Chapter 4

Integration Techniques

4.1 Transformation of Integrals

We defined the Riemann integral $\int_a^b f$ of a function $f : [a, b] \to \mathbb{R}$ through suprema and infima of lower and upper Darboux integrals (and through limits of Riemann sums). While this is a natural concept to begin with, and it can be used to evaluate a few elementary integrals, for more complex tasks, we need to understand how an integral transforms under a map. To get to this we need to replace the functional integrand $f : [a, b] \to \mathbb{R}$ of the integral $\int_a^b f$ by the associated form $\boldsymbol{\alpha} = f(x)dx$ (Section 2.7), and define

$$\int_a^b \boldsymbol{\alpha} = \int_a^b f(x)dx = \int_a^b f.$$

The justification of this more complex concept, the integral of a form, is contained in the following:

Proposition 4.1.1. *Let $g : [a, b] \to [c, d]$, $c = g(a)$, $d = g(b)$, be a differentiable function such that the derivative g' is Riemann integrable on $[a, b]$. Then, for any Riemann integrable form[1] α on $[g(a), g(b)]$, we have*

$$\int_a^b g^*(\alpha) = \int_{g(a)}^{g(b)} \alpha,$$

where the integrand on the left-hand side is the pull-back of α by g.

[1]By definition, a form $\alpha = f(x)\,dx$ is Riemann integrable if the function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof. Letting $\boldsymbol{\alpha} = f(x)dx$, $x \in [g(a), g(b)]$, we have $g^*(f(x)dx) = f(g(x))g'(x)dx$. The formula above is therefore equivalent to

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du,$$

where we used $u = g(x)$ with generic variables x and u. Let $F(x) = \int_a^x f(u)du$, so that $F'(x) = f(x)$. Then the formula is a direct consequence of the chain rule

$$\begin{aligned}\int_a^b f(g(x))g'(x)dx &= \int_a^b F'(g(x))g'(x)dx = \int_a^b (F(g(x)))'dx \\ &= F(g(x))\Big|_a^b = F(u)\Big|_{g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} F'(u)du = \int_{g(a)}^{g(b)} f(u)du.\end{aligned}$$

The formula in Proposition 4.1.1 is usually called the **substitution rule**, and the actual use of the formula is termed as **integration by substitution**. There are literally a miriad integration problems centered around the substitution rule (along with the FTA).

We now begin to enlist the integrals of various elementary functions in increasing complexity.
First, the integral formula for the power function gives the integral of any polynomial explicitly as

$$\int \sum_{k=0}^n a_k x^k \, dx = \sum_{k=0}^n \frac{a_k}{k+1} x^{k+1} + C, \quad a_0, a_1, \ldots, a_n \in \mathbb{R}.$$

Second, the integral of a rational function is always an elementary function. This follows from the partial fraction decomposition of rational functions[2], and from the fact that the integrals of partial fractions are elementary. We illustrate this by a few, somewhat non-standard, examples. We begin with the simplest case of distinct real roots in the denominator.

Example 4.1.1. For $n \in \mathbb{N}_0$, we have

$$\int \frac{dx}{x(x+1)\cdots(x+n)} = \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln|x+k| + C.$$

[2]For a detailed account, see *Elements of Mathematics - History and Foundations*, Section 9.2.

Indeed, this is a direct consequence of the partial fraction decomposition

$$\frac{1}{x(x+1)\cdots(x+n)} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k}.$$

This latter formula can be derived in the usual way, letting

$$\frac{1}{x(x+1)\cdots(x+n)} = \sum_{k=0}^{n}\frac{A_k}{x+k},$$

where the coefficients A_k, $k = 0, 1, \ldots, n$, are to be determined. Eliminating all the denominators by multiplying through $x(x+1)\cdots(x+n)$, and evaluating the obtained equality on the roots $-k$, $k = 0, 1, \ldots, n$, we obtain

$$1 = A_k(-k)\cdot(-k+1)\cdots(-1)\cdot 1\cdot 2\cdots(-k+n) = (-1)^k A_k\cdot k!(n-k)!$$

The stated formula follows.

Remark. The partial fraction decomposition of the previous example can also be derived inductively by letting $q(x) = 1/(x(x+1)\cdots(x+n))$, $x \in \mathbb{R}$, $n \in \mathbb{N}_0$, and noting that it satisfies the inductive formula

$$q_n(x) = \frac{1}{n}\left(q_{n-1}(x) - q_{n-1}(x+1)\right), \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

The next example is the case of irreducible quadratic factors:[3]

Example 4.1.2. Determine

$$\int \frac{dx}{x^4+1}.$$

The usual trick $x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2+1)^2 - (\sqrt{2}x)^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ yields the complete factorization. A simple evaluation gives the partial fraction decomposition

$$\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{1}{4}\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} - \frac{1}{4}\frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}.$$

Using the integral formula

$$\int \frac{dx}{(x+a)^2+b^2} = \frac{1}{b}\arctan\left(\frac{x+a}{b}\right), \quad b > 0,$$

[3] Or, using complex language, two pairs of conjugate complex roots.

a simple computation gives

$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\left(\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right)+C.$$

Oftentimes an integral can be "rationalized;" that is, it can be transformed into an integral of a rational function, which, in turn, can be integrated by using partial fractions. The next example is a typical, if somewhat technical, illustration to this.

Example 4.1.3. Determine the integral

$$\int \sqrt{\tan(x)}\, dx.$$

We first substitute $u = \sqrt{\tan(x)}$; that is, $u^2 = \tan(x)$. This gives $2udu = \sec^2(x)dx$. Using the Pythagorean identity $1+\tan^2(x) = \sec^2(x)$, we obtain $2udu/(u^4+1) = dx$. With this, we get

$$\int \sqrt{\tan(x)}\, dx = \int \frac{2u^2}{u^4+1}\, du.$$

We could proceed as in the previous example, but a simple trick works here better

$$\begin{aligned}
\int \sqrt{\tan(x)}\, dx &= \int \frac{u^2+1}{u^4+1}\, du + \int \frac{u^2-1}{u^4+1}\, du \\
&= \int \frac{1+1/u^2}{u^2+1/u^2}\, du + \int \frac{1-1/u^2}{u^2+1/u^2}\, du \\
&= \int \frac{1+1/u^2}{(u-1/u)^2+2}\, du + \int \frac{1-1/u^2}{(u+1/u)^2-2}\, du
\end{aligned}$$

Substituting $v = u - 1/u$, resp. $w = u + 1/u$ (and hence $dv = (1+1/u^2)du$, resp. $dw = (1-1/u^2)du$), we continue as

$$\begin{aligned}
&\int \sqrt{\tan(x)}\, dx = \int \frac{dv}{v^2+2} + \int \frac{dw}{w^2-2} \\
&= \frac{1}{\sqrt{2}}\arctan\left(\frac{v}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{w-\sqrt{2}}{w+\sqrt{2}}\right| + C \\
&= \frac{1}{\sqrt{2}}\arctan\left(\frac{u-1/u}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{u+1/u-\sqrt{2}}{u+1/u+\sqrt{2}}\right| + C \\
&= \frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{\tan(x)}-\sqrt{\cot(x)}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan(x)}+\sqrt{\cot(x)}-\sqrt{2}}{\sqrt{\tan(x)}+\sqrt{\cot(x)}+\sqrt{2}}\right| + C
\end{aligned}$$

The example follows.

Sometimes there is more emphasis on trigonometry than on rationalization. The following example illustrates this.

Example 4.1.4. Determine

$$\int \frac{dx}{1+\sin(x)}$$

It is well-known that all the six trigonometric functions can be expressed as rational functions of the tangent function with half of the respective angle.[4] In particular, we have

$$\sin(x) = \frac{2\tan(x/2)}{\tan^2(x/2)+1} \quad \text{and} \quad \cos(x) = \frac{1-\tan^2(x/2)}{\tan^2(x/2)+1}.$$

Hence

$$\begin{aligned}
\int \frac{dx}{1+\sin(x)} &= \int \frac{dx}{1+2\tan(x/2)/(\tan^2(x/2)+1)} \\
&= \int \frac{\tan^2(x/2)+1}{\tan^2(x/2)+2\tan(x/2)+1}\,dx \\
&= \int \frac{\sec^2(x/2)}{(\tan(x/2)+1)^2}\,dx = \int \frac{2}{(u+1)^2}\,du \\
&= -\frac{2}{u+1}+C = -\frac{2}{\tan(x/2)+1}+C,
\end{aligned}$$

where we used the substitution $u = \tan(x/2)$.

The next example is well-known to be notorious in resisting several standard methods:

Example 4.1.5. Show that

$$\int_0^1 \frac{\ln(1-x)}{x}dx = -\frac{\pi^2}{6}.$$

By the power series expansion of $\ln(1-x)$ with radius of convergence $\rho = 1$, the function under the integral sign is defined on $(-1,1)$, and can be expanded into the power series

$$\frac{\ln(1-x)}{x} = -\frac{1}{x}\cdot\sum_{n=1}^{\infty}\frac{x^n}{n} = -\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}, \quad |x|<1.$$

[4]See *Elements of Mathematics - History and Foundations*, Section 11.4.

By the corollary to Proposition 3.2.1, we can integrate term-by-term on any interval $[0, b]$, $0 < b < 1$, and obtain

$$\int_0^b \frac{\ln(1-x)}{x} dx = -\int_0^b \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} dx = -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^b x^{n-1} dx = -\sum_{n=1}^{\infty} \frac{b^n}{n^2}.$$

The crux here is that, by the Weierstrass M-test, the infinite series $\sum_{n=1}^{\infty} x^n/n^2$ converges uniformly on the **closed** interval $[0, 1]$ (with $M_n = 1/n^2$, $n \in \mathbb{N}$.) Thus, by Proposition 1.2.2, this infinite sum is left-continuous at $x = 1$. This finally gives

$$\int_0^1 \frac{\ln(1-x)}{x} dx = \lim_{b \to 1^-} \int_0^b \frac{\ln(1-x)}{x} dx = -\lim_{b \to 1^-} \sum_{n=1}^{\infty} \frac{b^n}{n^2} = -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{6},$$

where we used the Euler sum $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Example 4.1.6. Show that[5]

$$\int_0^{\pi/2} \ln(\sin(x))\, dx = \int_0^{\pi/2} \ln(\cos(x))\, dx = -\frac{\pi}{2} \ln 2.$$

First, we have

$$\begin{aligned} \int_0^{\pi/2} \ln(\sin(x))\, dx &= \lim_{a \to 0^+} \int_a^{\pi/2} \ln(\sin(x))\, dx \\ &= \lim_{a \to 0^+} \int_a^{\pi/2} \ln\left(\frac{\sin(x)}{x}\right) dx + \lim_{a \to 0^+} \int_a^{\pi/2} \ln(x)\, dx. \end{aligned}$$

The first integral is finite since $\lim_{x \to 0} \sin(x)/x = 1$ so that the function $f(x) = \ln(\sin(x)/x)$, $x \in (0, \pi/2]$ becomes continuous on $[0, \pi/2]$ by setting $f(0) = 0$. We calculate the second integral as

$$\lim_{a \to 0^+} \int_a^{\pi/2} \ln(x)\, dx = \lim_{a \to 0^+} [x \ln(x) - x]_a^{\pi/2} = \frac{\pi}{2} \ln\left(\frac{\pi}{2}\right) - \frac{\pi}{2} - \lim_{a \to 0^+} a \ln(a).$$

For the last limit, we have

$$\lim_{a \to 0^+} a \ln(a) = \lim_{a \to 0^+} \frac{\ln(a)}{1/a} = \lim_{a \to 0^+} \frac{1/a}{-1/a^2} = -\lim_{a \to 0^+} a = 0.$$

[5] Although well-known, this problem was in the William Lowell Putnam Mathematical Competition, 1953.

We conclude that our integral is finite.
We now use the trigonometric identity $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$, $\alpha \in \mathbb{R}$, and calculate

$$\begin{aligned}\int_0^{\pi/2} \ln(\sin(x))\,dx &= \int_0^{\pi/2} \ln(2\sin(x/2)\cos(x/2))\,dx \\ &= \frac{\pi}{2}\ln 2 + \int_0^{\pi/2} \ln(\sin(x/2))\,dx + \int_0^{\pi/2} \ln(\cos(x/2))\,dx \\ &= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin(u))\,du + 2\int_0^{\pi/4} \ln(\cos(u))\,du,\end{aligned}$$

where we performed the substitution $u = x/2$. Finally, performing yet another substitution $v = \pi/2 - u$ in the last integral, we obtain

$$\int_0^{\pi/4} \ln(\cos(u))\,du = -\int_{\pi/2}^{\pi/4} \ln(\cos(\pi/2 - v))\,dv = \int_{\pi/4}^{\pi/2} \ln(\sin(v))\,dv.$$

Substituting this into the previous computation, we obtain

$$\begin{aligned}\int_0^{\pi/2} \ln(\sin(x))\,dx &= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin(u))\,du + 2\int_{\pi/4}^{\pi/2} \ln(\sin(v))\,dv \\ &= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln(\sin(x))\,dx.\end{aligned}$$

Rearranging, the example follows.

Example 4.1.7. Let $0 < a \in \mathbb{R}$. Show that[6]

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x (1 + \sin(at))^{1/t}\ dt = e^a.$$

By the Euler limit $\lim_{x\to\infty}(1 + 1/x)^x = e$, for the integrand we have

$$\lim_{t\to 0^+} (1 + \sin(at))^{1/t} = \lim_{t\to 0^+} \left((1 + \sin(at))^{1/\sin(at)}\right)^{\sin(at)/t} = e^a.$$

Hence, FTC II gives

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x (1 + \sin(at))^{1/t}\ dt = \lim_{x\to 0^+} (1 + \sin(ax))^{1/x} = e^a.$$

The example follows.

[6]See the problem ($a = 2$) in the first William Lowell Putnam Mathematical Competition, 1938. The right-limit is of technical convenience.

Example 4.1.8. Show that[7] the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}.$$

We make use of the substitution $u = \pi - x$ with $du = -dx$, and calculate

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = -\int_\pi^0 (\pi - u)\frac{\sin u}{1+\cos^2 u}du = \pi\int_0^\pi \frac{\sin u}{1+\cos^2 u}du - \int_0^\pi \frac{u\sin u}{1+\cos^2 u}du,$$

where we used $\sin(\pi-u) = \sin(u)$ and $\cos(\pi-u) = -\cos(u)$. Rearranging, and using another substitution $v = \cos(u)$ with $dv = -\sin(u)$, we have

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin u}{1+\cos^2 u}du = -\frac{\pi}{2}\int_1^{-1} \frac{dv}{1+v^2} = \frac{\pi}{2}\left[\tan^{-1}(v)\right]_{-1}^{1} = \frac{\pi^2}{4}.$$

We finish this section by a well-known non-elementary integral, the arc length of an ellipse.

Example 4.1.9. We consider the arc length of the ellipse given in standard position by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 0 < b < a,$$

over an interval $[p, q]$ with $-a \le p < q \le a$. We claim that the arc length is given by

$$s(p,q) = ab\int_{\sqrt{1-(q/a)^2}}^{\sqrt{1-(p/a)^2}} \frac{1-k^2u^2}{\sqrt{(1-u^2)(1-k^2u^2)}}\,du,$$

where $k^2 = 1 - a^2/b^2$.
The arc length of the graph of a function[8] $f : \mathcal{D} \to \mathbb{R}$ over $[p, q] \subset \mathcal{D} \subset \mathbb{R}$ is given by

$$s(p,q) = \int_p^q \sqrt{1+f'(x)^2}\,dx.$$

We use the equation of the ellipse above with $y = f(x) = (b/a)\sqrt{a^2-x^2}$, $|x| \le a$. Differentiating $f(x)^2 = (b/a)^2(a^2 - x^2)$, we obtain $f(x)f'(x) = -(b/a)^2 x$. Hence

$$1 + f'(x)^2 = 1 + \frac{b^2}{a^2}\frac{x^2}{a^2-x^2} = \frac{a^4 + (b^2-a^2)x^2}{a^2(a^2-x^2)}.$$

[7] Several examples presented here and below are insipired by and are at the level of "integration bees" at various universities.

[8] We assume that f is differentiable and f' is integrable.

With this, the arc length integral takes the form

$$s(p,q) = \frac{1}{a}\int_p^q \frac{a^4 + (b^2-a^2)x^2}{\sqrt{(a^2-x^2)(a^4+(b^2-a^2)x^2)}}\,dx$$

We now perform the substitution $u^2 = (a^2-x^2)/a^2$ (or $x^2 = a^2(1-u^2)$) with $du = -x\,dx/(a\sqrt{a^2-x^2})$. We have $a^4+(b^2-a^2)x^2 = a^2b^2(1-k^2u^2)$, where $k^2 = 1-(a/b)^2$. Using these, the desired form above follows.

Remark. The arc length in the example above can be expressed by the **elliptic integral of the second kind**:

$$E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^{\sin\phi} \frac{\sqrt{1-k^2u^2}}{\sqrt{1-u^2}}\,du = \int_0^{\sin\phi} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}.$$

Exercises

1. Perform the induction to verify the partial fraction decomposition of Example 4.1.1 as indicated in the subsequent remark.

2. Evaluate the integral

$$\int_0^{\pi/2} \frac{\cos x}{1+\cos^2 x}dx.$$

3. Evaluate the integral

$$\int \left(\frac{x^4}{1+x^6}\right)^2 dx.$$

Solution: Use the substitution $u = x^3$ with $du = 3x^2dx$, and partial fractions as

$$\int \frac{(x^3)^2}{(1+x^6)^2}x^2\,dx = \frac{1}{3}\int \frac{u^2}{(1+u^2)^2}\,du = \frac{1}{3}\int \left(\frac{1}{1+u^2} - \frac{1}{(1+u^2)^2}\right) du.$$

Now use another substitution $u = \tan(\theta)$ with $du = \sec^2(\theta)d\theta$, and obtain

$$\begin{aligned}\frac{1}{3}\int \left(\frac{1}{1+u^2} - \frac{1}{(1+u^2)^2}\right) du &= \frac{1}{3}\int \left(\frac{1}{\sec^2(\theta)} - \frac{1}{\sec^4(\theta)}\right) \sec^2(\theta)\,d\theta \\ &= \frac{1}{3}\int (1-\cos^2\theta)\,d\theta = \frac{1}{3}\int \sin^2(\theta)\,d\theta = \frac{1}{6}(\theta - \sin(\theta)\cos(\theta)) + C \\ = \frac{1}{6}\left(\arctan(u) - \frac{u}{1+u^2}\right) + C &= \frac{1}{6}\left(\arctan(x^3) - \frac{x^3}{1+x^6}\right) + C.\end{aligned}$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \int_0^x t\left(1 + \cos\left(\frac{1}{t}\right)\right) dt.$$

Show that f is differentiable everywhere with critical points 0 and $1/((2n+1)\pi)$, $n \in \mathbb{N}$.

4.2 Integration by Parts

The product rule for differentials

$$d(u \cdot v) = v \cdot du \cdot +u \cdot dv$$

immediately gives the following:

Proposition 4.2.1. *We have*

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu.$$

Remark. The integration by parts formula above can also be proved by using the "summation by parts" technique on the respective Riemann sums (Section 2.3).

As an important application of the integration by parts technique, we briefly return to Taylor series and derive yet another form of the Taylor remainder, the so-called **intergral form of the Taylor remainder**, as follows:

Proposition 4.2.2. *Let $f : \mathcal{D} \to \mathbb{R}$, $(c-d, c+d) \subset \mathcal{D}$, $0 < d \in \mathbb{R}$, and $x \in (c-d, c+d) \setminus \{c\}$. Assume that f is differentiable up to order $n+1$ and $f^{(n+1)}$ integrable on the closed interval between x and c. Then, for the Taylor remainder $R_n(x) = f(x) - T_n(x)$, we have*

$$R_n(x) = \int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

PROOF. We use induction with repect to $n \in \mathbb{N}_0$. For $n = 0$, by the FTC I, we have

$$R_0(x) = f(x) - f(c) = \int_c^x f'(t)dt.$$

For the general induction step $n \Rightarrow n+1$, $n \in \mathbb{N}_0$, we use integration by parts (with $u = f^{(n+1)}(t)/(n+1)!$ and $v = -(x-t)^{n+1}$), and calculate

$$\begin{aligned} R_n(x) &= \int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt \\ &= -\left[\frac{f^{(n+1)}(t)}{(n+1)!}(x-t)^{n+1}\right]_c^x + \int_c^x \frac{f^{(n+2)}(t)}{(n+1)!}(x-t)^{n+1} dt \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + \int_c^x \frac{f^{(n+2)}(t)}{(n+1)!}(x-t)^{n+1} dt \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + R_{n+1}(x). \end{aligned}$$

The proposition follows.

We finish this section by a cadre of somewhat challenging integrals. Our first example is well-known. It is termed as **Dirichlet integral**, and is usually derived by using more advanced techniques (such as contour integrals on the complex plane). As usual we insist on elementary methods.

Example 4.2.1. Show that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

First, the integrand, the function under the integral sign, is continuous everywhere except at 0, where it has a removable discontinuity. Since $\lim_{x\to 0} \sin(x)/x = 1$, we remove this singularity by setting this function equal to 1 at 0. Now the integrand is continuous everywhere and hence integrable.
Second, we claim that the improper integral converges; that is, the limit

$$\lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x}\, dx = \int_0^1 \frac{\sin(x)}{x}\, dx + \lim_{R\to\infty} \int_1^R \frac{\sin(x)}{x}\, dx$$

exists;[9] where we split off the finite portion over $[0,1]$ for convenience. For the last limit, we use integration by parts via $u = 1/x$ and $dv = \sin(x)dx$ (and hence $du = -dx/x^2$ and $v = -\cos(x)$), and obtain

$$\int_1^R \frac{\sin(x)}{x}\, dx = \left[-\frac{\cos(x)}{x}\right]_1^R - \int_1^R \frac{\cos(x)}{x^2}\, dx = -\frac{\cos(R)}{R} + \cos 1 - \int_1^R \frac{\cos(x)}{x^2}\, dx.$$

[9] Another proof is by splitting the integral into the infinite series $\sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} (\sin(x)/x)\, dx$ and use the alternating series test; see also Section 4.7.

Letting $R \to \infty$, we have $\lim_{R\to\infty} \cos(R)/R = 0$, and the last integral is **absolutely** convergent since, by comparison, we have

$$\int_1^\infty \left|\frac{\cos(x)}{x^2}\right| dx \le \int_1^\infty \frac{dx}{x^2} = 1.$$

The claim follows.
(Warning: The original integral is not **absolutely** convergent: $\int_1^\infty |\sin(x)|/x\, dx > \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} |\sin(x)| dx/(n\pi) = (2/\pi) \cdot \sum_{n=1}^\infty 1/n = \infty$.)
The crux of this example is to estimate the transformed integral

$$\int_0^{(2n+1)\pi/2} \frac{\sin(x)}{x}\, dx = \int_0^{\pi/2} \frac{\sin((2n+1)t)}{t}\, dt, \quad n \in \mathbb{N},$$

(via the substitution $x = (2n+1)t$) by using the **Lagrange identity**[10]

$$\sum_{k=1}^{n} \cos(2kt) = -\frac{1}{2} + \frac{\sin((2n+1)t)}{2\sin(t)}.$$

To do this, we denote

$$I_n = \int_0^{\pi/2} \frac{\sin((2n+1)t)}{t}\, dt \quad \text{and} \quad J_n = \int_0^{\pi/2} \frac{\sin((2n+1)t)}{\sin(t)}\, dt, \quad n \in \mathbb{N}.$$

Clearly

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \lim_{n\to\infty} \int_0^{(2n+1)\pi/2} \frac{\sin(x)}{x}\, dx = \lim_{n\to\infty} I_n.$$

On the other hand, integrating both sides of the Langrange identity above, we have

$$\sum_{k=0}^{n} \int_0^{\pi/2} \cos(2kt)\, dt = -\frac{\pi}{4} + \frac{1}{2} J_n, \quad n \in \mathbb{N}.$$

Since each term in the sum on the left-hand side vanishes, we obtain

$$J_n = \frac{\pi}{2}, \quad n \in \mathbb{N}.$$

We now calculate the difference

$$\begin{aligned} J_n - I_n &= \int_0^{\pi/2} \left(\frac{1}{\sin(t)} - \frac{1}{t}\right) \sin((2n+1)t)\, dt \\ &= \frac{1}{2n+1} \int_0^{\pi/2} \left(\frac{1}{t^2} - \frac{\cos(t)}{\sin^2(t)}\right) \cos((2n+1)t)\, dt, \end{aligned}$$

[10] See *Elements of Mathematics - History and Foundations*, Exercise 11.3.6.

where we performed integration by parts with $1/\sin(t)-1/t = u$ and $\sin((2n+1)t)dt = dv$. Note that the boundary terms vanish since

$$\left[\left(\frac{1}{\sin(t)} - \frac{1}{t}\right)\frac{\cos((2n+1)t)}{2n+1}\right]_0^{\pi/2} = -\frac{1}{2n+1}\lim_{t\to 0}\left(\frac{1}{\sin(t)} - \frac{1}{t}\right) = 0.$$

(See Example 2.2.6.) Note also that, in the last integral above, the difference is the parentheses has removable discontinuity at 0, since

$$\lim_{t\to 0}\left(\frac{1}{t^2} - \frac{\cos(t)}{\sin^2(t)}\right) = \frac{1}{6}.$$

(See Example 2.2.7.) Since the last integral is bounded as $n \to \infty$, we conclude that $\lim_{n\to\infty}(J_n - I_n) = 0$, or equivalently

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} J_n.$$

Putting these together, we arrive at the following:

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} J_n = \frac{\pi}{2}.$$

The example follows.

Example 4.2.2. Derive the formula

$$\int_0^x t^n \cdot e^t\,dt = \left[\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\cdot x^k\right]\cdot e^x + (-1)^{n+1}n!, \quad n \in \mathbb{N}_0.$$

Letting $u = t^n$ and $v = e^t$, we use integration by parts:

$$\int_0^x t^n \cdot e^t\,dt = \int_0^x t^n d(e^t) = \left[t^n \cdot e^t\right]_0^x - \int_0^x e^t\,d(t^n) = x\cdot e^x - n\int_0^x t^{n-1}\cdot e^t\,dt,$$

since $d(e^t) = e^t\,dt$ and $d(t^n) = nt^{n-1}\,dt$ as forms. Letting $E_n(x)$ denote the integral on the left-hand side, this gives the inductive formula

$$E_n(x) = x^n \cdot e^x - nE_{n-1}(x), \quad n \in \mathbb{N},$$

and $E_0(x) = e^x - 1$. To derive the stated formula we now use induction with respect to $n \in \mathbb{N}_0$. Since the formula clearly holds for $n = 0$, we need only to perform the general induction step $n-1 \Rightarrow n$, $n \in \mathbb{N}$. We have

$$\begin{aligned} E_n(x) &= x^n \cdot e^x - nE_{n-1}(x) \\ &= x^n \cdot e^x - n\left[\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{(n-1)!}{k!}\cdot x^k\right]\cdot e^x - n(-1)^n(n-1)! \\ &= x^n \cdot e^x + \left[\sum_{k=0}^{n-1}(-1)^{n-k}\frac{n!}{k!}\cdot x^k\right]\cdot e^x + (-1)^{n+1}n! \\ &= \left[\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\cdot x^k\right]\cdot e^x + (-1)^{n+1}n!. \end{aligned}$$

The induction is complete, and the formula follows.

Example 4.2.3. Show that

$$\int_0^1 x^a(\ln(x))^n\,dx = (-1)^n\frac{n!}{(a+1)^{n+1}}, \quad -1 < a \in \mathbb{R},\ n \in \mathbb{N}_0.$$

We have the inductive formula for the integral[11]

$$\int x^a(\ln(x))^n\,dx = \frac{x^{a+1}(\ln(x))^n}{a+1} - \frac{n}{a+1}\int x^a(\ln(x))^{n-1}\,dx, \quad x > 0.$$

Indeed, this is a simple application of integration by parts with $u = (\ln(x))^n$ and $dv = x^a dx$. We then have $du = n(\ln(x))^{n-1}/x \cdot dx$ and $v = x^{a+1}/(a+1)$, and the formula follows.
Applying this formula inductively, we obtain

$$\begin{aligned} \int x^a(\ln(x))^n\,dx &= \frac{x^{a+1}}{a+1}\cdot\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!(a+1)^k}(\ln(x))^{n-k} + C, \\ &\qquad x > 0,\ -1 < a \in \mathbb{R},\ n \in \mathbb{N}_0. \end{aligned}$$

For the corresponding definite integral over $[0,1]$ we need to evaluate all boundary terms ($x = 0, 1$) on the right-hand side. All terms in the sum have vanishing right-limit at 0. Indeed, for $0 < b \in \mathbb{R}$ and $m \in \mathbb{N}$, we have the inductive formula

$$\lim_{x\to 0^+} x^b(\ln(x))^m = \lim_{x\to 0^+}\frac{(\ln(x))^m}{1/x^b} = \lim_{x\to 0^+}\frac{m(\ln(x))^{m-1}\cdot 1/x}{-b/x^{b+1}} = -\frac{m}{b}\lim_{x\to 0^+} x^b(\ln(x))^{m-1}.$$

[11] The only restriction for the validity of this formula is $a \neq -1$. (See Example 4.3.1 in the next session.) For $a = -1$, we have $\int (\ln(x))^n\,dx/x = (\ln(x))^{n+1}/(n+1) + C$, $n \neq -1$; and, finally, $\int dx/(x\ln(x)) = \ln(\ln(x)) + C$.

Applying this m times, the claim follows.
Finally, all terms but the last ($k = n$) in the sum vanish at $x = 1$.
The example follows.

The next example can be found in Euler's works:

Example 4.2.4. Show that

$$\int_0^1 \frac{\sin(\ln(x))}{\ln(x)}\,dx = \frac{\pi}{4}.$$

Removing the singularity at 0, we have the power series expansion

$$\frac{\sin(u)}{u} = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n}}{(2n+1)!}, \quad u \in \mathbb{R},$$

(with radius of convergence $\rho = \infty$). Hence

$$\int_0^1 \frac{\sin(\ln(x))}{\ln(x)}\,dx = \int_0^1 \sum_{n=0}^{\infty}(-1)^n \frac{\ln^{2n}(x)}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\int_0^1 \ln^{2n}(x)\,dx,$$

where the interchange of the sum and the (improper integral) is allowed due to uniform convergence of the power series expansion above on closed intervals (Proposition 3.2.1). By Example 4.2.3 just shown, we have

$$\int_0^1 \ln^n(x)\,dx = (-1)^n n!, \quad n \in \mathbb{N}_0.$$

Substituting, we obtain

$$\int_0^1 \frac{\sin(\ln(x))}{\ln(x)}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Here the last equality follows from the elementary integral

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}, \quad 0 \leq x \leq 1,$$

upon substituting $x = 1$, and noting Abel's theorem on power series (Proposition 2.3.3).

The integral in the next example is commonly called the "sophomore's dream."

Example 4.2.5. Show that

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n}.$$

Note that $x^{-x} = e^{-x\ln(x)}$, $x > 0$, and it is extended to 0 (with value $e^0 = 1$)using the right-limit $\lim_{x\to 0^+} x\ln(x) = 0$. Note, in addition, that the negative sign is due to the fact that $\ln(x) < 0$ for $0 < x < 1$.
Using the Taylor series of the natural exponential function (centered at 0), we obtain

$$x^{-x} = \sum_{n=0}^{\infty} \frac{(-x\ln(x))^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n(\ln(x))^n}{n!}, \quad x > 0.$$

Since the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = -x\ln(x)$, $0 < x \leq 1$, and $f(0) = 0$, is continuous, this series converges **uniformly** on $[0,1]$. Applying Proposition 2.1.9 with $f_n(x) = (-1)^n x^n (\ln(x))^n/n!$, $0 < x \leq 1$, and $f_n(0) = 0$, $n \in \mathbb{N}_0$, we obtain

$$\begin{aligned}
\int_0^1 x^{-x}\,dx &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^n(\ln(x))^n}{n!}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n(\ln(x))^n\,dx \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} n^{-n},
\end{aligned}$$

where we used the result of the previous example.
The example follows.

History. The formula in the previous example was dicovered by Johann Bernoulli in 1697, and we essentially followed the classical proof by intergration by parts (and filled the gaps of Bernoulli's proof such as the boundary behavior at 0). See also the modern account by Dunham.[12] Due to its beauty and simplicity, it was termed the "sophomore's dream" by Borwein, Bailey, and Girgensohn.[13] See also Exercise 5 at the end of this section. This is in contrast to the "freshman's dream," the erroneous formula $(a+b)^n = a^n + b^n$ which, however, is true for $a, b \in \mathbb{Z}$ when reduced to $\pmod{n}$. This is due to the elementary number theoretical fact that $n \,\big|\, \binom{n}{k}$ for $k = 1, 2, \ldots, n-1$.

[12]Dunham, W. *The calculus gallery, masterpieces from Newton to Lebesgue*, Princeton Univ. Press, pp. 46-51.
[13]See Borwein, J., Bailey, D.H. and Girgensohn, R., *Experimentation in mathematics: computational paths to discovery*, (2004) 4, 44.

Example 4.2.6. Evaluate the integral[14]

$$W_n = \int_0^{\pi/2} \sin^n(x) = \int_0^{\pi/2} \cos^n(x)\, dx, \quad n \in \mathbb{N}_0.$$

The last equality is because of the identity $\sin(x) = \cos(\pi/2 - x)$. It is therefore enough to consider the sine integral. We have $W_0 = \pi/2$ and $W_1 = 1$. Monotonicity of the integral gives $W_n \geq W_{n+1}$, $n \in \mathbb{N}_0$. We derive an inductive formula for W_n, $n \in \mathbb{N}$. For $2 \leq n \in \mathbb{N}$, we use integration by parts with the casting $u = \sin^{n-1} x$ and $v = -\cos x$; and hence $du = d(\sin^{n-1} x) = (n-1)\sin^{n-2} x \cdot \cos x\, dx$ and $dv = -d(\cos x) = \sin x\, dx$. We obtain

$$\begin{aligned}\int_0^{\pi/2} \sin^n(x)\, dx &= -\left[\sin^{n-1} x \cdot \cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \cdot \sin^{n-2} x\, dx \\ &= (n-1)\int_0^{\pi/2} (1 - \sin^2 x)\sin^{n-2} x\, dx \\ &= (n-1)\int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1)\int_0^{\pi/2} \sin^n x\, dx.\end{aligned}$$

This gives

$$W_n = \frac{n-1}{n} W_{n-2}, \quad 2 \leq n \in \mathbb{N}.$$

Splitting the even and odd cases, and using this formula inductively, for $n \in \mathbb{N}$, we arrive at the following

$$\begin{aligned}W_{2n} &= \int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\pi}{2} \\ W_{2n+1} &= \int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} = \frac{2^{2n}(n!)^2}{(2n+1)!}.\end{aligned}$$

Exercises

1. Derive the pair of integral formulas

$$\begin{aligned}\int e^{-ax} \sin(bx) dx &= -\frac{e^{-ax}}{a^2 + b^2} \left(a \sin(bx) + b \cos(bx)\right) \\ \int e^{-ax} \cos(bx) dx &= -\frac{e^{-ax}}{a^2 + b^2} \left(a \cos(bx) - b \sin(bx)\right).\end{aligned}$$

[14] This is sometimes called the **Wallis** integral in honor of John Wallis (1626–1703); see Section 4.11.

Solution: Use integration by parts twice.

2. Show that

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx.$$

Solution: Perform integration by parts via $\sin^2(x) = u$ and $dx/x^2 = dv$.

3. Determine

$$\int_0^\infty \frac{\sin(\sqrt{x^2+1})\cos(x)}{\sqrt{x^2+1}}\,dx.$$

Solution: Write the integral as

$$\frac{1}{2}\int_0^\infty \frac{\sin(\sqrt{x^2+1}-x)+\sin(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}}\,dx.$$

Split this into two integrals, and use the substitutions $\sqrt{x^2+1} \mp x = t$ to convert them to $\int_0^1 \sin(t)dt/t$ and $\int_1^\infty \sin(t)dt/t$. Finally, use Example 2.6.1.

4. Determine the integral

$$\int \sin^{-1} x\,dx.$$

Solution: Let $x = \sin(\theta)$; then $dx = \cos(\theta)d\theta$, and the integral transforms into $\int \theta\cdot\cos(\theta)\,d\theta$. Finally, use integration by parts.

5. Derive the "other sophomore's dream":

$$\int_0^1 x^x\,dx = -\sum_{n=1}^\infty (-n)^{-n}.$$

6. Use the "generating function technique" to derive the formula in Example 4.2.2 as follows. Let

$$\int_0^x e^{st}\cdot e^t\,dt = \int_0^x e^{(1+s)t}dt = \frac{e^{(s+1)x}-1}{1+s}, \quad x, s \in \mathbb{R}.$$

Expand all expressions in s in power series, integrate, and compare the coeffficients of the powers of s.

7. Derive the integral formulas

$$\int_0^1 (1-x^2)^n dx = \frac{2^{2n}(n!)^2}{(2n+1)!} \quad\text{and}\quad \int_0^1 \frac{dx}{(1-x^2)^n} = \frac{(2n-2)!}{2^{2n-2}((n-1)!)^2}\frac{\pi}{2}.$$

Solution: For the first integral, substitute $x = \cos(\theta)$; for the second, $x = \tan(\theta)$.

8. Use Abel's summation in parts in Section 2.3 to derive the formula

$$\sum_{k=m+1}^{n-1} A_k(b_{k+1} - b_k) = A_n b_n - A_m b_m - \sum_{k=m+1}^{n} b_k(A_k - A_{k-1}), \quad n > m,\ m, n \in \mathbb{N},$$

where $(A_k)_{k=m}^n$ and $(b_k)_{k=m}^n$ are finite sequences. Given a closed interval $[a, b]$, let $\boldsymbol{x} = (x_m, \ldots, x_n)$ be a partition of $[a, b]$ with $a = x_m$ and $b = x_n$. Define $A_k = f(x_k)$ and $b_k = g(x_k)$, $k = m, \ldots, n$, where $f, g : [a, b] \to \mathbb{R}$ are two functions. Rewrite the formula above as

$$\sum_{k=m+1}^{n-1} f(x_k)\frac{g(x_{k+1}) - g(x_k)}{x_{k+1} - x_k}(x_{k+1}-x_k) = [fg]_a^b - \sum_{k=m+1}^{n} \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} g(x_k)(x_k - x_{k-1}).$$

Finally, interpret this as a discrete version (with Riemann sums) of the intergration by parts formula.

4.3 Improper Integrals

A variation on the theme of Example 4.2.3 is the following:

Example 4.3.1. We have

$$\int_1^\infty \frac{(\ln(x))^n}{x^a}\, dx = \frac{n!}{(a-1)^{n+1}}, \quad n \in \mathbb{N},\ 1 < a \in \mathbb{R}.$$

We first perform the substitution $u = \ln(x)$ and obtain

$$\int_1^\infty \frac{(\ln(x))^n}{x^a}\, dx = \int_0^\infty u^n e^{-(a-1)u}\, du.$$

Second, integration by parts (with obvious roles) gives

$$\int_0^\infty u^n e^{-(a-1)u}\, du = \frac{n}{a-1}\int_0^\infty u^{n-1} e^{-(a-1)u}\, du,$$

where the boundary terms vanish:

$$-\frac{1}{a-1}\left[u^n e^{-(a-1)u}\right]_0^\infty = 0, \quad n \in \mathbb{N},\ 1 < a \in \mathbb{R}.$$

Now, a simple induction completes the proof.

We begin with this section with the following simple result:

Proposition 4.3.1. *Let $f : [a, \infty) \to \mathbb{R}$ be bounded on $[a, \infty)$ and integrable on every closed subinterval $[a, b]$, $b > a$, and let $g : [a, \infty) \to \mathbb{R}$ with absolutely convergent improper integral $\int_a^\infty g(x)\, dx$. Then $\int_a^\infty f(x)g(x)\, dx$ is absolutely convergent.*

PROOF. Let $M > 0$ be a bound for f on $[a, \infty)$; that is, $|f(x)| \leq M$, $x \geq a$. By assumption, the restriction $f|[a, b]$ to any interval $[a, b]$, $b > a$, is integrable. By Proposition 3.1.7 (5) and (3), $|f|$ and $|f|\,|g|$ are integable on $[a, b]$. We have

$$\int_a^b |f(x)g(x)|\, dx = \int_a^b |f(x)|\, |g(x)|\, dx \leq M \int_a^b |g(x)|\, dx.$$

Letting $b \to \infty$, we obtain that $\int_a^\infty |f(x)g(x)|\, dx < \infty$, and hence $\int_a^\infty f(x)g(x)\, dx$ is absolutely convergent.

Next we state Abel's theorem on improper integrals as follows:

Proposition 4.3.2. *Let $f : [a, \infty) \to \mathbb{R}$ with convergent improper integral $\int_a^\infty f(x)\, dx$. Let $g : [a, \infty) \to \mathbb{R}$ be bounded and monotonic. Then $\int_a^\infty f(x)g(x)\, dx$ is convergent.*

PROOF. First, note that, being bounded and monotonic, g is integrable over any finite interval $[b, c]$, $a \leq b < c$. Since f is also integrable on $[b, c]$, the second mean value theorem on integrals (Section 3.1) gives

$$\int_b^c f(x)g(x)\, dx = g(b) \int_b^{c_0} f(x)\, dx + g(c) \int_{c_0}^c f(x)\, dx,$$

for some $c_0 \in [b, c]$. We denote $M = \sup_{[a,\infty)} |g|$. Since $\int_a^\infty f(x)\, dx$ is convergent, by the Cauchy criterion of convergence, for every $0 < \epsilon \in \mathbb{R}$, there exists $a < N \in \mathbb{R}$ such that

$$\left| \int_b^c f(x)\, dx \right| < \frac{\epsilon}{2M}, \quad N < b < c.$$

Keeping $N < b < c$, we now estimate

$$\begin{aligned} \left| \int_b^c f(x)g(x)\, dx \right| &= \left| g(b) \int_b^{c_0} f(x)\, dx + g(c) \int_{c_0}^c f(x)\, dx \right| \\ &\leq |g(b)| \left| \int_b^{c_0} f(x)\, dx \right| + |g(c)| \left| \int_{c_0}^c f(x)\, dx \right| < M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

By the Cauchy criterion for convergence, the integral $\int_a^\infty f(x)g(x)\, dx$ converges.

Dirichlet's theorem on improper integrals is the following:

Proposition 4.3.3. *Let $f : [a, \infty) \to \mathbb{R}$ be such that, for all $b > a$, f is integrable on $[a, b]$ and $\left|\int_a^b f(x)\,dx\right| \leq M$, for some M (independent of b). Let $g : [a, \infty) \to \mathbb{R}$ be bounded and monotonic, and $\lim_{x\to\infty} g(x) = 0$. Then $\int_a^\infty f(x)g(x)\,dx$ is convergent.*

Proof. Since g is bounded and monotonic on $[a, \infty)$, it is integrable on any subinterval $[a, b]$. Also f is integrable on $[a, b]$. As in the previous proof, by the second mean value theorem, we have

$$\int_b^c f(x)g(x)\,dx = g(b)\int_b^{c_0} f(x)\,dx + g(c)\int_{c_0}^c f(x)\,dx,$$

for some $c_0 \in [b, c]$. Let $0 < \epsilon \in \mathbb{R}$, and $0 < N \in \mathbb{R}$ such that $|g(x)| < \epsilon/(4M)$ for $x > N$. For $N < b < c$, we estimate

$$\begin{aligned}\left|\int_b^c f(x)g(x)\,dx\right| &= |g(b)|\left|\int_b^{c_0} f(x)\,dx\right| + |g(c)|\left|\int_{c_0}^c f(x)\,dx\right| \\ &\leq \frac{\epsilon}{4M}2M + \frac{\epsilon}{4M}2M = \epsilon.\end{aligned}$$

Once again, Cauchy's criterion finishes the proof.

