{2}\,\overline{L}(\alpha), \quad \Gamma^1_{32} = \Gamma^1_{23} = \frac{\epsilon}{2}\,\xi(\omega),$$
$$\Gamma^2_{32} = \Gamma^2_{23} = -i\,\epsilon\,\alpha + \frac{\epsilon}{2}\, \left[\xi(b) + \overline{L}(\overline{\beta}) - L(\beta)\right],$$
$$\Gamma^3_{32} = \Gamma^3_{23} = \frac{\epsilon}{2}\, \left[\xi(v) + \overline{L}(u)\right],$$

and then

(114)
$$\Gamma^\lambda_{\lambda 2} = \epsilon\,\overline{L}(u+b).$$

Similarly

(115)
$$\Gamma^0_{33} = \frac{\epsilon}{2}\,\xi(\alpha),$$
$$\Gamma^1_{33} = \frac{\epsilon}{2}\, \left[2\,\xi(\beta) - \overline{L}(\alpha)\right], \quad \Gamma^2_{33} = \frac{\epsilon}{2}\, \left[2\,\xi(\overline{\beta}) - L(\alpha)\right],$$
$$\Gamma^3_{33} = \epsilon\,\xi(u),$$

and then

(116) $$\Gamma^\lambda_{\lambda 3} = \epsilon\, \xi(u+b).$$

Next (by (53) and $X_0(\Gamma^\alpha_{\alpha 0}) = 0$)

(117) $$R_{00} = X_\alpha\,(\Gamma^\alpha_{00}) + \Gamma^\sigma_{00}\Gamma^\alpha_{\alpha\sigma} - \Gamma^\sigma_{\alpha 0}\Gamma^\alpha_{\sigma 0}$$

where (by (102) and (106), (111), (114), respectively by (102)-(103))

$$\Gamma^\sigma_{00}\Gamma^\lambda_{\lambda\sigma} = \Gamma^j_{00}\Gamma^\lambda_{\lambda j} =$$

$$= \epsilon\, L(u+b)\,\Gamma^1_{00} + \epsilon\,\overline{L}(u+b)\,\Gamma^2_{00} + \epsilon\,\xi(u+b)\,\Gamma^3_{00} = O(\epsilon^2),$$

$$\Gamma^\sigma_{\lambda 0}\Gamma^\lambda_{\sigma 0} = -2[1+2\epsilon(u-b)] + 2i\epsilon\left[L(v) - \overline{L}(\overline{v})\right] + O(\epsilon^2),$$

$$X_\lambda\left(\Gamma^\lambda_{00}\right) = \frac{\epsilon}{2}\,\Delta_b a.$$

Consequently (117) becomes

$$R_{00} = \frac{\epsilon}{2}\,\Delta_b a + 2[1+2\epsilon(u-b)] - 2i\epsilon\left[L(v) - \overline{L}(\overline{v})\right] + O(\epsilon^2)$$

and (54) is proved. Next (by $X_0(\Gamma^\alpha_{\alpha 1}) = 0$)

$$R_{10} = X_\alpha\,(\Gamma^\alpha_{01}) + \Gamma^\sigma_{01}\Gamma^\alpha_{\alpha\sigma} - \Gamma^\sigma_{\alpha 0}\Gamma^\alpha_{\sigma 1}\,,$$

$$X_\alpha\,(\Gamma^\alpha_{01}) = \frac{\epsilon}{2}\left[L^2(v) - L\overline{L}(\overline{v}) + L\xi(a)\right] +$$

$$+ i\epsilon\, L(u-b) - i\epsilon\,\overline{L}(\overline{\omega}) - i\epsilon\,\xi(\overline{v}),$$

$$\Gamma^\sigma_{01}\Gamma^\lambda_{\lambda\sigma} = \epsilon\, i\, L(u+b) + O(\epsilon^2),$$

$$\Gamma^\sigma_{\lambda 0}\Gamma^\lambda_{\sigma 1} = -2\epsilon\,\overline{\beta} + \epsilon\, i\,\left[L(b) - \overline{L}(\overline{\omega})\right] + O(\epsilon^2),$$

hence (55) is proved. Collecting the information in (106), (111), (114) and (116) the contracted Christoffel symbols are given by