Remark. There is a close relationship between Abel's test for convergence for series (Proposition 2.3.1) and Abel's theorem on improper integrals (Proposition 4.3.2), as well as the analogous results of Dirichlet (Propositions 2.3.2 and 4.3.3). This will be exhibited in the first Euler-Maclaurin formula in Section 4.16.

Example 4.3.2. Let $s > 0$. Show that

$$\int_1^\infty \frac{\sin(x)}{x^s}\,dx$$

is convergent.
Letting $f(x) = \sin(x)$ and $g(x) = 1/x^s$, $x \geq 1$, $s > 0$, the conditions of Dirichlet's theorem are satisfied since

$$\left|\int_1^b \sin(x)\,dx\right| = \left|-\left[\cos(x)\right]_1^b\right| = |\cos(1) - \cos(b)| \leq 2, \quad 1 < b \in \mathbb{R}.$$

The example follows.

Example 4.3.3. Show that

$$\int_1^\infty \sin(x^2)\,dx$$

is convergent.
Letting $f(x) = 2x\sin(x^2)$ and $g(x) = 1/(2x)$, $x \geq 1$, the conditions of Dirichlet's theorem are satisfied since

$$\left|\int_1^b 2x\sin(x^2)\,dx\right| = \left|-\left[\cos(x^2)\right]_1^b\right| = \left|\cos(1) - \cos(b^2)\right| \leq 2, \quad 1 < b \in \mathbb{R}.$$

The example follows.

Exercises

1. Show that

$$\int_0^\infty \frac{\sin(x)}{x}\arctan(x)\,dx$$

exists.

2. Prove Dirichlet's theorem via integration by parts assuming that g is differentiable, and g' is integrable (on every integval $[a, b]$, $a < b$), using the following steps. (1) Let $F : [a, \infty) \to \mathbb{R}$ be defined by $F(x) = \int_a^x f(u)\,du$, and use integration by parts to show that

$$\int_a^b f(x)g(x)\,dx = \int_a^b F'(x)g(x)\,dx = F(b)g(b) + \int_a^b F(x)(-g'(x))\,dx, \quad a < b,$$

where $-g' \geq 0$. (2) Derive the estimate

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq Mg(b) + M(g(a) - g(b).$$

(3) Finally, let $b \to \infty$.

3. Prove Abel's theorem along the lines of the previous exercise for g differentiable and g' integrable.

4.4 Parametric Integrals

In this section we introduce the technique of interchanging differentiation and integration of a parametric integral (an integral depending on a parameter). This was one of the favorite methods of Feynman; and for this reason, is is sometimes termed as the "Feynman's trick;" a slight misnomer as this method is much older.[15]

[15] For the proofs of the results here, see Lewin, J.W., *Some applications of the bounded convergence theorem for an introductory course on analysis*, Amer. Math. Monthly, 94, 10 (December 1987) 988-993.

Proposition 4.4.1. *Let $I \subset \mathbb{R}$ be an open interval, and $f : [a,b] \times I \to \mathbb{R}$ a function such that, for every $t \in I$, the integral $\int_a^b f(x,t)\,dx$ exists. Let $t_0 \in I$, and assume that, for every $x \in [a,b]$, the derivative $(df/dt)(x,t_0)$ exists, and, as a function of $x \in [a,b]$, it is integrable on $[a,b]$; that is, the integral $\int_a^b (df/dt)(x,t_0)\,dx$ also exists. If*

$$\sup\left\{ \left| \frac{f(x,t) - f(x,t_0)}{t - t_0} \right| \,\middle|\, x \in [a,b],\, t_0 \neq t \in I \right\} < \infty,$$

then, we have

$$\frac{d}{dt} \int_a^b f(x,t)\,dx \bigg|_{t=t_0} = \int_a^b \frac{d}{dt} f(x,t)\bigg|_{t=t_0} dx.$$

Proof. By our assumptions, for a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = t_0$, we have the pointwise limit

$$\lim_{n\to\infty} \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} = \frac{df}{dt}(x,t_0), \quad x \in [a,b],$$

where all the particpating functions as well as the limit are integrable on $[a,b]$. In addition, by the assumption on the supremum, we also have

$$\left| \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \right| \leq M, \quad x \in [a,b],\ n \in \mathbb{N},$$

with some $0 < M \in \mathbb{R}$. We are now in the position to apply Arzelà's bounded convergence theorem (Section 3.2), and conclude

$$\begin{aligned} \frac{d}{dt} \int_a^b f(x,t)\,dx \bigg|_{t=t_0} &= \lim_{n\to\infty} \frac{\int_a^b f(x,t_n)\,dx - \int_a^b f(x,t_0)\,dx}{t_n - t_0} \\ &= \lim_{n\to\infty} \int_a^b \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}\,dx \\ &= \int_a^b \lim_{n\to\infty} \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}\,dx = \int_a^b \frac{df}{dt}(x,t_0)\,dx. \end{aligned}$$

The proposition follows.

Corollary. *Let $I \subset \mathbb{R}$ be an open interval, and $f : [a,b] \times I \to \mathbb{R}$ a function such that, for every $t \in I$, the integral $\int_a^b f(x,t)\,dx$ exists. Assume that, for every $(x,t) \in [a,b] \times I$, the derivative $(df/dt)(x,t)$ exists, and the integral $\int_a^b (df/dt)(x,t)\,dx$ also exists. If*

$$\left| \frac{df}{dt}(x,t) \right| \leq M, \quad (x,t) \in [a,b] \times I,$$

for some $M \in \mathbb{R}$ then, for every $t \in I$, we have

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{d}{dt}f(x,t)\,dx.$$

Applying Arzelà's dominated convergence theorem instead of the bounded convergence theorem, we obtain the analogous result for improper integrals:

Proposition 4.4.2. *Let $I \subset \mathbb{R}$ be an open interval, and $f : [a,\infty) \times I \to \mathbb{R}$ a function such that, for every $t \in I$, the improper integral $\int_a^\infty f(x,t)\,dx$ exists. Assume that, for every $(x,t) \in [a,\infty) \times I$, the derivative $(df/dt)(x,t)$ exists, and the integral $\int_a^\infty (df/dt)(x,t)\,dx$ also exists. If there is an improper integrable function $g : [a,\infty) \to \mathbb{R}$ such that*

$$\left|\frac{df}{dt}(x,t)\right| \leq g(x), \quad (x,t) \in [a,\infty) \times I,$$

then, for every $t \in I$, we have

$$\frac{d}{dt}\int_a^\infty f(x,t)\,dx = \int_a^\infty \frac{d}{dt}f(x,t)\,dx.$$

Remark. This proposition, with obvious modifications, holds for any improper integral, including finite intervals where the integrand is undefined at an end-point.

In Section 3.1 we briefly noted the so-called **Leibniz integral rule**. A more general form of this rule also holds for parametric integrals as follows. Given an open interval $I \subset \mathbb{R}$, let $f : [a_0,b_0] \times I \to \mathbb{R}$ be a continuous function such that the derivative df/dt, $t \in I$, is also continuous on $[a_0,b_0] \times I$. Let $a, b : I \to \mathbb{R}$ be two differentiable functions defined on an open interval $I \subset \mathbb{R}$ such that $a_0 \leq a(t) < b(t) \leq b_0$, $t \in I$. Then we have

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{d}{dt}f(x,t)\,dx, \quad t \in I.$$

For the proof, we define $g : I \to \mathbb{R}$ by

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx, \quad t \in I.$$

The left-hand side of the formula above is $g'(t)$, $t \in I$. For a fixed $t \in I$, we let $0 \neq h \in \mathbb{R}$ be such that $t + h \in I$. Using Proposition 3.1.5, we now calculate

$$g(t+h) - g(t) = \int_{a(t+h)}^{b(t+h)} f(x,t+h)\,dx - \int_{a(t)}^{b(t)} f(x,t)\,dx$$

$$= -\int_{a(t)}^{a(t+h)} f(x,t+h)\,dx + \int_{a(t)}^{b(t)} \left(f(x,t+h) - f(x,t)\right)\,dx + \int_{b(t)}^{b(t+h)} f(x,t+h)\,dx.$$

The MVT I for integrals asserts that

$$\int_{a(t)}^{a(t+h)} f(x,t+h)\,dx = (a(t+h)-a(t))\,f(c(t),t+h),$$

for some $c(t)$ **between** $a(t)$ and $a(t+h)$. Similarly, we have

$$\int_{b(t)}^{b(t+h)} f(x,t+h)\,dx = (b(t+h)-b(t))\,f(d(t),t+h),$$

for some $d(t)$ **between** $b(t)$ and $b(t+h)$. Substituting these, and dividing by $0 \neq h$, we obtain

$$\begin{aligned}\frac{g(t+h)-g(t)}{h} \quad &= \quad -\frac{a(t+h)-a(t)}{h} f(c(t),t+h) + \int_{a(t)}^{b(t)} \frac{f(x,t+h)-f(x,t)}{h}\,dx \\ &\quad + \frac{b(t+h)-b(t)}{h} f(d(t),t+h).\end{aligned}$$

Letting $h \to 0$, and using continuity, we obtain

$$g'(t) = -a'(t)f(a(t),t) + \lim_{h\to 0}\int_{a(t)}^{b(t)} \frac{f(x,t+h)-f(x,t)}{h}\,dx + b'(t)f(b(t),t), \quad t \in I.$$

By the Arzelà bounded convergence theorem (Section 3.2), the limit and the integral can be intechanged. The general Leibniz integral rule follows.

We close this section by a few examples.

Example 4.4.1. Show that

$$\int_0^\infty \frac{e^{-tx}-e^{-(t+a)x}}{x}\,dx = \ln\left(1+\frac{a}{t}\right), \quad t>0,\ t>-a.$$

Differentiating both sides with respect to t, and noting that the differentiation can be interchanged by integration (Proposition 4.4.2), we have

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \frac{e^{-tx}-e^{-(t+a)x}}{x}\,dx &= \int_0^\infty \frac{d}{dt}\frac{e^{-tx}-e^{-(t+a)x}}{x}\,dx \\ = -\int_0^\infty e^{-tx}\,dx + \int_0^\infty e^{-(t+a)x}\,dx &= \left[\frac{e^{-tx}}{t}\right]_0^\infty + \left[\frac{e^{-(t+a)x}}{t+a}\right]_0^\infty \\ = -\frac{1}{t}+\frac{1}{t+a} = \frac{d}{dt}\ln\left(\frac{t+a}{t}\right) &= \frac{d}{dt}\ln\left(1+\frac{a}{t}\right).\end{aligned}$$

Thus, the right- and left-hand sides of the formula to be proved have equal derivatives, and hence they differ by a constant. On the other hand, this constant must be zero since both sides tend to zero as $t \to \infty$. The example follows.

Example 4.4.2. Show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

For brevity, we denote this improper integral by I. We let $F:(0,\infty)\to\mathbb{R}$ be the parametric integral defined by

$$F(t) = \int_0^\infty \frac{e^{-t(1+x^2)}}{1+x^2}\,dx, \quad 0\le t\in\mathbb{R}.$$

Clearly, $F(0) = [\arctan(x)]_0^\infty = \pi/2$; in particular, $F(t)$ exists for all $t\ge 0$ as it is decreasing with $\lim_{t\to\infty} F(t) = 0$ (since $0\le F(t)\le e^{-t}F(0)$, $t\ge 0$.)
We wish to calculate the derivative $F'(t)$, $t>0$ by differentiating under the integral sign. We define $f:[0,\infty)\times(0,\infty)\to\mathbb{R}$ by $f(x,t) = e^{-t(1+x^2)}/(1+x^2)$, $x,t\ge 0$. Letting $\epsilon>0$, all the assumptions of Proposition 4.4.2 hold for f restricted to $[0,\infty)\times(\epsilon,0)$; namely, we have

$$\left|\frac{df}{dt}\right| = e^{-t(1+x^2)} \le e^{-\epsilon(1+x^2)}, \quad x\ge 0,$$

and the function on the right-hand side is integrable on $[0,\infty)$. Applying Proposition 4.4.2, for $t>\epsilon$, we calculate

$$F'(t) = \int_0^\infty \frac{df}{dt}(x,t)\,dx = -\int_0^\infty e^{-t(1+x^2)}\,dx = -e^{-t}\int_0^\infty e^{-tx^2}\,dx = -\frac{e^{-t}}{\sqrt{t}}I,$$

where, in the last equality, we used the substitution $u=\sqrt{t}x$. Since this holds for all $\epsilon>0$, this formula is valid for all $t>0$.
Working backwards, by the FTC II, we have

$$F(t) = I\cdot\int_t^\infty \frac{e^{-s}}{\sqrt{s}}\,ds = 2I\int_{\sqrt{t}}^\infty e^{-u^2}\,du, \quad t>0,$$

where, in the last equality, we used the substitution $u^2=s$. Letting $t\to 0$ (as both sides are continuous at 0), we obtain $F(0)=\pi/2=2I^2$. The example follows.

History. In probability theory, the **normal distribution** is defined by the probability density function

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x\in\mathbb{R}.$$

Here σ is called the **standard deviation** and μ the **variance**. A simple substitution in the integral in the example above shows that the integral of this function over the entire $\mathbb{R}=(-\infty,\infty)$ is equal to 1. The **probability** of a **random variable** to be in an interval (a, b) is equal to the integral of this function over (a, b) (where $a = -\infty$ and $b = \infty$ are also allowed). In his *Doctrine of chances*, the French mathematician Abraham de Moivre (1667−1754) observed first that probabilities of discretely generated random variables (e.g. rolling a die or flipping a coin) can be approximated by what he called the "curve of errors," that is, the normal distribution. This observation was elevated to what was the first version of the **central limit theorem** by Pierre Simon Laplace in his *Théorie analitique des probabilités* published in 1812. The integral of the normal distribution is nowadays called the Gauss-Laplace-Poisson integral. Gauss, in his study of the errors made in astronomical observations, over half a century after de Moivre in 1809, developed his law of observational error, a 2-parameter family of normal distributions. He himself attributed the invention of the normal distribution to Laplace and Poisson. It is sometimes contemplated that the fact that de Moivre's name receded to oblivion in this matter may had something to do with the fact that he was a hugenot, an immigrant of difficult position in French society of the time.[16]

Example 4.4.3. [17] Determine

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx.$$

We let $F : (0, \infty) \to \mathbb{R}$ be the parametric integral defined by

$$F(t) = \int_0^1 \frac{\ln(1+tx)}{1+x^2}\, dx, \quad t > 0.$$

As in the previous example, we wish to calculate $F'(t)$ by differentiating under the integral sign. Clearly, all the assumptions of the corollary to Proposition 4.4.1 hold. (The derivative $(d/dt)(\ln(1 + tx)/(1 + x^2)) = x/((1 + tx)(1 + x^2))$ is bounded for $0 \le x \le 1$ and $t > 0$.) Differentiating (under the integral sign), we obtain

$$F'(t) = \int_0^1 \frac{x\, dx}{(1+tx)(1+x^2)}, \quad t > 0.$$

The integrand can be decomposed into partial fractions (with respect to x) as

$$\frac{x}{(1+tx)(1+x^2)} = -\frac{t}{(1+t^2)(1+tx)} + \frac{x+t}{(1+t^2)(1+x^2)}.$$

[16] For a more complete story, see Stahl, S., *The evolution of the normal distribution*, Mathematics Magazine, Vol. 79, No. 2, (April 2006) 96-113.

[17] This was a problem in the William Lowell Putnam Mathematical Competition, 2005. For another solution, see Math. Magazine, 79 (2006) 76-79.

Integrating, we obtain

$$\begin{aligned} F'(t) &= -\frac{t}{(1+t^2)}\int_0^1 \frac{dx}{1+tx} + \frac{1}{1+t^2}\int_0^1 \frac{x+t}{1+x^2}\,dx \\ &= -\frac{\ln(1+t)}{1+t^2} + \frac{\ln(2)}{2}\frac{1}{1+t^2} + \frac{\pi}{4}\frac{t}{1+t^2}. \end{aligned}$$

This gives

$$F(t) = -\int_0^t \frac{\ln(1+x)}{1+x^2}\,dx + \frac{\ln(2)}{2}\arctan(t) + \frac{\pi}{8}\ln(1+t^2).$$

Evaluating at $t = 1$ and rearranging, we arrive at

$$F(1) = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln(2).$$

Exercises

1. Use Example 4.4.1 to derive the integral

$$\int_0^\infty \frac{e^{-x} - e^{-tx}}{x}\,dx = \ln(t), \quad 0 < t \in \mathbb{R}.$$

Solution: Change the variable from x to tx.

4.5 Double Integrals and Fubini's Theorem

The concept of double integral of a function is a natural extension of that of the integral (in a single variable). While this topic belongs to multivariate calculus, in rare instances the double integral is a powerful tool in deriving some statements in single variable calculus and real analysis. In this section we give a rapid course on this discussing only matters that are absolutely necessary for future developments.

A **rectangle** R in the plane $\mathbb{R}^2$ is the (Cartesian) product of two intervals. A closed, resp. open, rectangle is the product of two closed, resp. open, intervals. If the boundary needs to be specified, we let $R = [a, b] \times [c, d]$, resp. $R = (a, b) \times (c, d)$, denote a generic closed, resp. open, rectangle. Finally, note that $(u, v) \in R$ is an interior point of R if it is contained in an open rectangle that, itself, is contained in R. Equivalently, $(u - \delta_0, u + \delta_0) \times (v - \delta_0, v + \delta_0) \subset R$ for some $\delta_0 > 0$. Finally, a

function $f : R \to \mathbb{R}$ is continous on R if it is continuous at every point of R.
The concepts of limit superior, limit inferior, limit, and continuity of a function $f : R \to \mathbb{R}$ at a point are completely analogous to their one variable counterparts. We only note here that $f : R \to \mathbb{R}$ is continuous at an interior point $(u, v) \in R$ if, for every $\epsilon > 0$, there exists $0 < \delta \leq \delta_0$ such that $|x - u| < \delta$ and $|y - v| < \delta$ imply[18] $|f(x, y) - f(u, v)| < \epsilon$. Finally, f is continuous at a boundary point $(u, v) \in R$ if there exists a function $\tilde{f} : R_0 \to \mathbb{R}$ with $R_0 = (u - \delta_0, u + \delta_0) \times (v - \delta_0, v + \delta_0)$ which is continuous at $(u, v) \in R_0$ such that we have $\tilde{f}|_{R \cap R_0} = f|_{R \cap R_0}$. The basic properties of continuous functions in one variable naturally extend to our multivariate setting.

Let $R = [a, b] \times [c, d]$, $a < b$, $c < d$, $a, b, c, d \in \mathbb{R}$. A **partition** of R is a pair $(\boldsymbol{x}; \boldsymbol{y})$ of partitions $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \Pi(a, b)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_m) \in \Pi(c, d)$, $n, m \in \mathbb{N}$. Geometrically, a partition can be viewed as being composed of the subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The set of all partitions of R are denoted by $\Pi(a, b; c, d)$. The usual concept of refinement and its properties extend from one variable to this setting.

Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a **bounded** real function. For a partition $(\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a, b; c, d)$ as above. we define the **lower**, resp. **upper Darboux sums** of f as

$$\underline{S}_f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \cdot \inf_{R_{ij}} f,$$

resp.

$$\overline{S}_f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \cdot \sup_{R_{ij}} f.$$

With these, we define the **upper**, resp. **lower Darboux integrals**[19] of f by

$$\underline{I}_f = \sup_{(\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a,b;c,d)} \underline{S}_f(\boldsymbol{x}, \boldsymbol{y}), \text{ resp. } \overline{I}_f = \inf_{(\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a,b;c,d)} \overline{S}_f(\boldsymbol{x}, \boldsymbol{y}).$$

By the monotonicity property of the Darboux sums, the supremum and infimum exist, and we have

$$\left(\underline{S}_f(\boldsymbol{x}, \boldsymbol{y}) \leq\right) \underline{I}_f \leq \overline{I}_f \left(\leq \overline{S}_f(\boldsymbol{x}, \boldsymbol{y})\right), \quad (\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a, b; c, d).$$

We say that a **bounded** function $f : [a, b] \times [c, d] \to \mathbb{R}$ is **Darboux integrable**, or simply **integrable**, if equality holds; that is, if we have $\underline{I}_f = \overline{I}_f$. In this case, this common value is called the **(Darboux) integral** of $f : [a, b] \times [c, d] \to \mathbb{R}$, and it is denoted by $\int_R f = \int_{[a,b] \times [c,d]} f$.

[18]The customary condition $\sqrt{(x - u)^2 + (y - v)^2} < \delta$ is clearly equivalent to this. Our emphasis here is on rectangles.

[19]These are also called upper, resp. lower Riemann integrals.

As in single variable, a simple but useful criterion of integrability is the following:

Proposition 4.5.1. *A bounded function $f : [a, b] \times [c, d] \to \mathbb{R}$ is integrable if and only if, for any $\epsilon > 0$, there exists a partition $(\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a, b; c, d)$ such that $\overline{S}_f(\boldsymbol{x}, \boldsymbol{y}) - \underline{S}_f(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$.*

As in the single variable case, we have:

Proposition 4.5.2. *A continuous function $f : [a, b] \times [c, d] \to \mathbb{R}$ is integrable.*

The main result in this section is Fubini's theorem on iterated integrals:

Proposition 4.5.3. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be an integrable function, and assume that, for any $x \in [a, b]$, the function $y \mapsto f(x, y)$, $y \in [c, d]$, is integrable, and that $x \mapsto \int_c^d f(x, y)\, dy$, $x \in [a, b]$, defines an integrable function on $[a, b]$. Then, we have*

$$\int_{[a,b]\times[c,d]} f = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

Remark. The integrand on the left-hand side is not written in the differential form $\int_R f(x, y)\, dx \wedge dy$ as this would require an introduction of the concept of differential 2-forms on $\mathbb{R}^2$. Some authors also use the double integral sign $\int\int$ instead of a single $\int$. The right-hand side is often called an **iterated integral.** We usually suppress the parentheses when there is no danger of confusion.

Proof of Proposition 4.5.3. Let $(\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a, b; c, d)$ be a partition with $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \Pi(a, b)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_m) \in \Pi(c, d)$, $n, m \in \mathbb{N}$. We let $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, be the respective subrectangles. We start with

$$\inf_{R_{ij}} f \le f(x, y) \le \sup_{R_{ij}} f, \quad (x, y) \in R_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

By assumption, the function $x \mapsto f(x, y)$, $y \in [c, d]$, is integrable on $[c, d]$, and hence it is also integrable on any subinterval of $[c, d]$ (Proposition 3.1.5). Thus, integrating the inequalities above over $[y_{j-1}, y_j]$, $j = 1, \ldots, m$, we obtain

$$(y_j - y_{j-1}) \inf_{R_{ij}} f \le \int_{y_{j-1}}^{y_j} f(x, y)\, dy \le (y_j - y_{j-1}) \sup_{R_{ij}} f,$$
$$x \in [x_{i-1}, x_i],\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

Summing over with respect to $j = 1, \ldots, m$, and using Proposition 3.1.5 again, we get

$$\sum_{j=1}^{m}(y_j - y_{j-1}) \inf_{R_{ij}} f \le \int_c^d f(x,y)\,dy \le \sum_{j=1}^{m}(y_j - y_{j-1}) \sup_{R_{ij}} f,\ x \in [x_{i-1}, x_i],\ i = 1, \ldots, n.$$

By assumption, the function $x \mapsto \int_c^d f(x,y)\,dy$, $x \in [a,b]$, is integrable. Integrating over $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, again by Proposition 3.1.5, for $i = 1, \ldots, n$, we have

$$\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \inf_{R_{ij}} f \le \int_{x_{i-1}}^{x_i} \int_c^d f(x,y)\,dy \le \sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \sup_{R_{ij}} f.$$

Summing over with respect to $j = 1, \ldots, m$, and using Proposition 3.1.5 for the last time, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \inf_{R_{ij}} f \le \int_a^b \int_c^d f(x,y)\,dy\,dx \le \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \sup_{R_{ij}} f.$$

Using Darboux sums, this can be written as

$$\underline{S}_f(\boldsymbol{x}, \boldsymbol{y}) \le \int_a^b \int_c^d f(x,y)\,dy\,dx \le \overline{S}_f(\boldsymbol{x}, \boldsymbol{y}).$$

Since the partition $(\boldsymbol{x}, \boldsymbol{y}) \in \Pi(a, b; c, d)$ was arbitrary, we arrive at

$$\underline{I}_f \le \int_a^b \int_c^d f(x,y)\,dy\,dx \le \overline{I}_f.$$

By assumption, f is integrable. This means that $\underline{I}_f = \overline{I}_f = \int_{[a,b]\times[c,d]} f$. The proof is complete.

Remark. The assumptions in the Fubini theorem can be compactly expressed as the integrals

$$\int_c^d f(x,y)\,dy \text{ for all } x \in [a,b]$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

exist.

Corollary. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be an integrable function, and assume that the following integrals exist*

$$\begin{aligned}
&\int_c^d f(x, y)\, dy \text{ for all } x \in [a, b],\\
&\int_a^b f(x, y)\, dx \text{ for all } y \in [c, d],\\
&\int_a^b \int_c^d f(x, y)\, dy\, dx,\\
&\int_c^d \int_a^b f(x, y)\, dx\, dy.
\end{aligned}$$

Then, we have

$$\left(\int_{[a,b]\times[c,d]} f =\right) \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Note that, by Proposition 4.5.2, all assumptions of Proposition 4.5.3 and the subsequent corollary hold if $f : [a, b] \to \mathbb{R}$ is continuous.

Remark. Note that the corollary above automatically extends to improper integrals.

As a final note, all considerations in this section directly extend to multivariate functions $f : [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$, $2 \leq n \in \mathbb{N}$.

Part III

Special Functions

4.6 The Gamma Function

In this introductory section we define the gamma function, show that it is analytic everywhere except having simple poles at non-positive integers, and derive its functional equation.

We define the **gamma function** $\Gamma : (0, \infty) \to \mathbb{R}$ by the integral

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \quad t > 0.$$

We first show that the integral is absolutely convergent, so that the gamma function is well-defined. Let $t > 0$. We first split the integral:

$$\Gamma(t) = \int_0^1 x^{t-1} e^{-x}\, dx + \int_1^\infty x^{t-1} e^{-x}\, dx.$$

We estimate the first integral as

$$\int_0^1 x^{t-1} e^{-x}\, dx \le \int_0^1 x^{t-1}\, dx = \left[\frac{x^t}{t} \right]_0^1 = \frac{1}{t} < \infty.$$

For the second integral, letting $t < m$, $m \in \mathbb{N}$, we estimate

$$\int_1^\infty x^{t-1} e^{-x}\, dx \le m! \int_1^\infty \frac{dx}{x^{m-t+1}} = \frac{m!}{m-t},$$

where we used $e^x \ge x^m/m!$, $x \ge 0$, $m \in \mathbb{N}$. Absolute convergence of the integral follows for any $t > 0$.

Example 4.6.1. Let $-1 < a \in \mathbb{R}$ and $0 < b, c \in \mathbb{R}$. Show that

$$\int_0^\infty t^a e^{-b\,t^c}\, dt = \frac{\Gamma\left(\frac{a+1}{c}\right)}{b^{(a+1)/c} \cdot c}.$$

The substitution $x = b \cdot t^c$ and $bc \cdot t^{c-1} dt = dx$ transforms the integral on the left-hand side into

$$\int_0^\infty t^a e^{-b\,t^c}\, dt = \frac{1}{b^{(a+1)/c} \cdot c} \int_0^\infty x^{(a+1)/c-1} e^{-x}\, dx.$$

The example now follows from the definition of the gamma function.

A related special case of the example above is the improper integral

$$\int_0^1 x^a (-\ln(x))^b\, dx = \frac{\Gamma(b+1)}{(a+1)^{b+1}}, \quad -1 < a, b \in \mathbb{R},$$

where, in the last integral, we first need to perform the substitution $u = -\ln(x)$. For a, b integers, comparing this with the integral of Example 4.3.1, we obtain

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}.$$

History. It was for this last property that the gamma function was introduced by Leonhard Euler in 1729; that is, to extend the factorial to non-integer values. This question was raised earlier by Daniel Bernoulli and Euler's friend, Christian Goldbach. In a letter to Goldbach dated in October 6, 1729, Bernoulli gave the extension as the limit

$$\lim_{n\to\infty} \left(n + 1 + \frac{x}{2}\right) \prod_{k=1}^{n} \frac{k+1}{k+x}, \quad x \in \mathbb{R} \setminus \mathbb{N},$$

along with several numerical examples. Within a few days later, once again in a letter to Goldbach, Euler gave his first extension as the infinite product

$$\Gamma(x+1) = \prod_{n=1}^{\infty} \frac{n^{1-x}(n+1)^x}{n+x} = \prod_{n=1}^{\infty} \frac{(1+1/n)^x}{1+x/n}, \quad x \in \mathbb{R} \setminus \mathbb{N}.$$

For $x = N$, $N \in \mathbb{N}_0$, by inner cancellations, the infinite product reduces to the value $N!$. Moreover, for $x = 1/2$, it gives

$$\Gamma(1/2+1) = \prod_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{n+1/2} = \sqrt{\frac{2 \cdot 4^2 \cdot 6^2 \cdots}{1 \cdot 3^2 \cdot 5^2 \cdots}} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2},$$

where we used the Wallis product formula (Section 4.8; see also below). The appearance of π provided Euler a hint to look for a suitable integral expression. He soon arrived at the formula $\Gamma(t) = \int_0^1 (-\ln(x))^{t-1}\, dx$, $t > 0$; and this appears in a letter[20] to Goldbach, in January 8, 1730. Performing the change of variables $u = -\ln(x)$ (as above) we obtain the integral that we adopted in our definition. It was termed as the Eulerian integral of the **second kind** by Legendre in his *Exercises de Calcul Intégral*, I. (1817) p. 221 (as opposed to the Eulerian integral of the **first kind** defining the beta function; see below in Section 4.11). The use of Γ for the gamma function is due to Legendre in 1814. For a detailed history of the Gamma function, see Godefroy, M., *La fonction Gamma; Théorie, Histoire, Bibliographie*, Gauthier-Villars, Paris (1901). For a more modern account, see Artin, E., *The Gamma Function*, New York, Holt, Rinehart and Winston 1964, and Dover 2015.

Remark. By Example 4.4.2, we have

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx = 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi},$$

[20]Both results submitted to the St. Petersburg Academy on November 28, 1729, under the title *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt.* See also Fuss, N. *Correspondance Mathématique et Physique de Quelques Célebrès Géomètres du XVIII$^{\text{ème}}$ siècle*, 1, Saint-Pétersburg, 1843.

where we used the substitution $u = \sqrt{x}$ (and $du = dx/(2\sqrt{x})$). (Note that this will also be a special numerical case of Euler's reflection formula in Proposition 4.9.1.)

We now turn to the study of the derivatives of the gamma function. As before, we split the defining integral as

$$\Gamma(t) = \int_0^1 x^{t-1} e^{-x} \, dx + \int_1^\infty x^{t-1} e^{-x} \, dx, \quad t > 0.$$

We wish to apply Proposition 4.4.2 which would allow differentiation (d/dt) under the integral sign. We define $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ by $f(x, t) = x^{t-1} e^{-x}$, $x, t > 0$. For $n \in \mathbb{N}$, differentiating n times, we obtain

$$\frac{d^n f}{dt^n}(x, t) = x^{t-1} e^{-x} (\ln(x))^n, \quad x, t > 0.$$

For the first integral, $0 < x \leq 1$, we estimate

$$\left| \frac{d^n f}{dt^n}(x, t) \right| \leq x^{t-1} (-\ln(x))^n, \quad t > 0.$$

The function on the right-hand side of this inequality is (improperly) integrable on $[0, 1]$; in fact, by Example 4.2.3, we have

$$\int_0^1 x^{t-1} (-\ln(x))^n \, dx = \frac{n!}{t^{n+1}}, \quad n \in \mathbb{N}_0, \quad t > 0.$$

Letting $\epsilon > 0$, and applying this to $t \in (\epsilon, \infty)$, by the remark following Proposition 4.4.2, differentiation under the integral sign is allowed. We obtain

$$\frac{d^n}{dt^n} \int_0^1 x^{t-1} e^{-x} \, dx = \int_0^1 x^{t-1} e^{-x} (\ln(x))^n \, dx.$$

Since $\epsilon > 0$ is arbitrary, this formula holds for any $t > 0$.
For the second integral, $x \geq 1$, we let $t + n < m$, $m \in \mathbb{N}$. We estimate

$$\left| \frac{d^n f}{dt^n}(x, t) \right| \leq x^{t-1} e^{-x} (\ln(x))^n \leq m! \frac{x^{t-1} (\ln(x))^n}{x^m} = m! \frac{(\ln(x))^n}{x^{m-t+1}} \leq \frac{m!}{x^{m-n-t+1}},$$

where we used $e^x \geq x^m/m!$, $x \geq 0$, $m \in \mathbb{N}$. Since the functions here are (improperly) integrable over $[1, \infty)$, Proposition 4.4.2 applies. We obtain

$$\frac{d^n}{dt^n} \int_1^\infty x^{t-1} e^{-x} \, dx = \int_1^\infty x^{t-1} e^{-x} (\ln(x))^n \, dx.$$

Putting these together, we obtain that the gamma function is differentiable up to any order, and we have

$$\Gamma^{(n)}(t) = \int_0^\infty x^{t-1}e^{-x}(\ln(x))^n\,dx, \quad t > 0.$$

To estimate the growth rate of $\Gamma^{(n)}(t)$, $t > 0$, in $n \in \mathbb{N}$, we need to refine the upper bound for the second integral above. We fix $1 < b \in \mathbb{R}$, and assume $0 < t \leq b$. With this, we choose $0 < K \in \mathbb{R}$ such that

$$x^{t-1} \leq x^{b-1} \leq Ke^{x/2}, \quad x \geq 1,$$

or equivalently

$$x^{t-1}e^{-x} \leq Ke^{-x/2}, \quad x \geq 1.$$

For $x \geq 1$, $t > 0$, and $n \in \mathbb{N}$, we then have the estimate

$$x^{t-1}e^{-x}(\ln(x))^n \leq Ke^{-x/2}(\ln(x))^n \leq K\,2^n\,n!\left(\frac{\ln(x)}{x}\right)^n,$$

where we also employed the inequality $e^{x/2} \geq (x/2)^n/n!$, $x > 0$, $n \in \mathbb{N}_0$. With this, assuming $2 \leq n \in \mathbb{N}$, we return to the second integral as

$$\int_1^\infty x^{t-1}e^{-x}(\ln(x))^n\,dx \leq K\,2^n\,n!\int_1^\infty \left(\frac{\ln(x)}{x}\right)^n dx \leq K\,2^n\,n!\frac{n!}{(n-1)^{n+1}} \leq K\,2^{n+1}\,n!,$$

where we used Example 4.3.1 (with $a = n$), and the trivial inequality $n! \leq 2(n-1)^{n+1}$, $2 \leq n \in \mathbb{N}$.

Finally, putting this together with the previous estimate of the first integral, we obtain

$$\Gamma^{(n)}(t) \leq \frac{n!}{t^{n+1}} + K\,2^{n+1}\,n!, \quad 2 \leq n \in \mathbb{N},$$

where the constant $K > 0$ depends on the upper bound b of $t > 0$. In particular, for any fixed $t > 0$, there exists $C > 0$ such that

$$\Gamma^{(n)}(t) \leq C^{n+1}\,n!, \quad n \in \mathbb{N}.$$

We obtain that Γ is **analytic** on $(0, \infty)$ (Section 2.4).

The formula $\Gamma(n+1) = n!$, $n \in \mathbb{N}_0$, obtained previously actually holds for non-integer values, and its is known as the **functional equation for the gamma function**:

$$\Gamma(t+1) = t\Gamma(t), \quad t > 0.$$

Indeed, a simple integration by parts in the parametric integral for $\Gamma(t+1)$ (with $x^t = u$ and $e^{-x}dx = dv$) gives

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x}\,dx = \left[-x^t e^{-x}\right]_0^\infty + t\int_0^\infty x^{t-1}e^{-x}\,dx = t\Gamma(t), \quad t > 0.$$

Inductively, we have

$$\Gamma(t+n) = t(t+1)\cdots(t+n-1)\Gamma(t), \quad t > 0,\ n \in \mathbb{N}.$$

In particular, $\Gamma(1/2) = \sqrt{\pi}$ derived above gives

$$\Gamma(1/2+n) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi} \quad \text{and} \quad \Gamma(1/2-n) = (-1)^n\frac{2^{2n}n!}{(2n)!}\sqrt{\pi}, \quad n \in \mathbb{N}.$$

Remark. No closed formulas are known for $\Gamma(1/3)$, resp. $\Gamma(1/4)$, but these numbers are known to be **transcendental**, proved in 1983 by Le Lionnais, resp. by Chudnovsky in 1984. In addition, it is known that, if t is positive rational but non-integral then either $\Gamma(t)$ or $\Gamma(2t)$ is transcendental.[21]

Example 4.6.2. For $a > 0$, we have the power series expansion

$$(1-x)^{-a} = \sum_{n=0}^\infty \frac{\Gamma(a+n)}{n!\Gamma(a)}x^n,$$

with radius of convergence $\rho = 1$.
Indeed, this is a special case of Example 2.3.5, as

$$(1-x)^{-a} = 1 + \sum_{n=1}^\infty \frac{(-1)^n a(a+1)\cdots(a+n-1)}{n!}(-x)^n = \sum_{n=0}^\infty \frac{\Gamma(a+n)}{n!\Gamma(a)}x^n,$$

where we used the functional equation above (inductively).

The functional equation for the gamma function allows to define it for negative values by setting

$$\Gamma(t) = \frac{\Gamma(t+1)}{t}, \quad -1 < t < 0,$$

and then, inductively, by setting

$$\Gamma(t) = \frac{\Gamma(t+n)}{t(t+1)\cdots(t+n-1)}, \quad -n < t < 0,\ -t \notin \mathbb{N}.$$

[21] See Siegel, C.L., *Transcendental numbers*, Ann. Math. Studies, Princeton (1950).

The domain of definition of the extended Γ is $\mathbb{R}\setminus(-\mathbb{N}_0) = \{t \in \mathbb{R} \,|\, t \neq 0, -1, -2, \ldots\}$. Note that the validity of the functional equation for the gamma function above automatically extends to this domain. Finally, note that the the extended gamma function is **analytic** on $\mathbb{R}\setminus(-\mathbb{N}_0)$, since the quotient of analytic functions is analytic (Section 2.4).

A simple consequence of the extension of the gamma function is the limit

$$\lim_{t\to 0} t\Gamma(t) = \Gamma(1) = 1,$$

or equivalently, the asymptotic relation

$$\Gamma(t) \sim \frac{1}{t}, \quad \text{as} \quad t \to 0.$$

By induction, we have

$$\lim_{t\to -n} (t+n)\Gamma(t) = \lim_{t\to -n} \frac{\Gamma(t+n+1)}{t(t+1)\cdots(t+n-1)} = \frac{(-1)^n}{n!},$$

or equivalently

$$\Gamma(t) \sim \frac{(-1)^n}{n!}\frac{1}{t+n}, \quad \text{as} \quad t \to -n, \quad n \in \mathbb{N}_0.$$

We finish this section with a somewhat long winded example of an important pair of improper integrals:

Example 4.6.3. For $0 < s \in \mathbb{R}$, $0 < a \in \mathbb{R}$ and $b \in \mathbb{R}$, have[22]

$$\begin{aligned}\int_0^\infty x^{s-1}e^{-ax}\sin(bx)\,dx &= \Gamma(s)\frac{\sin(s\arctan(b/a))}{(a^2+b^2)^{s/2}}\\ \int_0^\infty x^{s-1}e^{-ax}\cos(bx)\,dx &= \Gamma(s)\frac{\cos(s\arctan(b/a))}{(a^2+b^2)^{s/2}}.\end{aligned}$$

We begin by replacing x by x/a, and letting $t = b/a$. These give

$$\begin{aligned}I(t,s) = \int_0^\infty x^{s-1}e^{-x}\sin(tx)\,dx &= \Gamma(s)\frac{\sin(s\arctan(t))}{(1+t^2)^{s/2}}\\ J(t,s) = \int_0^\infty x^{s-1}e^{-x}\cos(tx)\,dx &= \Gamma(s)\frac{\cos(s\arctan(t))}{(1+t^2)^{s/2}},\end{aligned}$$

[22]The known proofs invariably use complex analysis at various levels. We present here a completely elementary derivation using only real calculus. A proof based on similar ideas is known in the special case $a = \cos\theta$, $b = \sin\theta$, $\theta \in (-\pi/2, \pi/2)$; see the remark at the end of this section.

where we also introduced shorthand notations for the integrals. To derive these, we first claim[23]

$$\begin{aligned} \frac{dI}{dt} &= \frac{s}{1+t^2}\,(J - tI) \\ \frac{dJ}{dt} &= -\frac{s}{1+t^2}\,(I + tJ)\,. \end{aligned}$$

Interchanging the integration and the differentiation (Corollary to Proposition 4.4.1), we obtain

$$\begin{aligned} \frac{dI}{dt}(t,s) &= \frac{d}{dt}\int_0^\infty x^{s-1}e^{-x}\sin(tx)\,dx = \int_0^\infty x^s e^{-x}\cos(tx)\,dx = J(t,s+1) \\ \frac{dJ}{dt}(t,s) &= \frac{d}{dt}\int_0^\infty x^{s-1}e^{-x}\cos(tx)\,dx = -\int_0^\infty x^s e^{-x}\sin(tx)\,dx = -I(t,s+1). \end{aligned}$$

We now continue the computations for the first formula only, the proof of the second is analogous. We use integration by parts via

$$u = x^s \quad \text{and} \quad dv = e^{-x}\cos(tx)\,dx,$$

and hence

$$du = sx^{s-1} \quad \text{and} \quad v = -\frac{e^{-x}}{1+t^2}\,(\cos(tx) - t\sin(tx))\,.$$

(See Exercise 1 at the end of Section 4.2) We now calculate

$$\begin{aligned} \frac{dI}{dt} &= \int_0^\infty x^s e^{-x}\cos(tx)\,dx \\ &= \frac{s}{1+t^2}\left(\int_0^\infty x^{s-1}e^{-x}\cos(tx)\,dx - t\int_0^\infty x^{s-1}e^{-x}\sin(tx)\,dx\right) = \frac{s}{1+t^2}\,(J - tI)\,, \end{aligned}$$

where the boundary terms vanish:

$$\left[-\frac{1}{1+t^2}\,x^s e^{-x}\,(\cos(tx) - t\sin(tx))\right]_0^\infty = 0.$$

The claimed formula follows.

We now introduce

$$A(t,s) = I(t,s)\frac{(1+t^2)^{s/2}}{\sin(s\arctan(t))} \quad \text{and} \quad B(t,s) = J(t,s)\frac{(1+t^2)^{s/2}}{\cos(s\arctan(t))}.$$

[23] The partial derivatives $\partial I/\partial t$ and $\partial J/\partial t$ are meant here. Since, in this passage, the variable s is kept constant, we use regular derivatives for simplicity. In addition, we suppress the variables (t,s) whenever there is no danger of confusion.

We claim

$$\begin{aligned}\frac{dA}{dt} &= -\frac{s}{1+t^2}\cot(s\arctan(t))\,(A-B)\\ \frac{dB}{dt} &= -\frac{s}{1+t^2}\tan(s\arctan(t))\,(A-B)\,.\end{aligned}$$

Once again, we do the computations for the first formula only, the proof of the second is analogous. We calculate

$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}\left(I\frac{(1+t^2)^{s/2}}{\sin(s\arctan(t))}\right) = \frac{dI}{dt}\frac{(1+t^2)^{s/2}}{\sin(s\arctan(t))}\\ &+ tsI\frac{(1+t^2)^{s/2-1}}{\sin(s\arctan(t))} - sI(1+t^2)^{s/2-1}\frac{\cos(s\arctan(t)}{\sin^2(s\arctan(t))}\\ &= \frac{(1+t^2)^{s/2}}{\sin(s\arctan(t))}\left(\frac{s}{1+t^2}(J-tI) + \frac{ts}{1+t^2}I - \frac{s}{1+t^2}I\frac{\cos(s\arctan(t)}{\sin(s\arctan(t))}\right)\\ &= \frac{s}{1+t^2}\frac{\cos(s\arctan(t)}{\sin(s\arctan(t))}\left(J\frac{(1+t^2)^{s/2}}{\cos(s\arctan(t))} - I\frac{(1+t^2)^{s/2}}{\sin(s\arctan(t))}\right),\end{aligned}$$

where we used the formula for the derivative of I obtained above. The claim follows. Subtracting, we obtain

$$\frac{d(A-B)}{dt} = (\tan(s\arctan(t)) - \cot(s\arctan(t)))\,\frac{s}{1+t^2}(A-B).$$

This can be easily resolved as

$$A(t,s) - B(t,s) = C(s)\cdot\exp\left(s\int(\tan(s\arctan(t)) - \cot(s\arctan(t)))\,\frac{dt}{1+t^2}\right),$$

where the constant $C(s)$ depends only on s, and also on the bounds of the indefinite integral. By the substitution $u = \arctan(t)$, the exponent is

$$\begin{aligned}& s\int(\tan(s\arctan(t)) - \cot(s\arctan(t)))\,\frac{dt}{1+t^2} = \int(\tan(u) - \cot(u))\;du\\ &\quad = -\ln|\sin(2u)| = -\ln|\sin(2s\arctan(t)),\end{aligned}$$

where we suppressed the constants. Adjusting the constant $C(s)$, this gives

$$A(t,s) - B(t,s) = \frac{C(s)}{\sin(2s\arctan(t))}.$$

Playing this back to I and J, we obtain

$$I(t,s)\cos(s\arctan(t)) - J(t,s)\sin(s\arctan(t)) = \frac{C(s)}{2(1+t^2)^{s/2}}.$$

Substituting $t=0$ we get $C(s)=0$ (since $I(0,s)=0$). Therefore, we have

$$I(t,s)\cos(s\arctan(t)) = J(t,s)\sin(s\arctan(t)).$$

This gives $A=B$. Hence, $dA/dt = dB/dt = 0$, so that

$$A(t,s) = B(t,s) = G(s),$$

for some function G depending only on s. Once again, playing this back to I and J, we obtain

$$I(t,s) = G(s)\frac{\sin(s\arcsin(t))}{(1+t^2)^{s/2}} \quad \text{and} \quad J(t,s) = G(s)\frac{\cos(s\arcsin(t))}{(1+t^2)^{s/2}}.$$

It remains to show that G is the gamma function. But this follows immediately by substituting $t=0$ into the second formula:

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx = J(0,s) = G(s).$$

The example follows.

Remark. A notable special case of Example 4.6.3 is obtained[24] by setting $a=\cos\theta$, $b=\sin\theta$, $-\pi/2<\theta<\pi/2$. The integral formulas then reduce to

$$\begin{aligned}\int_0^\infty x^{s-1}e^{-x\cos\theta}\sin(x\sin\theta)\,dx &= \Gamma(s)\sin(s\theta)\\ \int_0^\infty x^{s-1}e^{-x\cos\theta}\cos(x\sin\theta)\,dx &= \Gamma(s)\cos(s\theta).\end{aligned}$$

In particular, for $0<s<1$, we have

$$\int_0^\infty x^{s-1}\sin(x)\,dx = \Gamma(s)\sin\left(\frac{\pi s}{2}\right) \text{ and } \int_0^\infty x^{s-1}\cos(x)\,dx = \Gamma(s)\cos\left(\frac{\pi s}{2}\right).$$

[24]See Farrel, O.J. and Ross, B., *Solved problems in analysis*, Dover, 2013; Problems II-22-23, and also Andrews, G.E., Askey, R. and Roy, R. *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge University Press, 1999, Exercise 20 at the end of Chapter 1.

Note that we obtain the correct answer for Example 4.2.1 in the limiting sense for $s \to 0^+$, since

$$\lim_{s\to 0^+} \Gamma(s)\sin\left(\frac{\pi s}{2}\right) = \lim_{s\to 0^+} \Gamma(s+1)\frac{\sin(\pi s/2)}{s} = \frac{\pi}{2}.$$

(For $s = 1$, the integrals are not convergent.) Finally, replacing s by $1-s$, the formulas above take the equivalent form

$$\int_0^\infty \frac{\sin(x)}{x^s}\,dx = \Gamma(1-s)\cos\left(\frac{\pi s}{2}\right) \text{ and } \int_0^\infty \frac{\cos(x)}{x^s}\,dx = \Gamma(1-s)\sin\left(\frac{\pi s}{2}\right),\ 0<s<1.$$

Exercises

1. Use Example 4.4.3 to show that

$$\int_0^1 \frac{\arctan(x)}{1+x}\,dx = \frac{\pi}{8}\ln(2).$$

Solution: Integrate $\int_0^1 \ln(1+x)/(1+x^2)\,dx$ by parts.

2. Show that

$$\int_0^1 \frac{x^t-1}{\ln(x)}\,dx = \ln(1+t), \quad t>0.$$

Solution: The integrand is continuous on $(0,1)$, and can be extended continuously to $[0,1]$ by setting it equal to 0 at $x=0$, and equal to t at $x=1$. In addition, we have $(d/dt)(x^t-1)/\ln(x) = x^t$.

3. Show that

$$\int_0^\infty e^{-x^4}\,dx = \Gamma\left(\frac{5}{4}\right).$$

Solution: Let $u = x^4$.

4. Derive the equivalent representation of the Gamma function

$$\Gamma(t) = 2\int_0^\infty x^{2t-1}e^{-x^2}\,dt, \quad t>0.$$

Solution: Use the substitution $u^2 = -\ln(x)$ in the original definition of the gamma function by Euler in the historical insert.

5. Show that

$$\int_0^\infty u^a \cdot r^{-u}\,du = \frac{\Gamma(a+1)}{(\ln(r))^{a+1}}, \quad -1<a\in\mathbb{R},\ 0<r\in\mathbb{R}.$$

6. Calculate

$$\frac{\Gamma(t+n)}{\Gamma(t-n)}, \quad n \in \mathbb{N},\ 0 < t \in \mathbb{R},\ t-n \neq (-\mathbb{N}_0).$$

Solution: Using the formula $\Gamma(t+1) = t\Gamma(t)$ inductively, this is equal to $(t+n-1)(t+n-2)\cdots(t-n)$.

7. Show that

$$\int_0^\infty e^{-x^2}\cos(tx)\,dx = \frac{\sqrt{\pi}}{2}e^{-t^2/4}, \quad t > 0.$$

Solution: Letting $F : (0,\infty) \to \mathbb{R}$ be the parametric integral $F(t)$ defined by the right-hand side, differentiating under the indefinite integral followed by integration by parts gives $F'(t) = -(t/2)F(t)$, and hence $F(t) = Ce^{-t^2}$. The constant, given by Example 4.4.2, is $C = \sqrt{\pi}/2$.

8. Show that

$$\int_0^\infty \frac{1-e^{-tx^2}}{x^2}\,dx = \sqrt{\pi t}, \quad t > 0.$$

4.7 The Fresnel Integrals

The **Fresnel integrals** are defined as

$$C = \int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\int_0^\infty \frac{\cos(u)}{\sqrt{u}}\,du \quad \text{and} \quad S = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin(u)}{\sqrt{u}}\,du.$$

(Note that the equalities follow by the substitution $u = x^2$.) We will show the convergence of these improper integrals below.

History. The convergence of the improper integral $\int_0^\infty \sin(x^2)\,dx$ was observed by Dirichlet in his paper in the Journal für Math. (1837), p. 60. His primary interest was to exhibit a convergent improper integral $\int_0^\infty f(x)\,dx$ for which $\lim_{x\to\infty} f(x) \neq 0$.

We introduce the parametric integrals[25] $C, S : [0,\infty) \to \mathbb{R}$ defined by

$$C(t) = \int_0^\infty \frac{\cos(t(x^2+1))}{x^2+1}\,dx \quad \text{and} \quad S(t) = \int_0^\infty \frac{\sin(t(x^2+1))}{x^2+1}\,dx, \quad t \geq 0.$$

Clearly, the integrals are convergent, and we have

$$C(0) = \frac{\pi}{2} \quad \text{and} \quad S(0) = 0.$$

[25] This example follows van Yzeren, J., *Moivre's and Fresnel's integrals by simple integration*, Amer. Math. Monthly, Vol. 86, No. 8 (Oct. 1979) 690-693.

It will be of technical convenience to use the trigonometric addition formulas for sine and cosine, and write

$$C(t) = \cos(t)\cdot c(t) - \sin(t)\cdot s(t) \quad \text{and} \quad S(t) = \cos(t)\cdot s(t) + \sin(t)\cdot c(t),$$

where

$$c(t) = \int_0^\infty \frac{\cos(tx^2)}{x^2+1}\,dx = \frac{1}{2}\int_0^\infty \frac{\cos(tu)}{(u+1)\sqrt{u}}\,du$$

and

$$s(t) = \int_0^\infty \frac{\sin(tx^2)}{x^2+1}\,dx = \frac{1}{2}\int_0^\infty \frac{\sin(tu)}{(u+1)\sqrt{u}}\,du.$$

We first claim that

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} S(t) = 0.$$

In view of the linear combinations above, it is enough to derive these limit relations for c and s. We will show that $\lim_{t\to\infty} s(t) = 0$; the proof for c is analogous. Using the second integral for $s(t)$, we observe that the integrand can be interpreted as a damped sine wave with period $2\pi/t$. We therefore write this as an alternating series

$$s(t) = \sum_{n=0}^{\infty} \int_{n\pi/t}^{(n+1)\pi/t} \frac{\sin(tu)}{(u+1)\sqrt{u}}\,du = \sum_{n=0}^{\infty} (-1)^n a_n,$$

where

$$a_n = \int_{n\pi/t}^{(n+1)\pi/t} \frac{|\sin(tu)|}{(u+1)\sqrt{u}}\,du, \quad n\in\mathbb{N}_0.$$

Now the crux is that $(a_n)_{n\in\mathbb{N}_0}$ is a strictly decreasing null-sequence (since the denominator of the integrand is strictly increasing). Hence, by the alternating series test,[26] the series converges. Moreover, once again because this is an alternating series, we have

$$0 < \int_0^\infty \frac{\sin(tu)}{(u+1)\sqrt{u}}\,du < \int_0^{\pi/t} \frac{\sin(tu)}{(u+1)\sqrt{u}}\,du < \int_0^{\pi/t} \frac{du}{\sqrt{u}}\,du = 2\sqrt{\frac{\pi}{t}}, \quad t>0.$$

Letting $t\to\infty$, the claim follows.

Note that the same argument in the use of the alternating series shows that the original integrals C and S (in the second integral form) are convergent; for example, we have

$$S = \frac{1}{2}\int_0^\infty \frac{\sin(u)}{\sqrt{u}}\,du = \frac{1}{2}\sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi} \frac{\sin(u)}{\sqrt{u}}\,du < \frac{1}{2}\int_0^\pi \frac{\sin(u)}{\sqrt{u}}\,du < \frac{1}{2}\int_0^\pi \frac{du}{\sqrt{u}} = \sqrt{\pi}.$$

[26] This simple test, used by Leibniz, is based on the fact that the the odd partial sums increase while even partial sums decrease, and the infinite sum is between the two.

Next, we wish to calculate the derivatives $C'(t)$ and $S'(t)$, $t > 0$, by differentiating under the integral sign. To do this, we first claim that

$$\frac{d}{dt}\int_0^\infty \frac{\cos(tx^2)}{x^2+1}\,dx = \int_0^\infty \frac{d}{dt}\frac{\cos(tx^2)}{x^2+1}\,dx = -\int_0^\infty \frac{x^2\sin(tx^2)}{x^2+1}\,dx = -\frac{1}{2}\int_0^\infty \frac{u\sin(tu)}{(u+1)\sqrt{u}}\,du$$

and

$$\frac{d}{dt}\int_0^\infty \frac{\sin(tx^2)}{x^2+1}\,dx = \int_0^\infty \frac{d}{dt}\frac{\sin(tx^2)}{x^2+1}\,dx = \int_0^\infty \frac{x^2\cos(tx^2)}{x^2+1}\,dx = \frac{1}{2}\int_0^\infty \frac{u\cos(tu)}{(u+1)\sqrt{u}}\,du.$$

The difficulty here is that the integrals on the right-hand sides are not absolutely convergent, so that we cannot use Proposition 4.4.2 to interchange the integrals with differentiation. We therefore need to proceed with the definition of the derivative as a limit. We will do the computations for the second integral, the first is analogous. We have

$$\frac{d}{dt}\int_0^\infty \frac{\sin(tx^2)}{x^2+1}\,dx = \lim_{h\to 0}\frac{1}{h}\int_0^\infty \frac{\sin((t+h)x^2)-\sin(tx^2)}{x^2+1}\,dx$$

The crux here is to reshape the first term in the numerator to resemble the second by a change of variables. We let $t+h = ts^2$, $s > 0$, $h > -t$, for the limit, and then $u = sx$ and $du = sdx$ for the integral. Thus, for the integral of the first term, we have

$$\frac{1}{h}\int_0^\infty \frac{\sin((t+h)x^2)}{x^2+1}\,dx = \frac{1}{t(s^2-1)}\int_0^\infty \frac{\sin(t(sx)^2)}{x^2+1}\,dx = \frac{1}{t(s^2-1)}\int_0^\infty \frac{s\sin(tu^2)}{u^2+s^2}\,du$$

We now rebaptize the variable u back to x, substitute this integral back to the limit above, and calculate

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty \frac{\sin(tx^2)}{x^2+1}\,dx &= \lim_{s\to 1}\frac{1}{t(s^2-1)}\int_0^\infty \left(\frac{s\sin(tx^2)}{x^2+s^2} - \frac{\sin(tx^2)}{x^2+1}\right)dx \\
&= \lim_{s\to 1}\frac{1}{t(s+1)}\int_0^\infty \frac{(x^2-s)\sin(tx^2)}{(x^2+s^2)(x^2+1)}\,dx \\
&= \frac{1}{2t}\lim_{s\to 1}\int_0^\infty \frac{(x^2-s)\sin(tx^2)}{(x^2+s^2)(x^2+1)}\,dx.
\end{aligned}$$

We are now in the position to apply Arzelà's dominated convergence theorem (for a sequence$(s_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} s_n = 1$) since the integrand

$$\left|\frac{(x^2-s)\sin(tx^2)}{(x^2+s^2)(x^2+1)}\right| < \frac{1}{x^2+1}.$$

Interchanging the limit with the integral, we arrive at

$$\frac{d}{dt}\int_0^\infty \frac{\sin(tx^2)}{x^2+1}\,dx = \frac{1}{2t}\int_0^\infty \frac{(x^2-1)\sin(tx^2)}{(x^2+1)^2}\,dx$$

Finally, we perform integration by parts for the integral on the right-hand side with

$$u=\sin(tx^2) \quad \text{and} \quad dv = \frac{x^2-1}{(x^2+1)^2}\,dx,$$

and hence

$$du = 2tx\cos(tx^2) \quad \text{and} \quad v=-\frac{x}{x^2+1}.$$

We obtain

$$\frac{1}{2t}\int_0^\infty \frac{(x^2-1)\sin(tx^2)}{(x^2+1)^2}\,dx = \int_0^\infty \frac{x^2\cos(tx^2)}{x^2+1}\,dx,$$

since the boundary terms vanish:

$$\left[-\frac{x\sin(tx^2)}{x^2+1}\right]_0^\infty = 0.$$

The claim now follows.
We rewrite these differentiation formulas as

$$\begin{aligned} c'(t) &= -\int_0^\infty \frac{x^2\sin(tx^2)}{x^2+1}\,dx = -\int_0^\infty \sin(tx^2)\,dx + \int_0^\infty \frac{\sin(tx^2)}{x^2+1}\,dx = -\frac{S}{\sqrt{t}} + s(t) \\ s'(t) &= \int_0^\infty \frac{x^2\cos(tx^2)}{x^2+1}\,dx = .\int_0^\infty \cos(tx^2)\,dx - \int_0^\infty \frac{\cos(tx^2)}{x^2+1}\,dx = \frac{C}{\sqrt{t}} - c(t) \end{aligned}$$

Applying these differentiation rules to calculate to the linear combinations for C and S above, we have

$$C'(t) = -\sin(t)c(t) + \cos(t)c'(t) - \cos(t)s(t) - \sin(t)s'(t) = -S\frac{\cos(t)}{\sqrt{t}} - C\frac{\sin(t)}{\sqrt{t}}$$

and

$$S'(t) = -\sin(t)s(t) + \cos(t)s'(t) + \cos(t)c(t) + \sin(t)c'(t) = C\frac{\cos(t)}{\sqrt{t}} - S\frac{\sin(t)}{\sqrt{t}}.$$

Finally, integrating over $[0,\infty)$ and using the limit relations for C and S, we obtain

$$-\frac{\pi}{2} = -2SC - 2CS \quad \text{and} \quad 0 = 2C^2 - 2S^2.$$

These give

$$C = S = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

History. The **Fresnel functions** are defined by

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos(v^2)\, dv = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos(u)}{\sqrt{u}}\, du$$

and

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin(v^2)\, dv = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin(u)}{\sqrt{u}}\, du.$$

The "normalizing constants" are chosen such that

$$\lim_{x\to\infty} C(x) = \lim_{x\to\infty} S(x) = \frac{1}{2}.$$

The power series expansion of cosine and sine give

$$C(x) = \sqrt{\frac{2}{\pi}} x \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!(4k+1)} \qquad S(x) = \sqrt{\frac{2}{\pi}} x \sum_{k=0}^{\infty} (-1)^k \frac{x^{2(2k+1)}}{(2k+1)!(4k+3)}$$

both with radius of convergence $\rho = \infty$.

We close this section by revisiting Example 4.2.1.

Example 4.7.1. Show that

$$\int_0^{\infty} \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

By the alternating series test method of the previous example, splitting the integral into the sum

$$\int_0^{\infty} \frac{\sin(x)}{x}\, dx = \sum_{n=0}^{\infty} (-1)^n \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}\, dx,$$

the improper integral converges.

We consider the parametric integral $F : (0, \infty) \to \mathbb{R}$ given by

$$F(t) = \int_0^{\infty} e^{-tx} \frac{\sin(x)}{x}\, dx, \quad t > 0.$$

First, we wish to apply Proposition 4.4.2 to determine F. (Note that this is a limiting case of Example 4.6.3.) We need to verify that the assumptions of this proposition hold.

We let $f : [0, \infty) \times (0, \infty)$ be defined by $f(x, t) = e^{-tx} \sin(x)/x$, $x, t > 0$, and $f(0, t) =$

1 (the right-limit at $x = 0$). For any $t > 0$, this improper integral is absolutely convergent, since

$$\int_0^\infty |f(x,t)|\,dx = \int_0^\infty e^{-tx}\frac{|\sin(x)|}{x}\,dx \le \int_0^\infty e^{-tx}\,dx = \frac{1}{t}.$$

Moreover, the same holds for the integral

$$\int_0^\infty \left|\frac{df}{dt}(x,t)\right|\,dx \le \int_0^\infty e^{-tx}|\sin(x)|\,dx \le \int_0^\infty e^{-tx}\,dx = \frac{1}{t}.$$

For any $\epsilon > 0$, on the interval (ϵ, ∞), we have the upper estimate

$$\left|\frac{df}{dt}(x,t)\right| = e^{-tx}|\sin(x)| \le e^{-tx} < e^{-\epsilon x}, \quad x \ge 0,\ t > \epsilon.$$

Now we can apply Proposition 4.4.2 with $I = (\epsilon, \infty)$ and $g : [0, \infty) \to \mathbb{R}$ given by $g(x) = e^{-\epsilon x}$, $x \ge 0$. For $t > \epsilon$, this gives

$$\frac{d}{dt}\int_0^\infty e^{-tx}\frac{\sin(x)}{x}\,dx = \frac{d}{dt}\int_0^\infty f(x,t)\,dx = \int_0^\infty \frac{df}{dt}(x,t)\,dx = -\int_0^\infty e^{-tx}\sin(x)\,dx.$$

Since $\epsilon > 0$ was arbitrary, this formula holds for **all** $t > 0$.
Now, a simple integration by parts (twice), gives

$$\int_0^\infty e^{-tx}\sin(x)\,dx = \frac{1}{1+t^2}, \quad t > 0.$$

Putting everything together, we obtain

$$F'(t) = \frac{dF}{dt} = -\frac{1}{1+t^2}, \quad t > 0.$$

Integrating, we get

$$F(t) = -\arctan(t) + C, \quad t > 0.$$

To calculate the value of C, we let $t \to \infty$, and obtain

$$\lim_{t\to\infty}\int_0^\infty e^{-tx}\frac{\sin(x)}{x}\,dx = \lim_{t\to\infty} F(t) = -\frac{\pi}{2} + C,$$

On the other hand, we can evaluating the limit on the left-hand side on a divergent sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \ge 1$, $n \in \mathbb{N}$, say. By Arzelà's dominated convergence theorem with $f_n(x) = e^{-t_n x}\sin(x)/x$, $x > 0$, and $f_n(0) = 1$, $n \in \mathbb{N}$, this limit is zero

since $\lim_{n\to\infty} f_n = 0$ pointwise[27] on $(0, \infty)$ (and the theorem applies as $|f_n(x)| \le e^{-x}$, $x \ge 0$, $n \in \mathbb{N}$).
The formula above then reduces to $C = \pi/2$. Using this, we get

$$F(t) = \int_0^\infty e^{-tx} \frac{\sin(x)}{x}\, dx = -\arctan(t) + \frac{\pi}{2} = \arctan\left(\frac{1}{t}\right), \quad t > 0.$$

(Once again, note that this is a limiting case of Example 4.6.3.) Since the right-hand side is continuous at $t = 0$, it remains to show that

$$\lim_{t\to 0^+} \int_0^\infty e^{-tx} \frac{\sin(x)}{x}\, dx = \int_0^\infty \lim_{t\to 0^+} e^{-tx} \frac{\sin(x)}{x}\, dx = \int_0^\infty \frac{\sin(x)}{x}\, dx.$$

It is enough to show this on the positive null-sequence $(t_n)_{n\in\mathbb{N}}$.
As usual, we split the integral as

$$\int_0^\infty e^{-t_n x} \frac{\sin(x)}{x}\, dx = \int_0^1 e^{-t_n x} \frac{\sin(x)}{x}\, dx + \int_1^\infty e^{-t_n x} \frac{\sin(x)}{x}\, dx, \quad n \in \mathbb{N}.$$

For the first integral on the right-hand side, we apply Arzelà's bounded convergence theorem with $f_n(x) = e^{-t_n x} \sin(x)/x$, $0 < x \le 1$, and $f_n(0) = 1$. (Note that $|f_n| \le 1$, $n \in \mathbb{N}$, as $\sin(x) \le x$, $x \in [0, 1]$.) We obtain

$$\lim_{n\to\infty} \int_0^1 e^{-t_n x} \frac{\sin(x)}{x}\, dx = \int_0^1 \lim_{n\to\infty} e^{-t_n x} \frac{\sin(x)}{x}\, dx = \int_0^1 \frac{\sin(x)}{x}\, dx.$$

For the second integral, we first perform integration by parts with $u = 1/x$ and $dv = e^{-t_n x} \sin(x)\, dx$, and hence $du = -dx/x^2$ and

$$v = -e^{-t_n x} \frac{t_n \sin(x) + \cos(x)}{1 + t_n^2}.$$

This gives

$$\int_1^\infty e^{-t_n x} \frac{\sin(x)}{x}\, dx = e^{-t_n} \frac{t_n \sin(1) + \cos(1)}{1 + t_n^2} - \frac{1}{1 + t_n^2} \int_1^\infty e^{-t_n x} \frac{t_n \sin(x) + \cos(x)}{x^2}\, dx,$$

since the boundary terms

$$\left[-\frac{e^{-t_n x}}{x} \frac{t_n \sin(x) + \cos(x)}{1 + t_n^2} \right]_1^\infty = e^{-t_n} \frac{t_n \sin(1) + \cos(1)}{1 + t_n^2}.$$

[27] For $x = 0$, this limit is 1, but the limit is still integrable on $[0, \infty)$ with zero integral.

Letting $n \to \infty$, up to this point, we have

$$\lim_{n\to\infty} \int_1^\infty e^{-t_n x} \frac{\sin(x)}{x}\,dx = \cos(1) - \lim_{n\to\infty} \int_1^\infty e^{-t_n x} \frac{t_n \sin(x) + \cos(x)}{x^2}\,dx.$$

Finally, Arzelà's dominated convergence theorem can be applied to the last integral with $f_n(x) = e^{-t_n x}(t_n \sin(x) + \cos(x))/x^2$, $x \geq 1$, $n \in \mathbb{N}$. (Note that $|f_n(x)| \leq M/x^2$, $x \in [1, \infty)$, $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}}(1 + t_n)$.) We obtain

$$\begin{aligned}\lim_{n\to\infty} &\int_1^\infty e^{-t_n x} \frac{t_n \sin(x) + \cos(x)}{x^2}\,dx \\ &= \int_1^\infty \lim_{n\to\infty} \left(e^{-t_n x} \frac{t_n \sin(x) + \cos(x)}{x^2} \right) dx = \int_1^\infty \frac{\cos(x)}{x^2}\,dx.\end{aligned}$$

Putting this together, we get

$$\lim_{n\to\infty} \int_1^\infty e^{-t_n x} \frac{\sin(x)}{x}\,dx = \cos(1) - \int_1^\infty \frac{\cos(x)}{x^2}\,dx = \int_1^\infty \frac{\sin(x)}{x}\,dx.$$

(For the last equality, see the beginning of Example 4.2.1.) The example follows.

Exercises.

1. For $1 < a \in \mathbb{R}$, determine the integrals

$$\int_0^\infty \sin(x^a)\,dx \quad \text{and} \quad \int_0^\infty \cos(x^a)\,dx.$$

Solution: We have

$$\int_0^\infty \sin(x^a)\,dx = \frac{1}{a} \int_0^\infty \frac{\sin(u)}{u^{1-1/a}}\,du,$$

and similarly for the cosine.

4.8 The Wallis Product and Euler's Sine Product

As a classical example of the integration by parts technique, we derive here the Wallis product formula[28]

$$\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n!)^2(2n+1)} = \frac{\pi}{2},$$

where the first equality is the definition of the infinite product, the second is a simple algebraic equivalent via

$$\frac{4k^2}{4k^2-1} = \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2^4k^4}{(2k-1)(2k)(2k)(2k+1)}, \quad k \in \mathbb{N},$$

and the third is the Wallis product formula.

History. This product formula was discovered and published by John Wallis in his *Arithmetica Infinitorum* in 1656.

To obtain the Wallis product formula, we return to the **Wallis integral** $W_n = \int_0^{\pi/2} \sin^n x\, dx$, $n \in \mathbb{N}$, in Example 4.2.6. The inductive formula for even indices can be written as

$$\frac{W_{2n+2}}{W_{2n}} = \frac{2n+1}{2n+2}, \quad n \in \mathbb{N}.$$

Using monotonicity of the sequence, we obtain

$$\frac{2n+1}{2n+2} = \frac{W_{2n+2}}{W_{2n}} \le \frac{W_{2n+1}}{W_{2n}} \le 1,$$

Letting $n \to \infty$, we obtain the limit

$$\lim_{n\to\infty} \frac{W_{2n+1}}{W_{2n}} = 1.$$

On the other hand, the explicit formulas at the end of Example 4.2.6 give

$$\frac{W_{2n+1}}{W_{2n}} = \frac{2^2 \cdot 4^2 \cdots (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2(2n+1)} \frac{2}{\pi} = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \frac{2}{\pi}.$$

[28]There are a handful of proofs of the Wallis product formula that do not require integration by parts, but use elementary if sophisticated arguments. See, for example, the proof of the Yaglom brothers in A.M. Yaglom and I.M. Yaglom, *An elementary derivation of the formulas of Walllis, Leibniz and Euler for the number* π, Uspechi matematiceskich nauk. (N. S.) 57 (1953) 181-187 (in Russian). Another simpler proof is given by J. Wästlund, *An elementary proof of the Wallis product formula for* π, Linköping studies in Mathematics, No. 2, February 21, 2005.

Substituting this to the limit above, the Wallis formula now follows.

As an interesting byproduct, combining the last estimates, we have

$$\frac{2n+1}{2n+2} \leq \frac{W_{2n+1}}{W_{2n}} = \frac{2^2 \cdot 4^2 \cdots (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2(2n+1)} \frac{2}{\pi} \leq 1$$

Rearranging, this gives

$$\frac{\pi}{4} \frac{1}{n+1} \leq \left(\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \right)^2 \leq \frac{\pi}{4} \frac{1}{n+1/2}.$$

Taking the square root, and using the formula for W_{2n+1} in Example 4.2.6, we obtain the estimate

$$\frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{n+1}} \leq \int_0^{\pi/2} \sin^{2n+1}(x)\, dx \leq \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{n+1/2}}, \quad n \in \mathbb{N}_0.$$

Changing parity, starting from the ineqalities $W_{2n+1} \leq W_{2n} \leq W_{2n-1}$, $n \in \mathbb{N}$, in an entirely analogous manner we can derive the estimate

$$\frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{n+1/2}} \leq \int_0^{\pi/2} \sin^{2n}(x)\, dx \leq \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

These last two estimates can be combined to give

$$\sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{m+1}} \leq W_m = \int_0^{\pi/2} \sin^m(x)\, dx \leq \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{m}}, \quad m \in \mathbb{N}.$$

As a beautiful (direct) application of the Wallis formula (actually, its consequence, the integral estimates above), we now derive the "Gaussian" integral in Example 4.4.2:

Example 4.8.1. We have

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

We will derive sharp upper and lower bounds for the integral using the following simple inequalities[29]

$$e^{-x^2} \leq \frac{1}{1+x^2}, \; x \geq 0, \quad \text{and} \quad 1 - x^2 \leq e^{-x^2}, \; 0 \leq x \leq 1.$$

[29]See *Elements of Mathematics - History and Foundations*, Section 10.1.

For the upper bound, for $2 \le n \in \mathbb{N}$, we estimate

$$\begin{aligned}\int_0^\infty e^{-x^2}\,dx &= \sqrt{n}\int_0^\infty e^{-nx^2}\,dx = \int_0^\infty \frac{1}{(1+x^2)^n}\,dx = \sqrt{n}\int_0^{\pi/2}\frac{\sec^2(u)}{(1+\tan^2(u))^n}\,du\\ &= \sqrt{n}\int_0^{\pi/2}\cos^{2n-2}(u)\,du = \sqrt{n}\int_0^{\pi/2}\sin^{2n-2}(u)\,du \le \frac{\sqrt{\pi}}{2}\sqrt{\frac{n}{n-1}}.\end{aligned}$$

For the lower bound, for $n \in \mathbb{N}$, we split the integral as

$$\begin{aligned}&\int_0^\infty e^{-x^2}\,dx \ge \int_0^{\sqrt{n}} e^{-x^2}\,dx = \sqrt{n}\int_0^1 e^{-nx^2}\,dx \ge \sqrt{n}\int_0^1 (1-x^2)^n\,dx\\ &= \sqrt{n}\int_0^{\pi/2}(1-\cos^2(u))^n\sin(u)\,du = \sqrt{n}\int_0^{\pi/2}\sin^{2n+1}(u)\,du \ge \frac{\sqrt{\pi}}{2}\sqrt{\frac{n}{n+1/2}}.\end{aligned}$$

Combining these, we obtain

$$\frac{\sqrt{\pi}}{2}\sqrt{\frac{n}{n+1/2}} \le \int_0^\infty e^{-x^2}\,dx \le \frac{\sqrt{\pi}}{2}\sqrt{\frac{n}{n-1}}, \quad 2 \le n \in \mathbb{N}.$$

Letting $n \to \infty$, the example follows.

Euler Infinite Product Formulas.[30] *For $t \in \mathbb{R}$, we have*

$$\begin{aligned}\sin(\pi t) &= \pi t\prod_{k=1}^\infty\left(1-\frac{t^2}{k^2}\right)\\ \cos(\pi t) &= \prod_{k=0}^\infty\left(1-\frac{4t^2}{(2k+1)^2}\right).\end{aligned}$$

PROOF.[31] We first claim the inductive formula

$$\int_0^{\pi/2}\cos^n(x)\cos(2tx)\,dx = \frac{n-1}{n}\left(1-\frac{4t^2}{n^2}\right)^{-1}\int_0^{\pi/2}\cos^{n-2}(x)\cos(2tx)\,dx,\ 2 \le n \in \mathbb{N}.$$

[30]An infinite product $\prod_{n=1}^\infty b_n$, $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, is said to **converge** if there exists $m \in \mathbb{N}$ such that $b_n \ne 0$ for $n \ge m$, and $\lim_{n\to\infty}\prod_{k=m}^n b_k$ converges to a non-zero number. In this case, we write $\prod_{n=1}^\infty b_n = b_1 \cdot b_2 \cdots b_{m-1} \cdot \lim_{n\to\infty}\prod_{k=m}^n b_k$. It is an elementary fact that an infinite product $\prod_{n=1}^\infty (1+a_n)$, $0 \le a_n \in \mathbb{R}$, $n \in \mathbb{N}$, converges if and only if the infinite series $\sum_{n=1}^\infty a_n$ converges.

[31]The proof follows Salwinski, D., *Euler's Sine product Formula: An Elementary Proof*, Coll. Math. J., March 2018, which, in turn, closely follows the outline in Spivak, M., *Calculus*, 4th ed. Publish or Perish, Houston, TX, p. 395. For another elementary proof, see Ciaurri, O. *Euler's product expansion for the sine: An elementary proof*, Amer. Math. Monthly, Vol. 122, No. 7 (August-September 2015) 693-695.

We assume, initially, that $t \neq 0, \pm n/2$. Using integration by parts, with obvious cast, we have

$$\int_0^{\pi/2} \cos^n(x)\cos(2tx)\,dx = \frac{n}{2t}\int_0^{\pi/2} \cos^{n-1}(x)\sin(x)\sin(2tx)\,dx,$$

since the boundary terms vanish:

$$\left[\frac{\cos^n(x)\sin(2tx)}{2t}\right]_0^{\pi/2} = 0.$$

We perform yet another integration by parts, and obtain

$$\begin{aligned}\int_0^{\pi/2} &\cos^{n-1}(x)\sin(x)\sin(2tx)\,dx\\ &= \frac{1}{2t}\int_0^{\pi/2}\left(-(n-1)\cos^{n-2}(x)\sin^2(x) + \cos^n(x)\right)\cos(2tx)\,dx,\end{aligned}$$

since, once again, the boundary terms vanish:

$$\left[-\frac{\cos^{n-1}(x)\sin(x)\cos(2tx)}{2t}\right]_0^{\pi/2} = 0.$$

Using the identity $\sin^2(x) = 1 - \cos^2(x)$, putting everything together and rearranging, the inductive formula above follows.

Following the main structure of the proof of the Wallis formula, we now split the cases according to the parity of $2 \leq n \in \mathbb{N}$. The case $n = 2m$, $m \in \mathbb{N}$, will give the (first) infinite product formula for the sine function, the second case $n = 2m+1$, $m \in \mathbb{N}$, will result in the (second) infinite product formula for the cosine. We will give details only in the first case as the proof of the second case is analogous. For $n = 2m$, $m \in \mathbb{N}$, the inductive formula takes the form

$$\int_0^{\pi/2} \cos^{2m}(x)\cos(2tx)\,dx = \frac{2m-1}{2m}\left(1 - \frac{t^2}{m^2}\right)^{-1}\int_0^{\pi/2}\cos^{2m-2}(x)\cos(2tx)\,dx,\ m \in \mathbb{N},$$

where $t \neq \pm m$. Assuming $t \neq \pm k$, $k = 1, \ldots, m$, we now use this formula inductively, and obtain

$$\begin{aligned}\int_0^{\pi/2}\cos^{2m}(x)\cos(2tx)\,dx &= \prod_{k=1}^m \frac{2k-1}{2k}\prod_{k=1}^m\left(1 - \frac{t^2}{k^2}\right)^{-1}\int_0^{\pi/2}\cos(2tx)\,dx\\ &= \frac{\pi}{2}\prod_{k=1}^m\frac{2k-1}{2k}\prod_{k=1}^m\left(1 - \frac{t^2}{k^2}\right)^{-1}\frac{\sin(\pi t)}{\pi t}.\end{aligned}$$

We now replace the first product on the right-hand side of the last expression, using the formula

$$\int_0^{\pi/2} \cos^{2m}(x)\,dx = \int_0^{\pi/2} \sin^{2m}(x)\,dx = I_{2m} = \frac{\pi}{2}\prod_{k=1}^{m}\frac{2k-1}{2k}$$

in Example 4.2.6, where the first equality here follows from the identity $\sin(x) = \cos(\pi/2 - x)$.
We obtain

$$\sin(\pi t) = \pi t \prod_{k=1}^{m}\frac{2k-1}{2k}\prod_{k=1}^{n}\left(1 - \frac{t^2}{k^2}\right)\frac{\int_0^{\pi/2}\cos^{2m}(x)\cos(2tx)\,dx}{\int_0^{\pi/2}\cos^{2m}(x)\,dx}, \quad t \in \mathbb{R},$$

where we observed that this formula obviously holds for the previously excluded cases $t = \pm k$, $k = 1, \ldots, m$.
It remains to show that

$$\lim_{m\to\infty}\frac{\int_0^{\pi/2}\cos^{2m}(x)\cos(2tx)\,dx}{\int_0^{\pi/2}\cos^{2m}(x)\,dx} = 1.$$

We show this under more general conditions as follows:

Lemma. *Let $f : [0, \pi/2] \to \mathbb{R}$ be integrable, and assume that $|f(x) - f(0)| \le Mx$, $x \in [0, \pi/2]$. Then, we have*

$$\lim_{n\to\infty}\frac{\int_0^{\pi/2} f(x)\cos^n(x)\,dx}{\int_0^{\pi/2}\cos^n(x)\,dx} = f(0).$$

Proof. We derive suitable estimates for the numerator and denominator in the limit. First, note that the trivial inequality $0 \le x \le \tan(x)$, $x \in [0, \pi/2)$, combined with the identity $\sec^2(x) = 1 + \tan^2(x)$ gives $\cos(x) \le 1/\sqrt{1+x^2}$, $x \in [0, \pi/2]$. Using this, for $3 \le m \in \mathbb{N}$, we estimate

$$\begin{aligned}\left|\int_0^{\pi/2} f(x)\cos^n(x)\,dx - f(0)\cos^n(x)\,dx\right| &\le \int_0^{\pi/2}|f(x) - f(0)|\cos^n(x)\,dx \\ \le M\int_0^{\pi/2}\frac{x\,dx}{(1+x^2)^{n/2}} &= \frac{1}{2}\int_1^{1+\pi^2/4}\frac{du}{u^{n/2}} \\ = \frac{M}{n-2}\left(1 - \frac{1}{(1+\pi^2/4)^{n/2-1}}\right) &\le \frac{M}{n-2}.\end{aligned}$$

Dividing by $\int_0^{\pi/2} |f(x)-f(0)| \cos^n(x)\,dx = \int_0^{\pi/2} |f(x)-f(0)| \cos^n(x)\,dx$ and estimating this below by the last formula at the beginning of this section, we obtain

$$\left| \frac{\int_0^{\pi/2} f(x)\cos^n(x)\,dx}{\int_0^{\pi/2} \cos^n(x)\,dx} - 1 \right| \leq \frac{M}{n-2}\frac{\sqrt{n+1}}{\sqrt{\pi/2}} = M\sqrt{\frac{2}{\pi}}\frac{\sqrt{n+1}}{n-2}, \quad 3 \leq n \in \mathbb{N}.$$

Letting $n \to \infty$, the lemma follows.

Remark. Setting $t = 1/2$ is Euler's infinite product for sine, we recover the Wallis formula.

In the Euler product formula for sine, taking the natural logarithm of both sides, we have

$$\ln \sin(\pi t) = \ln(\pi t) + \sum_{k=1}^{\infty} \ln\left(1 - \frac{t^2}{k^2}\right) \quad 0 < t < 1.$$

Differentiating both sides (which is legitimate by Proposition 2.2.4), after rearranging, we obtain the **expansion of the cotangent**:

$$\frac{\pi \cot(\pi t)}{2t} = \frac{1}{2t^2} - \sum_{k=1}^{\infty} \frac{1}{k^2 - t^2}, \quad 0 < t < 1.$$

Similary, for the infinite product of the cosine, we obtain the **expansion of the tangent**:

$$\frac{\pi \tan(\pi t)}{8t} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - 4t^2}, \quad -\frac{1}{2} < t < \frac{1}{2}.$$

Exercises

1. Show that the hypothesis of the Lemma holds for functions $f : [0, \pi/2] \to \mathbb{R}$ such that f is continuous on $[0, \pi/2]$ and all Dini derivatives are finite on $(0, \pi/2)$.
Solution: Use Proposition 2.1.7.

2. Derive the formula

$$\sqrt{2} = \prod_{k=0}^{\infty} \frac{(4k+2)^2}{(4k+1)(4k+3)}.$$

Solution: Evaluate Euler's infinite product for cosine at $t = 1/4$.

3. Derive the estimates

$$\begin{aligned} 1-\frac{2|t|}{\sqrt{\pi}}\frac{\sqrt{n+1/2}}{n-1} &\le \frac{\sin(\pi t)}{\pi t\prod_{k=1}^{\infty}(1-t^2/k^2)}\le 1,\ t\in\mathbb{R}\setminus\mathbb{Z},\ 2\le n\in\mathbb{N} \\ 1-\frac{2|t|}{\sqrt{\pi}}\frac{\sqrt{n+1}}{n-1/2} &\le \frac{\cos(\pi t)}{\prod_{k=1}^{\infty}(1-4t^2/(2k+1)^2)}\le 1,\ t\in\mathbb{R}\setminus(\mathbb{Z}+1/2)\,,\ n\in\mathbb{N}. \end{aligned}$$

Solution: Letting $f(x)=\cos(2tx)$ and $M=2|t|$ in the final estimate in the proof of the lemma, we obtain

$$\left|1-\frac{\int_0^{\pi/2}\cos^n(x)\cos(2tx)\,dx}{\int_0^{\pi/2}\cos^n(x)\,dx}\right|\le\frac{2|t|}{\sqrt{\pi}}\frac{\sqrt{2n+2}}{n-2},\ 3\le n\in\mathbb{N}.$$

Now notice that the expression in the absolute value is non-negative. Finally, split into two cases according to the parity of n.