(118) $$\Gamma^\lambda_{\lambda\sigma} = \begin{cases} 0, & \sigma = 0, \\ \epsilon\, L(u+b), & \sigma = 1, \\ \epsilon\,\overline{L}(u+b), & \sigma = 2, \\ \epsilon\,\xi(u+b), & \sigma = 3, \end{cases}$$

hence

$$R_{20} = X_\alpha\,(\Gamma^\alpha_{02}) + \Gamma^\sigma_{02}\Gamma^\alpha_{\alpha\sigma} - \Gamma^\sigma_{\alpha 0}\Gamma^\alpha_{\sigma 2}\,,$$

$$X_\alpha\,(\Gamma^\alpha_{02}) = \frac{\epsilon}{2}\left[\overline{L}^2(\overline{v}) - L\overline{L}(v) + \xi\overline{L}(a)\right] + i\epsilon\,\left[L(\omega) - \overline{L}(u-b) + \xi(v)\right],$$

$$\Gamma^\sigma_{02}\Gamma^\alpha_{\alpha\sigma} = -i\,\epsilon\,\overline{L}(u+b) + O(\epsilon^2),$$

$$\Gamma^\sigma_{\alpha 0}\Gamma^\alpha_{\sigma 2} = i\,\epsilon\,\left[L(\omega) - \overline{L}(b)\right] - 2\epsilon\beta + O(\epsilon^2),$$

and (56) is shown to hold as well. Similarly

$$R_{30} = X_\lambda \left(\Gamma^\lambda_{03}\right) + \Gamma^\sigma_{03}\Gamma^\lambda_{\lambda\sigma} - \Gamma^\sigma_{\lambda 0}\Gamma^\lambda_{\sigma 3}\,,$$

$$X_\lambda \left(\Gamma^\lambda_{03}\right) = \frac{\epsilon}{2}\left\{\Delta_b u + L\xi(v) + \overline{L}\xi(\overline{v}) + \xi^2(a)\right\},$$

$$\Gamma^\sigma_{03}\Gamma^\lambda_{\lambda\sigma} = O(\epsilon^2),$$

$$\Gamma^\sigma_{\lambda 0}\Gamma^\lambda_{\sigma 3} = i\,\epsilon\,\left[L(\beta) - \overline{L}(\overline{\beta})\right] - 2\,\epsilon\,\alpha + O(\epsilon^2),$$

$$R_{11} = X_\lambda \left(\Gamma^\lambda_{11}\right) - X_1\left(\Gamma^\lambda_{\lambda 1}\right) + \Gamma^\sigma_{11}\Gamma^\lambda_{\lambda\sigma} - \Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 1}\,,$$

$$X_\lambda \left(\Gamma^\lambda_{11}\right) = \epsilon\,\left[L^2(b) + \xi L(\overline{v})\right] + i\,\epsilon\,\left[2\,L(\overline{\beta}) + \xi(\overline{\omega})\right],$$

$$X_1\left(\Gamma^\lambda_{\lambda 1}\right) = \epsilon\,L^2(u+b),$$

$$\Gamma^\sigma_{11}\Gamma^\lambda_{\lambda\sigma} = O(\epsilon^2), \quad \Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 1} = 2\,i\epsilon\,L(\overline{\beta}) + O(\epsilon^2),$$

thus leading to (57)-(58). The calculation of R_{21} is a bit trickier and a few computational details are provided below. By (109) and (118)

$$\Gamma^\sigma_{12}\Gamma^\lambda_{\lambda\sigma} = -i\,\epsilon\,\xi(u+b) + O(\epsilon^2). \tag{119}$$

Moreover

$$\Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 2} = \Gamma^\sigma_{01}\Gamma^0_{\sigma 2} + \Gamma^\sigma_{11}\Gamma^1_{\sigma 2} + \Gamma^\sigma_{21}\Gamma^2_{\sigma 2} + \Gamma^\sigma_{31}\Gamma^3_{\sigma 2} =$$

[by (102), (107), (109) and (110)]

$$= \left\{i\,\left[1 + \epsilon(u-b)\right] + \frac{\epsilon}{2}\left[L(v) - \overline{L}(\overline{v})\right]\right\}\Gamma^0_{12} +$$

$$+\frac{\epsilon}{2}\,\left[\overline{L}(\overline{\beta}) + L(\beta) - \xi(b)\right]\,\Gamma^2_{02} + \left\{i + \frac{\epsilon}{2}\,\left[\overline{L}(\overline{v}) + L(v)\right]\right\}\,\Gamma^2_{32} +$$

$$+\left\{i\,\epsilon\,\alpha + \frac{\epsilon}{2}\,\left[\xi(b) + L(\beta) - \overline{L}(\overline{\beta})\right]\right\}\,\Gamma^3_{12} + O(\epsilon^2)$$

or

$$\Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 2} = i\,\epsilon\,\left[\overline{L}(\overline{\beta}) - L(\beta)\right] + 2\,\epsilon\,\alpha + O(\epsilon^2). \tag{120}$$

Next [by (109) and (118)]

$$X_\lambda\left(\Gamma^\lambda_{12}\right) = \frac{\epsilon}{2}\left[\xi\overline{L}(\overline{v}) + \xi L(v) + L^2(\omega) + \overline{L}^2(\overline{\omega})\right] + \\ +i\,\epsilon\,\left[L(\beta) - \overline{L}(\overline{\beta})\right], \tag{121}$$

$$X_1\left(\Gamma^\lambda_{\lambda 2}\right) = \epsilon\,L\overline{L}(u+b). \tag{122}$$

Finally, substitution from (119)-(122) into

$$R_{21} = X_\lambda\left(\Gamma^\lambda_{12}\right) - X_1\left(\Gamma^\lambda_{\lambda 2}\right) + \Gamma^\sigma_{12}\Gamma^\lambda_{\lambda\sigma} - \Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 2}$$

leads to

$$R_{21} = \frac{\epsilon}{2}\,\left[-2\,L\overline{L}\,(u+b) + \xi L(v) + \xi\overline{L}(\overline{v}) + L^2(\omega) + \overline{L}^2(\overline{\omega})\right] +$$

$$+2\,i\,\epsilon\,\left[L(\beta)-\overline{L}(\overline{\beta})\right]-i\,\epsilon\,\xi(u+b)-2\,\epsilon\,\alpha$$

which is (61) in Lemma 1. Similar calculations

$$R_{12}=X_\lambda\left(\Gamma^\lambda_{21}\right)-X_2\left(\Gamma^\lambda_{\lambda 1}\right)+\Gamma^\sigma_{21}\Gamma^\lambda_{\lambda\sigma}-\Gamma^\sigma_{\lambda 2}\Gamma^\lambda_{\sigma 1}\,,$$
$$X_\lambda\left(\Gamma^\lambda_{21}\right)=X_\lambda\left(\Gamma^\lambda_{12}\right),\quad X_2\left(\Gamma^\lambda_{\lambda 1}\right)=\epsilon\,\overline{L}L(u+b),$$
$$\Gamma^\sigma_{21}\Gamma^\lambda_{\lambda\sigma}=i\Gamma^\lambda_{\lambda 3}=i\,\epsilon\,\xi(u+b)+O(\epsilon^2),$$
$$\Gamma^\sigma_{\lambda 2}\Gamma^\lambda_{\sigma 1}=\Gamma^\lambda_{\sigma 2}\Gamma^\sigma_{\lambda 1}=i\,\epsilon\,\left[\overline{L}(\overline{\beta})-L(\beta)\right]+2\,\epsilon\,\alpha+O(\epsilon^2),$$

$$R_{31}=X_\lambda\left(\Gamma^\lambda_{13}\right)-X_1\left(\Gamma^\lambda_{\lambda 3}\right)+\Gamma^\sigma_{13}\Gamma^\lambda_{\lambda\sigma}-\Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 3}\,,$$
$$X_\lambda\left(\Gamma^\lambda_{13}\right)=\frac{\epsilon}{2}\,\left[L\xi(u+b)+\xi^2(\overline{v})+L^2(\beta)-L\overline{L}(\overline{\beta})+\overline{L}\xi(\overline{\omega})\right]+i\,\epsilon\,L(\alpha),$$
$$X_1\left(\Gamma^\lambda_{\lambda 3}\right)=\epsilon\,L\xi(u+b),$$
$$\Gamma^\sigma_{13}\Gamma^\lambda_{\lambda\sigma}=O(\epsilon^2),\quad \Gamma^\sigma_{\lambda 1}\Gamma^\lambda_{\sigma 3}=i\,\epsilon\,\xi(\overline{\beta})+O(\epsilon^2),$$