4.[32] (a) For $2\le n\in\mathbb{N}$, derive the inductive formula

$$\int_0^{\pi/2}\cos^n(x)\cosh(2tx)\,dx=\frac{n-1}{n}\left(1+\frac{4t^2}{n^2}\right)^{-1}\int_0^{\pi/2}\cos^{n-2}(x)\cosh(2tx)\,dx.$$

(b) Split the inductive formula in (a) into two cases according to the parity of n to obtain the infinite product formulas for the hyperbolic sine and cosine functions:

$$\begin{aligned} \sinh(\pi t) &= \pi t\prod_{k=1}^{\infty}\left(1+\frac{t^2}{k^2}\right) \\ \cosh(\pi t) &= \prod_{k=0}^{\infty}\left(1+\frac{4t^2}{(2k+1)^2}\right). \end{aligned}$$

(c) Take the logarithm and differentiate (as for the sine and cosine products) to obtain the following expansions of the hyperbolic cotangent and hyperbolic tangent functions:

$$\begin{aligned} \frac{\pi\coth(\pi t)}{2t} &= \frac{1}{2t^2}+\sum_{k=1}^{\infty}\frac{1}{k^2+t^2}, \quad t>0 \\ \frac{\pi\tanh(\pi t)}{8t} &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2+4t^2}, \quad t\neq 0. \end{aligned}$$

(d) As a byproduct, calculate the limit

$$\lim_{t\to 0^+}\left(\frac{\pi\coth(\pi t)}{2t}-\frac{1}{2t^2}\right)=\frac{\pi^2}{6},$$

and obtain Euler's solution to the Basel problem.

[32] See the previous footnote.

4.9 The Gamma Function Revisited

The gamma function has several equivalent representations for different purposes. We begin here with the original definition of the gamma function due to Euler in 1729:

$$\Gamma(t) = \lim_{n\to\infty} n^t \frac{n!}{t(t+1)\cdots(t+n)} = \lim_{n\to\infty} \frac{n^t}{t}\prod_{k=1}^{n}\frac{1}{1+t/k}, \quad t \in \mathbb{R}\setminus(-\mathbb{N}_0).$$

This is usually written as a single infinite product since we have

$$\begin{aligned}\Gamma(t) &= \lim_{n\to\infty}\frac{(n+1)^t}{t}\prod_{k=1}^{n}\frac{1}{1+t/k} \\ &= \frac{1}{t}\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{k+1}{k}\right)^t\prod_{k=1}^{n}\frac{1}{1+t/k} \\ &= \frac{1}{t}\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(1+1/k)^t}{1+t/k},\end{aligned}$$

where we replaced n^t by $(n+1)^t$ without affecting the limit. This gives Euler's representation of the gamma function as an **infinite product**:

$$\Gamma(t) = \frac{1}{t}\prod_{n=1}^{\infty}\frac{(1+1/n)^t}{1+t/n}, \quad t \in \mathbb{R}\setminus(-\mathbb{N}_0)$$

Note the obvious advantage that the product converges for all $t \in \mathbb{R}\setminus(-\mathbb{N}_0)$.

History. This formula appears in a letter of Euler written to Goldbach in October 13, 1729, and thus, it predates the formula that we adopted in Section 4.6; and, in fact, this is the first formula that defines the gamma function.

We now show that this infinite product representation of the gamma function is the same as the integral representation we started with in Section 4.6. We first set $t > 0$, and define $f_n : [0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, by $f_n(x) = (1 - x/n)^n$ for $0 \le x \le n$, and $f_n(x) = 0$ for $x > n$. We observe that, by Euler's limit relation, $\lim_{n\to\infty} f_n(x) = e^{-x}$, $x \in \mathbb{R}$, pointwise on $[0,\infty)$. We thus have

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx = \int_0^\infty \lim_{n\to\infty}\left(x^{t-1}f_n(x)\right)\,dx$$

To interchange the improper integral with the limit we use Arzelà's dominated convergence theorem (Section 3.2). Since[33] $(1 - x/n)^n \le e^{-x}$, $0 \le x \le n$, we have

[33] See *Elements of Mathematics - History and Foundations*, Section 10.5.

$|x^{t-1}f_n(x)| \le x^{t-1}(1-x/n)^n \le x^{t-1}e^{-x}$, $x, t > 0$. The latter (dominating) function, being the integrand for original definition of the gamma function (Section 4.6), is improperly integrable on $[0,\infty)$; and hence the assumptions of the theorem hold. We obtain[34]

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx = \lim_{n\to\infty}\int_0^\infty x^{t-1}f_n(x)\,dx = \lim_{n\to\infty}\int_0^n x^{t-1}\left(1-\frac{x}{n}\right)^n dx.$$

(Note the change of the upper bound for the integral.) We now perform integration by parts on the last integral with $u = (1-x/n)^n$ and $dv = x^{t-1}dx$ (and hence $du = -(1-x/n)^{n-1}dx$ and $v = x^t/t$), and obtain

$$\int_0^n x^{t-1}\left(1-\frac{x}{n}\right)^n dx = \frac{1}{t}\int_0^n x^t\left(1-\frac{x}{n}\right)^{n-1} dx,$$

since the boundary terms vanish:

$$\left[\frac{x^t}{t}\left(1-\frac{x}{n}\right)^n\right]_0^n = 0.$$

We perform this inductively(n times), and arrive at

$$\int_0^n x^{t-1}\left(1-\frac{x}{n}\right)^n dx = \frac{n}{nt}\frac{n-1}{n(t+1)}\frac{n-2}{n(t+2)}\cdots\frac{1}{n(t+n-1)}\int_0^n x^{t+n-1}\,dx$$
$$= \frac{1}{n^n}\frac{n!}{t(t+1)\cdots(t+n-1)}\frac{n^{t+n}}{t+n} = n^t\frac{n!}{t(t+1)\cdots(t+n)}.$$

The stated formula follows for $t > 0$.
The extension to negative $t < 0$, $-t \neq \mathbb{N}_0$, can be shown by compatibility with the relation $\Gamma(t) = \Gamma(t+1)/t$. Indeed, we have

$$\begin{aligned}\frac{\Gamma(t+1)}{t} &= \lim_{n\to\infty} n^{t+1}\frac{n!}{t(t+1)\cdots(t+n+1)}\\ &= \lim_{n\to\infty}\frac{n}{t+n+1}\cdot\lim_{n\to\infty} n^t\frac{n!}{t(t+1)\cdots(t+n)} = \Gamma(t).\end{aligned}$$

The formula now follows for all $t \in \mathbb{R}\setminus(-\mathbb{N}_0)$.

---

[34]It is instructive to compare this with 12.2 in Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 3rd ed. Dover, 2020, where the argument is based on estimates of the exponential function used in our Example 4.16.3 below.

Example 4.9.1. We have[35]

$$(\Gamma'(1) =) \int_0^\infty e^{-x} \ln(x)\, dx = \lim_{n\to\infty} (\ln(n) - H_n) = -\gamma,$$

where $H_n = \sum_{k=1}^n 1/k$ and $0 < \gamma < 1$ is the Euler-Mascheroni constant[36] (defined by the last equality).
We first note that

$$\int_0^\infty e^{-x} \ln(x)\, dx = \int_0^1 e^{-x} \ln(x)\, dx + \int_1^\infty e^{-x} \ln(x)\, dx,$$

whereas (1) the first improper integral on the right-hand side converges since

$$\left| \int_0^1 e^{-x} \ln(x)\, dx \right| \le - \int_0^1 \ln(x)\, dx = - \left[x\ln(x) - x \right]_0^1 = 1 + \lim_{x\to 0^+} x\ln(x) = 1,$$

and (2) performing integration by parts on the second integral, we get

$$\int_1^\infty e^{-x} \ln(x)\, dx = \int_1^\infty \frac{e^{-x}}{x} dx < \infty,$$

since $[-e^{-x}\ln(x)]_1^\infty = -\lim_{x\to\infty} e^{-x}\ln(x) = 0$.

Turning to the main computation, we define $f_n : [0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, as above. With this, by Arzelà's dominated convergence theorem, we have

$$\int_0^\infty e^{-x} \ln(x)\, dx = \int_0^\infty \lim_{n\to\infty} f_n(x) \ln(x)\, dx = \lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \ln(x)\, dx.$$

For the integral, we use the substitution $u = 1 - x/n$

$$\int_0^n \left(1 - \frac{x}{n}\right)^n \ln(x)\, dx = n \int_0^1 u^n \ln(n(1-u))\, du = \frac{n}{n+1} \ln(n) + n \int_0^1 u^n \ln(1-u)\, du,$$

where split the logarithm as $\ln(n(1-u)) = \ln(n) + \ln(1-u)$.
It remains to treat the last integral. We have the power series expansion $-\ln(1-u) = \sum_{k=1}^\infty u^k/k$ with radius of convergence $\rho = 1$. Therefore, the power series converges uniformly on any closed interval contained in $(-1,1)$. Actually, for $u \in [0,1)$, the

[35]For a direct proof using the mean value theorem, see Bagby, R., *A simple proof that* $\Gamma'(1) = -\gamma$, Amer. Math. Monthly, Vol. 117, No 1. (January 2010) 83-85.
[36]See *Elements of Mathematics - History and Foundations*, Section 10.3.

partial sums of the series are increasing, and the monotone convergence theorem (Section 3.2) applies. We obtain

$$\int_0^1 u^n \ln(1-u)\,du = -\lim_{b\to 1^-}\int_0^b u^n \sum_{k=1}^{\infty}\frac{u^k}{k}\,du = -\lim_{b\to 1^-}\int_0^b \sum_{k=1}^{\infty}\frac{u^{k+n}}{k}\,du$$
$$= -\lim_{b\to 1^-}\sum_{k=1}^{\infty}\int_0^b \frac{u^{k+n}}{k}\,du = -\lim_{b\to 1^-}\sum_{k=1}^{\infty}\frac{b^{k+n+1}}{k(k+n+1)} = -\sum_{k=1}^{\infty}\frac{1}{k(k+n+1)}.$$

The last sum is telescopic (with strings of $n+1$ terms consecutively canceling), since

$$\frac{1}{k(k+n+1)} = \frac{1}{n+1}\left(\frac{1}{k} - \frac{1}{k+n+1}\right), \quad k \in \mathbb{N}.$$

We obtain

$$\int_0^1 u^n \ln(1-u)\,du = -\frac{1}{n+1}\sum_{k=1}^{n=1}\frac{1}{k} = -\frac{1}{n+1}H_{n+1}.$$

Putting everything together, we arrive at

$$\int_0^{\infty} e^{-x}\ln(x)\,dx = \lim_{n\to\infty}\left(\frac{n}{n+1}\ln(n) - \frac{n}{n+1}H_{n+1}\right) = \lim_{n\to\infty}\left(\ln(n) - H_n\right),$$

since $H_{n+1} = H_n + 1/(n+1)$. The example follows.

Example 4.9.2. Show that

$$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\,dx = -\gamma.$$

We use the substitution $u = e^{-x}$ and $du = -e^{-x}dx$ in the previous example.

We use the new representation of the gamma function obtained above to derive **Euler's reflection formula**.[37]

Proposition 4.9.1. *We have*[38]

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}, \quad t \notin \mathbb{Z}.$$

[37] Sometimes also called Euler's functional equation.

[38] Most of the proofs of this formula are non-elementary, and use some basic complex analysis.

Proof. We first use the functional equation for the gamma function, and calculate

$$\begin{aligned}\Gamma(t)\Gamma(1-t) &= -t\Gamma(t)\Gamma(-t) = -t\lim_{n\to\infty}\left(\frac{n^{-t}}{-t}\prod_{k=1}^{n}\frac{1}{1-t/k}\cdot\frac{n^t}{t}\prod_{k=1}^{n}\frac{1}{1+t/k}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{t}\prod_{k=1}^{n}\frac{1}{1-t^2/k^2}\right) = \frac{1}{t}\prod_{k=1}^{\infty}\frac{1}{1-t^2/k^2} = \frac{\pi}{\sin(\pi t)},\end{aligned}$$

where, in the last equality, we used Euler's infinite product formula for the sine. The proposition follows.

Corollary. *We have*

$$\Gamma(t-n) = (-1)^{n+1}\frac{\Gamma(-t)\Gamma(1+t)}{\Gamma(n+1-t)}, \quad n\in\mathbb{Z}.$$

Proof. Applying Euler's reflection formula for $t-n$ and $-t$, we have

$$\frac{\Gamma(t-n)\Gamma(n+1-t)}{\Gamma(-t)\Gamma(1+t)} = -\frac{\pi}{\sin(\pi(t-n))}\frac{\sin(\pi t)}{\pi}.$$

The corollary follows since $\sin(\pi(t-n)) = \sin(\pi t)\cos(\pi n) = (-1)^n\sin(\pi t)$.

Example 4.9.3. [39] For $a>0$ and $0<t<1$, we have

$$\int_0^\infty \frac{\sin(ax)}{x^t}\,dx = a^{t-1}\frac{\pi}{2\Gamma(t)}\csc\left(\frac{\pi t}{2}\right) \text{ and } \int_0^\infty \frac{\cos(ax)}{x^t}\,dx = a^{t-1}\frac{\pi}{2\Gamma(t)}\sec\left(\frac{\pi t}{2}\right).$$

We derive only the first formula as the proof of the second is entirely analogous. By the remark at the end of Section 4.6, Euler's reflection formula, and the duplication formula for sine, we calculate

$$\begin{aligned}\int_0^\infty \frac{\sin(x)}{x^t}\,dx &= \Gamma(1-t)\cos\left(\frac{\pi t}{2}\right)\\ &= \frac{\pi}{\Gamma(t)\sin(\pi t)}\cos\left(\frac{\pi t}{2}\right) = \frac{\pi}{2\Gamma(t)}\csc\left(\frac{\pi t}{2}\right), \quad 0<t<1.\end{aligned}$$

The example now follows by simple scaling $x\mapsto ax$.

---

[39] See also Andrews, G.E., Askey, R. and Roy, R., *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge Uniersity Press, 1999, Exercise 19 at the end of Chapter 1.

Exercises

1. Use Euler's reflection formula ($t = 1/2$) to derive the "Gausssian" integral of Examples 4.4.2 and 4.8.1:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Soluion: Use the substitution $u = x^2$ with $du/\sqrt{u} = 2dx$, and calculate

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,du = \frac{1}{2}\int_0^\infty u^{-1/2}e^{-u}\,du = \frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

4.10 The Stirling Formula and Multiplicative Properties of the Gamma Function

**

We will use the fact that the natural logarithmic function $y = \ln(x)$, $0 < x \in \mathbb{R}$, is **concave** in two ways.[40] (1) The graph of $y = \ln(x)$, $0 < x \in \mathbb{R}$, is below[41] any of its tangent lines; (2) Any part of the graph of $y = \ln(x)$, $0 < x \in \mathbb{R}$, cut out by a secant lies above[42] the corresponding line segment of the secant.

We fix $1 \leq t \in \mathbb{R}$, and let $\mathcal{D}_t \subset \mathbb{R}$ be the region under the graph of the natural logarithm function $y = \ln(x)$ for $t \leq x \leq t+1$. The crucial step in deriving the Stirling formula amounts to estimate the area $\int_t^{t+1}\ln(x)\,dx$ of $\mathcal{D}_t$.
To do this, we let $X_0 = (t, 0)$, $X_1 = (t+1, 0)$, $Y_0 = (t, \ln(t))$, $Y_1 = (t+1, \ln(t+1))$. Moreover, we let Z_0 be the intersection of the tangent line to the graph of ln at Y_1 and the vertical line through X_0; and Z_1 the intersection of the tangent line to the graph of ln at Y_0 and the vertical line through X_1. The differentiation formula $(\ln|x|)' = 1/x$, $0 \neq x \in \mathbb{R}$ easily implies that $Z_0 = (t, \ln(t+1) - 1/(t+1))$ and $Z_1 = (t+1, \ln(t) + 1/t)$.
By concavity of the natural logarithm, the trapezoid $[X_0, X_1, Y_0, Y_1]$ is contained in $\mathcal{D}_t$; whereas the two trapezoids $[X_0, X_1, Y_1, Z_0]$ and $[X_0, X_1, Y_0, Z_1]$ contain $\mathcal{D}_t$. The

[40] The following proof is a variant of the classical approach; see A.J. Coleman, *A simple proof of Stirling's formula*, The American Mathematical Monthly, Vol. 58, No. 5 (1951) 334-336.

[41] This follows from the analogous statement for the natural exponential function $y = e^x$, $x \in \mathbb{R}$, via the fundamental inequality $e^x \geq 1 + x$, $x \in \mathbb{R}$. See *Elements of Mathematics - History and Foundations*, Section 10.1.

[42] This follows form the analogous property of the natural exponential function $y = e^x$, $x \in \mathbb{R}$, via the Bernoulli inequality. See *Elements of Mathematics - History and Foundations*, Section 10.1.

area of the first trapezoid $[X_0, X_1, Y_0, Y_1]$ is

$$\frac{1}{2}\left(\ln(t) + \ln(t+1)\right) < \int_t^{t+1} \ln(x)\, dx.$$

Letting $2 \leq n \in \mathbb{N}$, and summing up for $t = k = 1, \ldots, n-1$, we obtain

$$\frac{1}{2}\sum_{k=1}^{n-1}(\ln(k) + \ln(k+1)) = \sum_{k=1}^{n} \ln(k) - \frac{1}{2}\ln(n) = \ln(n!) - \frac{1}{2}\ln(n) < \int_1^n \ln(x)dx.$$

On the other hand, for $0 < t \in \mathbb{R}$, with the areas of the trapezoids $(\mathcal{D}_t \subset) [X_0, X_1, Y_1, Z_0]$ and $(\mathcal{D}_t \subset) [X_0, X_1, Y_0, Z_1]$, we have

$$\int_t^{t+1} \ln(x)dx < \ln(t+1) - \frac{1}{2(t+1)} \quad \text{and} \quad \int_t^{t+1} \ln(x)dx < \ln(t) - \frac{1}{2t}.$$

We now take the **arithmetic sum** of these

$$\int_t^{t+1} \ln(x)dx < \frac{1}{2}\left(\ln(t) + \ln(t+1)\right) + \frac{1}{4}\left(\frac{1}{t} - \frac{1}{t+1}\right).$$

As before, letting $2 \leq n \in \mathbb{N}$, and summing up for $t = k = 1, \ldots, n-1$, a similar computation as above gives

$$\int_1^n \ln(x)dx < \frac{1}{2}\sum_{k=1}^{n-1}(\ln(k)+\ln(k+1)) + \frac{1}{4}\sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \ln(n!) - \frac{1}{2}\ln(n) + \frac{1}{4} - \frac{1}{4n},$$

where the second sum is telescopic.
With these estimates in place, for $2 \leq n \in \mathbb{N}$, we define

$$\begin{aligned} d_n &= \int_1^n \ln(x)\, dx - \frac{1}{2}\sum_{k=1}^{n-1}(\ln(k) + \ln(k+1)) \\ &= \int_1^n \ln(x)\, dx - \sum_{k=1}^{n} \ln(k) + \frac{1}{2}\ln(n) \\ &= \int_1^n \ln(x)\, dx - \ln(n!) + \frac{1}{2}\ln(n). \end{aligned}$$

By the first equality, d_n, $2 \leq n \in \mathbb{N}$, is the difference of the area under the graph of $y = \ln(x)$, for $1 \leq n$, and the **inscribed** first set of trapezoids. Hence, the sequence $(d_n)_{2 \leq n \in \mathbb{N}}$ has positive terms (see also our first (lower) estimate above), and it is strictly increasing. By the second (upper) estimate, we have

$$(0 <)\, d_n < \frac{1}{4} - \frac{1}{4n} < \frac{1}{4}, \quad 2 \leq n \in \mathbb{N}.$$

Using the monotone convergence theorem, we conclude that the limit $\lim_{n\to\infty} d_n = d$ exists with $0 < d \le 1/4$.
Finally, we can calculate the integral

$$\int_1^n \ln(x)\, dx = [x \cdot \ln(x) - x]_1^n = n\ln(n) - n + 1.$$

Substituting this into the definition of d_n above, we obtain

$$\begin{aligned} d_n &= \int_1^n \ln(x)\, dx - \ln(n!) + \frac{1}{2}\ln(n) \\ &= \left(n + \frac{1}{2}\right)\ln(n) - \ln(n!) - n + 1 \\ &= \ln\left(\frac{n^{n+1/2}}{n! \cdot e^n}\right) + 1 \\ &= -\ln\left(\frac{n! \cdot e^n}{n^{n+1/2}}\right) + 1. \end{aligned}$$

Summarizing, these give the Stirling limit

$$\lim_{n\to\infty} \frac{n!}{\sqrt{n} \cdot n^n \cdot e^{-n}} = a,$$

where $a = e^{1-d}$ with $e^{3/4} \le a < e$.
To obtain the value of a we use the Wallis product formula

$$\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

We replace the factorials in this limit by the two equivalent forms of the Stirling limit above

$$\lim_{n\to\infty} \frac{\sqrt{2n} \cdot (2n)^{2n} \cdot e^{-2n}}{(2n)!} = \frac{1}{a} \quad \text{and} \quad \lim_{n\to\infty} \frac{(n!)^2}{n^{2n+1} \cdot e^{-2n}} = a^2.$$

Upon substitution and simplification, we obtain

$$a \cdot \lim_{n\to\infty} \frac{2^{2n} \cdot n^{2n+1}}{\sqrt{2n+1} \cdot \sqrt{2n} \cdot (2n)^{2n}} = \frac{a}{2} = \sqrt{\frac{\pi}{2}}.$$

This gives

$$a = \sqrt{2\pi}.$$

Summarizing, we obtain the **Stirling formula**

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2} \cdot e^{-n}} = \sqrt{2\pi},$$

or equivalently

$$n! \sim \sqrt{2\pi n} \cdot n^n \cdot e^{-n} \quad \text{as} \quad n \to \infty.$$

History. The Stirling formula with a generic constant in place of $\sqrt{2\pi}$ was discovered by Abraham de Moivre. Stirling then found this constant to be $\sqrt{2\pi}$. It was also de Moivre who encouraged Stirling to publish the more precise formula above that is subsequently named after the latter.[43] A lesser known but slightly better asymptotics[44] is $n! \sim \sqrt{(2n+1/3)\pi} \cdot n^n \cdot e^{-n}$.

As an application of the Stirling formula, we now derive Legendre's duplication formula for the gamma function.

Proposition 4.10.1. *We have*

$$\Gamma(t)\Gamma(t+1/2) = 2^{1-2t}\sqrt{\pi}\Gamma(2t),$$

Proof. We use Euler's representation for the gamma function derived at the beginning of Section 4.9. For brevity, we write $\Gamma(t) = \lim_{n\to\infty} q_n(t)$, where

$$q_n(t) = \frac{n^t \cdot n!}{t(t+1)\cdots(t+n)}, \quad n \in \mathbb{N},\ t \in \mathbb{R} \setminus (-\mathbb{N}_0).$$

We calculate

$$\begin{aligned} q_n(t)q_n(t+1/2) &= \frac{n^t \cdot n!}{t(t+1)\cdots(t+n)} \cdot \frac{2^{n+1}n^{t+1/2} \cdot n!}{(2t+1)(2t+3)\cdots(2t+2n+1)} \\ &= \frac{2^{2n+2}n^{2t+1/2} \cdot (n!)^2}{2t(2t+1)\cdots(2t+2n+1)} \\ &= \frac{2^{2n+2}n^{1/2} \cdot (n!)^2}{2^{2t}(2n)!} \cdot \frac{1}{2t+2n+1} \cdot q_{2n}(2t). \end{aligned}$$

We now use the Stirling formula as

$$\frac{2^{2n+2}n^{1/2} \cdot (n!)^2}{(2n)!} \sim \frac{2^{2n+2}n^{1/2} \cdot 2\pi n \cdot n^{2n}e^{-2n}}{\sqrt{4\pi n}\, 2^{2n}\, n^{2n}e^{-2n}} = 4\sqrt{\pi}\, n, \quad n \to \infty.$$

Putting everything together, we arrive at

$$q_n(t)q_n(t+1/2) \sim \frac{\sqrt{\pi}}{2^{2t-1}} \cdot \frac{2n}{2t+2n+1} \cdot q_{2n}(2t) \sim 2^{1-2t}\sqrt{\pi}q_{2n}(2t), \quad n \to \infty.$$

[43] See Tweedle, I., *James Stirling*, Scottish Academic Press, Edinburgh, 1988.

[44] See Gosper, R.W., *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA, 75 (1978) 40-42.

Letting $n \to \infty$, the proposition follows.

History. As the name suggests, this formula is due to Legendre in 1809.

Remark. In an alternative development,[45] the Legendre duplication formula can be used to derive Euler's reflection formula. This is expounded in Exercise 1.

The following is a "multiplicative" generalization of the Legendre duplication formula ($m = 2$) usually termed as the **Legendre-Gauss formula**. It follows from a general **multiplication theorem**[46] obeyed by several special functions. As usual, we prefer a short and direct proof.

Proposition 4.10.2. *We have*

$$\prod_{k=0}^{m-1} \Gamma\left(\frac{t+k}{m}\right) = (2\pi)^{(m-1)/2} m^{1/2-t}\Gamma(t), \quad m \in \mathbb{N},\ t \in \mathbb{R} \setminus (-\mathbb{N}_0)$$

Proof. Let $m \in \mathbb{N}$. By Euler's representation of the gamma function, the product

$$\prod_{k=1}^{m} \Gamma\left(\frac{t+k}{m}\right), \quad t \notin (-\mathbb{N}_0)$$

(with modified ends) is the limit (as $n \to \infty$) of the expression

$$Q_n(t,m) = \prod_{k=1}^{m} \frac{n^{\frac{t+k}{m}} \cdot n!}{\left(\frac{t+k}{m}\right)\left(\frac{t+k}{m}+1\right)\cdots\left(\frac{t+k}{m}+n\right)} = \frac{n^{t+(m+1)/2} \cdot m^{(n+1)m} \cdot (n!)^m}{(t+1)(t+2)\cdots(t+(n+1)m)},$$

where we used $\sum_{k=1}^{m} k = m(m+1)/2$. For $t = 0$, this reduces to

$$Q_n(0,m) = \frac{n^{(m+1)/2} \cdot m^{(n+1)m} \cdot (n!)^m}{((n+1)m)!}.$$

It is convenient to consider the ratio

$$\begin{aligned}\frac{Q_n(t,m)}{Q_n(0,m)} &= \frac{n^t \cdot ((n+1)m)!}{(t+1)(t+2)\cdots(t+(n+1)m)} \\ &= tm^{-t}\left(\frac{n}{n+1}\right)^t \frac{((n+1)m)^t((n+1)m)!}{t(t+1)(t+2)\cdots(t+(n+1)m)} \sim tm^{-t}\Gamma(t), \text{ as } n \to \infty.\end{aligned}$$

[45]See Artin, E., *The Gamma Function*, New York, Holt, Rinehart and Winston 1964, and Dover 2015.

[46]See Bourbaki, N., *Élements de Mathématique, Fonctions d'une variable réelle*, Springer, 2007, VII.21, Exercise 1) a).

On the other hand, by Euler's reflection formula, we have

$$\begin{aligned}
\lim_{n\to\infty} Q_n(0,m) &= \prod_{k=1}^{m-1}\Gamma\left(\frac{k}{m}\right) = \prod_{k=1}^{m-1}\Gamma\left(1-\frac{k}{m}\right) \\
&= \sqrt{\prod_{k=1}^{m-1}\Gamma\left(\frac{k}{m}\right)\Gamma\left(1-\frac{k}{m}\right)} \\
&= \sqrt{\frac{\pi^{m-1}}{\prod_{k=1}^{m-1}\sin(k\pi/m)}}.
\end{aligned}$$

The trigonometric product in the denominator is well-known. For completeness, we include a proof here.[47]

Lemma. *For $2 \leq m \in \mathbb{N}$, we have*

$$\prod_{k=1}^{m-1}\sin\left(\frac{k\pi}{m}\right) = \frac{m}{2^{m-1}} \quad \textit{and} \quad \prod_{k=1}^{m-1}\cos\left(\frac{k\pi}{m}\right) = \frac{\sin(m\pi/2)}{2^{m-1}}.$$

PROOF. let T_m and U_m, $m \in \mathbb{N}_0$ denote the **Chebyshev polynomials**[48] of degree m. By definition, we have

$$T_m(\cos(\alpha)) = \cos(m\alpha) \quad \text{and} \quad U_{m-1}(\cos(\alpha)) = \frac{\sin(m\alpha)}{\sin(\alpha)}.$$

An easy consequence of the recurrence relations[49] for $T_m(x)$ and $U_{m-1}(x)$ (as polynomials in x) is that their leading coefficients are equal to 2^{m-1}.
Now, the crucial observation is the simple fact that $\cos(k\pi/m)$, $k = 1, \ldots, m-1$ are the roots of U_{m-1}. With this, we have the factorization

$$U_{m-1}(x) = 2^{m-1}\prod_{k=1}^{m-1}\left(x - \cos\left(\frac{k\pi}{m}\right)\right).$$

[47] A typical proof uses (complex) roots of unity. Since we steer clear of complex arithmetic, we need to recourse to Chebyshev polynomials.

[48] See *Elements of Mathematics - History and Foundations*, Section 11.3.

[49] See ibid.

To derive the first formula, we use this at $x = \pm 1$, and calculate

$$\begin{aligned} U_{m-1}(1)U_{m-1}(-1) &= (-1)^{m-1}2^{2m-2}\prod_{k=1}^{m-1}\left(1-\cos^2\left(\frac{k\pi}{m}\right)\right) \\ &= (-1)^{m-1}2^{2m-2}\prod_{k=1}^{m-1}\sin^2\left(\frac{k\pi}{m}\right). \end{aligned}$$

On the other hand

$$U_{m-1}(1) = U_{m-1}(\cos(0)) = \lim_{\alpha\to 0}\frac{\sin(m\alpha)}{\sin(\alpha)} = m,$$

and

$$U_{m-1}(-1) = U_{m-1}(\cos(\pi)) = \lim_{\alpha\to \pi}\frac{\sin(m\alpha)}{\sin(\alpha)} = (-1)^{m-1}m.$$

Putting these together, we obtain

$$m^2 = 2^{2m-2}\prod_{k=1}^{m-1}\sin^2\left(\frac{k\pi}{m}\right),$$

The first formula follows.
For the second formula, we first note that, by the last Viète relation,[50] the product of the roots of $U_{m-1}(x)$ is equal to

$$\prod_{k=1}^{m-1}\cos(k\pi/m) = (-1)^{m-1}\frac{U_{m-1}(0)}{2^{m-1}},$$

where the numerator in the fraction is the constant term of the polynomial $U_{m-1}(x)$. On the other hand, we have

$$U_{m-1}(x) = \frac{\sin(m\arccos(x))}{\sin(\arccos(x))}.$$

At $x = 0$, this gives $U_{m-1}(0) = \sin(m\pi/2)$. The second formula follows. (The sign $(-1)^{m-1}$ does not come into play since $\sin(m\pi/2)$ vanishes for m even.)

Remark. Taking the natural logarithm of both sides of the first formula, we obtain

$$\frac{\pi}{m}\sum_{k=1}^{m-1}\ln\left(\sin\left(\frac{k\pi}{m}\right)\right) = \pi\frac{\ln(m)}{m} - \pi\frac{m-1}{m}\ln 2.$$

[50]See *Elements of Mathematics - History and Foundations*, Section 6.6.

Interpreting the left-hand side as a middle Riemann sum (on the interval $[\pi/(2m), (2m-1)\pi/(2m)]$), we recover Example 4.1.6

$$\int_0^\pi \ln(\sin(x))\,dx = \lim_{m\to\infty}\frac{\pi}{m}\sum_{k=1}^{m-1}\ln\left(\sin\left(\frac{k\pi}{m}\right)\right) = \pi\lim_{m\to\infty}\frac{\ln(m)}{m} - \pi\lim_{m\to\infty}\frac{m-1}{m}\ln 2 = -\pi\ln 2.$$

Returning to the main line of the proof, the last step gives

$$\lim_{n\to\infty} Q_n(0,m) = (2\pi)^{(m-1)/2}\cdot m^{-1/2}.$$

Hence, we arrive at

$$\prod_{k=1}^{m}\Gamma\left(\frac{t+k}{m}\right) = \lim_{n\to\infty} Q_n(t,m) = (2\pi)^{(m-1)/2}\cdot m^{-1/2}\cdot t\cdot m^{-t}\Gamma(t).$$

In the final step, we convert the last factor of the product in the left-hand side as

$$\Gamma\left(\frac{t+m}{m}\right) = \Gamma\left(\frac{t}{m}\right) = \frac{t}{m}\Gamma\left(\frac{t}{m}\right).$$

The Gauss-Legendre formula follows.

History. It is worth noting that the special case

$$\prod_{k=1}^{m-1}\Gamma\left(\frac{k}{m}\right) = \frac{(2\pi)^{(m-1)/2}}{\sqrt{m}}, \quad m\in\mathbb{N},$$

obtained in the proof above is due to Euler.

Remark. There is yet another "additive" generalization of the Legendre duplication formula due to Schlömilch[51] as follows

$$2^{t-1}\Gamma\left(\frac{t+m+1}{2}\right)\Gamma\left(\frac{t-m}{2}\right) = \sqrt{\pi}\sum_{k=0}^{m}\frac{\Gamma(t-k)}{2^k\cdot k!}(m-k+1)_{2k},\ t>m\in\mathbb{N}_0,$$

where $(x)_n = \prod_{j=0}^{n-1}(x-j)$ is the so-called **Pochhammer symbol**.
Finally, note a generalization of the Legendre-Gauss formula due to Schobloch.[52] The **Schobloch reciprocity formula** is a "symmetrized" Legendre-Gauss formula:

$$(2\pi)^{-n/2}n^{t+\frac{mn-m-n}{2}}\prod_{k=0}^{n-1}\Gamma\left(\frac{t+km}{n}\right) = (2\pi)^{-m/2}m^{t+\frac{mn-m-n}{2}}\prod_{k=0}^{m-1}\Gamma\left(\frac{t+kn}{m}\right),\ m,n\in\mathbb{N}.$$

[51] See Schlömilch, O., *Analitische Studien: Erste Abtheilung*, Leipzig, 1848. For an elementary proof as well as yet another generalization, see Goenka, R. and Srinivasan, G.K., *Gamma function and its functional equations*, Resonance 26 (2021) 367-386.

[52] See Srinivasan, G.K., *The gamma function: an eclectic tour*, Amer. Math. Monthly, Vol. 114, No. 4 (Apr. 2007) 297-315, and the references therein.

Returning to the main line, in view of the formula $\Gamma(n+1) = n!$, $n \in \mathbb{N}_0$, it is natural to expect that the Stirling formula derived above extends to an asymptotic formula for the gamma function. Indeed, we have

$$\Gamma(t+1) \sim \sqrt{2\pi t} \cdot t^t \cdot e^{-t}, \quad t \to \infty.$$

History. The extension of the Stirling formula to the gamma function is due to Pierre Simon Laplace in his *Mémoire sur la probabilité des courses par les événemens*, Mémoires de matématique et de physique presentés a L'Académie royale des sciences, par divers savans, & lus dans ses assemblées 6 (1774) 621-656. (Reprinted in Laplace's Œuvres complètes 8 27-65.)

For the proof,[53] we need some preparations, notably yet another infinite product formula for the gamma function. It is commonly termed as the Weierstrass representation of the gamma function, albeit it has been discovered by Schlömilch and Newman earlier.[54] To derive this, we begin with Euler's representation of the gamma function discussed in the previous section, and calculate

$$\begin{aligned}
\Gamma(t) &= \lim_{n\to\infty} \frac{n!n^t}{t(t+1)\cdots(t+n)} \\
&= \frac{1}{t}\lim_{n\to\infty} \frac{e^{t\ln(n)}}{(1+t/1)(1+t/2)\cdots(1+t/n)} \\
&= \frac{1}{t}\lim_{n\to\infty} \frac{e^{t(\ln(n)-H_n)}e^{tH_n}}{(1+t/1)(1+t/2)\cdots(1+t/n)} \\
&= \frac{e^{-\gamma t}}{t}\lim_{n\to\infty} \frac{e^{t/1+t/2+\cdots+t/n}}{(1+t/1)(1+t/2)\cdots(1+t/n)} \\
&= \frac{e^{-\gamma t}}{t}\prod_{n=1}^{\infty}\left(1+\frac{t}{n}\right)^{-1} e^{t/n},
\end{aligned}$$

where we inserted the Euler-Mascheroni number $\gamma = \lim_{n\to\infty}(H_n - \ln(n))$, $H_n = 1 + 1/2 + \cdots + 1/n$, $n \in \mathbb{N}$. The last formula is the Weierstrass representation of the gamma function.

[53] There are many proofs of Stirling's formula; see Diaconis, P. Freedman, D., *An elementary proof of Stirling's formula*, Amer. Math. Monthly, Vol. 93 (1986) 123-125; Patin, J.M. *A very short proof of Stirling's formula*, Amer. Math. Monthly, 96 (1989) 41-42; Lou, H. *A short proof of Stirling's formula*, Amer. Math. Monthly, Vol. 121, No. 2 (February 2014) 154-157; Nichel, R., *On Stirling's formula*, Amer. Math. Monthly, 109 (2002) 388-390; and Neuschel, Th., *A new proof of Stirling's formula*, Amer. Math. Monthly, Vol 121, No. 4 (April 2014) 350-352.

[54] See Nielsen, N., *Handbuch der Theorie der Gamma Funktion*, B.B. Teubner, Leipzig, 1906. Note that this work contains a comprehensive account on the gamma function until the end of the eighteenth century. Note finally that the Weierstrass representation of the gamma function is the beginning of the **function theoretic approach** to the gamma function.

History. The Euler-Mascheroni number was introduced by Euler in 1734 by the limit above. Euler used C to denote this number, while the notation γ is probably due to Lorenzo Mascheroni (1750 – 1800) in 1790. It is not known whether γ is rational or irrational.

Taking the natural logarithm, we obtain

$$\ln \Gamma(t) = -\ln(t) - \gamma t + \sum_{n=1}^{\infty} \left(\frac{t}{n} - \ln\left(1 + \frac{t}{n}\right)\right).$$

We now use logarithmic differentiation to calculate

$$\frac{\Gamma'(t)}{\Gamma(t)} = \frac{d}{dt}\left(\ln(\Gamma(t))\right) = -\frac{1}{t} - \gamma + \frac{d}{dt}\sum_{n=1}^{\infty} \left(\frac{t}{n} - \ln\left(1 + \frac{t}{n}\right)\right).$$

We need to interchange the differentiation and the infinite summation. To apply Proposition 2.2.4, we let $f_n : (0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, be defined by $f_n(t) = t/n - \ln(1 + t/n)$, $t > 0$. Clearly, $\lim_{n\to\infty} f_n(t) = 0$ (pointwise) on $(0,\infty)$. In addition, we have

$$f_n'(t) = \frac{1}{n} - \frac{1/n}{1+t/n} = \frac{t}{n(t+n)} \leq \frac{t}{n^2}, \quad t > 0,\ n \in \mathbb{N}.$$

By the Weierstrass M-test, the series $\sum_{n=1}^{\infty} f_n'(t)$ converges uniformly on every closed interval $[\epsilon, R]$, $0 < \epsilon < R$. Proposition 1.3.10 applies, and we obtain

$$\begin{aligned}
\frac{\Gamma'(t)}{\Gamma(t)} &= -\frac{1}{t} - \gamma + \sum_{n=1}^{\infty} \frac{d}{dt}\left(\frac{t}{n} - \ln\left(1 + \frac{t}{n}\right)\right) \\
&= -\frac{1}{t} - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{t+n}\right) \\
&= -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{t+n-1}\right),
\end{aligned}$$

where the last equality follows by rearrangement of the absolutely convergent series.

Note the obvious consequence that Γ is **strictly decreasing** on $(0,1]$ (since $\Gamma'(t) < 0$ for $0 < t < 1$).

Remark. The formula above for the logarithmic derivative of the gamma function can be used to give another proof for the Legendre duplication formula, and, in general, for the Legendre-Gauss formula as well. We give details to the first.[55] We

[55] For the second, see Srinivasan, G.K. *The gamma function: an eclectic tour*, Amer. Math. Monthly, Vol. 114, No. 4 (Apr. 2007) 297-315, and the references therein.

calculate

$$\begin{aligned}
&\frac{\Gamma'(t)}{\Gamma(t)} - \frac{1}{2}\frac{\Gamma'(t/2)}{\Gamma(t/2)} - \frac{1}{2}\frac{\Gamma'((t+1)/2)}{2\Gamma((t+1)/2)} \\
= & \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{t+k-1}\right) - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{t/2+k-1}\right)\right. \\
&\qquad \left. -\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{(t+1)/2+k-1}\right)\right) \\
= & \lim_{n\to\infty}\left(-\sum_{k=1}^{n}\frac{1}{t+k-1} + \sum_{k=1}^{n}\frac{1}{t+2k-2} + \sum_{k=1}^{n}\frac{1}{t+2k-1}\right) \\
= & \lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{t+k-1} = \lim_{n\to\infty}\sum_{k=n+1}^{2n}\left(\frac{1}{t+k-1} - \frac{1}{k}\right) + \lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k} = \ln 2,
\end{aligned}$$

where, due to convergence, the first limit on the right-hand side is zero; and the second is well known[56] to be equal to $\ln 2$. Rewriting the entire formula into logarithmic derivatives, and integrating, we obtain

$$\frac{\Gamma(t)}{\Gamma(t/2)\Gamma((t+1)/2)} = C \cdot 2^t.$$

Finally, letting $t = 1$, the value of the constant is $C = 1/(2\sqrt{\pi})$. The Legendre duplication formula follows.

Example 4.10.1. Show that[57]

$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(1/2)}{\Gamma(1/2)} = 2\ln 2.$$

This is a special case ($t = 1$) of the computation in the remark above.

Returning to the main line, differentiating one more time, we obtain

$$\frac{d^2}{dt^2}\ln\Gamma(t) = \frac{d}{dt}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{t+n-1}\right) = \sum_{n=0}^{\infty}\frac{1}{(t+n)^2}, \quad t > 0,$$

---

[56]See *Elements of Mathematics - History and Foundations*, Example 4.10.2. Indeed, $\sum_{k=n}^{2} 1/k = (H_{2n} - \ln(2n)) - (H_n - \ln(n)) + \ln(2)$, and let $n \to \infty$.

[57]See Whittaker, E.T. and Watson, G.N. *A Course in Modern Analysis*, 3rd ed. Dover, 2020; Exercise 3, p. 259.

where we note that, by the Weierstrass M-test, the last infinite sum is uniformly convergent on closed subintervals in $(0, \infty)$, and therefore, by Proposition 1.2.10, differentiation can be brought under the infinite sum.
In particular, the second derivative of $\ln \Gamma$ is positive on $(0, \infty)$, a property that we term as the gamma function is (strictly) **logarithmically convex** on $(0, \infty)$. More about this later.

Remark. In some developments, the formula above is used to the effect that the gamma function is the **unique** solution of the differential equation $d^2 \ln(y(t))/dt^2 = \sum_{n=0}^{\infty} 1/(t+n)^2$ with initial values $y(1) = 1$ and $y'(1) = -\gamma$.

After these preparations,[58] we define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \ln \left(\frac{\Gamma(t+1)e^t}{t^{t+1/2}} \right), \quad t > 0.$$

By the Stirling formula, we have

$$\lim_{n \to \infty} f(n) = \ln \left(\sqrt{2\pi} \right).$$

We need to show that

$$\lim_{t \to \infty} f(t) = \ln \left(\sqrt{2\pi} \right).$$

To do this, by Proposition 2.2.2 (with $a_n = n$, $n \in \mathbb{N}$, and $L = \ln(\sqrt{2\pi})$) we need to prove that

$$\lim_{t \to \infty} f'(t) = 0.$$

We now use logarithmic differentiation and calculate

$$f'(t) = \frac{\Gamma'(t+1)}{\Gamma(t+1)} - \ln(t) - \frac{1}{2t} = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{t+n} \right) - \ln(t) - \frac{1}{2t}, \quad t > 0.$$

With this, we have

$$\begin{aligned} f'(t) &= \lim_{n \to \infty} \left(\ln(n) - H_n + \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{t+k} \right) - \ln(t) - \frac{1}{2t} \right) \\ &= \lim_{n \to \infty} \left(\ln \left(\frac{n}{t} \right) - \sum_{k=0}^{n} \frac{1}{t+k+1} \right) - \frac{1}{2t}. \end{aligned}$$

[58] We follow here Dutkay, D.E., Niculascu, C.P., Popovici, F., *A note on Stirling's formula for the gamma function*, Journal of Prime Research in Mathematics, Vol. 8 (2012) 1-4.

(Note the shift in the summation.)
To give suitable bounds for the sum here, we first note that the trivial estimate

$$\int_u^{u+1} \frac{dx}{x} < \frac{1}{u} < \int_{u-1}^{u} \frac{dx}{x}, \quad u > 1,$$

gives

$$\ln(u+1) - \ln(u) < \frac{1}{u} < \ln(u) - \ln(u-1), \quad u > 1.$$

In particular, we have

$$\ln(t+k+2) - \ln(t+k+1) < \frac{1}{t+k+1} < \ln(t+k+1) - \ln(t+k), \quad t > 0,\ k \in \mathbb{N}_0.$$

Summing, we obtain

$$\ln(t+n+2) - \ln(t+1) < \sum_{k=0}^{n} \frac{1}{t+k+1} < \ln(t+n+1) - \ln(t), \quad x > 0,\ n \in \mathbb{N}_0.$$

Hence

$$\ln\left(\frac{t+n+2}{n}\right) + \ln\left(\frac{t}{t+1}\right) < \sum_{k=0}^{n} \frac{1}{t+k+1} - \ln\left(\frac{n}{t}\right) < \ln\left(\frac{t+n+1}{n}\right)$$

Letting $n \to \infty$ and using the formula for $f'(t)$ above, we arrive at

$$-\frac{1}{2t} < f'(t) < \ln\left(\frac{t+1}{t}\right) - \frac{1}{2t}, \quad t > 0.$$

This gives $\lim_{t\to\infty} f'(t) = 0$. The extension of the Stirling formula for the gamma function follows.

We now briefly return to the Legendre-Gauss formula in Proposition 4.10.2. Replacing the variable $t > 0$ by mt, $m \in \mathbb{N}$, taking the natural logarithm, and dividing by m, the left-hand side of the formula becomes

$$\frac{1}{m} \sum_{k=0}^{m-1} \ln \Gamma\left(t + \frac{k}{m}\right).$$

We notice that this is a (left-)Riemann sum of the integral $\int_t^{t+1} \ln(\Gamma(x))\, dx$. This is the principal observation leading to the **Raabe integral**, as stated in the following:

Proposition 4.10.3. *We have*

$$\int_t^{t+1} \ln\Gamma(x)\,dx = t\,(\ln(t)-1) + \frac{1}{2}\ln(2\pi), \quad t>0.$$

PROOF. We first note that the formula makes sense for $t=0$ in the limiting sense ($t\to 0^+$):

$$\int_0^1 \ln\Gamma(x)\,dx = \frac{1}{2}\ln(2\pi).$$

Indeed, the improper integral converges, since, by the Weierstrass representation of the gamma function, we have[59]

$$\int_0^1 \ln\Gamma(x)\,dx = -\int_0^1 \ln(x)\,dx - \gamma + \int_0^1 \sum_{n=1}^\infty \left(\frac{x}{n} - \ln\left(1+\frac{x}{n}\right)\right) dx$$
$$\le 1-\gamma + \int_0^1 \sum_{n=1}^\infty \frac{x^2}{n(n+x)}\,dx < 1-\gamma+\frac{1}{3}\sum_{n=1}^\infty \frac{1}{n^2}\,dx = 1-\gamma+\frac{\pi^2}{18},$$

where we used the fundamental estimate for the natural logarithm[60] $x/(1+x) \le \ln(1+x) \le x$, $-1<x\in\mathbb{R}$. In particular, we have

$$\lim_{\epsilon\to 0^+}\int_0^\epsilon \ln\Gamma(x)\,dx = 0.$$

As noted above, Γ is strictly decreasing on $(0,1]$. This implies that, for any $\epsilon\in(0,1]$, we have

$$\frac{1}{n}\sum_{k=1}^{[n\epsilon]} \ln\Gamma\left(\frac{k}{n}\right) \le \int_0^\epsilon \ln\Gamma(x)\,dx, \quad n\in\mathbb{N}.$$

Here the greatest integer $[n\epsilon]$ is the largest integer $m\in\mathbb{N}_0$ such that $m/n\le\epsilon$. Therefore, by continuity, for $\epsilon\in(0,1)$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=[n\epsilon]+1}^{n} \ln\Gamma\left(\frac{k}{n}\right) = \int_\epsilon^1 \ln\Gamma(x)\,dx.$$

Combining these, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} \ln\Gamma\left(\frac{k}{n}\right) = \int_0^1 \ln\Gamma(x)\,dx.$$

[59]Note that, by Arzelà's bounded convergence theorem, we could intechange the summation with the integral, integrate, and finally expand $\ln(1+1/n)$, but this does not result in a better estimate.

[60]See *Elements of Mathematics - History and Foundations*, Section 10.3.

We now bring in (the last product of the proof of) the Legendre-Gauss theorem:

$$\prod_{k=1}^{n} \Gamma\left(\frac{t+k}{n}\right) = (2\pi)^{(n-1)/2} \cdot n^{-1/2} \cdot n^{-t}\Gamma(t+1), \quad n \in \mathbb{N},$$

where we used $\Gamma(t+1) = t\Gamma(t)$. Setting $t = 0$, we obtain

$$\prod_{k=1}^{n} \Gamma\left(\frac{k}{n}\right) = (2\pi)^{(n-1)/2} \cdot n^{-1/2}.$$

Hence

$$\frac{1}{n}\sum_{k=1}^{n} \ln\Gamma\left(\frac{k}{n}\right) = \frac{n-1}{2n}\ln(2\pi) - \frac{\ln(n)}{2n}.$$

Taking the limit as $n \to \infty$, we arrive at

$$\int_0^1 \ln\Gamma(x)\, dx = \frac{1}{2}\ln(2\pi).$$

For the general case, we use the identity $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, to the effect that

$$\int_1^{t+1} \ln\Gamma(x)\, dx = \int_0^t \ln\Gamma(x+1)\, dx = \int_0^t \ln\Gamma(x)\, dx + \int_0^t \ln(x)\, dx, \quad t > 0.$$

Therefore

$$\begin{aligned}\int_t^{t+1} \ln\Gamma(x)\, dx &= \int_0^1 \ln\Gamma(x)\, dx + \int_1^{t+1} \ln\Gamma(x)\, dx - \int_0^t \ln\Gamma(x)\, dx \\ &= \frac{1}{2}\ln(2\pi) + \int_0^t \ln(x)\, dx, \ t > 0.\end{aligned}$$

The proposition follows since $\int \ln(x)\, dx = x(\ln(x) - 1) + C$.

As noted in a previous history, Euler introduced the gamma function to extend the concept of factorial to non-integer values. The question naturally arises as to whether the property $\Gamma(t+1) = t\Gamma(t)$, $t > 0$ (along with the normalizing condition $\Gamma(1) = 1$) uniquely determine the gamma function or not. The answer is clearly "no" as any functional multiple $g \cdot \Gamma$ with periodic g (with period 1 and $g(1) = 1$) also satisfies this property. According to the Bohr-Mollerup theorem,[61], unicity holds, however, if one adds the property of logarithmic convexity.

[61]See Artin, E., *The Gamma Function*, Holt, Rinehart, Winston, 1964.

Bohr-Mollerup Theorem. *The gamma function is uniquely characterized as a function* $f : (0, \infty) \to (0, \infty)$ *satisfying the following three properties: (1)* $f(1) = 1$*; (2)* $f(t+1) = tf(t)$, $t > 0$*; and (3)* f *is* ***logarithmically convex****; that is,* $\ln \circ f$ *is convex.*

First, note that we defined strict logarithmic convexity of the gamma function by requiring that the second derivative of the composition $\ln \circ \Gamma$ is positive. More generally, we say that a function[62] $f : (0, \infty) \to \mathbb{R}$ is **convex** if, for every fixed $0 < x \in \mathbb{R}$, the difference quotient

$$\mathfrak{m}_f(x+h, x) = \frac{f(x+h) - f(x)}{h}, \quad -x < h \neq 0, \; h \in \mathbb{R},$$

(as a function of h) is increasing on $(-x, \infty) \setminus \{0\}$. Thus, for $0 < u < x < v$, letting $h = u - x < 0$ and $k = v - x > 0$, we have

$$\frac{f(u) - f(x)}{u - x} = \mathfrak{m}_f(x+h, x) \leq \mathfrak{m}_f(x+k, x) = \frac{f(v) - f(x)}{v - x}.$$

Rearranging, this is equivalent to

$$f(x) \leq \frac{v - x}{v - u} f(u) + \frac{x - u}{v - u} f(v), \quad 0 < u < x < v.$$

The transparent geometric interpretation of this is that, for any closed interval $[u, v] \subset (0, \infty)$, the graph of the function f restricted to $[u, v]$ is below its secant line segment connecting $(u, f(u))$ and $(v, f(v))$. Note that this property, for all $[u, v] \subset (0, \infty)$, implies that a convex function is continuous everywhere.[63]
Assume now that f is twice continuously differentiable. Differentiating with respect to h, we obtain

$$\frac{d}{dh}\mathfrak{m}_f(x+h, x) = \frac{d}{dh}\frac{f(x+h) - f(x)}{h} = \frac{hf'(x+h) - f(x+h) + f(x)}{h^2}$$

Using the substitution $u = x + h$, $u > 0$, the Taylor formula (centered at u) gives

$$(f(x) =)\, f(u - h) = f(u) - hf'(u) + \frac{h^2}{2} f''(u - (1-s)h)$$

for some $s \in [0, 1]$, where we used the Lagrange form of the remainder (Section 2.3). (The points $u - (1-s)h$ parametrized by $s \in [0, 1]$ fill the interval with end-points u

[62] We chose the domain interval as $(0, \infty)$ to fit to our purposes; clearly convexity can be defined on any interval.

[63] Letting $x \to u$, we have $f(u+) \leq f(u)$, and letting $v \to x$, we have $f(x) \leq f(x+)$, so that $f(x+) = f(x)$ for all $x > 0$, etc.

and $u-h$; the choice of $1-s$, instead of s is of technical convenience.) Substituting this back to the derivative of the difference quotient above, we obtain

$$\frac{d}{dh}\mathfrak{m}_f(x+h,x)=\frac{1}{2}f''(x+sh),\quad -x<h\neq 0,\ 0\leq s\leq 1,\ h,s\in\mathbb{R}.$$

After these preparations, still assuming that f is twice continuously differentiable on $(0,\infty)$, we claim that $f:(0,\infty)\to\mathbb{R}$ is convex if and only if $f''\geq 0$ on $(0,\infty)$.
Indeed, by the formula just derived, if $f''\geq 0$ on $(0,\infty)$ then $d\mathfrak{m}_f(x+h,x)/dh\geq 0$, $-x<h$, $0\neq h\in\mathbb{R}$, and so, by Proposition 2.1.5, the difference quotient $\mathfrak{m}_f$ is increasing (in the variable h), and convexity of f follows. Conversely, if $\mathfrak{m}_f(x+h,x)$, $-x<h\neq 0$, is increasing (in h), then, again by Proposition 2.1.5, $d\mathfrak{m}_f(x+h,x)/dh\geq 0$, $-x<h$, $0\neq h\in\mathbb{R}$. If $f''<0$ at some point in $(0,\infty)$ then, by the assumed continuity, f'' is negative on an open interval. Choosing x and $x+h$ (with h small enough) in this interval, we get a contradiction to the formula above.
Finally, we say that a function $f:(0,\infty)\to\mathbb{R}$ is **strictly convex** if sharp inequalities hold throughout; that is, if the difference quotient above is strictly increasing, or if, for any closed interval $[x',x'']\subset(0,\infty)$, the graph of the function f restricted to $[x',x'']$ is strictly below its secant line segment connecting $(x',f(x'))$ and $(x'',f(x''))$ except these end-points. If f is twice continuously differentiable then strict convexity is equivalent to positivity of the second derivative.
Finally, we call a function $f:(0,\infty)\to\mathbb{R}$ (strictly) **concave** if $-f$ is (strictly) convex.

Since Γ is differentiable up to any order on $(0,\infty)$, it follows that the composition $\ln\circ\Gamma$ is strictly convex according to our (more general) definition. Adopting this, we say that Γ is strictly logarithmically convex on $(0,\infty)$.

Remark. Using the Hölder inequality (Proposition 3.4.3 naturally extended to improper integrals), a quick proof of the logarithmic convexity of Γ on $(0,\infty)$ (by the original definition of convexity as well as the Gamma function) can be given as follows. For $1/p+1/q=1$, $0<p,q<1$, and $u,v>0$, we have

$$\begin{aligned}\Gamma\left(\frac{u}{p}+\frac{v}{q}\right) &= \int_0^\infty x^{(u-1)/p}e^{-x/p}x^{(v-1)/q}e^{-x/q}\,dx\\ &\leq \left(\int_0^\infty x^{u-1}e^{-x}\,dx\right)^{1/p}\left(\int_0^\infty x^{v-1}e^{-x}\,dx\right)^{1/q}=\Gamma(u)^{1/p}\Gamma(v)^{1/q}.\end{aligned}$$

Note that strict inequality holds iff $u\neq v$. Hence

$$\ln\Gamma\left(\frac{1}{p}\,u+\frac{1}{q}\,v\right)\leq\frac{1}{p}\ln\Gamma(u)+\frac{1}{q}\ln\Gamma(u),\quad \frac{1}{p}+\frac{1}{q}=1,\ p,q>0.$$

Strict logarithmic convexity follows.

Proof of the Bohr-Mollerup Theorem. Assume $f : (0, \infty) \to \mathbb{R}$ satisfies properties (1)-(3) in the theorem. Let $t \in (0, 1]$ and $2 \le n \in \mathbb{N}$. For $n - 1 < n < n + t \le n + 1$, logarithmic convexity of f gives

$$\frac{\ln f(n-1) - \ln f(n)}{(n-1) - n} \le \frac{\ln f(n+t) - \ln f(n)}{(n+t) - n} \le \frac{\ln f(n+1) - \ln f(n)}{(n+1) - n}.$$

On the other hand, by (1) and (3), we have $f(k) = (k-1)!$, $k \in \mathbb{N}$, and

$$f(n+t) = t(t+1)(t+2)\cdots(t+n-1)f(t).$$

Substituting these into the inequalities above and rearranging, we obtain

$$\ln(n-1)^t \le \ln \frac{t(t+1)(t+2)\cdots(t+n-1)f(t)}{(n-1)!} \le \ln n^t.$$

Taking natural exponents, and rearranging, we have

$$\frac{(n-1)^t(n-1)!}{t(t+1)(t+2)\cdots(t+n-1)} \le f(t) \le \frac{n^t(n-1)!}{t(t+1)(t+2)\cdots(t+n-1)}$$

Moving up the value $n-1$ to n in the first inequality (as $f(t)$ does not depend on n), we get

$$\frac{n^t n!}{t(t+1)(t+2)\cdots(t+n)} \le f(t) \le \frac{n^t n!}{t(t+1)(t+2)\cdots(t+n)} \frac{t+n}{n}.$$

Finally, letting $n \to \infty$, we obtain

$$f(t) = \lim_{n\to\infty} \frac{n^t n!}{t(t+1)(t+2)\cdots(t+n)}, \quad t \in (0, 1].$$

The last limit is Euler's representation of $\Gamma(t)$ for $t \in (0, 1]$. Finally, replacing t by $t + 1$ in the right-hand side, the limit gets multiplied by t. This shows that the limit satisfies the functional equation for the gamma function. Hence the equality above holds for all $t \in \mathbb{R}$. The theorem follows.

History. The Bohr-Mollerup theorem was proved by the Danish mathematicians Harald Bohr and Johannes Mollerup.

As an application, we now derive the following inequalities:

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+s)^{1-s}, \quad x > 0,\ 0 < s < 1.$$

Indeed, according to strict logarithmic convexity of the gamma function, we have

$$\Gamma(tu+(1-t)v) < \Gamma(u)^t\Gamma(v)^{1-t}, \quad u \neq v, u, v > 0,\ 0 < t < 1.$$

Letting $u = x$, $v = x+1$, and $t = 1-s$, we get

$$x^{1-s} = \left(\frac{\Gamma(x+1)}{\Gamma(x)}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)}.$$

Letting $u = x+s$, $v = x+s+1$, and $t = s$, we get

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(\frac{\Gamma(x+s+1)}{\Gamma(x+s)}\right)^{1-s} = (x+s)^{1-s}.$$

The inequalities follow.
Note the obvious consequence

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^{1-s}\Gamma(x+s)} = 1, \quad 0 < s < 1.$$

Remark. The inequalities above are due to Wendel[64] in 1948. In about a decade later Gautschi independently derived two inequalities for the ratio of the gamma function with lower bounds identical to Wendel's, and, depending on the values of $x > 0$ and $0 < s < 1$, the two upper bounds can be stronger or weaker than Wendel's. In 1983, Kershaw proved stronger inequalities[65]

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right)^{1-s}, \quad x > 0,\ 0 < s < 1.$$

There is a proliferation of estimates for various ratios of the gamma function.[66]

Exercises

1. Use the following steps to derive Euler's reflection formula from the Legendre duplication formula. Consider the function $\phi : \mathbb{R}\setminus\mathbb{Z} \to \mathbb{R}$ defined by

$$\phi(t) = \Gamma(t)\Gamma(1-t)\sin(\pi t), \quad t \in \mathbb{R}\setminus\mathbb{Z}.$$

[64]See Wendel, J.G., *Note on the Gamma function*, Amer. Math. Monthly 55 (9) (1948) 563-564.
[65]See Kershaw, D., *Some extensions of W. Gautsch's inequalities for the gamma function*, Math. Comp. 41 (1983) 607-611.
[66]For a survey article, see Qi, F., *Bounds for the ratio of two gamma functions*, Journal of Inequalities and Applications, Vol. 2010.

(a) Show that ϕ is periodic with period 1. (b) Write

$$\phi(t) = \pi\Gamma(1+t)\Gamma(1-t)\frac{\sin(\pi t)}{\pi t}, \quad t \in \mathbb{R} \setminus \mathbb{Z},$$

define $\phi(n) = \pi$, $n \in \mathbb{Z}$, and show that $\phi > 0$ is infinitely many times differentiable at 0, and then, by periodicity, everywhere on $\mathbb{R}$. (c) Use the Legendre duplication formula (in various settings) to derive the formula

$$\phi\left(\frac{t}{2}\right)\phi\left(\frac{t+1}{2}\right) = \pi\phi(t), \quad t \in \mathbb{R}.$$

(d) Letting $\psi = (\ln \circ \phi)''$ and $M = \sup_{[0,1]} |\psi| = \sup_{\mathbb{R}} |\psi|$, use (c) to obtain $|\psi| \leq M/2$; and thereby conclude $M = 0$, and hence $\psi = 0$. (e) Finally, show that (d) implies that ϕ is constant with the value of the constant equal to π.

2. Use the Weierstrass representation of the gamma function and Euler's reflection formula to derive Euler's infinite product formula for the sine.

3. Show that

$$\prod_{k=1}^{3m} \Gamma\left(\frac{k}{3}\right) = \frac{(2\pi)^m}{\sqrt{3}^{m(3m-2)}} \prod_{\ell=1}^{m-1} (3\ell)!, \quad m \in \mathbb{N}.$$

Solution: Use the Legendre-Gauss formula for $m = 3$ and split the product on the left-hand side into groups of consecutive three's.

4. Recall the formula

$$n! = \int_0^\infty x^n e^{-x}\, dx = \int_0^\infty e^{n\ln(x) - x} dx, \quad n \in \mathbb{N}.$$

Replace the exponent $n\ln(x) - x$ by its **quadratic** Taylor polynomial at its **maximum** $c = n$. Evaluate the improper integral, and show that it becomes the Stirling approximation of $n!$.
Solution: The quadratic Taylor polynomial is $p(x) = n\ln(n) - n - (x-n)^2/(2n)$. The improper integral becomes

$$\int_0^\infty e^{p(x)} dx = n^n \cdot e^{-n} \int_0^\infty e^{-(x-n)^2/(2n)} dx = \sqrt{2n} \cdot n^n \cdot e^{-n} \int_{-\sqrt{n/2}}^\infty e^{-u^2}\, du.$$

Now use Example 2.9.2: $\int_{-\infty}^\infty e^{-u^2}\, du = 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}$.

5. Use the Weierstrass representation of the gamma function to obtain the formula[67]

$$\prod_{n=1}^{\infty}\left(1-\frac{t}{s+n}\right)e^{t/n}=e^{\gamma t}\frac{\Gamma(s+1)}{\Gamma(s-t+1)},\quad s,t\in\mathbb{R}\setminus\mathbb{Z}.$$

Solution: This is a direct computation in the use of the Weierstrass representation twice.

6. Show that

$$\Gamma'(m+1)=m!\left(-\gamma+H_m\right),\quad m\in\mathbb{N}.$$

Solution: We have

$$\Gamma'(m+1)=\Gamma(m+1)\left(-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{m+n}\right)\right)=m!\left(-\gamma+\sum_{n=1}^{m}\frac{1}{n}\right),$$

since the middle sum is telescopic.

7. Show that the trigonometric identity in the lemma for the proof of the Legendre-Gauss formula (Proposition 4.10.2) follows from the special case of the Legendre-Gauss formula

$$\prod_{k=1}^{m-1}\Gamma\left(\frac{k}{m}\right)=(2\pi)^{(m-1)/2}m^{1/2}$$

used in the proof of the Raabe integral (Proposition 4.10.3) and Euler's reflection fomula

$$\Gamma\left(\frac{k}{m}\right)\Gamma\left(1-\frac{k}{m}\right)=\frac{\pi}{\sin(k\pi/m)},\quad k=1,\ldots,m-1.$$

8. Use the improper integral formula at the end of Section 4.6 as well as Euler's infinite product formula for the sine and the Weierstrass representation of the gamma function to derive the following[68]

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin(x)}{x^{1-t}}\,dx=e^{-\gamma t}\frac{\prod_{n=1}^{\infty}\left(1-\frac{t}{2n}\right)e^{t/(2n)}}{\prod_{n=1}^{\infty}\left(1+\frac{t}{2n-1}\right)e^{t/(2n-1)}}$$

[67]See Whittaker, E.T. and Watson, G.N. *A Course in Modern Analysis*, 3rd ed. Dover, 2020; 12.1, Example 3.

[68]See ibid. Example 4 in 12.22.

4.11 The Beta Function

The **beta function** $B : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is defined by

$$B(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}\,dx, \quad 0 < t, s \in \mathbb{R},$$

The integral is improper[69] for $0 < t, s < 1$; its existence needs justification. We choose $c \in (0,1)$, and split the integral as

$$\int_0^1 x^{t-1}(1-x)^{s-1}\,dx = \int_0^c x^{t-1}(1-x)^{s-1}\,dx + \int_c^1 x^{t-1}(1-x)^{s-1}\,dx.$$

Now, in the first integral on the left-hand side, the factor $(1-x)^{s-1}$ of the integrand is continuous on $[0,c]$, and hence the first integral equiconverges with $\int_0^c x^{t-1}\,dx$. The latter is equal to $[x^t/t]_0^c = c^t$ for $t > 0$. Similarly, the second integral equiconverges with $\int_c^1 (1-x)^{s-1}\,dx = [-(1-x)^s/s]_c^1 = (1-c)/s$ for $s > 0$. The existence of the beta function is established.
The change of variables $x = \sin^2(u)$ and $dx = 2\sin(u)\cos(u)du$ transforms the definition to the equivalent

$$B(t,s) = 2\int_0^{\pi/2} \sin^{2t-1}(u)\cos^{2s-1}(u)\,du, \quad t, s > 0.$$

History. As early as 1656, Wallis was pursuing an idea to obtain the value of π by evaluating the integral $\int_0^1 \sqrt{1-x^2}\,dx$ (giving the area of a quarter disk). Although he could only evaluate the non-trivial cases of the integrals $\int_0^1 x^m(1-x)^n\,dx$, $m, n \in \mathbb{Z}$, after some heuristic arguments, and using modern notations, he essentially ended up with the formula

$$\int_0^1 \sqrt{1-x^2}\,dx = \lim_{n\to\infty} n\left(\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n+1)}\right)^2 = \left(\Gamma\left(\frac{3}{2}\right)\right)^2.$$

This may have prompted Euler (some seven decades later) to seek a relation between the gamma function and integrals of the type $\int_0^1 x^m(1-x)^n\,dx$. The beta function appears first in his works in *Nov. Comm. Petrop.* XVI. (1772), and those of Legendre in *Exercises de Calcul Intégral*, I. (1817) p. 221; the latter called it "Eulerian integral of the first kind."

We now derive a few properties of the beta function. Symmetry in the variables is obvious:

$$B(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}\,dx = -\int_1^0 (1-y)^{t-1}y^{s-1}\,dy = B(s,t),$$

[69] The integrand has removable discontinuities at 0 for $t = 0$ and at 1 for $s = 1$.

where we used the substitution $x = 1 - y$, $dx = -dy$.

The change of variables $x = y/(y+1)$ with $dx = dy/(y+1)^2$ transforms the integral in the definition of the beta function into the following

$$B(t,s) = \int_0^\infty \left(\frac{y}{y+1}\right)^{t-1} \left(\frac{1}{y+1}\right)^{s-1} \frac{dy}{(y+1)^2} = \int_0^\infty \frac{y^{t-1}}{(y+1)^{t+s}}\, dy.$$

This last representation of the beta function has an important consequence, the pair of inductive formulas

$$B(t+1,s) = \frac{t}{t+s} B(t,s) \quad \text{and} \quad B(t,s+1) = \frac{s}{t+s} B(t,s), \quad t,s > 0.$$

By symmetry, it is sufficient to derive the first formula. We perform integration by parts

$$B(t+1,s) = \int_0^\infty \frac{y^t}{(y+1)^{t+s+1}}\, dy = \int_0^\infty \frac{t y^{t-1}}{(t+s)(y+1)^{t+s}}\, dy = \frac{t}{t+s} B(t,s),$$

where $u = y^t$ and $dv = dy/(y+1)^{t+s+1}$ (therefore $du = t y^{t-1} dy$ and $v = -1/((t+s)(y+1)^{t+s})$). Note that the boundary terms vanish:

$$\left[-\frac{y^t}{(t+s)(y+1)^{t+s}} \right]_0^\infty = -\frac{1}{t+s} \lim_{y\to\infty} \frac{y^t}{(y+1)^{t+s}} = -\frac{1}{t+s} \lim_{y\to\infty} \frac{1}{(y+1)^s} = 0, \ t,s > 0.$$

The inductive formulas follow.

The same substitution as above gives the following:

Example 4.11.1. Show that[70]

$$\int_0^1 \frac{x^{t-1}(1-x)^{s-1}}{(x+a)^{t+s}}\, dx = \frac{B(t,s)}{(1+a)^t a^s}, \quad t,s,a > 0.$$

Indeed, the substitution $x = y/(y+1)$ with $dx = dy/(y+1)^2$ transforms the integral as

$$\begin{aligned}
\int_0^1 \frac{x^{t-1}(1-x)^{s-1}}{(x+a)^{t+s}}\, dx &= \int_0^\infty \frac{y^{t-1}}{(y+a(y+1))^{t+s}}\, dy \\
&= \frac{1}{a^{t+s}} \int_0^\infty \frac{y^{t-1}}{((1+a)y/a+1)^{t+s}}\, dy \\
&= \left(\frac{a}{1+a}\right)^t \frac{1}{a^{t+s}} \int_0^\infty \frac{z^{t-1}}{(z+1)^{t+s}}\, dz,
\end{aligned}$$

where $z = (1+a)y/a$. Now the example follows from the discussion above.

[70]See Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020; Exercise 28, p. 261.

Example 4.11.2. We have

$$\int_0^\infty \frac{\cosh(2sx)}{\cosh^{2t}(x)}\, dx = 4^{t-1} B(t+s, t-s), \quad |s| < t.$$

We work backwards from the beta function (with obvious substitution) as

$$\begin{aligned} B(t+s,t-s) &= \int_0^\infty \frac{t+s-1}{(y+1)^{2t}}\, dy = 2\int_{-\infty}^\infty \frac{\left(e^{2x}\right)^{t+s}}{\left(e^{2x}+1\right)^{2t}}\, dx \\ &= 2\int_{-\infty}^\infty \frac{\left(e^{2x}\right)^{s}}{\left(e^{x}+e^{-x}\right)^{2t}}\, dx = 2^{1-2t}\int_{-\infty}^\infty \frac{e^{2sx}}{\cosh^{2t}(x)}\, dx \\ &= 2^{1-2t}\int_0^\infty \frac{e^{2sx}}{\cosh^{2t}(x)}\, dx + 2^{1-2t}\int_0^\infty \frac{e^{-2sx}}{\cosh^{2t}(x)}\, dx \\ &= 2^{1-2t}\int_0^\infty \frac{e^{2sx}+e^{-2sx}}{\cosh^{2t}(x)}\, dx = 2^{2-2t}\int_0^\infty \frac{\cosh(2sx)}{\cosh^{2t}(x)}\, dx. \end{aligned}$$

The example follows.

The previous considerations can be considered preparatory for the following so-called **beta gamma relation** expressing the beta function in terms of the gamma function:

$$B(t,s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)} \quad t, s > 0.$$

To derive this, we introduce a new parameter $z > 0$ in the definition of the Gamma function via the substitution $x = zy$ as

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx = z^t \int_0^\infty y^{t-1} e^{-zy}\, dy,$$

and obtain

$$\frac{\Gamma(t)}{z^t} = \int_0^\infty y^{t-1} e^{-zy}\, dy, \quad t, z > 0.$$

Replacing t by $t+s$, $t, s > 0$, and z by $z+1$, $z > 0$, this gives

$$\frac{\Gamma(t+s)}{(z+1)^{t+s}} = \int_0^\infty y^{t+s-1} e^{-(z+1)y}\, dy.$$

Using this and the representation of the beta function derived above, we now calculate

$$\Gamma(t+s)B(t,s) = \Gamma(t+s)\int_0^\infty \frac{z^{t-1}}{(z+1)^{t+s}}dz = \int_0^\infty z^{t-1}\frac{\Gamma(t+s)}{(z+1)^{t+s}}dz$$
$$= \int_0^\infty \int_0^\infty z^{t-1}y^{t+s-1}e^{-(z+1)y}\,dy\,dz = \int_0^\infty \int_0^\infty z^{t-1}y^{t+s-1}e^{-(z+1)y}\,dz\,dy$$
$$= \int_0^\infty y^{t+s-1}e^{-y}\int_0^\infty z^{t-1}e^{-yz}\,dz\,dy = \int_0^\infty y^{t+s-1}e^{-y}\frac{\Gamma(t)}{y^t}dy$$
$$= \Gamma(t)\int_0^\infty y^{s-1}e^{-y}dy = \Gamma(t)\Gamma(s).$$

In the computation we reversed the order of the iterated integral.[71] This holds if the integrand $z^{t-1}y^{t+s-1}e^{-(z+1)y}$ is **continuous** in the respective domain; that is, if $t > 1$. Thus, this computation, and hence our proof of the formula above, holds if $t > 1$. Finally, the validity of our formula extends to $t > 0$ via the inductive property of the beta function, since, for $t, s > 0$, we have

$$B(t,s) = \frac{t+s}{t}B(t+1,s) = \frac{t+s}{t}\frac{\Gamma(t+1)\Gamma(s)}{\Gamma(t+s+1)} = \frac{t+s}{t}\frac{t\Gamma(t)\Gamma(s)}{(t+s)\Gamma(t+s)} = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}.$$

The beta gamma relation follows.

Remark. The Bohr-Molleup theorem can also be used to derive the beta gamma relation as follows.[72] Fix $0 < s \in \mathbb{R}$, and define $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = B(t,s)\frac{\Gamma(t+s)}{\Gamma(s)}, \quad 0 < t \in \mathbb{R}.$$

Now, by simple evaluation, we have $B(1, s) = 1/s$, $s > 0$, and hence $f(1) = 1$. Thus, condition (1) in the Bohr-Mollerup theorem holds.
Next, we calculate

$$f(t+1) = B(t+1,s)\frac{\Gamma(t+s+1)}{\Gamma(s)} = \frac{t}{t+s}B(t,s)\frac{(t+s)\Gamma(t+s)}{\Gamma(s)} = tf(t),$$

so that condition (2) also holds.
Finally, we have

$$\ln f(t) = \ln B(t,s) + \ln\Gamma(t+s) - \ln\Gamma(s),$$

[71]The customary proof here is to rewrite the iterated integral into a double integral, and change the Cartesian to polar coordinates. As usual, we preferred to keep the level of exposition to as elementary as possible.

[72]This approach is pursued in Artin, E., *The Gamma Function*, New York, Holt, Rinehart and Winston 1964, and Dover 2015.

where the first term on the right-hand side is convex (by inspecting the dependence in t of the beta integral), the second by logarithmic convexity of the gamma function, and the third term is constant. Hence condition (3) in the Bohr-Mollerup theorem also holds. This theorem now gives $f(t) = \Gamma(t)$, $t > 0$. The beta gamma relation follows.

History. The original integral representation of the gamma function, the Legendre duplication formula, the Stirling formula for the gamma function, the beta gamma relation, and a host of other identities comprise what is nowadays termed the **Eulerian approach** to the gamma function. The 3-volume work of Legendre[73] gives the first comprehensive treatise on this. For latter works, the most noteworthy are those of Dirichlet[74] and Schlömlich.[75]

Example 4.11.3. [76] We have

$$\int_0^1 \frac{x^{t-1}(1-x)^{s-1}}{ax+b(1-x)}\,dx = \frac{\Gamma(t)\Gamma(s)}{a^t b^s \Gamma(t+s)}, \quad 0 < t, s, a, b \in \mathbb{R}.$$

In the formula

$$\int_0^1 x^{t-1}(1-x)^{s-1}\,dx = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)},$$

just derived, we use the substitution

$$x = \frac{au}{au+b(1-u)}, \quad 1-x = \frac{b(1-u)}{au+b(1-u)}, \quad dx = \frac{ab\,du}{(au+b(1-u))^2}.$$

We obtain

$$\int_0^1 \frac{(au)^{t-1}(b(1-u))^{s-1}}{(au+b(1-u))^{t-1+s-1+2}}\,ab\,du = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}.$$

Rearranging, the formula follows.

The next example is a simple consequence of the beta gamma relation and the Stirling formula.

[73] Legendre, A.M., *Exercises de calcul intégral sur divers ordres de transcendantes et sur les quadratures*, Mme Ve Courcier, Paris, I (1811), II (1817), III (1817).

[74] Dirichlet, L.G., *Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen*, Friedrich Vieweg und Sohn, Braunschweig, 1904, pp. 100-124.

[75] Schlömlich, O., *Compendium der höheren Analysis*, vol. 2, Friedrich Vieweg und Sohn, Braunschweig, 1866, pp. 239-280.

[76] See also Andrews, G.E., Askey, R. and Roy, R., *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge Uniersity Press, 1999, Exercise 17 at the end of Chapter 1.

Example 4.11.4. We have

$$\lim_{n\to\infty} n^x B(x,n) = \Gamma(x), \quad 0 < x \in \mathbb{R}.$$

Using the Stirling formula twice, we calculate

$$\begin{aligned} n^x B(x,n) &= n^x \frac{\Gamma(n)\Gamma(x)}{\Gamma(x+n)} = \frac{n^x (x+n)\Gamma(n+1)}{n\Gamma(x+n+1)}\Gamma(x) \\ &\sim \frac{n^x (x+n)\sqrt{2\pi n}\, n^n e^{-n}}{n\sqrt{2\pi(x+n)}(x+n)^{x+n} e^{-(x+n)}}\Gamma(x) \quad \text{as} \quad n\to\infty \\ &= \left(\frac{n}{x+n}\right)^{x+n-1/2} \cdot e^x \cdot \Gamma(x) \end{aligned}$$

Finally, by the Euler limit, we have

$$\left(\frac{x+n}{n}\right)^{x+n-1/2} = \left(1+\frac{x}{n}\right)^n \left(1+\frac{x}{n}\right)^{x-1/2} \sim e^x \quad \text{as} \quad n\to\infty.$$

The example follows.

Example 4.11.5. Derive the integral formula[77]

$$\int_0^\infty \frac{y^{t-1}}{y+1}\, dy = \frac{\pi}{\sin(\pi t)}, \quad 0 < t < 1.$$

The right-hand side is suggestive of using Euler's reflection formula. We use this backward as

$$\frac{\pi}{\sin(\pi t)} = \Gamma(t)\Gamma(1-t) = B(t, 1-t) = \int_0^\infty \frac{y^{t-1}}{y+1}\, dy,$$

where, in the middle equality, we used the beta gamma relation. The example follows.

History. Dedekind's Inauguraldissertation on the gamma function (under the supervision of Gauss) contains an interesting proof of Euler's reflection formula based on showing that the function $f : (0,1) \to \mathbb{R}$ defined by $f(t) = B(t, 1-t)$ satisfies the non-linear (second-order) differential equation $f''(t)f(t) - f'(t)^2 = f(t)^4$ with initial conditions $f(1/2) = \pi$ and $f'(1/2) = 0$. See Dedekind, R., *Über die Elemente der Theorie der Eulerschen Integrale*, dissertation, Göttingen, 1852, Gesammelte mathematische Werke, Bd I, 1-31; and also *Über ein Eulersches Integral*, J. reine angewandte Math. (1853) 370-374. See also Exercise 8 at the end of this section.

A notable consequence of the previous example is the following:

[77] The (indefinite) integral can, at least in principle, be calculated for $t \in \mathbb{Q}$. See Dirichlet, G.L., *Vorlesungen über die Lehre von der einfachen und mehrfachen bestimmten Integralen*, Friedrich Vieweg und Sohn, Braunswieg, 1904.

Example 4.11.6. We have

$$\int_0^{\pi/2} \tan^a(x)\, dx = \frac{\pi}{2\cos(\pi a/2)}, \quad 0 < a < 1.$$

To show this, we rewrite the integral into a beta integral as

$$\int_0^{\pi/2} \tan^a(x)\, dx = \int_0^{\pi/2} \sin^a(x)\cos^{-a}(x)\, dx = \frac{1}{2}B\left(\frac{1+a}{2}, \frac{1-a}{2}\right) = \frac{\pi}{2\sin(\pi(1+a)/2)},$$

where we used the last computation of the previous example with $t = (1+a)/2$ (and hence $1 - t = (1-a)/2$). The example follows.

Remark. It is instructive to compare this with Example 4.1.3; they both give the correct value $\pi/\sqrt{2}$ for $a = 1/2$.

Among the many applications of the beta gamma relation, we discuss the case $t = s$; this will give a short proof of the Legendre duplication formula. We first calculate

$$\begin{aligned} B(t,t) &= 2\int_0^{\pi/2} (\sin(u)\cos(u))^{2t-1}\, du = 2^{2-2t}\int_0^{\pi/2} \sin^{2t-1}(2u)\, du \\ &= 2^{1-2t}\int_0^{\pi} \sin^{2t-1}(v)\, dv = 2^{2-2t}\int_0^{\pi/2} \sin^{2t-1}(v)\, dv = \\ &= 2^{1-2t}B(t, 1/2) = 2^{1-2t}\frac{\Gamma(t)\Gamma(1/2)}{\Gamma(t+1/2)}. \end{aligned}$$

Since $B(t,t) = \Gamma(t)^2/\Gamma(2t)$ (and $B(1/2,1/2) = 2\int_0^{\pi/2} du = \pi = \Gamma(1/2)^2$), we recover the Legendre duplication formula in Proposition 4.10.1:

$$\Gamma(t)\Gamma(t+1/2) = 2^{1-2t}\sqrt{\pi}\,\Gamma(2t), \quad t > 0.$$

In the previous computation the Wallis integral of Example 4.2.6 was present. In fact, we can automatically extend the definition of the **Wallis integral** for any real parameter $-1 < a \in \mathbb{R}$, and the computation above amounts to the following

$$W(a) = \int_0^{\pi/2} \sin^a(x)\, dx = \frac{1}{2}B\left(\frac{a+1}{2}, \frac{1}{2}\right) = 2^{a-1}B\left(\frac{a+1}{2}, \frac{a+1}{2}\right), \quad a > -1.$$

Example 4.11.7. Derive the integral formulas

$$\int_0^{\pi/2} \sqrt{\sin(x)}\, dx = \int_0^{\pi/2} \sqrt{\cos(x)}\, dx = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2},$$

and
$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin(x)}} = \int_0^{\pi/2} \frac{dx}{\sqrt{\cos(x)}} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}.$$
We have
$$\int_0^{\pi/2} \sqrt{\sin(x)}\, dx = W_{1/2} = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)}.$$
Euler's reflection formula gives $\Gamma(1/4)\Gamma(3/4) = \pi/\sin(\pi/4) = \sqrt{2}\pi$, and we also have $\Gamma(5/4) = \Gamma(1/4)/4$. Putting these together, the first integral follows.
Sinilarly, we have
$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin(x)}} = W_{-1/2} = \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}.$$
The example follows.

Remark. The substitution $t^2 = \sin(x)$ transforms the first integral in the example above as
$$\int_0^{\pi/2} \sqrt{\sin(x)}\, dx = 2\int_0^1 \frac{t^2}{\sqrt{1-t^4}}\, dt = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}.$$
Similarly, the second integral becomes
$$\int_0^{\pi/2} \frac{1}{\sqrt{\sin(x)}}\, dx = 2\int_0^1 \frac{1}{\sqrt{1-t^4}}\, dt = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}.$$
These integrals were noted by Euler who put them together in the product
$$\int_0^1 \frac{1}{\sqrt{1-t^4}}\, dt \cdot \int_0^1 \frac{t^2}{\sqrt{1-t^4}}\, dt = \frac{\pi}{4}.$$
We also have
$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{\sin(x)}} &= 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+t^2)}} \\ &= 2\int_0^{\pi/2} \frac{d\phi}{\sqrt{2-\cos^2(\phi)}} = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2(\theta)}}, \end{aligned}$$
where we used the substitution $t = \sin(\phi)$ (with $dt = \cos(\phi)d\phi$) followed by yet another substitution $\phi = \pi/2 - \theta$ (with $d\phi = -d\theta$).