lead to (62)-(63). Indeed

$$\Gamma^\lambda_{31}=\Gamma^\lambda_{13}\,,\quad X_3\left(\Gamma^\lambda_{\lambda 1}\right)=\epsilon\,\xi L(u+b),$$

hence $R_{13}=R_{31}$. Finally

$$R_{22}=X_\lambda\left(\Gamma^\lambda_{22}\right)-X_2\left(\Gamma^\lambda_{\lambda 2}\right)+\Gamma^\sigma_{22}\Gamma^\lambda_{\lambda\sigma}-\Gamma^\sigma_{\lambda 2}\Gamma^\lambda_{\sigma 2}\,,$$
$$X_\lambda\left(\Gamma^\lambda_{22}\right)=\epsilon\,\left[\overline{L}^2(b)+\xi\overline{L}(v)\right]-i\,\epsilon\,\left[2\,\overline{\beta}+\xi(\omega)\right],$$
$$X_2\left(\Gamma^\lambda_{\lambda 2}\right)=\epsilon\,\overline{L}^2\,(u+b)\,,$$
$$\Gamma^\sigma_{22}\Gamma^\lambda_{\lambda\sigma}=O(\epsilon^2),\quad \Gamma^\sigma_{\lambda 2}\Gamma^\lambda_{\sigma 2}=-2\,i\,\epsilon\,\overline{L}(\beta)+O(\epsilon^2),$$

$$R_{32}=X_\lambda\left(\Gamma^\lambda_{23}\right)-X_2\left(\Gamma^\lambda_{\lambda 3}\right)+\Gamma^\sigma_{23}\Gamma^\lambda_{\lambda\sigma}-\Gamma^\sigma_{\lambda 2}\Gamma^\lambda_{\sigma 3}\,,$$
$$X_\lambda\left(\Gamma^\lambda_{23}\right)=\frac{\epsilon}{2}\,\left[\overline{L}\xi\,(u+b)+\xi^2(v)+\right.$$
$$\left.+\,\overline{L}^2(\overline{\beta})-\overline{L}L(\beta)+\xi L(\omega)\right]-i\,\epsilon\,\overline{L}(\alpha),$$
$$X_2\left(\Gamma^\lambda_{\lambda 3}\right)=\epsilon\,\overline{L}\xi(u+b),$$
$$\Gamma^\sigma_{23}\Gamma^\lambda_{\lambda\sigma}=O(\epsilon^2),\quad \Gamma^\sigma_{\lambda 2}\Gamma^\lambda_{\sigma 3}=-i\,\epsilon\,\xi(\beta)+O(\epsilon^2),$$

$$R_{33}=X_\lambda\left(\Gamma^\lambda_{33}\right)-X_3\left(\Gamma^\lambda_{\lambda 3}\right)+\Gamma^\sigma_{33}\Gamma^\lambda_{\lambda\sigma}-\Gamma^\sigma_{\lambda 3}\Gamma^\lambda_{\sigma 3}\,,$$
$$X_\lambda\left(\Gamma^\lambda_{33}\right)=\frac{\epsilon}{2}\,\Delta_b\alpha+\epsilon\,\left[\xi^2(u)+L\xi(\beta)+\overline{L}\xi(\overline{\beta})\right],$$
$$X_3\left(\Gamma^\lambda_{\lambda 3}\right)=\epsilon\,\xi^2(u+b),$$
$$\Gamma^\sigma_{33}\Gamma^\lambda_{\lambda\sigma}=\Gamma^\sigma_{\lambda 3}\Gamma^\lambda_{\sigma 3}=O(\epsilon^2),$$

lead to (59)-(60) and (64) in Lemma 1. Q.e.d.

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