For the other integral, we employ the same substitutions, and obtain

$$\begin{aligned}\int_0^{\pi/2}\sqrt{\sin(x)}\,dx &= 2\int_0^1 \frac{t^2\,dt}{\sqrt{1-t^4}} = 2\int_0^{\pi/2}\frac{\sin^2(\phi)\,d\phi}{\sqrt{2-\cos^2(\phi)}}\\ &= 2\int_0^{\pi/2}\frac{\cos^2(\theta)\,d\theta}{\sqrt{2-\sin^2(\theta)}} = 2\int_0^{\pi/2}\frac{1-\sin^2(\theta)\,d\theta}{\sqrt{2-\sin^2(\theta)}}\\ &= 2\sqrt{2}\int_0^{\pi/2}\sqrt{1-\frac{1}{2}\sin^2(\theta)}\,d\theta - \sqrt{2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2(\theta)}}.\end{aligned}$$

In these two examples, as well as the period integral of the pendulum in Exercise 9 at the end of this section, we see the same type of integrals to occur. These are called **elliptic integrals**.[78]

Example 4.11.8. We have[79]

$$\int_0^{\pi}\frac{dx}{\sqrt{3-\cos(x)}}\,dx = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}.$$

We use the substitution

$$\cos(x) = 1 - 2\tan\left(\frac{u}{2}\right) \quad \text{with} \quad \sin(x)\,dx = \left(1+\tan^2\left(\frac{u}{2}\right)\right)\,du.$$

Since

$$1-\cos^2(x) = 4\tan\left(\frac{u}{2}\right)\left(1-\tan\left(\frac{u}{2}\right)\right),$$

upon substitution, we find

$$\begin{aligned}\int_0^{\pi}\frac{dx}{\sqrt{3-\cos(x)}}\,dx &= \frac{1}{2}\int_0^{\pi/2}\frac{1+\tan^2(u/2)}{\sqrt{1-\tan^2(u/2)}\sqrt{2\tan(u/2)}}\,du\\ &= \frac{1}{2}\int_0^{\pi/2}\sqrt{\frac{1+\tan^2(u/2)}{1-\tan^2(u/2)}}\sqrt{\frac{1+\tan^2(u/2)}{2\tan(u/2)}}\,du\\ &= \frac{1}{2}\int_0^{\pi/2}\sqrt{\sec(u)\csc(u)}\,du = \frac{1}{2}\int_0^{\pi/2}(\sin(u))^{-1/2}(\cos(u))^{-1/2}\,du\\ &= \frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{4}\frac{\Gamma(1/4)^2}{\Gamma(1/2)} = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}.\end{aligned}$$

[78]The name comes from the fact that the perimeter of an ellipse is given by an elliptic integral.

[79]See Whittaker, E.T. and Watson, G.N. *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020; Exercise 33, p. 262.

Exercises

1. Derive the formula

$$\int_{-1}^{1}(1+x)^{t-1}(1-x)^{s-1}\,dx = 2^{s+t-1}\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}.$$

2. Show that

$$B(t,n) = \frac{n!}{t(t+1)\cdots(t+n)}, \quad n \in \mathbb{N}.$$

3. Let $a < b$, $a, b \in \mathbb{R}$. Derive the formula

$$\int_{a}^{b}(x-a)^{t-1}(b-x)^{s-1}\,dx = (b-a)^{t+s-1}B(t,s), \quad 0 < s, t \in \mathbb{R}.$$

Solution: Use the substitution $y = (x-a)/(b-a)$ in the definition of the beta function.

4. Show that

$$B(t,t)B(t+1/2, t+1/2) = \frac{\pi}{2^{4t-1}t}, \quad t > 0.$$

Solution: Use the Legendre duplication formula.

5. Show that

$$\int_0^\infty \frac{x^a\,dx}{(x^b+c)^d} = \frac{c^{(a+1)/b-d}}{b}\frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(d-\frac{a+1}{b}\right)}{\Gamma(d)}, \ a > -1\,, b > 0, \ c > 0, \ d > (a+1)/b.$$

6. Derive the integral formula

$$\int_0^a x^b(a^c - x^c)^d dx = \frac{a^{b+cd+1}}{b+cd+1}\frac{\Gamma((b+1)/c)\Gamma(d+1)}{\Gamma((b+cd+1)/c)}, \quad a, c > 0, \ b, d > -1, \ b+cd > -1.$$

Solution: Use the substitution $x^c = a^c u$ to convert the integral into a beta integral. Specialize this to the following

$$\int_0^1 \frac{x^b}{\sqrt{1-x^c}}\,dx = \frac{\sqrt{\pi}}{c}\frac{\Gamma((b+1)/c)}{\Gamma((b+1)/c+1/2)}, \quad c > 0, \ b > -1.$$

7. Use the previous exercise to calculate the integrals

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}}\,dx \quad \text{and} \quad \int_0^1 \frac{dx}{\sqrt{1-\sqrt[n]{x}}}, \quad n \in \mathbb{N}.$$

8. Use the following steps to obtain Dedekind's proof of Euler's reflection formula: (a) Let $f : (0,1) \to \mathbb{R}$ be the function defined by $f(t) = \int_0^\infty x^{t-1}/(1+x)\,dx$, $0 < t < 1$. Split the integral at $x = 1$ to derive the formula

$$f(t) = \int_0^1 \frac{x^{t-1} + x^{-t}}{1+x}\,dx = \frac{1}{2}\int_0^\infty \frac{x^{t-1} + x^{-t}}{1+x}\,dx.$$

(b) Differentiate the identity $f(t) = f(1-t)$ to show that f has a local minimum at $t = 1/2$ with minimum value π. (f' is strictly increasing.) (c) Use scaling to derive the formulas

$$u^{t-1} f(t) = \int_0^\infty \frac{x^{t-1}}{x+u}\,dx \quad \text{and} \quad u^{-t} f(t) = \int_0^\infty \frac{x^{t-1}}{1+xu}\,dx.$$

(d) Add the equations in (c), divide by $1+u$ and integrate with respect to u in $[0,\infty)$ to obtain

$$f(t)^2 = \int_0^\infty \frac{x^{t-1}\ln(x)}{x-1}\,dx.$$

(e) Letting $0 < t < 1/2$, subtract the equations in (c), divide by $u-1$ and integrate with respect to u in $[0,\infty)$ to obtain

$$f(t)\int_0^\infty \frac{u^{t-1} - u^{-t}}{u-1}\,du = 2\int_0^\infty \frac{x^{t-1}\ln(x)}{x+1}\,dx = 2f'(t).$$

(f) Use (d) to show that the integral on the left-hand side is $\int_{1-t}^t f^2(s)\,ds$. Conclude from (e) that

$$f(t)\int_{1-t}^t f^2(s)\,ds = 2f'(t);$$

and hence $f(t)\int_{1/2}^t f^2(s)\,ds = f'(t)$ via the symmetry $f(s) = f(1-s)$. Differentiate this to obtain $f(t)f''(t) = f'(t)^2 + f(t)^4$. (g) Finally, solve this differential equation with the intitial conditions $f(1/2) = \pi$ and $f'(1/2) = 0$.

8. In this exercise we discuss the classical example of the motion of the **pendulum**. A point-mass m is suspended from a fixed point by a rigid massless rod of (constant) length ℓ, and it swings under the sole influence of (uniform) gravity (in a two dimensional vertical plane); its weight (on the surface of the Earth) is mg, where g is the (constant) magnitude of the gravitational field. The motion is described by the time (t) dependent angle θ measured from the vertical (stable) equilibrium position of the pendulum. Since the gravitational field is conservative, the total enery of the pendulum is constant. The total energy is the sum of the potential energy $mg(\ell - \ell\cos(\theta))$ (with the stable equilibrium corresponding to zero potential energy) and the kinetic energy $mv^2/2 = m\left(d(\ell\theta)/dt\right)^2/2$, where $\ell\theta$ is the arc length, and the derivative is with respect to time. We thus have[80]

$$\frac{g}{\ell}(1-\cos(\theta)) + \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = C,$$

where we incorporated m and ℓ^2 into the constant C. The motion also depends on the maximum θ_0 of θ, and we we set the time $t = 0$ when this maximum is attained. Thus, we have the intitial condition $\theta(0) = \theta_0$. The kinetic energy at the maximum position is zero, and this gives $C = (g/\ell)(1-\cos(\theta_0))$. Incorporating this into the conservation of energy formula above, we obtain

$$\frac{d\theta}{dt} = -\sqrt{\frac{2g}{\ell}}\sqrt{\cos(\theta)-\cos(\theta_0)}.$$

Equivalently, as (differential) forms, we have

$$dt = -\sqrt{\frac{\ell}{2g}}\frac{d\theta}{\sqrt{\cos(\theta)-\cos(\theta_0)}}.$$

We assume that $0 < \theta_0 < \pi$ since otherwise we have the stable and unstable equilibrium positions with θ being constants.
Derive the following integral formula for the full **period** $T = T(\theta_0)$ of the pendulum; that is, the time elapsed when the pedulum returns to its initial position:

$$T(\theta_0) = 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2(x)}}, \quad k = \sin\left(\frac{\theta_0}{2}\right).$$

Solution: By symmetry, $T/4$ is the time elapsed from the maximum position to the stable equilibrium. This gives

$$T(\theta_0) = 2\sqrt{\frac{2\ell}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos(\theta)-\cos(\theta_0)}}.$$

Now make the substitution

$$\sin\left(\frac{\theta}{2}\right) = k\sin(x), \quad k = \sin\left(\frac{\theta_0}{2}\right),$$

[80] Alternatively, using Newton's second law of motion, we have $d^2\theta/dt^2 = -(g/\ell)\sin(\theta)$. Multiplying through by $d\theta/dt$ and integrating, we obtain conservation of the total energy.

where $0 \leq \theta \leq \theta_0 < \pi$ and $0 \leq x \leq \pi/2$.
Squaring, the half angle formula for sine[81] gives

$$1 - \cos(\theta) = 2k^2 \sin^2(x), \quad 2k^2 = 1 - \cos(\theta_0),$$

and hence

$$\sqrt{\cos(\theta) - \cos(\theta_0)} = \sqrt{2}k\sqrt{1 - \sin^2(x)} = \sqrt{2}k\cos(x).$$

For the differentials, we have

$$\cos\left(\frac{\theta}{2}\right) d\theta = \sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)}\, d\theta = 2k\cos(x)\, dx,$$

or equivalently

$$d\theta = \frac{2k\cos(x)}{\sqrt{1 - k^2\sin^2(x)}}\, dx.$$

Putting these together and simplifying, the example follows.

4.12 The Bernoulli Numbers and Bernoulli Polynomials

In this section, preparatory to the next, we summarize some basic properties and estimates of the Bernoulli numbers and the associated polynomials. Recall[82] that the **Bernoulli numbers** B_m, $m \in \mathbb{N}_0$, can be defined concisely by the generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Multiplying out by the exponential denominator $e^x - 1 = \sum_{l=1}^{\infty} x^l/l!$, we get

$$x = \sum_{l=1}^{\infty} \frac{x^l}{l!} \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

The coefficients of the linear term give $B_0 = 1$. For $m \in \mathbb{N}$, the coefficients of the x^{m+1} term on the right-hand side are obtained by setting $l + k = m + 1$, $k = 0, 1, \ldots, m$,

[81]See *Elements of Mathematics - History and Foundations*, Section 11.3.
[82]See *Elements of Mathematics - History and Foundations*, Section 10.2.

and multiplying the respective terms of the two sums. We obtain

$$\sum_{k=0}^{m} \frac{B_k}{k!(m-k+1)!} = 0.$$

Multiplying through by $(m+1)!$ allows to convert the factorials into binomials. The resulting equality gives the well known inductive formula for the Bernoulli numbers:

$$B_m = -\frac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k} B_k, \quad m \in \mathbb{N}.$$

As a byproduct, it follows that all Bernoulli numbers are **rational**.

Another simple fact is that, with the exception of $B_1 = -1/2$, the odd Bernoulli numbers B_{2k+1}, $k \in \mathbb{N}$, are zero. Indeed, this follows from the fact that the function $x/(e^x - 1) + x/2$ is even:

$$\frac{-x}{e^{-x}-1} - \frac{x}{2} = \frac{xe^x}{e^x-1} - \frac{x}{2} = \frac{x}{e^x-1} + \frac{x}{2}.$$

The first few Bernoulli numbers are tabulated as follows:

k	B_k	k	B_k
0	1	12	$-691/2730$
1	$-1/2$	14	$7/6$
2	$1/6$	16	$-3617/510$
4	$-1/30$	18	$43867/798$
6	$1/42$	20	$-174611/330$
8	$-1/30$	22	$854513/138$
10	$5/66$	24	$-236364091/2730$

History. As Jacob Bernoulli (1655 – 1705) first realized in his “Ars Conjectandi” (Latin for “The Art of Conjecturing” published posthoumously in 1713 by his nephew Niklaus Bernoulli), these numbers (that have subsequently been named after him) appear naturally in evaluating the power sums $\sum_{i=1}^{n} i^k$, $n \in \mathbb{N}$. The problem of calculating power sums has been considered by many mathematicians of antiquity, notably by Archimedes in ancient Greece, and it occurs in the works of the Indian mathematicians Aryabhatta (476 – 550 CE), the Persian Abū Bakr Al-Karajī in 1019, and the Muslim Arab mathematician Hasan Ibn Al-Haytham (965 – 1040). While the English mathematician and astronomer Thomas Harriot (1560 – 1621) is believed to be the first to develop symbolic formulas for these sums of powers, albeit only up to the fourth powers, it was the German mathematician Johann Faulhaber (1580 – 1635) who derived these formulas up to the seventeenth power but he did not obtain a general pattern. We quote here Bernoulli's well-known comment upon the moment of compiling the first table of these numbers as follows: "With the help of this table, it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum $91,409,924,241,424,243,424,241,924,242,500$." The Bernoulli numbers have also been discovered and tabulated by the Japanese mathematician Seki Takakazu (also known as Seki Kōwa) (c. 1642 – 1708).

We define the associated **Bernoulli polynomials** $B_m(x)$, $m \in \mathbb{N}$, through the generating function

$$\frac{ye^{xy}}{e^y - 1} = \frac{y}{e^y - 1} \cdot e^{xy} = \sum_{m=0}^{\infty} B_m(x) \frac{y^m}{m!}.$$

The connection with the Bernoulli numbers can be shown by applying the Cauchy product rule to the product

$$\sum_{k=0}^{\infty} B_k \frac{y^k}{k!} \cdot \sum_{l=0}^{\infty} \frac{(xy)^l}{l!},$$

and, setting $k + l = m$, comparing coefficients. We obtain the following simple expansion of the Bernoulli polynomials in terms of the Bernoulli numbers:

$$B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}, \quad m \in \mathbb{N}_0.$$

In particular, we have $B_0(x) = B_0 = 1$ and $B_m(0) = B_m$. Moreover, for $m \in \mathbb{N}$, differentiating, we calculate

$$\begin{aligned} B_m'(x) &= \sum_{k=0}^{m-1} \binom{m}{k} (m-k) B_k x^{m-k-1} \\ &= m \sum_{k=0}^{m-1} \binom{m-1}{k} B_k x^{m-1-k} = m B_{m-1}(x). \end{aligned}$$

This gives the important formula

$$B_m'(x) = m B_{m-1}(x), \quad m \in \mathbb{N}.$$

In particular, integrating over $[0, x]$, $x \in \mathbb{R}$, we obtain

$$B_m(x) = B_m + m \int_0^x B_{m-1}(t)\, dt, \quad m \in \mathbb{N}.$$

This formula and the previous list of Bernoulli numbers give the following table

k	$B_k(x)$
0	1
1	$x - 1/2$
2	$x^2 - x + 1/6$
3	$x^3 - 3x^2/2 + x/2$
4	$x^4 - 2x^3 + x^2 - 1/30$
5	$x^5 - 5x^4/2 + 5x^3/3 - x/6$
6	$x^6 - 3x^5 + 5x^4/2 - x^2/2 + 1/42$
7	$x^7 - 7x^6/2 + 7x^5/2 - 7x^3/6 + x/6$
8	$x^8 - 4x^7 + 14x^6/3 - 7x^4/3 + 2x^2/3 - 1/30$

Note that

$$\int_0^1 B_m(x)\,dx = \sum_{k=0}^{m} \binom{m}{k} B_k \int_0^1 x^{m-k}\,dx = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k = 0, \quad m \in \mathbb{N}.$$

This, along with the previous formula, gives

$$B_m(1) = B_m(0) = B_m, \quad 2 \le m \in \mathbb{N}.$$

We finish this sequence of formulas with one of a different genre:

$$B_m(x+1) - B_m(x) = mx^{m-1}, \quad m \in \mathbb{N}.$$

This shows that Bernoulli polynomials play the same role in the calculus of finite-differences than power functions in differential calculus.

The simplest proof is by induction with respect to $m \in \mathbb{N}$; the formula being immediate for $m = 1$. For the general induction step $m \Rightarrow m+1$, we let $C_m(x) = B_m(x+1) - B_m(x) - mx^{m-1}$, $m \in \mathbb{N}$, and assume $C_m(x) = 0$. We have $C'_{m+1} = (m+1)C_m(x) = 0$, so that $C_{m+1}(x)$ is constant. Using the definition (at $x = 0$), this constant must be $B_{m+1}(1) - B_{m+1}(0)$ which, by the above, is zero since $m \in \mathbb{N}$. We conclude that $C_{m+1}(x) = 0$. The induction is complete.

History. The Bernoulli polynomials first appeared in Euler's "Institutiones Calculi Differentialis." The actual term was coined by J.L. Raabe in 1851.

There is a simple but important **unicity** principle concerning the Bernoulli polynomials; namely, the properties

$$B_m(x+1) - B_m(x) = mx^{m-1} \quad \text{and} \quad \int_0^1 B_m(x)\,dx = 0, \quad m \in \mathbb{N},$$

uniquely determine the sequence $(B_m(x))_{m\in\mathbb{N}_0}$ (along with $B_0(x)=1$).
Indeed, assume that a sequence of polynomials $(\bar{B}_m(x))_{m\in\mathbb{N}_0}$ satisfies these two properties (along with $\bar{B}_0(x)=1$). Letting $C_m(x)=B_m(x)-\bar{B}_m(x)$, $m\in\mathbb{N}$, by the first property, we have $C_m(x+1)=C_m(x)$, $x\in\mathbb{R}$. Since a non-constant polynomial can assume only finitely many given values, we obtain that $C_m(x)$ is constant. By the second property, $\int_0^1 C_m(x)\,dx=0$, so this constant must be zero. Unicity follows.

Remark. The previous unicity can be put into a more elegant linear algebraic framework. We let $\mathbb{R}[x]$ denote the vector space of all polynomials of the single variable x. For $m\in\mathbb{N}_0$, the linear subspace $\mathbb{R}[x]_m\subset\mathbb{R}[x]$ of degree $\leq m$ polynomials is finite dimensional; $\dim\mathbb{R}[x]_m=m+1$. We define the linear map $L:\mathbb{R}[x]\to\mathbb{R}[x]\times\mathbb{R}$ by

$$L(p(x))=\left(\Delta(p(x)),\int_0^1 p(t)\,dt\right),\quad \Delta(p(x))=p(x+1)-p(x),\ \ p(x)\in\mathbb{R}[x],$$

with $\Delta:\mathbb{R}[x]_{m+1}\to\mathbb{R}[x]_m$, $m\in\mathbb{N}_0$, (and $\Delta|\mathbb{R}=0$, $\mathbb{R}=\mathbb{R}[x]_0$), the **forward difference operator**. As above, it follows that the kernel of Δ is $\mathbb{R}=\mathbb{R}[x]_0$, and hence the kernel of L is zero; that is, L is injective. For reasons of dimensions, it is also onto since $\dim L(\mathbb{R}[x]_{m+1})=\dim(\mathbb{R}[x]_{m+1})=\dim(\mathbb{R}[x]_m\times\mathbb{R})$, $m\in\mathbb{N}_0$. Finally, since L is invertible, the sequence of Bernoulli polynomials $(B_m(x))_{m\in\mathbb{N}_0}$ is uniquely defined by $B_0(x)=L^{-1}(0,1)$ and $B_m(x)=L^{-1}(mx^{m-1},0)$, $m\in\mathbb{N}$.

Armed with this new unicity principle of the Bernoulli polynomials, we now derive several new formulas needed in the sequel.

First, we claim

$$B_m(1-x)=(-1)^m B_m(x),\quad m\in\mathbb{N}_0.$$

Indeed, setting $C_m=(-1)^m B_m(1-x)$, $m\in\mathbb{N}$, we calculate

$$\begin{aligned}\Delta C_m(x) &= (-1)^m\left(B_m(-x)-B_m(1-x)\right)=(-1)^{m-1}\Delta B_m(-x)\\ &= (-1)^{m-1}m(-x)^{m-1}=mx^{m-1},\end{aligned}$$

and

$$\int_0^1 C_m(t)\,dt=(-1)^m\int_0^1 B_m(1-t)\,dt=(-1)^m\int_0^1 B_m(t)\,dt=0.$$

By unicity, we have $C_m(x)=B_m(x)$, $m\in\mathbb{N}_0$, and the claim follows.

As a direct consequence of the formula just derived, we see that the graphs of the even Bernoulli polynomials are (reflectionally) symmetric about the vertical line given by $x=1/2$, and the graphs of the odd Bernoulli polynomials are (centrally) symmetric about the point $(1/2, 0)$. Moreover, for $2 \leq m \in \mathbb{N}$, we have

$$\begin{aligned}(-1)^m B_m - B_m &= (-1)^m B_m(0) - B_m = B_m(1) - B_m(0) \\ &= \int_0^1 B'_m(x)\,dx = m\int_0^1 B_{m-1}(x)\,dx = 0.\end{aligned}$$

We obtain $B_m(1) = (-1)^m B_m = B_m$, $2 \leq m \in \mathbb{N}$. In particular, for all odd Bernoulli numbers, we have $B_{2m+1}(1) = B_{2m+1}(0) = B_{2m+1} = 0$, $m \in \mathbb{N}$; and, for even Bernoulli numbers, we have $B_{2m}(1) = B_{2m}(0) = B_{2m}$, $m \in \mathbb{N}$.

Second, we claim

$$\frac{1}{n}\sum_{k=0}^{n-1} B_m\left(\frac{x+k}{n}\right) = \frac{1}{n^m}B_m(x), \quad n \in \mathbb{N}, \;\; m \in \mathbb{N}_0.$$

Once again, the proof of this is based on the unicity principle. Indeed, for $n \in \mathbb{N}$, we let $C_m(x) = n^{m-1}\sum_{k=0}^{n-1} B_m\left((x+k)/n\right)$, $m \in \mathbb{N}_0$. Clearly, $C_0(x) = 1$, and the formula holds. For $m \in \mathbb{N}$, we first calculate

$$\begin{aligned}\Delta C_m &= n^{m-1}\sum_{k=0}^{n-1} B_m\left(\frac{x+k+1}{n}\right) - n^{m-1}\sum_{k=0}^{n-1} B_m\left(\frac{x+k}{n}\right) \\ &= n^{m-1}\left(\sum_{k=1}^{n} B_m\left(\frac{x+k}{n}\right) - n^{m-1}\sum_{k=0}^{n-1} B_m\left(\frac{x+k}{n}\right)\right) \\ &= n^{m-1}\left(B_m\left(\frac{x+n}{n}\right) - B_m\left(\frac{x}{n}\right)\right) \\ &= n^{m-1}\Delta B_m\left(\frac{x}{n}\right) = n^{m-1}\cdot m\left(\frac{x}{n}\right)^{m-1} = mx^{m-1}.\end{aligned}$$

Second, for $m \in \mathbb{N}$, we have

$$\begin{aligned}\int_0^1 C_m(t)\,dt &= n^{m-1}\sum_{k=0}^{n-1}\int_0^1 B_m\left(\frac{t+k}{n}\right)\,dt \\ &= n^{m-1}\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} B_m(t)\,dt = n^{m-1}\int_0^1 B_m(t)\,dt = 0.\end{aligned}$$

Again by unicity, the claim follows.

Once again, a direct consequence of this ($n = 2$) is

$$B_m\left(\frac{x}{2}\right) + B_m\left(\frac{x+1}{2}\right) = 2^{1-m}B_m(x), \quad m \in \mathbb{N}_0.$$

Substituting $x = 0$, we obtain

$$B_m\left(\frac{1}{2}\right) = \left(2^{1-m} - 1\right) B_m, \quad m \in \mathbb{N}_0.$$

We now study the behavior of the Bernoulli polynomials on the unit interval $[0,1]$. Since the first Bernoulli polynomial $B_1(x) = x - 1/2$ is negative on $[0, 1/2)$ and positive on $(1/2, 1]$, the second Bernoulli polynomial $B_2(x) = x^2 - x + 1/6$ is strictly decreasing on $[0, 1/2]$ and strictly increasing on $[1/2, 1]$. Furthermore, it takes opposite signs on the end-points of these intervals, and so it follows that $B_2(x)$ vanishes exactly once on $(0, 1/2)$ and also on $(1/2, 1)$. This is a characteristic behavior of all the Bernoulli polynomials on $[0,1]$ as stated in the following:

Proposition 4.12.1. *For $m \in \mathbb{N}$, we have the following:*
$\mathbf{I}_m$*: The polynomial $(-1)^m B_{2m}(x)$ is strictly increasing on $[0, 1/2]$ and strictly decreasing on $[1/2, 1]$, and therefore, it vanishes exactly once in $(0, 1/2)$ and also on $(1/2, 1)$.*
$\mathbf{II}_m$*: The polynomial $(-1)^m B_{2m+1}(x)$ is negative on $(0, 1/2)$ and positive on $(1/2, 1)$ and has simple zeros at $0, 1/2, 1$.*

Proof. $\mathbf{I}_m \Rightarrow \mathbf{II}_m$. Assume $\mathbf{I}_m$ holds. Consider the polynomial $C_m(x) = (-1)^m B_{2m+1}(x)$. Since the odd Bernoulli numbers vanish, our earlier formulas imply that $C_m(0) = C_m(1/2) = C_m(1) = 0$. Moreover $C'_m(x) = (2m+1)(-1)^m B_{2m}(x)$, and $\mathbf{I}_m$ implies that $C'_m(x)$ has a unique zero $0 < r_m < 1/2$ and another unique zero $1/2 < s_m < 1$; and elsewhere $C'(x)$ is negative on $[0, r_m) \cup (s_m, 1]$ and positive on (r_m, s_m). These imply that $C_m(x)$ itself is strictly decreasing on $[0, r_m]$ and $[s_m, 1]$ and strictly increasing on $[r_m, s_m]$. Now $\mathbf{II}_m$ follows.
$\mathbf{II}_m \Rightarrow \mathbf{I}_{m+1}$. Assume $\mathbf{II}_m$ holds. Consider the polynomial $D_m(x) = (-1)^{m+1} B_{2m+2}(x)$. We have $D'_m(x) = -(2m+2)(-1)^m B_{2m+1}(x)$ which, by $\mathbf{II}_m$, is positive on $(0, 1/2)$ and negative on $(1/2, 1)$ (and has simple zeros at $0, 1/2, 1$). Hence $D_m(x)$ itself is strictly increasing on $[0, 1/2]$ and strictly decreasing on $[1/2, 1]$. Moreover,

$$\begin{aligned} D_m(1/2) &= (-1)^{m+1} B_{2m+2}(1/2) = (-1)^m (1 - 2^{-(2m+1)}) B_{2m+2} \\ &= -(1 - 2^{-(2m+1)}) D_m(0) = -(1 - 2^{-(2m+1)}) D_m(1). \end{aligned}$$

By the monotonicity concluded above, we have $D_m(1/2) > 0$ and $D_m(0) = D_m(1) < 0$. In particular, $D_m(x)$ takes opposite values at the end-points of the intervals $[0, 1/2]$ and $[1/2, 1]$. Now $\mathbf{I}_{m+1}$ follows.

Remark. As in the proof, let $0 < r_m < 1/2$ and $1/2 < s_m < 1$ be the unique zeros of $B_{2m}(x)$, $m \in \mathbb{N}$, on $[0,1]$. It is known[83] that

$$\frac{1}{4} - \frac{1}{2^{2m+1}\pi} < r_m < r_{m+1} < \frac{1}{4} \quad \text{and} \quad \frac{3}{4} < s_{m+1} < s_m < \frac{3}{4} + \frac{1}{2^{2m+1}\pi}.$$

Note that $\mathbf{I}_m$ (second part of the proof) implies $(-1)^{m+1}B_{2m} > 0$, $m \in \mathbb{N}$. It also shows that, on $[0,1]$, the polynomial $B_{2m}(x)$ attains its extrema at $0, 1/2, 1$ and $B_{2m}(0) = B_{2m}(1) = B_{2m}$, and $B_{2m}(1/2) = (2^{1-2m} - 1)B_{2m}$. As a byproduct, we obtain

$$\sup_{x\in[0,1]} |B_{2m}(x)| = |B_{2m}| \quad m \in \mathbb{N}.$$

It is desirable to obtain a bound for $|B_{2m+1}(x)|$ for $x \in [0,1]$ as well. A simple bound follows from the estimate just derived for $B_{2m}(x)$ above, and the fact that $0, 1/2, 1$ are zeros of $B_{2m+1}(x)$. Indeed, for $x \in [0, 1/4]$, we have

$$B_{2m+1}(x) = B_{2m+1}(x) - B_{2m+1}(0) = \int_0^x B'_{2m+1}(t)\,dt = (2m+1)\int_0^x B_{2m}(t)\,dt.$$

This and the estimate above gives

$$|B_{2m+1}(x)| \le (2m+1)\int_0^x |B_{2m}(t)|\,dt < (2m+1)|B_{2m}|\,x \le \frac{2m+1}{4}|B_{2m}|, \ x \in [0, 1/4].$$

Similarly, for $x \in [1/4, 1/2]$, we have

$$B_{2m+1}(x) = \int_{1/2}^x B'_{2m+1}(t)\,dt = (2m+1)\int_{1/2}^x B_{2m}(t)\,dt.$$

This, for $x \in [1/4, 1/2]$, gives

$$|B_{2m+1}(x)| \le (2m+1)\int_x^{1/2} |B_{2m}(t)|\,dt < (2m+1)|B_{2m}|\left(\frac{1}{2} - x\right) \le \frac{2m+1}{4}|B_{2m}|.$$

Combining these two estimates, along with the fact that, over the interval $[1/2, 1]$, the same estimates hold due to the symmetry of $B_{2m+1}(x)$ about $x = 1/2$, we finally arrive at the following

$$\sup_{x\in[0,1]} |B_{2m+1}(x)| < \frac{2m+1}{4}|B_{2m}|, \quad m \in \mathbb{N}.$$

[83] See Lehmer, D.H., *On the maxima and minima of Bernoulli polynomials*, Amer. Math. Monthly, 47, 8 (1940), 533-538; and Dilcher, K. *Zeros of Bernoulli, generalized Bernoulli and Euler polynomials*, Mem. Amer. Math. Soc. 386 (1988).

4.13 Bernoulli Numbers and Polynomials: Estimates and Asymptotics

In this section we will derive some sharp estimates and asymptotic relations on the Bernoulli numbers and Bernoulli polynomials. These will follow from the expansions

$$\begin{aligned} B_{2m}(x) &= (-1)^{m+1}\frac{2(2m)!}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^{2m}}, \\ B_{2m+1}(x) &= (-1)^{m+1}\frac{2(2m+1)!}{(2\pi)^{2m+1}}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k^{2m+1}}, \quad m\in\mathbb{N},\ x\in[0,1]. \end{aligned}$$

The sums on the right-hand sides are actually the (uniformly convergent) **Fourier series expansions**[84] of the **periodized Bernoulli polynomials** $P_m:\mathbb{R}\to\mathbb{R}$ given by $P_m(x)=B_m(x-[x])$, $x\in\mathbb{R}$, $m\in\mathbb{N}$. The second formula also holds for the somewhat exceptional case $m=0$:

$$B_1(x) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k}, \quad x\in[0,1],$$

where $B_1(x)$ is redefined to vanish at the points of discontinuity $x=0,1$. The convergence of the series is not uniform on $[0,1]$ as the Gibbs phenomenon amply illustrates, although uniform convergence holds on any closed subintevals $[a,b]\subset(0,1)$. We will derive these formulas in the next section via a thoroughly elementary treatment that does not use the Fourier convergence theorem nor the Riemann-Lebesgue lemma; the only tools are the formulas for the Bernoulli numbers and polynomials in the previous section, integration by parts, and Chebyshev polynomials. (An indication of how to derive these expansions using Fourier series is expounded in Exercise 5 at the end of this section.)

To begin with, as immediate consequences of these expansions, for $m\in\mathbb{N}$ and $x\in[0,1]$, we have the estimates

$$\begin{aligned} \left|(-1)^{m+1}\frac{(2\pi)^{2m}}{2\cdot(2m)!}B_{2m}(x)-\cos(2\pi x)\right| &\le \sum_{k=2}^{\infty}\frac{1}{k^{2m}} < \left(1+\frac{2}{2m-1}\right)\frac{1}{2^{2m}} \\ \left|(-1)^{m+1}\frac{(2\pi)^{2m+1}}{2\cdot(2m+1)!}B_{2m+1}(x)-\sin(2\pi x)\right| &\le \sum_{k=2}^{\infty}\frac{1}{k^{2m+1}} < \left(1+\frac{1}{m}\right)\frac{1}{2^{2m+1}}, \end{aligned}$$

[84] There are several classical texts on Fourier analysis; see for example Zygmund, A., *Trigonometric Series*, 3rd ed. Cambridge University Press, 2002; Katznelson, Y., *An Introduction to Harmonic Analysis*, 2nd ed. Dover, New York 1976; Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill, Inc. New York, 1976.

where, for the last inequalities, we used the trivial estimate

$$\sum_{k=2}^{\infty} \frac{1}{k^n} < \frac{1}{2^n} + \int_2^{\infty} \frac{dx}{x^{2n}} = \left(1 + \frac{2}{n-1}\right) \frac{1}{2^n}, \quad 2 \leq n \in \mathbb{N}.$$

Note that these imply the stated uniform convergence of the Fourier expansions.

Combining the expansion of $B_{2m+1}(x)$ above with this last formula we obtain the estimate

$$\sup_{x\in[0,1]} |B_{2m+1}(x)| \leq \frac{2(2m+1)!}{(2\pi)^{2m+1}} \sum_{k=1}^{\infty} \frac{1}{k^{2m+1}} < \frac{2(2m+1)!}{(2\pi)^{2m+1}} \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{2m+1}{2\pi} |B_{2m}|.$$

This gives a sharper upper bound than the one obtained at the end of the previous section.

Substituting $x = 0$ into the first expansion, and rearranging, we obtain **Euler's summation formula**[85]

$$\sum_{k=1}^{\infty} \frac{1}{k^{2m}} = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}, \quad m \in \mathbb{N}.$$

Observe that the first case, $m = 1$, is Euler's solution to the Basel problem:[86] $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. Finally, note also that the infinite sum above is the value of the **zeta function** $\zeta(2m)$, $m \in \mathbb{N}$. We will treat the zeta function in Section 4.17 in detail, where we will derive this formula as a byproduct of the functional equation using a different path.[87]
In the next example we will give yet another proof of Euler's summation formula using the product expansion of the sine hyperbolic function.[88]

Example 4.13.1. We start with the infinite product formula

$$\frac{\sinh(t)}{t} = \prod_{k=1}^{\infty} \left(1 + \frac{t^2}{\pi^2 \cdot k^2}\right),$$

[85]First published by Euler in 1740.
[86]For history and yet another elementary proof using trigonometry, see *Elements of Mathematics - History and Foundations*, Section 11.7.
[87]Proofs of Euler's summation formula abound; see, for example, De Amo, E., Carrillo, M.D. and Hernandez-Sanchez, J., *Another proof of Euler's formula for $\zeta(2k)$*, Proc. Amer. Math. Soc. 139 (2011) 1441-1444; and also below.
[88]See Ribeiro, P., *Another proof of the famous formula for the zeta function at positive even integers*, Amer. Math. Monthly, Vol. 125, No. 9 (2018) 839-841.

of Exercise 4 at the end of Section 4.8. We take the natural logarithm of both sides, expand of the logarithm into a power series, and calculate

$$\begin{aligned}\ln\left(\frac{\sinh(t)}{t}\right) &= \sum_{k=1}^{\infty}\ln\left(1+\frac{t^2}{\pi^2\cdot k^2}\right) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\frac{t^{2m}}{\pi^{2m}\cdot k^{2m}}\\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\frac{t^{2m}}{\pi^{2m}}\sum_{k=1}^{\infty}\frac{1}{k^{2m}},\end{aligned}$$

where the interchange of the two summations is clearly allowed.
On the other hand, letting $s = 2t$, we have

$$\begin{aligned}\ln\left(\frac{\sinh(t)}{t}\right) &= \ln\left(\frac{e^{s/2}-e^{-s/2}}{s}\right) = \ln\left(\frac{e^{s/2}(1-e^{-s})}{s}\right)\\ &= -\ln(s)+\frac{s}{2}+\ln(1-e^{-s}) = \int_0^s\left(-\frac{1}{x}+\frac{1}{2}+\frac{1}{e^x-1}\right)dx.\end{aligned}$$

We now notice that the integrand in the last integral can be written in terms of the Bernoulli numbers as

$$\frac{1}{x}\left(-1+\frac{x}{2}+\frac{x}{e^x-1}\right) = \frac{1}{x}\left(-B_0-B_1x+\sum_{k=0}^{\infty}\frac{x^k}{k!}B_k\right) = \frac{1}{x}\sum_{k=2}^{\infty}\frac{x^k}{k!}B_k = \sum_{m=1}^{\infty}\frac{x^{2m-1}}{(2m)!}B_{2m},$$

where we used the fact that all odd Bernoulli numbers except $B_1 = -1/2$ vanish. Substituting, and performing the integration, we obtain

$$\ln\left(\frac{\sinh(t)}{t}\right) = \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}\int_0^s x^{2m-1}\,dx = \sum_{m=1}^{\infty}\frac{2^{2m}t^{2m}}{2m(2m)!}B_{2m},$$

where, in the last step, we reverted back to $t = s/2$. Comparing these two formulas for the natural logarithm of the sine hyperbolic function, Euler's summation formula follows.

Euler's summation formula above can be used do derive a sharp (asymptotic) estimate on the growth of the even Bernoulli numbers $|B_{2m}|$, $m \in \mathbb{N}$. To get to this, we begin by replacing the infinite sum with the (equiconvergent) improper integral as follows:

$$\sum_{k=3}^{\infty}\frac{1}{k^{2m}} < \sum_{k=3}^{\infty}\int_{k-1}^{k}\frac{dx}{x^{2m}} = \int_2^{\infty}\frac{dx}{x^{2m}} = \frac{1}{(2m-1)2^{2m-1}}.$$

(Note the missing first two terms in the sum.) This gives

$$1 < \sum_{k=1}^{\infty}\frac{1}{k^{2m}} < 1+\frac{1}{2^m}+\frac{1}{(2m-1)2^{2m-1}} = 1+\frac{2m+1}{2m-1}\cdot\frac{1}{2^{2m}}.$$

Using the explicit formula for the infinite sum derived above, along with $(-1)^{m+1}B_{2m} = |B_{2m}| > 0$, $m \in \mathbb{N}$, we obtain

$$\frac{2(2m)!}{(2\pi)^{2m}} < |B_{2m}| \leq \left(1 + \frac{2m+1}{2m-1} \cdot \frac{1}{2^{2m}}\right) \frac{2(2m)!}{(2\pi)^{2m}} \leq \left(1 + \frac{3}{2^{2m}}\right) \frac{2(2m)!}{(2\pi)^{2m}}$$

This gives the asymptotic formula

$$|B_{2m}| \sim \frac{2(2m)!}{(2\pi)^{2m}} \sim 4\sqrt{\pi m} \left(\frac{m}{e\pi}\right)^{2m}, \quad m \to \infty,$$

where we also used the Stirling formula (Section 4.10).

We close this section by showing that the upper bound in the estimate for

$$\sup_{x \in [0,1]} |B_{2m+1}(x)|$$

is asymptotically the best possible (as $m \to \infty$). We begin with substituting $x = 1/4$ in the series expansion of $B_{2m+1}(x)$ above, and obtain

$$B_{2m+1}\left(\frac{1}{4}\right) = (-1)^{m+1} \frac{2(2m+1)!}{(2\pi)^{2m+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}}.$$

The series on the right-hand side is altermating, and so we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}} > 1 - \frac{1}{3^{2m+1}}.$$

Substituting, we obtain the lower estimate

$$\left|B_{2m+1}\left(\frac{1}{4}\right)\right| > \left(1 - \frac{1}{3^{2m+1}}\right) \frac{2(2m+1)!}{(2\pi)^{2m+1}} \geq \left(\frac{1 - 1/3^{2m+1}}{1 + 3/2^{2m}}\right) \frac{2m+1}{2\pi} |B_{2m}|.$$

Combining this with the earlier estimate, we obtain

$$\left(\frac{1 - 1/3^{2m+1}}{1 + 3/2^{2m}}\right) \frac{2m+1}{2\pi} |B_{2m}| \leq \sup_{x \in [0,1]} |B_{2m+1}(x)| < \frac{2m+1}{2\pi} |B_{2m}|.$$

This shows that asymptotically, we have

$$\sup_{x \in [0,1]} |B_{2m+1}(x)| \sim \frac{2m+1}{2\pi} |B_{2m}|, \quad m \to \infty.$$

Exercises

1. Show that
$$\int_x^{x+1} B_m(t)\,dt = x^m, \quad 2 \le m \in \mathbb{N}.$$

2. Derive the identity $B'_{m+1}(x) = (m+1)B_m(x)$, $m \in \mathbb{N}_0$, using the unicity principle in the text.

3. Derive the formula
$$B_m(x+y) = \sum_{k=0}^{m} \binom{m}{k} B_{m-k}(x) y^k, \quad m \in \mathbb{N}_0.$$

4. Use the previous exercise to show that
$$x^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k(x), \quad m \in \mathbb{N}_0.$$

5. This and the next exercises use some facts on Fourier series. Use induction on $m \in \mathbb{N}$ to derive, for $x \in [0,1]$, the following series expansions for the periodized $P_m(x) = B_m(x - [x])$, $m \in \mathbb{N}$, as follows
$$\begin{aligned} P_{2m}(x) &= (-1)^{m+1} \frac{2(2m)!}{(2\pi)^{2m}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2m}} \\ P_{2m+1}(x) &= (-1)^{m+1} \frac{2(2m+1)!}{(2\pi)^{2m+1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2m+1}}. \end{aligned}$$

6. Use the Parseval identity to derive, for $m, n \in \mathbb{N}$, the formulas
$$\begin{aligned} \int_0^1 B_m(x) B_n(x)\,dx &= (-1)^{m+1} \frac{B_{m+n}}{\binom{m+n}{m}} \\ \int_0^1 |B_m(x)|^2\,dx &= \frac{(m!)^2}{(2m)!} |B_{2m}|. \end{aligned}$$

7. Use Euler's summation formula to derive the following
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2m}} = (-1)^{m+1} \left(2^{2m-1} - 1\right) \frac{\pi^{2m}}{(2m)!} B_{2m}, \quad m \in \mathbb{N}.$$

Solution: Note that the sum on the left-hand side is the value of the alternating zeta function on $2m$, $m \in \mathbb{N}$. See also the end of Section 1.2.

8. Use the formula $(1/n)\sum_{k=0}^{n-1} B_m((x+k)/n) = (1/n^m)B_m(x)$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, with $x = 0$ and $n = 4$ to derive the following

$$B_{2m}\left(\frac{1}{4}\right) = \frac{1}{2^{2m}} B_{2m}\left(\frac{1}{2}\right).$$

Conclude that we have $0 < r_m < 1/4$ and $3/4 < s_m < 1$, for the two roots of $B_{2m}(x)$, $m \in \mathbb{N}$, in $[0, 1]$.

4.14 Fourier Expansions of the (Periodized) Bernoulli Polynomials

In this section we derive the Fourier series expansions[89]

$$\begin{aligned} B_{2m}(x) &= (-1)^{m+1}\frac{2(2m)!}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^{2m}}, \\ B_{2m+1}(x) &= (-1)^{m+1}\frac{2(2m+1)!}{(2\pi)^{2m+1}}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k^{2m+1}}, \quad m \in \mathbb{N},\ x \in [0,1]; \end{aligned}$$

along with $(m = 0)$:

$$B_1(x) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k}, \quad x \in [0,1],$$

to be treated here separately ($B_1(0) = B_1(1) = 0$).

History. The Fourier expansion for $B_{2m}(x)$ above was first derived by Hurwitz in 1890.

To begin with, we denote by $\tilde{B}_{2m}(x)$ resp. $\tilde{B}_{2m+1}(x)$, $m \in \mathbb{N}$, the sums on the right-hand sides of the expansions above. Now, termwise differentiation of the infinite sum in $\tilde{B}_{2m}(x)$ gives $2m\tilde{B}_{2m-1}(x)$, and termwise differentiation of the infinite sum in $\tilde{B}_{2m+1}(x)$ gives $(2m+1)\tilde{B}_{2m}(x)$. In each case, the convergence is uniform so that Proposition 2.2.4 applies, and we obtain

$$\tilde{B}_{2m}'(x) = 2m\tilde{B}_{2m-1}(x) \quad \text{and} \quad \tilde{B}_{2m+1}'(x) = (2m+1)\tilde{B}_{2m}(x), \quad x \in [0,1],\ m \in \mathbb{N}.$$

[89]Here and below, the term Fourier expansion is tacitly understood to be valid on the stated interval of period, or equivalently, for the respective periodized function.

We conclude that the inductive formula for the Bernoulli polynomials (Section 4.13) is satisfied:

$$\tilde{B}'_m(x) = m\tilde{B}'_{m-1}(x), \quad m \in \mathbb{N}.$$

Now the crux is that this uniquely defines the Bernoulli polynomials provided that we can show the following two relations:

$$\tilde{B}_1(x) = B_1(x), \quad x \in [0,1],$$

and

$$\tilde{B}_{2m}(0) = B_{2m}(0) = B_{2m}, \quad m \in \mathbb{N}.$$

(The corresponding odd-indexed relation $\tilde{B}_{2m+1}(0) = B_{2m+1}(0) = B_{2m+1} = 0$, $m \in \mathbb{N}$, is obvious as the infinite sum reduces to zero.) Once these are proved, by unicity, we obtain $B_m(x) = \tilde{B}_m(x)$, $x \in [0,1]$, $m \in \mathbb{N}$, and all the Fourier expansions stated above follow.

The first relation to be proved is equivalent to the Fourier expansion

$$\sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{\pi k} = \frac{1}{2} - x, \quad 0 < x < 1.$$

(Here we omitted the end-points for convenience as the limit is a mismatch to the actual values of the respective function.) After scaling, this takes the form

$$\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n} = \frac{\pi - \alpha}{2}, \quad 0 < \alpha < 2\pi,$$

where the convergence of the sum is pointwise, and uniform on every closed intervals in $(0, 2\pi)$. As noted above and as expected, the usual proof requires the Fourier convergence theorem. As usual, we insist on an elementary approach via a recourse to Chebyshev polynomials.

The second relation to be proved can be written as **Euler's summation formula**

$$\sum_{k=1}^{\infty} \frac{1}{k^{2m}} = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m} = \frac{(2\pi)^{2m}}{2(2m)!} |B_{2m}|, \quad m \in \mathbb{N}.$$

We now derive the Fouries series expansion above. We first note that, by the Dirichlet test for convergence of a series (Proposition 2.3.2), the sum on the left-hand side converges pointwise. Indeed, given $\alpha \in (0, 2\pi)$, we let $a_n = 1/n$ and $b_n = \sin(n\alpha)$,

$n \in \mathbb{N}$. Then the sequence $(1/n)_{n\in\mathbb{N}}$ is (strictly) decreasing, and, for every $n \in \mathbb{N}$, by the Lagrange identity for sine,[90] we have

$$\left|\sum_{k=1}^{n} \sin(k\alpha)\right| = \frac{|\sin((n+1)\alpha/2)\sin(n\alpha/2)|}{\sin(\alpha/2)} \leq \frac{1}{\sin(\alpha/2)}, \quad 0 < \alpha < 2\pi.$$

Since the upper bound here is independent of n, the conditions in the Dirichlet test for convergence are satisfied, and convergence follows. The statement on uniform convergence also holds since $1/\sin(\alpha/2)$ is bounded on every closed interval in $(0, 2\pi)$.

To derive the stated equality, we may assume that $0 < \alpha < \pi$ since the formula stays the same by replacing α by $2\pi - \alpha$ (and it holds trivially for $\alpha = \pi$). We first claim

$$\sum_{n=0}^{\infty} \frac{\sin((n+1)\alpha)}{\sin(\alpha)}\, t^n = \frac{1}{1 - 2t\cdot\cos(\alpha) + t^2}, \quad 0 < \alpha < \pi,\ |t| < 1.$$

For fixed $\alpha \in (0, \pi)$, the left-hand side is a power series (in t) with radius of convergence $\rho = 1$. This formula is actually the **generating function formula** for the second Chebyshev polynomials U_n, $n \in \mathbb{N}_0$, (which we briefly met in the lemma following the Legendre-Gauss formula for the gamma function in Proposition 4.10.2):[91]

$$\sum_{n=0}^{\infty} \frac{\sin((n+1)\alpha)}{\sin(\alpha)}\, t^n = \sum_{n=0}^{\infty} U_n(\cos(\alpha))t^n = \frac{1}{1 - 2t\cdot\cos(\alpha) + t^2},\ 0 < \alpha < \pi,\ |t| < 1.$$

Equivalently

$$\sum_{n=0}^{\infty} U_n(x)\, t^n = \frac{1}{1 - 2tx + t^2}, \quad |x| < 1,\ |t| < 1.$$

where the sum is absolutely convergent since $\max_{[-1,1]} U_n = n + 1$, $n \in \mathbb{N}_0$, with the maxima and minima attained at the end-points.[92] We briefly indicate how to derive this last formula. The (2-step) inductive formula for the Chebyshev polynomials is

$$U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x), \quad n \in \mathbb{N},\ x \in \mathbb{R},$$

which can be easily verified by converting U_n, $n \in \mathbb{N}_0$, into rational expressions in the sine function. Multiplying through by t^{n+1} and summing up with respect to n, we obtain

$$\sum_{n=1}^{\infty} U_{n+1}(x)t^{n+1} = 2xt \sum_{n=1}^{\infty} U_n(x)t^n - t^2 \sum_{n=1}^{\infty} U_{n-1}(x)t^{n-1}$$

[90] See *Elements of Mathematics - History and Foundations*, Example 11.3.6.

[91] For a brief account of the Chebyshev polynomials, see also *Elements of Mathematics - History and Foundations*, Section 11.3.

[92] See Exercise 11.3.14 in *Elements of Mathematics - History and Foundations*.

Using $U_0(x) = 1$ and $U_1(x) = 2x$, after simplifying and rearranging, we arrive at

$$(t^2 - 2tx + 1) \sum_{n=0}^{\infty} U_n(x)\, t^n = 1.$$

The claim follows.
Returning to the main line of the proof, as above, we regard the left-hand side of the formula

$$\sum_{n=0}^{\infty} \sin((n+1)\alpha)\, t^n = \frac{\sin(\alpha)}{1 - 2t \cdot \cos(\alpha) + t^2},\ \ 0 < \alpha < \pi,\ \ |t| < 1.$$

as a power series expansion in the variable $|t| < 1$ with radius of convergence $\rho = 1$. Using the corollary to Proposition 3.2.1, for fixed $0 < \alpha < \pi$, we integrate over $[0, b]$, $0 < b < 1$, and obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\sin((n+1)\alpha)}{n+1}\, b^{n+1} &= \sin(\alpha) \int_0^b \frac{dt}{1 - 2t \cdot \cos(\alpha) + t^2} = \left[\arctan\left(\frac{t - \cos(\alpha)}{\sin(\alpha)}\right)\right]_0^b \\ &= \left(\arctan\left(\frac{b - \cos(\alpha)}{\sin(\alpha)}\right) + \arctan\left(\frac{\cos(\alpha)}{\sin(\alpha)}\right)\right) = \arctan\left(\frac{b \sin(\alpha)}{1 - b\cos(\alpha)}\right),\end{aligned}$$

where we used the identity

$$\arctan(u) + \arctan(v) = \arctan\left(\frac{u+v}{1-uv}\right), \quad uv < 1.$$

For fixed $0 < \alpha < \pi$, the left-hand side is a power series in the variable b with radius of convergence $\rho = 1$. Now the crux is that, at the beginning of this proof, we showed that it is convergent at the boundary $b = 1$ (with the index shifted by 1). By Proposition 2.3.3, this means that this power series is uniformly convergent on $[0, 1]$, and hence its left limit as $b \to 1^-$ is equal to its value at $b = 1$. Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n} = \sum_{n=0}^{\infty} \frac{\sin((n+1)\alpha)}{n+1} &= \lim_{b \to 1^-} \arctan\left(\frac{b\sin(\alpha)}{1 - b\cos(\alpha)}\right) \\ = \arctan\left(\frac{\sin(\alpha)}{1 - \cos(\alpha)}\right) &= \arctan\left(\cot\left(\frac{\alpha}{2}\right)\right) = \frac{\pi - \alpha}{2},\end{aligned}$$

where we used the half-angle conversions

$$\sin(\alpha) = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} \quad \text{and} \quad \cos(\alpha) = \frac{1 - \tan^2\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)}.$$

The Fourier expansion follows.

It remains to prove the second relation, Euler's summation formula above.[93] We begin with the Fourier coefficients:[94]

$$I(m,k) = \int_0^1 B_{2m}(x)\cos(k\pi x)\,dx, \quad m \in \mathbb{N}_0,\ k \in \mathbb{N}.$$

Clearly, $I(0,k) = 0$, $k \in \mathbb{N}$, since $B_0(x) = B_0 = 1$. For $m, k \in \mathbb{N}$, we claim

$$I(m,k) = \begin{cases} 0 & \text{if k is odd} \\ (-1)^{m-1}(2m)!/(k^{2m}\pi^{2m}) & k \text{ is even.} \end{cases}$$

To derive this, we use integration by parts (with obvious choices) twice and the inductive formula for the Bernoulli polynomials, and calculate

$$\begin{aligned} I(m,k) &= -\frac{2m}{k\pi}\int_0^1 B_{2m-1}(x)\sin(k\pi x)\,dx \\ &= \frac{2m}{k^2\pi^2}\left[B_{2m-1}(x)\cos(k\pi x)\right]_0^1 - \frac{2m(2m-1)}{k^2\pi^2}I(m-1,k). \end{aligned}$$

For $m = 1$ (and $k \in \mathbb{N}$), this gives

$$I(1,k) = \frac{2}{k^2\pi^2}\left[B_1(x)\cos(k\pi x)\right]_0^1 = \frac{1}{k^2\pi^2}(\cos(k\pi)+1) = \begin{cases} 0 & \text{if k is odd} \\ 2/(k^2\pi^2) & k \text{ is even.} \end{cases}$$

For $2 \le m \in \mathbb{N}$, since the $B_{2m-1} = 0$, we obtain the inductive formula

$$I(m,k) = -\frac{2m(2m-1)}{k^2\pi^2}I(m-1,k), \quad 2 \le m \in \mathbb{N},\ k \in \mathbb{N}.$$

Applying this inductively, and combining the case $m = 1$, the claim follows.

We now make a minor change of technical importance, and replace the Bernoulli polynomials $B_m(x)$ by $B_m^0(x) = B_m(x) - B_m(0) = B_m(x) - B_m$, $m \in \mathbb{N}$; these having the effect of deleting the constant terms. The corresponding integral clearly stays the same:

$$I^0(m,k) = \int_0^1 B_{2m}^0(x)\cos(k\pi x)\,dx = I(m,k), \quad m \in \mathbb{N},\ k \in \mathbb{N}.$$

---

[93] There are several elementary proofs of this. We follow here Ciaurri, Ó., Navas, L.M., Ruiz, F.J., Varona, J.L., *A simple computation of* $\zeta(2k)$, Amer. Math Monthly, Vol. 122, No. 5 (May 2015) 444-451. See also Osler, T., *Finding* $\zeta(2p)$ *from a product of sines*, Amer. Math. Monthly, 111 (2004) 52-54; and Sittinger, B.D., *Computing* $\zeta(2m)$ *by using telescopic sums*, Amer. Math. Monthly, 123 (August-September 2016) 710-715.

[94] These integrals are the Fourier coefficients of the Bernoulli polynomials $B_{2m}(x)$, $m \in \mathbb{N}$.

Summing over $k \in \mathbb{N}$ and using the result of our computations above, for $m \in \mathbb{N}$, we obtain

$$\sum_{k=1}^{\infty} I^0(m,k) = \sum_{k=1}^{\infty} I^0(m,2k) = (-1)^{m+1}\frac{(2m)!}{\pi^{2m}}\sum_{k=1}^{\infty}\frac{1}{(2k)^{2m}} = (-1)^{m+1}\frac{(2m)!}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{1}{k^{2m}}.$$

Rearranging, we obtain

$$\sum_{k=1}^{\infty}\frac{1}{k^{2m}} = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}\sum_{k=1}^{\infty} I^0(m,k), \quad m \in \mathbb{N}.$$

For the Euler summation formula, it remains to show that

$$\left(\sum_{k=1}^{\infty} I^0(m,k) = \right) \sum_{k=1}^{\infty}\int_0^1 B^0_{2m}(x)\cos(k\pi x)\,dx = \frac{B_{2m}}{2}, \quad m \in \mathbb{N}.$$

The key technical step is to use the product to sum trigonometric identity

$$2\sin\left(\frac{u}{2}\right)\cos(ku) = \sin\left(\frac{2k+1}{2}u\right) - \sin\left(\frac{2k-1}{2}u\right), \quad k \in \mathbb{N},$$

and rewrite the integrands (with $u = \pi x$) so that the infinite sum becomes telescopic. We calculate

$$\begin{aligned}\sum_{k=1}^{\infty}\int_0^1 B^0_{2m}(x)\cos(k\pi x)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1 B^0_{2m}(x)\cos(k\pi x)\,dx \\ &= \lim_{n\to\infty}\int_0^1 B^0_{2m}(x)\frac{\sin\left((2n+1)\pi x)/2\right)}{2\sin\left(\pi x/2\right)} - \frac{1}{2}\int_0^1 B^0_{2m}(x)dx.\end{aligned}$$

The second term calculates as

$$\frac{1}{2}\int_0^1 B^0_{2m}(x)dx = \frac{1}{2}\int_0^1 \left(B_{2m}(x) - B_{2m}\right)\,dx = -\frac{B_{2m}}{2},$$

since the integral $\int_0^1 B_{2m}(x)\,dx$ vanishes (Section 4.12).
It remains to show that

$$\lim_{n\to\infty}\int_0^1 \frac{B^0_{2m}(x)}{2\sin\left(\pi x/2\right)}\sin\left(\frac{(2n+1)\pi x}{2}\right) = 0.$$

The fraction in the integrand is a function $f : (0,1] \to \mathbb{R}$ given by

$$f(x) = \frac{B^0_{2m}(x)}{2\sin\left(\pi x/2\right)} = \frac{B^0_{2m}(x)}{\pi x}\cdot\frac{\pi x}{2\sin(\pi x/2)}.$$

The first factor is a polynomial (since the constant term in $B^0_{2m}(x)$ is (by setup) missing), and the second (positive) factor is the reciprocal of $\sin(u)/u$, $u = \pi x/2$, that extends continuously to $u = 0$, and is infinitely many times differentiable for $0 \le u \le \pi/2$. We now perform integration by parts, and our integral becomes

$$\int_0^1 f(x)\sin\left(\frac{(2n+1)\pi x}{2}\right)dx = -\frac{2}{(2n+1)\pi}\int_0^1 f(x)\left(\cos\left(\frac{(2n+1)\pi x}{2}\right)\right)' dx$$
$$= -\frac{2}{(2n+1)\pi}\left[f(x)\cos\left(\frac{(2n+1)\pi x}{2}\right)\right]_0^1 + \frac{2}{(2n+1)\pi}\int_0^1 f(x)\cos\left(\frac{(2n+1)\pi x}{2}\right)dx$$

All boundary expressions vanish, except

$$f(0) = \lim_{x\to 0}\frac{B^0_{2m}(x)}{\pi x} = \lim_{x\to 0}\frac{B'_{2m}(x)}{\pi} = 2m\lim_{x\to 0}\frac{B_{2m-1}(x)}{\pi} = \frac{2m}{\pi}B_{2m-1}(0) = \frac{2m}{\pi}B_{2m-1}$$

which also vanishes for $2 \le m \in \mathbb{N}$, but equals $-1/\pi$ for $m = 1$. The last integral estimates as

$$\left|\int_0^1 f(x)\cos\left(\frac{(2n+1)\pi x}{2}\right)dx\right| \le \int_0^1 |f(x)|\,dx,$$

in particular, it is independent of $n \in \mathbb{N}$. We obtain that, due to the presence of the factor $2/(2n+1)$, the limit as $n \to \infty$ is zero. Euler's summation formula, and therefore the Fourier expansions follow.

Exercises

1. Derive the generating function formula

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x},$$

for the harmonic number $H_n = \sum_{k=1}^n 1/k$, $n \in \mathbb{N}$.

2. Derive the formula[95]

$$\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}} = (-1)^{m+1}\frac{(2\pi)^{2m+1}}{2(2m+1)!}\int_0^1 B_{2m+1}(x)\cot\left(\frac{\pi x}{2}\right)dx, \quad m \in \mathbb{N},$$

using the following steps: (1) Calculate the integrals

$$J(m,k) = \int_0^1 B_{2m+1}(x)\sin(k\pi x)\,dx = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (-1)^{m-1}(2m+1)!/(k^{2m+1}\pi^{2m+1}) & k \text{ is even.}\end{cases}$$

[95]See ibid. Note that the left-hand side is the value of the zeta function on odd itegers; see Section 4.17.

(2) Use the trigonometric identity

$$2\sin\left(\frac{u}{2}\right)\sin(ku) = -\cos\left(\frac{2k+1}{2}u\right) + \cos\left(\frac{2k-1}{2}u\right), \quad k \in \mathbb{N},$$

to obtain the formula

$$\sum_{k=1}^{\infty} J(m,k) = (-1)^{m+1}\frac{(2m+1)!}{\pi^{2m+1}}\sum_{k=1}^{\infty}\frac{1}{(2k)^{2m+1}}$$
$$= -\lim_{n\to\infty}\int_0^1 B_{2m+1}(x)\frac{\cos\left((2n+1)\pi x)/2\right)}{2\sin\left(\pi x/2\right)}dx + \frac{1}{2}\int_0^1 B_{2m+1}(x)\cot\left(\frac{\pi x}{2}\right)dx.$$

(3) Letting $n \to \infty$ argue as the the text to conclude that the first limit is zero.

3. Use the Fourier expansion

$$\sum_{n=1}^{\infty}\frac{\sin(n\alpha)}{n} = \frac{\pi-\alpha}{2}, \quad 0 < \alpha < 2\pi,$$

(proved in the text) to derive the analogous expansion

$$\sum_{n=1}^{\infty}\frac{\cos(n\alpha)}{n} = -\ln\left(2\sin\left(\frac{\alpha}{2}\right)\right), \quad 0 < \alpha < 2\pi,$$

via the following steps. (1) Letting $n \in \mathbb{N}$, integrate both sides of the trigonometric identity

$$\sum_{k=1}^{n}\sin(kx) = \frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}, \quad 0 < x < 2\pi,$$

over $[\pi, \alpha]$, $0 < \alpha < 2\pi$, to obtain

$$\sum_{k=1}^{n}\frac{\cos(k\alpha)}{k} = \sum_{k=1}^{n}\frac{(-1)^k}{k} - \int_{\pi}^{\alpha}\frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}\,dx.$$

(2) Rewrite this as

$$\sum_{k=1}^{n}\frac{\cos(k\alpha)}{k} = -\ln(2) - \ln\sin\left(\frac{\alpha}{2}\right) + \int_{\pi}^{\alpha}\frac{\cos((n+1/2)x)}{2\sin(x/2)}\,dx.$$

(3) Use integration by parts (with $u = 1/(2\sin(x/2))$ and $dv = \cos((n+1/2)x)\,dx$) in the last integral to show that it converges to zero as $n \to \infty$.

4.15 The Euler-Maclaurin Summation Formula

The derivation of the Stirling formula in Section 4.10 was based on the estimate of the difference $\int_1^n \ln(x)\,dx - \sum_{k=1}^n \ln(k)$, $2 \leq n \in \mathbb{N}$. There is a more general method, discovered independently by Euler and Maclaurin, to develop more sophisticated estimates for the difference $\int_1^n f(x)\,dx - \sum_{k=1}^n f(k)$, $2 \leq n \in \mathbb{N}$, for more general functions $f : [1, \infty) \to \mathbb{R}$. In this section we present this method and illustrate it with a number of examples.

We fix $2 \leq n \in \mathbb{N}$, and let $f : [1, n] \to \mathbb{R}$ a continuously differentiable function. We first write

$$\int_1^n f(x)\,dx - \sum_{k=1}^{n-1} f(k) = \sum_{k=1}^{n-1} \int_k^{k+1} (f(x) - f(k))\,dx.$$

For $k = 1, \ldots, n-1$, we perfom integration by parts in the last integral with $u = f(x) - f(k)$ and $v = x - k - 1/2$, where the choice of the constant will be given below. We obtain

$$\begin{aligned}
&\int_k^{k+1} (f(x) - f(k))\,dx \\
&\qquad = \left[(f(x) - f(k))\left(x - k - \frac{1}{2}\right)\right]_k^{k+1} - \int_k^{k+1} \left(x - k - \frac{1}{2}\right) \cdot f'(x)\,dx \\
&\qquad = \frac{1}{2}(f(k+1) - f(k)) - \int_k^{k+1} \left(x - [x] - \frac{1}{2}\right) \cdot f'(x)\,dx,
\end{aligned}$$

where, in the last integral, we used the greatest integer function $[\cdot]$. Substituting this into the sum in the formula above, and noticing that the first terms are telescopic, after rearraging we arrive at the following

$$\sum_{k=1}^n f(k) = \int_1^n f(x)\,dx + \int_1^n P_1(x) \cdot f'(x)\,dx + \frac{1}{2}(f(n) + f(1)),$$

where $P_1 : \mathbb{R} \to \mathbb{R}$, $P_1(x) = x - [x] - 1/2$, $x \in \mathbb{R}$. (Note that we moved the upper limit of the summation from $n-1$ to n.) This is the **first Euler-Maclaurin summation formula**.

We now assume that (the extended) function $f : [1, \infty) \to \mathbb{R}$ is continuously differentiable and that the improper integral $\int_1^\infty P_1(x) \cdot f'(x)\,dx$ exists. Since P_1 is bounded, in fact $|P_1(x)| \leq 1/2$, $x \in \mathbb{R}$, the latter holds if the improper integral[96] $\int_1^\infty |f'(x)|\,dx < \infty$. In this case, the first Euler-Maclaurin summation formula can also be written as

$$\sum_{k=1}^n f(k) = \int_1^n f(x)\,dx + C_f + E_f(n),$$

where

$$C_f = \frac{1}{2}f(1) + \int_1^\infty P_1(x)\cdot f'(x)\,dx \quad \text{and} \quad E_f(n) = \frac{1}{2}f(n) - \int_n^\infty P_1(x)\cdot f'(x)\,dx.$$

Here the constant C_f depends only on f, and the improper integral in the "error term" $E_f(n)$ converges to zero. If, in addition, $\lim_{n\to\infty} f(n) = 0$ then the entire error term $E_f(n)$ converges to zero as $n \to \infty$.

Example 4.15.1. For $f(x) = 1/x$, $1 \leq x \in \mathbb{R}$, we have

$$\left|\int_n^\infty \frac{P_1(x)}{x^2}\,dx\right| \leq \int_n^\infty \frac{|P_1(x)|}{x^2}\,dx < \frac{1}{2}\int_n^\infty \frac{dx}{x^2} = \frac{1}{2n}, \qquad n \in \mathbb{N}.$$

Suppressing the functional subscript, for the nth harmonic number $H_n = \sum_{k=1}^n 1/k$, $n \in \mathbb{N}$, we obtain[97]

$$H_n = \ln(n) + C + E(n),$$

where

$$C = \frac{1}{2} - \int_1^\infty \frac{P_1(x)}{x^2}\,dx \quad \text{and} \quad E(n) = \frac{1}{2n} + \int_n^\infty \frac{P_1(x)}{x^2}\,dx.$$

As $0 < E(n) < 1/n$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} E(n) = 0$. Hence the limit $\lim_{n\to\infty}(H_n - \ln(n))$ exists; this is the classical **Euler-Mascheroni constant** γ. As a byproduct, we obtain

$$\gamma = \frac{1}{2} - \int_1^\infty \frac{P_1(x)}{x^2}\,dx.$$

Example 4.15.2. For $f(x) = \ln(x)$, $1 \leq x \in \mathbb{R}$, the Euler-Maclaurin summation formula takes the form

$$\sum_{k=1}^n \ln(k) = \ln(n!) = (n + 1/2)\ln(n) - n + 1 + \int_1^n \frac{P_1(x)}{x}\,dx,$$

[96] This is certainly the case if f is decreasing and $\lim_{x\to\infty} f(x) = 0$; the Euler-Maclaurin summation formula then asserts the **integral test for convergence**: the infinite sum $\sum_{k=1}^\infty f(k)$ and the improper integral $\int_1^\infty f(x)\,dx$ equiconverge. Note that the integral test for convergence is due to Maclaurin in his *Fluxions*, I., pp. 289-290, and Cauchy in *Oeuvres* (2), VII. p. 269.

[97] For a direct elementary geometric derivation of a somewhat sharper estimate, see Young, R.M., *Euler's constant*, Math. Gazette 75, No. 472 (1991), 187-190.

In this example we will calculate the improper integral (in the error term) as[98]

$$\int_1^\infty \frac{P_1(x)}{x}\,dx = \int_0^{1/2}\left(\frac{8x^2}{1-4x^2} - \pi x\tan(\pi x)\right)dx = \ln(\sqrt{2\pi}) - 1.$$

With this, supressing the functional subscript, we have

$$C = \ln(\sqrt{2\pi}) - 1 \quad \text{and} \quad E(n) = \frac{\ln(n)}{2} - \int_n^\infty \frac{P_1(x)}{x}\,dx.$$

Rearranging, we find

$$\ln\left(\frac{n!}{\sqrt{2\pi n}\cdot n^n\cdot e^{-n}}\right) = -\int_n^\infty \frac{P_1(x)}{x}\,dx.$$

Since the right-hand side converges to zero as $n\to\infty$, we recover the classical Stirling formula (Section 4.10).
We now turn to the evaluation of the improper integral

$$\int_1^\infty \frac{P_1(x)}{x}\,dx = \int_0^\infty \frac{P_1(x)}{1+x}\,dx$$

where we shifted the variable in the last integral by one (and used periodicity of P_1) for technical convenience. Using $P_1(x) = x - [x] - 1/2$, $x\in\mathbb{R}$, we have

$$\begin{aligned}
\int_0^\infty \frac{P_1(x)}{1+x}\,dx &= \sum_{n=0}^\infty\left(\int_n^{n+1/2}\frac{x-n-1/2}{1+x}\,dx + \int_{n+1/2}^n \frac{x-n-1/2}{1+x}\,dx\right)\\
&= \sum_{n=0}^\infty\left(\int_0^{1/2}\frac{x-1/2}{1+n+x}\,dx + \int_0^{1/2}\frac{x}{3/2+n+x}\,dx\right)\\
&= \sum_{n=0}^\infty\left(-\int_0^{1/2}\frac{x}{3/2+n-x}\,dx + \int_0^{1/2}\frac{x}{3/2+n+x}\,dx\right)\\
&= -\sum_{n=0}^\infty\int_0^{1/2}\frac{2x^2}{(3/2+n)^2-x^2}\,dx = -\int_0^{1/2}\sum_{n=0}^\infty\frac{2x^2}{(3/2+n)^2-x^2}\,dx\\
&= -\int_0^{1/2}\sum_{n=0}^\infty\frac{8x^2}{(2n+3)^2-4x^2}\,dx
\end{aligned}$$

[98]We follow here Neuschel, Th., *A new proof of Stirling's formula*, Amer. Math. Monthly, Vol 121, No. 4 (April 2014) 350-352. Note also that, as we will see below, the improper integral converges albeit not absolutely.

where we performed obvious linear substitutions, and, in the last step we interchanged the integral and the infinite sum (which is allowed due to uniform convergence of the latter). Comparing this with the expansion of the tangent (Section 4.8)

$$\pi x \tan(\pi x) = \sum_{n=0}^{\infty} \frac{8x^2}{(2n+1)^2 - 4x^2}, \qquad -\frac{1}{2} < x < \frac{1}{2},$$

we obtain

$$\int_1^\infty \frac{P_1(x)}{x}\, dx = \int_0^{1/2} \left(\frac{8x^2}{1-4x^2} - \pi x \tan(\pi x) \right) dx.$$

It remains to evalue the last integral on the right-hand side. Using the partial fraction decomposition

$$\frac{8x^2}{1-4x^2} = \frac{1}{1-2x} + \frac{1}{1+2x} - 2,$$

and integrating the trivial terms, this reduces to

$$\int_1^\infty \frac{P_1(x)}{x}\, dx = \ln(\sqrt{2}) - 1 + \int_0^{1/2} \left(\frac{1}{1-2x} - \pi x \tan(\pi x) \right) dx.$$

The last integral is improper (at the upper bound $1/2$). Letting $0 < b < 1/2$, we first use integration by parts (with obvious choices) for the second term in the integrand as

$$\int_0^b \pi x \tan(\pi x)\, dx = -b \ln(\cos(\pi b) + \int_0^b \ln(\cos(\pi x))\, dx$$

Putting this together with the first term, we calculate

$$\begin{aligned}\int_0^b \left(\frac{1}{1-2x} - \pi x \tan(\pi x) \right) dx &= -\frac{1}{2} \ln(1-2b) + b \ln(\cos(\pi b)) - \int_0^b \ln(\cos(\pi x))\, dx \\ &= \left(b - \frac{1}{2} \right) \ln(\cos(\pi b)) + \frac{1}{2} \ln\left(\frac{\cos(\pi b)}{1-2b} \right) - \int_0^b \ln(\cos(\pi x))\, dx\end{aligned}$$

We now let $b \to 1/2^-$ and evaluate each of the three terms on the right-hand side separately. For the first term, we have

$$\begin{aligned}\lim_{b \to 1/2^-} \left(b - \frac{1}{2} \right) \ln(\cos(\pi b)) &= -\frac{1}{\pi} \lim_{u \to 0^-} u \ln(\sin(u)) \\ &= -\frac{1}{\pi} \lim_{u \to 0^-} \frac{\ln(\sin(u))}{1/u} = \frac{1}{\pi} \lim_{u \to 0^-} \frac{\cos(u)/\sin(u)}{1/u^2} \\ &= \frac{1}{\pi} \lim_{u \to 0^-} \frac{u^2}{\sin(u)} \cos(u) = 0,\end{aligned}$$

where we used the limit rule for indeterminate forms. For the second term, we calculate

$$\frac{1}{2}\lim_{b\to 1/2^-}\ln\left(\frac{\cos(\pi b)}{1-2b}\right)=\frac{1}{2}\ln\left(\lim_{b\to 1/2^-}\frac{\cos(\pi b)}{1-2b}\right)=\frac{1}{2}\ln\left(\lim_{b\to 1/2^-}\frac{\pi\sin(\pi b)}{2}\right)=\frac{1}{2}\ln\left(\frac{\pi}{2}\right).$$

The third term is the improper integral

$$\int_0^{1/2}\ln(\cos(\pi x))\,dx=\frac{1}{\pi}\int_0^{\pi/2}\ln(\cos(u))\,du=-\ln(\sqrt{2}),$$

treated in Example 4.1.6.
Finally, putting everything together, we obtain

$$\int_1^\infty\frac{P_1(x)}{x}\,dx=\ln(\sqrt{2})-1+\frac{1}{2}\ln\left(\frac{\pi}{2}\right)+\ln(\sqrt{2})=\ln(\sqrt{2\pi})-1.$$

The example follows.

Assuming higher order differentiability of the function f, repeated integrations by part now yield all the subsequent Euler-Maclaurin summation formulas. The key is to introduce higher order analogues of the function P_1 that will appear next to the higher order derivatives of f. We view P_1 as the **periodized** linear polynomial $B_1(x)=x-1/2$; that is $P_1(x)=B_1(x-[x])$, $x\in\mathbb{R}$. We streamline the integration by parts first by imposing the inductive relation $B_m'(x)=mB_{m-1}(x)$, $m\in\mathbb{N}$, with $B_0(x)=1$. Equivalently, we set

$$B_m(x)=m\int_0^x B_{m-1}(t)\,dt+B_m,\quad B_m=B_m(0),\quad m\in\mathbb{N}.$$

Periodizing, that is, setting $P_m(x)=B_m(x-[x])$, $x\in\mathbb{R}$, $m\in\mathbb{N}_0$, results a periodic function with period 1. (Clearly, all P_m, $m\in\mathbb{N}_0$, are differentiable up to any order away from the integers.) Second, to ensure continuity and progressive smoothness of P_m, $2\leq m\in\mathbb{N}$, at the **integers**, we impose the condition $B_m(1)=B_m(0)=B_m$, or equivalently

$$\int_0^1 B_m(x)\,dx=0,\quad m\in\mathbb{N}.$$

Now, P_1 is discontinuous at the integers, but this gives $B_1=-1/2$, explaining the choice of constant in the first integration by parts above. For $2\leq m\in\mathbb{N}$, this condition gives continuity of P_m at the integers; and, by the first condition, for $3\leq n\in\mathbb{N}$, the function P_m is differentiable up to order $n-2$ at the integers.

As shown in Section 4.12, the two relations above define the sequence $(B_m(x))_{m\in\mathbb{N}}$

of **Bernoulli polynomials**, and the sequence $(B_m)_{m\in\mathbb{N}_0}$ of **Bernoulli numbers** (with $B_0 = 1$).

We now state the main result of this section.

Euler-Maclaurin Summation Theorem. *Let* $2 \le n \in \mathbb{N}$ *and* $m \in \mathbb{N}_0$, *and* $f : [1, n] \to \mathbb{R}$ *a continuously differentiable function up to order* $2m + 1$. *We have*

$$\begin{aligned}\sum_{k=1}^{n} f(k) &= \int_1^n f(x)\,dx + \frac{1}{2}(f(n) + f(1)) + \sum_{\ell=1}^{m} \frac{B_{2\ell}}{(2\ell)!}\left(f^{(2\ell-1)}(n) - f^{(2\ell-1)}(1)\right) \\ &+ \frac{1}{(2m+1)!}\int_1^n P_{2m+1}(x)\cdot f^{(2m+1)}(x)\,dx.\end{aligned}$$

(For $m = 0$, *the sum on the right-hand side is absent.)*

PROOF. We proceed by induction with respect to $m \in \mathbb{N}_0$. For $m = 0$ the stated formula reduces to the first Euler-Maclaurin summation fomula derived above. For the general induction step $m - 1 \Rightarrow m$, $m \in \mathbb{N}$, (assuming that the formula holds for m replaced by $m - 1$) we perform integration by parts on the integral

$$I_{m,n} = \int_1^n P_{2m-1}(x)\cdot f^{(2m-1)}(x)\,dx$$

with $u = f^{(2m-1)}(x)$ and $v = \int_0^x P_{2m-1}(t)\,dt = (P_{2m}(x) - B_{2m})/(2m)$. We have $du = f^{(2m)}(x)dx$ and $dv = P_{2m-1}(x)dx$. We calculate

$$\begin{aligned}I_{m,n} &= \frac{1}{2m}\left[(P_{2m}(x) - B_{2m})\cdot f^{(2m-1)}(x)\right]_1^n - \frac{1}{2m}\int_1^n (P_{2m}(x) - B_{2m})\cdot f^{(2m)}(x)\,dx \\ &= -\frac{1}{2m}\int_1^n P_{2m}(x)\cdot f^{(2m)}(x)\,dx + \frac{B_{2m}}{2m}\int_1^n f^{(2m)}(x)\,dx \\ &= -\frac{1}{2m}\int_1^n P_{2m}(x)\cdot f^{(2m)}(x)\,dx + \frac{B_{2m}}{2m}\left(f^{(2m-1)}(n) - f^{(2m-1)}(1)\right).\end{aligned}$$

where, by periodicity, $P_{2m}(n) = P_{2m}(1) = B_{2m}$, $m \in \mathbb{N}$. We perform yet another integration by parts on the last integral with $u = f^{(2m)}(x)$ and $v = \int_0^x P_{2m}(t)\,dt = P_{2m+1}(x)/(2m+1)$, where $B_{2m+1} = 0$, $m \in \mathbb{N}$. We obtain

$$\int_1^n P_{2m}(x)\cdot f^{(2m)}(x)\,dx = -\frac{1}{2m+1}\int_1^n P_{2m+1}(x)\cdot f^{(2m+1)}(x)\,dx.$$

Putting everything together, we arrive at

$$\begin{aligned}\int_1^n P_{2m-1}\cdot f^{(2m-1)}(x)\,dx &= \frac{1}{(2m)(2m+1)}\int_1^n P_{2m+1}(x)\,f^{(2m+1)}(x)\,dx \\ &+ \frac{B_{2m}}{2m}\left(f^{(2m-1)}(n) - f^{(2m-1)}(1)\right).\end{aligned}$$

Substituting this back to the original formula, the induction is complete, and the proposition follows.

History. The Euler-Maclaurin formula has a bit of convoluted history.[99] Euler published it in the Comm. Acad. Imp. Petrop. VI with year of publication 1732 (but with actual appearance in 1738), and four years later, in June 9, 1736, he jotted it down in a letter to Stirling. The latter responded somewhat belatedly, in April 16, 1738, stating that it was a generalization of one of his own results, and also noted that a more general formula had already been discovered by Maclaurin somewhat earlier. In response to this Euler rescinded the primary authorship of the formula, but Maclaurin's work appeared only in 1742 in his *Treatise on Fluxions*, p. 672. Finally, note that a common generalization of this and the Taylor expansion was given by Darboux well over a century later.[100]

As before, letting $f : [1, \infty) \to \mathbb{R}$ and assuming that the improper integral $\int_1^\infty P_{2m+1}(x) \cdot f^{(2m+1)}\, dx$ exists, we obtain

$$\sum_{k=1}^n f(k) = \int_1^n f(x)\, dx + C_f^m + E_f^m(n),$$

where

$$C_f^m = \frac{1}{2} f(1) - \sum_{\ell=1}^m \frac{B_{2\ell}}{(2\ell)!} f^{(2l-1)}(1) + \frac{1}{(2m+1)!} \int_1^\infty P_{2m+1}(x) \cdot f^{(2m+1)}(x)\, dx.$$

and

$$E_f^m(n) = \frac{1}{2} f(n) + \sum_{\ell=1}^m \frac{B_{2\ell}}{(2\ell)!} f^{(2l-1)}(n) - \frac{1}{(2m+1)!} \int_n^\infty P_{2m+1}(x) \cdot f^{(2m+1)}(x)\, dx.$$

By the estimates of the Bernoulli polynomial (on $[0, 1]$) along with periodicity, for $1 \leq x \in \mathbb{R}$, we have

$$|P_{2m}(x)| \leq |B_{2m}| \quad \text{and} \quad |P_{2m+1}(x)| \leq \frac{2m+1}{2\pi} |B_{2m}|, \quad m \in \mathbb{N}.$$

(As noted above, we also have $|B_1(x)| \leq 1/2$, $x \in [0, 1]$, and hence $|P_1(x)| \leq 1/2$ for all $x \in \mathbb{R}$.)

---

[99] See Barnes, Proc. London Math. Soc. (2), III. (1905), p. 258. For complete accounts, see Hardy, G.H., *Divergent Series*, Cambridge University Press, 1949, pp. 318-348; and Olver, F.W.J., *Asymptotics and Special Functions*,Academic press, New York, 1974, 279-289.

[100] See Darboux, Journal de Math. (3), II (1876), p. 271; and also Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020; 7.1.

Finally, note that, as a consequence of the second estimate, the improper integrals in the formulas above exist if

$$\lim_{n\to\infty}\int_n^\infty |f^{(2m+1)}(x)|\,dx = 0.$$

History. Finding a continuous function that "interpolates" the harmonic numbers $H_n = 1 + 1/2 + \cdots + 1/n$, $n \in \mathbb{N}$, has been a central problem of Euler, Goldbach, and Daniel Bernoulli. While the latter two have been unsuccesful, Euler, in a letter to Goldbach dated October 13, 1729, hinted that not only did he find such a function but also that the value of this function at $1/2$ is equal to $2 - 2\ln(2)$. A few years later Euler did publish this discovery in *De summatione innumerabilium progressionum*, Commentarii academiae scientiarium imperialis Petropolitanae 5 (1730/31) 1738, pp. 91-105. Reprinted in *Opera Omnia* I.14 pp. 42-72. The function in question is the integral[101]

$$\int_0^1 \frac{1-x^t}{1-x}\,dx, \quad 0 < t \in \mathbb{R}.$$

For $t = n \in \mathbb{N}$, using the geometric series fomula, the integral reduces to H_n. For $t = 1/2$, the integral calculates as

$$\int_0^1 \frac{1-\sqrt{x}}{1-x}\,dx = \int_0^1 \frac{1}{1+\sqrt{x}}\,dx = \left[2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right)\right]_0^1 = 2 - 2\ln(2),$$

as Euler claimed.

Example 4.15.3. Derive the following formulas for the Euler-Mascheroni constant[102]

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\int_0^1 \left(1 - \left(1 - \frac{t}{n}\right)^n\right)\frac{dt}{t} - \int_1^n \left(1-\frac{t}{n}\right)^n \frac{dt}{t}\right)\\ &= \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt = \int_0^1 \frac{1-e^{-t}-e^{-1/t}}{t}\,dt.\end{aligned}$$

Indeed, the first equality follows directly from the definition $\gamma = \lim_{n\to\infty}(H_n - \ln(n))$ via

$$H_n = \int_0^1 \frac{1-x^n}{1-x}\,dx = \int_0^1 \frac{1-(1-t)^n}{t}\,dt = \int_0^n \left(1 - \left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t}.$$

(see the historical insert above) and $\ln(n) = \int_1^n dt/t$. The third equality follows from the second by replacing t by $1/t$ in the second integral.
Finally, the second equality is via interchanging the limit with the integration, and the latter hinges on the estimates

$$0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le t^2 e^{-t/n}, \quad 0 \le t \le n.$$

[101]Note that the integrand has removable discontinuity at $x = 1$ since $\lim_{x\to 1^-}(1-x^t)/(1-x) = t$.

[102]See also Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020; Example 2 in 12.1, and Example 4 in 12.2.

The first inequality in this estimate is well known.[103] For the second, we calculate

$$\begin{aligned} e^{-t}-\left(1-\frac{t}{n}\right)^n &\le e^{-t}\left(1-e^t\left(1-\frac{t}{n}\right)^n\right) \\ &\le e^{-t}\left(1-\left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n\right) \\ &= e^{-t}\left(1-\left(1-\frac{t^2}{n^2}\right)^n\right) \le e^{-t}\frac{t^2}{n}. \end{aligned}$$

Here the last inequality is the Bernoulli inequality[104] in disguise

$$1-n\left(\frac{t^2}{n^2}\right) \le \left(1-\frac{t^2}{n^2}\right)^n$$

The second equality, and the example now follows.

We now revisit Example 4.15.1.

Example 4.15.4. Let $f(x) = 1/x$, $1 \le x \in \mathbb{R}$. Suppressing the functional subscript, for $m \in \mathbb{N}_0$, a simple computation gives

$$H_n = \ln(n) + C^m + E^m(n),$$

where

$$C^m = \frac{1}{2} + \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell} - \int_1^\infty \frac{P_{2m+1}(x)}{x^{2m+2}}dx$$

and

$$E^m(n) = \frac{1}{2n} - \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell}\frac{1}{n^{2\ell}} + \int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+2}}dx.$$

We denote the last improper integral by $\tilde{E}^m(n)$, and estimate as

$$\left|\tilde{E}^m(n)\right| \le \int_n^\infty \frac{|P_{2m+1}(x)|}{x^{2m+2}}dx \le \frac{2m+1}{2\pi}|B_{2m}| \int_n^\infty \frac{dx}{x^{2m+2}} = \frac{1}{2\pi}\frac{|B_{2m}|}{n^{2m+1}}.$$

It follows that the improper integrals in the formulas above exist, and, for $m \in \mathbb{N}_0$, we have $\lim_{n\to\infty} E^m(n) = 0$. Since $\lim_{n\to\infty}(H_n - \ln(n)) = \gamma$, the Euler-Mascheroni constant, we obtain that $C^m = C^0 = \gamma$ for all $m \in \mathbb{N}_0$. Summarizing, for $m \in \mathbb{N}_0$, we arrive at

$$H_n - \ln(n) = \gamma + \frac{1}{2n} - \sum_{l=1}^{m} \frac{B_{2\ell}}{2\ell}\frac{1}{n^{2\ell}} + \tilde{E}^m(n),$$

[103] See *Elements of Mathematics - History and Foundations*, Section 10.5.
[104] See ibid. Section 3.2.

where

$$\left|\tilde{E}^m(n)\right| \leq \frac{1}{2\pi}\frac{B_{2m}}{n^{2m+1}}.$$

Remark. Note that, by Section 4.13, $\sum_{\ell=0}^{\infty} B_{2\ell}/(2\ell) = \infty$, so that, albeit tempting, letting $m \to \infty$ in the formulas above do not yield feasible results.

History. In 1736 Euler used the formula above to obtain the approximation

$$\gamma \sim H_n - \ln(n) - \frac{1}{2n} + \sum_{l=1}^{m} \frac{B_{2\ell}}{2\ell}\frac{1}{n^{2\ell}}.$$

For $n = 10$ and $m = 7$, he obtained the value of γ in 16 decimal precisison as

$$\gamma \sim 0.5772156649015328\ldots$$

with the upper bound of the error term as above being $1.856807669405446 \cdot 10^{-16}$. In 1809, in his *Adnotationes ad calculum integrale Euleri* Mascheroni made similar calculations, and went up to 40 decimal places. As dicovered later by Johann von Soldner (1766 – 1833), Mascheroni's calculations contained an error in the 20th decimal place. Urged by Gauss, a young mathematical prodigy, Nicolai (1793 – 1846), redid the calculations up to 40 decimals, and found agreement with Soldner's value:

$$\gamma \sim 0.5772156649015328606065120900824024310422\ldots$$

There is a proliferation of various refinements of estimates for the Euler-Mascheroni constant γ above, especially by replacing $\ln(n)$, $n \in \mathbb{N}$, by an equiconvergent expression as $n \to \infty$. We give here one example;[105] see also Exercise 2 at the end of this section.

Example 4.15.5. We have

$$\frac{1}{24(n+1)^2} < H_n - \ln\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24n^2}, \quad n \in \mathbb{N}.$$

To derive this, we define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = -\frac{1}{x+1} - \ln\left(x + \frac{1}{2}\right) + \ln\left(x + \frac{3}{2}\right), \quad x > 0.$$

Clearly, $\lim_{x\to\infty} f(x) = 0$. The choice of f is justified by the simple observation that

$$f(n) = E_n - E_{n+1}, \quad n \in \mathbb{N},$$

[105] See DeTemple, D. W., *A quicker convergence to Euler's constant*, Amer. Math. Monthly, Vol. 100, No. 5 (May 1993)0, 468-470.

where

$$E_n = H_n - \ln\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

By the definition of the Euler-Mascheroni constant, we have $\lim_{n\to\infty} E_n = \gamma$. Differentiating and simplifying, we obtain

$$f'(x) = -\frac{1}{4(x+1/2)(x+1)^2(x+3/2)}, \quad x > 0.$$

In particular, f is decreasing.
For the upper bound stated above, we need an estimate of the rate of decrease of f. First, the simple estimate

$$-f'(x) = \frac{1}{4(x+1/2)(x+1)^2(x+3/2)} < \frac{1}{4(x+1/2)^4}, \quad x > 0,$$

gives

$$\begin{aligned} f(n) &= -\int_n^\infty f'(x)\,dx < \frac{1}{4}\int_n^\infty \frac{dx}{(x+1/2)^4} = \frac{1}{12}\frac{1}{(n+1/2)^3} \\ &= \frac{1}{24}\frac{2n+1}{(n+1/2)^4} < \frac{1}{24}\frac{2n+1}{n^2(n+1)^2} = \frac{1}{12}\int_n^{n+1}\frac{dx}{x^3}. \end{aligned}$$

We use this to calculate as

$$E_n - \gamma = \sum_{k=n}^\infty (E_k - E_{k+1}) = \sum_{k=n}^\infty f(k) < \frac{1}{12}\int_n^\infty \frac{dx}{x^3} = \frac{1}{24n^2}.$$

The upper estimate follows.
For the lower bound stated above, we use the estimate

$$-f'(x) = \frac{1}{4(x+1/2)(x+1)^2(x+3/2)} > \frac{1}{4(x+1)^4}, \quad x > 0.$$

As before, we estimate

$$f(n) > \frac{1}{4}\int_n^\infty \frac{dx}{(x+1)^4} = \frac{1}{12}\frac{1}{(n+1)^3} > \frac{1}{24}\frac{2n+3}{(n+1)^2(n+2)^2} = \frac{1}{12}\int_{n+1}^{n+2}\frac{dx}{x^3}.$$

This gives

$$E_n - \gamma > \frac{1}{12}\int_{n+1}^\infty \frac{dx}{x^3} = \frac{1}{24(n+1)^2}.$$

The lower estimate also follows.

Example 4.15.6. Let $f(x) = x^p$, $p \in \mathbb{N}$. We apply the Euler-Maclaurin formula for $m = [p/2]$. Since $(x^p)^{(2m+1)}$ is zero for p even, and equals $p!$ for p odd, and $\int_1^n P_{2m+1}(x)\,dx = 0$ by periodicity and $\int_0^1 B_{2m+1}(x)\,dx = 0$, we see that the integral $I_{m+1,n} = \int_1^n P_{2m+1}(x)(x^p)^{(2m+1)}\,dx = 0$. The Euler-Maclaurin formula becomes

$$1^p + 2^p + \cdots + n^p = \sum_{k=1}^{n} k^p = \frac{n^{p+1}-1}{p+1} + \sum_{\ell=1}^{[p/2]} \binom{p}{2\ell-1} \frac{B_{2\ell}}{2\ell} \left(n^{p-2\ell+1} - 1\right) + \frac{n^p+1}{2},$$

where we used $\int_1^n x^p\,dx = (n^{p+1}-1)/(p+1)$, and the differentiation

$$(x^p)^{(2\ell-1)} = (2\ell-1)!\binom{p}{2\ell-1} x^{p-2\ell+1}, \quad \ell = 1, \ldots, [p/2].$$

We rewrite this using classical notation as

$$\begin{aligned} s_p(n) &= 1^p + 2^p + \cdots + (n-1)^p \\ &= \frac{n^{p+1}-1}{p+1} + \frac{1}{p+1}\sum_{\ell=1}^{[p/2]} \binom{p+1}{2\ell} B_{2\ell}\left(n^{p-2\ell+1} - 1\right) - \frac{n^p - 1}{2} \\ &= \frac{1}{p+1}\sum_{k=0}^{p} \binom{p+1}{k} B_k \left(n^{p-k+1} - 1\right), \end{aligned}$$

where in the last equality we used the fact that all odd Bernoulli numbers are zero except $B_1 = -1/2$. Finally, since

$$\sum_{k=0}^{p} \binom{p+1}{k} B_k = 0,$$

the constants (the terms that do not depend on n) cancel, and we arrive at the classical formula for the power sums

$$s_p(n) = 1^p + 2^p + \cdots + (n-1)^p = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k} B_k n^{p-k+1}.$$

Next, we treat a refinement of the Stirling formula.

Example 4.15.7. We let $f(x) = \ln(x)$, $1 \le x \in \mathbb{R}$. After a simple computation the Euler-Maclaurin formula takes the form

$$\begin{aligned} \sum_{k=1}^{n} \ln(k) &= \ln(n!) = (n+1/2)\ln(n) - n + 1 + \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\left(\frac{1}{n^{2\ell-1}} - 1\right) \\ &+ \frac{1}{2m+1}\int_1^n \frac{P_{2m+1}(x)}{x^{2m+1}} dx. \end{aligned}$$

Equivalently

$$\ln\left(\frac{n!}{n^{n+1/2}e^{-n}}\right) = \ln(n!) - (n+1/2)\ln(n) + n$$
$$= 1 + \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\left(\frac{1}{n^{2\ell-1}} - 1\right) + \frac{1}{2m+1}\int_1^n \frac{P_{2m+1}(x)}{x^{2m+1}}dx.$$

As in the previous section, the Wallis product formula gives $\lim_{n\to\infty} n!/(n^{n+1/2}e^{-n}) = \sqrt{2\pi}$. Extracting the constants form the right-hand side, by a simple computation, we then arrive at the **Euler-Maclaurin formula for the Stirling approximation**:

$$\ln\left(\frac{n!}{\sqrt{2\pi}\cdot n^{n+1/2}e^{-n}}\right) = \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{n^{2\ell-1}} - \frac{1}{2m+1}\int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}dx$$

We wish to estimate the (improper) integral on the left-hand side.[106] For $k \in \mathbb{N}_0$, we consider the integral

$$\begin{aligned}\int_k^{k+1} \frac{P_{2m+1}(x)}{x^{2m+1}}dx &= \int_k^{k+1} \frac{B_{2m+1}(x-[x])}{x^{2m+1}}dx = \int_0^1 \frac{B_{2m+1}(u)}{(u+k)^{2m+1}}du \\ &= \int_0^{1/2} \frac{B_{2m+1}(u)}{(u+k)^{2m+1}}du + \int_{1/2}^1 \frac{B_{2m+1}(u)}{(u+k)^{2m+1}}du \\ &= \int_0^{1/2} \frac{B_{2m+1}(u)}{(u+k)^{2m+1}}du - \int_0^{1/2} \frac{B_{2m+1}(v)}{(1-v+k)^{2m+1}}dv,\end{aligned}$$

where we made linear substitutions, and used the identity $B_{2m+1}(1-v) = -B_{2m+1}(v)$, $0 \le v \le 1$ (Section 4.12).

We now assume that $m \in \mathbb{N}$ is **odd**. We write the last difference of integrals as $a_k - b_k$, $k \in \mathbb{N}_0$. By Proposition 4.12.1 ($\mathbf{II}_m$), the numerators of the integrands are positive (on $(0, 1/2)$). Since the denominator of the integrands are increasing functions, we have $a_k > b_k > a_{k+1} > 0$, $k \in \mathbb{N}_0$, and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$.

The improper integral itself is thus an alternating sum

$$\int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}dx = a_n - b_n + a_{n+1} - b_{n+1} + \cdots = \sum_{k=n}^{\infty}(a_k - b_k).$$

Since $a_n - b_n < \sum_{k=n}^\infty (a_k - b_k) < a_n$, we obtain the estimate

$$\int_n^{n+1} \frac{P_{2m+1}(x)}{x^{2m+1}}dx < \int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}dx < \int_n^{n+1/2} \frac{P_{2m+1}(x)}{x^{2m+1}}dx$$

[106]Most authors consider the entire right-hand side as the "error term." Since the first sum is finite, and can be evaluated directly (at least for low values of m), we consider estimating the improper integral only.

Using the estimate for the Bernoulli polynomial in Section 4.13, we finally arrive at

$$0 < \frac{1}{2m+1}\int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}dx < \frac{|B_{2m}|}{2\pi}\int_0^{1/2}\frac{du}{(u+n)^{2m+1}}$$
$$= \frac{|B_{2m}|}{4\pi m}\left(\frac{1}{n^{2m}} - \frac{1}{(n+1/2)^{2m}}\right).$$

The case when $m \in \mathbb{N}$ is **even** is entirely analogous. We obtain

$$\frac{|B_{2m}|}{4\pi m}\left(\frac{1}{(n+1/2)^{2m}} - \frac{1}{n^{2m}}\right) < \frac{1}{2m+1}\int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}dx < 0.$$

The two cases (regardless the parity of m) can be summarized as

$$\frac{1}{2m+1}\left|\int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}dx\right| < \frac{|B_{2m}|}{4\pi m}\left(\frac{1}{n^{2m}} - \frac{1}{(n+1/2)^{2m}}\right).$$

For example, $m = 1$ gives

$$\left|\ln\left(\frac{n!}{\sqrt{2\pi}\, n^{n+1/2}e^{-n}}\right) - \frac{1}{12n}\right| < \frac{1}{24\pi}\frac{4n+1}{n^2(2n+1)^2}, \quad n \in \mathbb{N}.$$

Taking into account of the **sign** of the improper integral only (with respect to parity), as a simple byproduct, we obtain the following

$$\sum_{\ell=1}^{2m}\frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{n^{2\ell-1}} < \ln\left(\frac{n!}{\sqrt{2\pi}\cdot n^{n+1/2}e^{-n}}\right) < \sum_{\ell=1}^{2m+1}\frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{n^{2\ell-1}}.$$

As a final quest, and as a generalization of the previous example, we derive the Euler-Maclaurin refinement of the gamma function.

Example 4.15.8. In order to derive the Euler-Maclaurin formula for the gamma function, we need to consider the function $f(x,t) = \ln(t+x)$, where $1 \le x \in \mathbb{R}$, and $0 \le t \in \mathbb{R}$ is treated as a parameter. Differentiation gives

$$\frac{d^{2m+1}\ln(t+x)}{dx^{2m+1}} = \frac{(2m)!}{(t+x)^{2m+1}}, \quad m \in \mathbb{N}_0.$$

using this in the Euler-Maclaurin formula, after a simple integration and calculation, we arrive at the following

$$\begin{aligned}\sum_{k=1}^{n}\ln(t+k) &= (t+n+1/2)\ln(t+n) - (t+1/2)\ln(t+1) - n + 1\\ &+ \sum_{\ell=1}^{m}\frac{B_{2\ell}}{2\ell(2\ell-1)}\left(\frac{1}{(t+n)^{2\ell-1}} - \frac{1}{(t+1)^{2\ell-1}}\right)\\ &+ \frac{1}{2m+1}\int_1^n \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}\,dx, \quad -1 < t \in \mathbb{R},\ 2 \le n \in \mathbb{N}.\end{aligned}$$

To get any further, we now bring in Euler's original definition of the gamma function

$$\Gamma(t{+}1) = \lim_{n\to\infty} \frac{n!n^{t+1}}{(t+1)(t+2)\cdots(t+n+1)} = \lim_{n\to\infty} \frac{n!n^t}{(t+1)(t+2)\cdots(t+n)}, \ -1 < t \in \mathbb{R},$$

where we moved up the value of the parameter, and discarded $\lim_{n\to\infty} n/(t{+}n{+}1) = 1$. Taking the natural logarithms, we obtain

$$\ln\Gamma(t+1) = \lim_{n\to\infty}\left(\ln(n!) + t\ln(n) - \sum_{k=1}^{n}\ln(t+k)\right).$$

We replace the first and third terms in the limit on the right-hand side. For the first term, from Example 4.15.7 above, we have

$$\begin{aligned}\ln(n!) &= \ln\sqrt{2\pi} + (n+1/2)\ln(n) - n \\ &+ \sum_{\ell=1}^{m}\frac{B_{2\ell}}{2\ell(2\ell-1)n^{2\ell-1}} - \frac{1}{2m+1}\int_0^\infty \frac{P_{2m+1}(x)}{(x+n)^{2m+1}}\,dx,\end{aligned}$$

where we used periodicity of P and shifted the variable by n as

$$\int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+1}}\,dx = \int_0^\infty \frac{P_{2m+1}(x)}{(x+n)^{2m+1}}\,dx.$$

For the third term, we use the formula above. Substituting, after a short calculation in taking the limit as $n\to\infty$, we arrive at the following

$$\begin{aligned}\ln\Gamma(t+1) &= \ln\sqrt{2\pi} + (t+1/2)\ln(t+1) - (t+1) \\ &+ \sum_{\ell=1}^{m}\frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{(t+1)^{2\ell-1}} - \frac{1}{2m+1}\int_1^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}\,dx.\end{aligned}$$

In the calculation, we used the limit rule for indeterminate forms as

$$\begin{aligned}\lim_{n\to\infty}(t+n+1/2)\ln\left(\frac{n}{t+n}\right) &= \lim_{n\to\infty} n\ln\left(\frac{n}{t+n}\right) = \lim_{x\to\infty}\frac{\ln(x/(t+x))}{1/x} \\ &= \lim_{u\to\infty}\frac{(t+x)t/(x(t+x)^2)}{-1/x^2} = -\lim_{x\to\infty} tx/(t+x) = -t.\end{aligned}$$

We make two final adjustments. First, we add $\ln(t{+}1)$ to both sides. On the left-hand side this results in

$$\ln(t+1) + \ln\Gamma(t+1) = \ln\left((t+1)\Gamma(t+1)\right) = \ln\Gamma(t+2), \quad -1 < t \in \mathbb{R}.$$

Using this, we have

$$\begin{aligned}\ln\Gamma(t+2) &= \ln\sqrt{2\pi} + (t+3/2)\ln(t+1) - (t+1)\\ &+ \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{(t+1)^{2\ell-1}} - \frac{1}{2m+1}\int_1^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}\,dx.\end{aligned}$$

Second, we replace $t+1$ by t, and finally obtain the **Euler-Maclaurin formula for the gamma function**:

$$\begin{aligned}\ln\Gamma(t+1) &= \ln\sqrt{2\pi} + (t+1/2)\ln(t) - t\\ &+ \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{t^{2\ell-1}} - \frac{1}{2m+1}\int_0^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}\,dx, \quad 0 < t \in \mathbb{R}.\end{aligned}$$

(Notice that the lower limit in the last integral moved down to 0.)

Returning to the main line, the estimate of the (improper) integral on the right-hand side is entirely analogous to the process in Example 4.15.7. For $k \in \mathbb{N}_0$, we have

$$\int_k^{k+1} \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}\,dx = \int_0^{1/2} \frac{P_{2m+1}(u)}{(t+u+k)^{2m+1}}\,du - \int_0^{1/2} \frac{P_{2m+1}(v)}{(t+1-v+k)^{2m+1}}\,dv.$$

Assuming first that $m \in \mathbb{N}$ is odd, we write this as $a_k - b_k$, $k \in \mathbb{N}_0$. We then have $a_k > b_k > a_{k+1} > 0$, $k \in \mathbb{N}_0$, and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$; and the improper integral itself is an alternating sum

$$\int_0^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx = a_0 - b_0 + a_1 - b_1 + \cdots = \sum_{k=0}^\infty (a_k - b_k), \quad 0 < t \in \mathbb{R}.$$

Hence

$$\int_0^1 \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx < \int_0^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx < \int_0^{1/2} \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx.$$

Using the estimate for the Bernoulli polynomial in Section 4.15, we finally arrive at

$$\begin{aligned}0 < \frac{1}{2m+1}\int_0^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx &< \frac{|B_{2m}|}{2\pi}\int_0^{1/2} \frac{du}{(t+u)^{2m+1}}\\ &= \frac{|B_{2m}|}{4\pi m}\left(\frac{1}{t^{2m}} - \frac{1}{(t+1/2)^{2m}}\right), \quad 0 < t \in \mathbb{R}.\end{aligned}$$

The case when $m \in \mathbb{N}$ is **even** is entirely analogous. We obtain

$$\frac{|B_{2m}|}{4\pi m}\left(\frac{1}{(t+1/2)^{2m}} - \frac{1}{t^{2m}}\right) < \frac{1}{2m+1}\int_0^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx < 0.$$

The two cases (regardless the parity of m) can be summarized as

$$\frac{1}{2m+1}\left|\int_0^\infty \frac{P_{2m+1}(x)}{(t+x)^{2m+1}}dx\right| < \frac{|B_{2m}|}{4\pi m}\left(\frac{1}{t^{2m}} - \frac{1}{(t+1/2)^{2m}}\right).$$

The first case $m = 1$, for $0 < t \in \mathbb{R}$, gives

$$\left|\ln\Gamma(t+1) - \ln\sqrt{2\pi} - (t+1/2)\ln(t) + t - \frac{1}{12t}\right| < \frac{1}{24\pi}\frac{4t+1}{t^2(2t+1)^2}.$$

We close this section by a formula due to Darboux which is a common generalization of the Taylor formula and the Euler-Maclaurin formula.
We let p_n be a polynomial of degree n, $n \in \mathbb{N}_0$; and assume that $f : [0,\infty) \to \mathbb{R}$ is a function differentiable up to order $n+1$ with $f^{(n+1)}$ integrable on closed intervals in $[0,\infty)$. The **Darboux formula** is the following:

$$\begin{aligned} p_n^{(n)}(f(x) - f(c)) &= \sum_{k=1}^{n}(-1)^{k+1}(x-c)^k \left(p_n^{(n-k)}(1)f^{(k)}(x) - p_n^{(n-k)}(0)f^{(k)}(c)\right) \\ &+ (-1)^n(x-c)^{n+1}\int_0^1 p_n(t)f^{(n+1)}(c+t(x-c))\,dt, \quad c, x > 0. \end{aligned}$$

(Since p_n is a degree n polynomial, $p_n^{(n)}$ is constant.) Note that, using integration by parts, the last (nth) term in the sum cancels, and the Darboux formula takes the equivalent form

$$\begin{aligned} p_n^{(n)}(f(x) - f(c)) &= \sum_{k=1}^{n-1}(-1)^{k+1}(x-c)^k \left(p_n^{(n-k)}(1)f^{(k)}(x) - p_n^{(n-k)}(0)f^{(k)}(c)\right) \\ &+ (-1)^{n+1}(x-c)^{n+1}\int_0^1 p_n'(t)f^{(n)}(c+t(x-c))\,dt. \end{aligned}$$

For the proof, we first perform a reduction step. Setting $g(t) = f(c+t(x-c))$, the (first) Darboux formula simplifies as

$$\begin{aligned} p_n^{(n)}(g(1) - g(0)) &= \sum_{k=1}^{n}(-1)^{k+1}\left(p_n^{(n-k)}(1)g^{(k)}(1) - p_n^{(n-k)}(0)g^{(k)}(0)\right) \\ &+ (-1)^n\int_0^1 p_n(t)g^{(n+1)}(t)\,dt. \end{aligned}$$

For the proof of this, we first claim that

$$\frac{d}{dt}\left(\sum_{k=1}^{n}(-1)^k p_n^{(n-k)}g^{(k)}(t)\right) = -p_n^{(n)}g'(t) + (-1)^n p_n(t)g^{(n+1)}(t)$$

since the sum on the left-hand side is telescopic. Indeed, performing the differentation of the kth term of the sum

$$\frac{d}{dt}(-1)^k p_n^{(n-k)} g^{(k)}(t) = (-1)^k p_n^{(n-k+1)} g^{(k)}(t) + (-1)^k p_n^{(n-k)} g^{(k+1)}(t);$$

we see that the first term on the right-hand side becomes the opposite of the second by $k \mapsto k+1$; and the claim follows.
Once this holds, we integrate both sides over the interval $[0,1]$, and the Darboux formula follows.

As the first application of the Darboux formula, we set $p_n(t) = (t-1)^n$. Since $p_n^{(n-k)}(t) = n!/k! \cdot (t-1)^k$, $k = 1, \ldots, n$, a simple computation gives

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{(-1)^n}{n!}(x-c)^{n+1} \int_0^1 (t-1)^n f^{(n+1)}(c+t(x-c))\, dt.$$

For the integral remainder, we perform the substitution $u = c + t(x-c)$ (and hence $(x-u)^n = (1-t)^n(x-c)^n$ and $du = (x-c)dt$), and obtain

$$R_n(x) = \int_c^x \frac{f^{(n+1)}(u)}{n!}(x-u)^n\, du.$$

This is the Taylor remainder in Proposition 4.2.2. The Taylor formula follows.

As the second application of the Darboux formula, we set $p_n(t) = B_n(t)$, the nth Bernoulli polynomial, $n \in \mathbb{N}_0$. To evaluate the Bernoulli polynomials and their derivatives at 0 and 1, we need to recall two identities. First, we have $B_n'(t) = nB_{n-1}(t)$, $n \in \mathbb{N}$, so that

$$B_n^{(n-k)}(0) = \frac{n!}{k!} B_k(0) = \frac{n!}{k!} B_k, \quad k = 0, 1, \ldots, n.$$

In particular, we have $B_n^{(n)} = n!$ and $B_n^{(n-1)}(0) = -n!/2$.
Second, we have $B_n(t+1) = B_n(t) + nt^{n-1}$, $n \in \mathbb{N}$, and hence $B_n^{(n-k)}(t+1) = B_n^{(n-k)}(t) + n!/(k-1)! \cdot t^{k-1}$, $k = 1, \ldots, n$. This gives

$$B_n^{(n-k)}(1) = B_n^{(n-k)}(0) = \frac{n!}{k!} B_k, \quad k = 2, \ldots, n,$$

whereas

$$B_n^{(n-1)}(1) = B_n^{(n-1)}(0) + n! = -\frac{n!}{2} + n! = \frac{n!}{2}.$$

Using these, the second version of the Darboux formula, with the simplified substitution $g(t) = f(c + t(x - c))$, takes the form

$$\begin{aligned} g(1) - g(0) &= \frac{1}{2}\left(g'(1) + g'(0)\right) + \sum_{k=2}^{n-1}(-1)^{k+1}\frac{B_k}{k!}\left(g^{(k)}(1) - g^{(k)}(0)\right) \\ &+ \frac{(-1)^{n+1}}{(n-1)!}\int_0^1 B_{n-1}(t)g^{(n)}(t)\,dt, \end{aligned}$$

where we used $B_n'(t) = nB_{n-1}(t)$. We now set $n = 2m + 2$, $m \in \mathbb{N}_0$, and use the fact that all odd Bernoulli numbers vanish except $B_1 = -1/2$. We obtain

$$\begin{aligned} g(1) - g(0) &= \frac{1}{2}\left(g'(1) + g'(0)\right) - \sum_{\ell=1}^{m}\frac{B_{2\ell}}{(2\ell)!}\left(g^{(2\ell)}(1) - g^{(2\ell)}(0)\right) \\ &- \frac{1}{(2m+1)!}\int_0^1 B_{2m+1}(t)g^{(2m+2)}(t)\,dt. \end{aligned}$$

Recall that, up to this point, $c, x > 0$ were arbitrary. For $k \in \mathbb{N}$, we now set[107] $c = k$ and $x = k + 1$, and hence $g(t) = f(t + k)$. Playing everything back to f, this gives

$$\begin{aligned} f(k+1) - f(k) &= \frac{1}{2}\left(f'(k+1) + f'(k)\right) - \sum_{\ell=1}^{m}\frac{B_{2\ell}}{(2\ell)!}\left(f^{(2\ell)}(k+1) - f^{(2\ell)}(k)\right) \\ &- \frac{1}{(2m+1)!}\int_0^1 B_{2m+1}(t)f^{(2m+2)}(t+k)\,dt \\ &= \frac{1}{2}\left(f'(k+1) + f'(k)\right) - \sum_{\ell=1}^{m}\frac{B_{2\ell}}{(2\ell)!}\left(f^{(2\ell)}(k+1) - f^{(2\ell)}(k)\right) \\ &- \frac{1}{(2m+1)!}\int_k^{k+1} P_{2m+1}(t)f^{(2m+2)}(t)\,dt, \end{aligned}$$

where, in the last step, we had to replace the Bernoulli polynomial $B_{2m+1}(t)$ by its periodized $P_{2m+1}(t)$.

As a final substitution, we let $h = f'$. Using $\int_k^{k+1} h(t)\,dt = \int_k^{k+1} f'(t)\,dt = f(k+1) - f(k)$, and rearranging, we find

$$\begin{aligned} \int_k^{k+1} h(t)\,dt &= \frac{1}{2}\left(h(k+1) + h(k)\right) - \sum_{\ell=1}^{m}\frac{B_{2\ell}}{(2\ell)!}\left(h^{(2\ell-1)}(k+1) - h^{(2\ell-1)}(k)\right) \\ &- \frac{1}{(2m+1)!}\int_k^{k+1} P_{2m+1}(t)h^{(2m+1)}(t)\,dt. \end{aligned}$$

[107] Not to be confused with the previous index k.

Summing up with respect to $k = 1, \dots, n-1$, $n \in \mathbb{N}$, we obtain the Euler-Maclaurin formula

$$\begin{aligned}\int_1^n h(t)\,dt &= \sum_{k=1}^n h(k) - \frac{1}{2}\left(h(n)+h(1)\right) - \sum_{\ell=1}^m \frac{B_{2\ell}}{(2\ell)!}\left(h^{(2\ell-1)}(n) - h^{(2\ell-1)}(1)\right)\\ &- \frac{1}{(2m+1)!}\int_1^n P_{2m+1}(t)h^{(2m+1)}(t)\,dt.\end{aligned}$$

Example 4.15.9. Let $f : \mathbb{R} \to \mathbb{R}$ be odd and differentiable up to order $2n+1$, $n \in \mathbb{N}$, with $f^{(2n+1)}$ integrable on closed intervals. Then we have

$$f(x) = xf'(x) - \sum_{\ell=1}^n \frac{2^{2\ell}}{(2\ell)!}B_{2\ell}x^{2\ell}f^{(2\ell)}(x) + \frac{2^{2n}}{(2n)!}x^{2n+1}\int_0^1 B_{2n}(t)f^{(2n+1)}((2t-1)x)\,dt.$$

To derive this, we apply the Darboux formula for $c = -x$ and $p_n(t) = B_n(t)$. Replacing n by $2n$, and using that fact that the odd derivatives of f are even functions and the even derivatives of f are odd functions, by a computation similar to the one above, the example follows.

Exercises

1. Show that, for $f(x) = \ln(x)$, $1 \le x \in \mathbb{R}$, the first Euler-Maclaurin formula reduces to the identity

$$\frac{n!}{n^n} = \prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^k, \quad 2 \le n \in \mathbb{N}.$$

2.[108] Derive the following refinement of Example 4.15.5:

$$-\frac{1}{48n^3} < H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \gamma < -\frac{1}{48(n+1)^3}, \quad n \in \mathbb{N},$$

using the following steps: (a) Let $E_n = H_n - \ln(n + 1/2 + 1/(24n))$, $n \in \mathbb{N}$, and show that the sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n = E_n + 1/(48n^3)$, $n \in \mathbb{N}$, is decreasing with $\lim_{n\to\infty} a_n = \gamma$. For the difference $a_{n+1} - a_n$, consider the function $f : (0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x+1} - \ln\left(x + \frac{3}{2} + \frac{1}{24(x+1)}\right) + \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right) + \frac{1}{48(x+1)^3} - \frac{1}{48x^3}.$$

[108] See Negoi, T., *A faster convergence to Euler's constant*, The Math. Gazette, Vol. 83, No. 498 (Nov. 1999), 487-489.

Differentiate and obtain[109]

$$f(x) = \frac{2656x^6 + 10096x^5 + 15008x^4 + 10836x^3 + 3870x^2 + 652x + 37}{16x^4(x+1)^4(24x^2+60x+37)(24x^2+12x+1)} > 0, \quad x > 0.$$

Observe that f is strictly increasing, and hence negative, and therefore the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing. Finally, use $a_n > \gamma$, $n \in \mathbb{N}$, to conclude $E_n - \gamma = a_n - 1/(48n^3) - \gamma > -1/(48n^3)$, $n \in \mathbb{N}$, and hence we obtain the first inequality. (b) The treatment of the second inequality is analogous in considering the sequence $(b_n)_{n\in\mathbb{N}}$ given by $b_n = E_n + 1/(48(n+1)^3)$, $n \in \mathbb{N}$.

3. Derive the following version of the Euler-Maclaurin formula for the gamma function[110]

$$\ln\Gamma(t+1) = \ln\sqrt{2\pi} + (t+1/2)\ln(t) - t + \sum_{k=1}^{\infty}\int_0^{\infty}\frac{\sin(2k\pi x)}{k\pi}\frac{dx}{x+t}, \quad 0 < t \in \mathbb{R}.$$

Solution: Use the Fourier series expansion of P_1 in Section 4.14 in the Euler-Maclauring formula for the gamma function for $m = 0$.

4. Derive an Euler-Maclaurin formula for the function $f(x) = x \cdot \ln(x)$, $1 \le x \in \mathbb{R}$.

5. Let $f : [1, \infty) \to \mathbb{R}$ be a positive, decreasing, and continuous function with $\lim_{x\to\infty} f(x) = 0$. Show that the limit

$$\lim_{x\to\infty}\left(\sum_{k=1}^{n} f(k) - \int_1^k f(x)\,dx\right)$$

exists.

4.16 The Binet Formulas for the Gamma Function

In the Euler-Maclaurin formula for the gamma function, the last two "error terms" (the sum with the Bernoulli coefficients and the improper integral) can be more compactly expressed by single (exponential) integrals in two formulas due to Binet[111] The **(first) Binet formula for the gamma function** is

$$\ln\Gamma(t+1) = \ln\sqrt{2\pi} + \left(t+\frac{1}{2}\right)\ln(t) - t + \int_0^{\infty}\left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right)\frac{e^{-tx}}{x}\,dx, \quad t > 0.$$

[109] A computer algebra system is recommended here.
[110] Attributed to Bourguet by Stieltjes; see Journal de Math. v. p. 432.
[111] Journal de l'École Polytechnique, XVI. (1839) 123-143.

Remark 1. Since $\Gamma(t+1) = t\Gamma(t)$, this formula is often written in the equivalent form

$$\ln\Gamma(t) = \ln\sqrt{2\pi} + (t-1/2)\ln(t) - t + \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x-1}\right)\frac{e^{-tx}}{x}\,dx, \quad t>0.$$

In this section we give an elementary proof of this formula.[112]

Remark 2. As in Section 4.15, the expression in the parentheses in the improper integral can be written in terms of the Bernoulli numbers as

$$\frac{1}{x}\left(-1 + \frac{x}{2} + \frac{x}{e^x-1}\right) = \sum_{\ell=1}^{\infty} \frac{x^{2\ell-1}}{(2\ell)!}B_{2\ell}.$$

(We elaborate on this in Exercise 2 at the end of this section.) In particular, it follows that

$$\sup_{x>0}\left|\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x-1}\right|\frac{1}{x} = C < \infty.$$

Hence, we obtain the estimate

$$\left|\ln\Gamma(t) - \ln\sqrt{2\pi} - (t-1/2)\ln(t) + t\right| \le C\int_0^\infty e^{-tx}\,dx = \frac{C}{t}, \quad t>0.$$

This allows to estimate $\Gamma(t)$ for large $t>0$ with increasing accuracy.

The Binet formula above can be written in “Stirling form” as

$$\Gamma(t+1) = \left(\frac{t}{e}\right)^t \sqrt{2\pi t}e^{F(t)},$$

where

$$F(t) = \int_0^\infty \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right)\frac{e^{-tx}}{x}\,dx.$$

Note that, since $\lim_{t\to\infty} F(t) = 0$, this immediately gives the Stirling approximation of the Gamma function.
Turning to the proof, we write start with the definition of the gamma function

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x}\,dx = t^{t+1}\int_0^\infty x^t e^{-tx}\,dx, \quad t>0,$$

[112] We follow here Sasvári, Z., *An elementary proof of Binet's formula for the gamma function*, Amer. Math. Monthly, Vol. 106, No. 2 (February 1999) 156-158.

where we replaced the variable x by tx. We write this in "Stirling form" as

$$\Gamma(t+1) = \left(\frac{t}{e}\right)^t \sqrt{2\pi t} e^{G(t)},$$

where

$$e^{G(t)} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{t} \left(xe^{1-x}\right)^t \, dx.$$

For the proof we need to show

$$F(t) = G(t), \quad t > 0.$$

Lemma 1. *We have*

$$F(t) - F(t+1) = G(t) - G(t+1) = \left(t + \frac{1}{2}\right) \ln\left(1 + \frac{1}{t}\right) - 1, \quad t > 0.$$

PROOF. First, we have

$$t + 1 = \frac{\Gamma(t+2)}{\Gamma(t+1)} = \frac{((t+1)/e)^{t+1} \sqrt{2\pi(t+1)} e^{G(t+1)}}{(t/e)^t \sqrt{2\pi t} e^{G(t)}},$$

where we used $\Gamma(t+2) = (t+1)\Gamma(t+1)$. This simplifies to

$$e\left(\frac{t}{t+1}\right)^{t+1/2} = e^{G(t+1)-G(t)}.$$

Taking the natural logarithm of both sides and rearranging, we obtain

$$G(t) - G(t+1) = \left(t + \frac{1}{2}\right) \ln\left(1 + \frac{1}{t}\right) - 1.$$

Second, we take derivatives and calculate

$$\begin{aligned} F'(t) - F'(t+1) &= \frac{d}{dt} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \frac{e^{-tx} - e^{-(t+1)x}}{x} \, dx \\ &= -\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \left(e^{-tx} - e^{-(t+1)x}\right) dx \\ &= \int_0^\infty \frac{e^{tx} - e^{-(t+1)x}}{x} \, dx - \int_0^\infty \frac{e^{tx} + e^{-(t+1)x}}{2} \, dx, \end{aligned}$$

where the interchange of the differentiation and integration is allowed, and the last equality follows using the identity

$$\left(\frac{1}{e^x-1}+\frac{1}{2}\right)(1-e^{-x})=\frac{1+e^{-x}}{2}.$$

Recalling Example 4.4.1, we now finish the computation and obtain

$$F'(t)-F'(t+1)=\ln\left(1+\frac{1}{t}\right)-\frac{1}{2}\left(\frac{1}{t}+\frac{1}{t+1}\right)=\left(\left(t+\frac{1}{2}\right)\ln\left(1+\frac{1}{t}\right)-1\right)'.$$

Up to this point, we obtained that the formula to be proved

$$F(t)-F(t+1)=\left(t+\frac{1}{2}\right)\ln\left(1+\frac{1}{t}\right)-1$$

holds up to an additive constant. On the other hand, this constant must be zero as both sides tend to zero as $t\to\infty$. This is obvious for the left-hand side since $\lim_{t\to\infty}F(t)=\lim_{t\to\infty}F(t+1)=0$, and, for the right-hand side, it follows by an easy application of Proposition 2.2.5 as

$$\lim_{t\to\infty}\left(t+\frac{1}{2}\right)\ln\left(1+\frac{1}{t}\right)=\lim_{t\to\infty}\left(t+\frac{1}{2}\right)^2\frac{t}{t+1}\frac{1}{t^2}=1.$$

The lemma follows.

Lemma 2. *We have*

$$F\left(\frac{1}{2}\right)=G\left(\frac{1}{2}\right)=\frac{1}{2}-\frac{1}{2}\ln(2).$$

Proof. First, by the definition of G (evaluated at $t=1/2$ and using $\Gamma(3/2)=\sqrt{\pi}/2$, we have $\sqrt{e/2}=e^{G(1/2)}$, and hence $G(1/2)=(1-\ln(2))/2$.
Second, to evaluate $F(1/2)$, we start with[113]

$$F(1)=\int_0^\infty\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)\frac{e^{-x}}{x}\,dx=\int_0^\infty\left(\frac{1}{e^{x/2}-1}-\frac{2}{x}+\frac{1}{2}\right)\frac{e^{-x/2}}{x}\,dx.$$

[113]This is an idea due to A. Pringsheim, Math. Ann. XXXI. (1988) p. 473.

We use this to calculate

$$
\begin{aligned}
F(1/2) &= (F(1/2) - F(1)) - F(1) \\
&= \int_0^\infty \left(\frac{1}{x} + \frac{1}{e^x - 1} - \frac{1}{e^{x/2} - 1}\right) \frac{e^{-x/2}}{x}\, dx + \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \frac{e^{-x}}{x}\, dx \\
&= \int_0^\infty \left(\frac{e^{-x/2}}{x} - \frac{1}{e^x - 1}\right) \frac{dx}{x} + \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \frac{e^{-x}}{x}\, dx \\
&= \int_0^\infty \left(\frac{e^{-x/2} - e^{-x}}{x} - \frac{e^{-x}}{2}\right) \frac{dx}{x} \\
&= -\int_0^\infty \left(\frac{d}{dx}\left(\frac{e^{-x/2} - e^{-x}}{x}\right) + \frac{e^{-x/2} - e^{-x}}{2x}\right) dx \\
&= \lim_{x\to 0} \frac{e^{-x/2} - e^{-x}}{x} - \int_0^\infty \frac{e^{-x/2} - e^{-x}}{2x}\, dx = \frac{1}{2} - \frac{1}{2}\ln(2),
\end{aligned}
$$

where we used Example 4.4.1 (with $a = t = 1/2$). The lemma follows.

Proof of the First Binet Formula Letting $n \in \mathbb{N}_0$, and applying Lemma 1 to $x + k$, $k = 1, \ldots, n-1$, we obtain

$$F(t) - F(t+n) = G(t) - G(t+n), \quad t > 0,$$

as both differences are telescopic. We now note as above that $\lim_{n\to\infty} F(t+n) = 0$, $t > 0$. Hence, we get

$$F(t) = G(t) - g(t), \quad t > 0,$$

where $g : (0, \infty) \to \mathbb{R}$ is given by $g(t) = \lim_{n\to\infty} G(t+n)$, $t > 0$.

$$g(t) = \lim_{n\to\infty} G(t+n) = \lim_{n\to\infty} \ln \int_0^\infty \sqrt{t+n}\left(xe^{1-x}\right)^{t+n}\, dx - \ln\sqrt{2\pi}.$$

First, the function g is periodic with period 1:

$$
\begin{aligned}
g(t) - g(t+1) &= \lim_{n\to\infty} (G(t+n) - G(t+n+1)) \\
&= \lim_{n\to\infty} \left(t + n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{t+n}\right) - 1 = 0.
\end{aligned}
$$

Second, we claim that g is decreasing. Indeed, for $0 \le s < t$, and $0 \le a \le 1$, we have

$$\sqrt{t+n}\, a^{t+n} - \sqrt{s+n}\, a^{s+n} \le \sqrt{t+n}\, a^{s+n} - \sqrt{s+n}\, a^{s+n} = (\sqrt{t+n} - \sqrt{s+n})\, a, \ n \in \mathbb{N}.$$

Applying this to $a = xe^{1-x}$, $x \geq 0$, we calculate

$$\begin{aligned} e^{G(t+n)} - e^{G(s+n)} &\leq \frac{1}{\sqrt{2\pi}}(\sqrt{t+n} - \sqrt{s+n})) \int_0^\infty xe^{1-x}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{t-s}{\sqrt{t+n}+\sqrt{s+n}} \int_0^\infty xe^{1-x}\,dx. \end{aligned}$$

Letting $n \to \infty$, we obtain $e^{g(t)} - e^{g(s)} \leq 0$, and hence $g(t) \leq g(s)$. The claim follows. We conclude that g must be constant. On the other hand, the equality above connecting F and G at $t = 1/2$ gives

$$F(1/2) = G(1/2) - g(1/2).$$

By Lemma 2, $F(1/2) = G(1/2)$ so that $g = g(1/2) = 0$. Thus, $F(t) = G(t)$, $t > 0$, and the first Binet formula for the gamma function follows.

The **second Binet formula for the gamma function** is

$$\ln\Gamma(t+1) = \ln\sqrt{2\pi} + (t+1/2)\ln(t) - t + 2\int_0^\infty \frac{\arctan(x/t)}{e^{2\pi x}-1}\,dx, \quad t > 0.$$

The proof is preceded by the following:

Example 4.16.1. We have

$$\int_0^\infty \frac{\sin(ux)}{e^{2\pi x}-1}\,dx = -\frac{1}{2}\left(\frac{1}{u} - \frac{\coth(u/2)}{2}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}\right), \quad u > 0.$$

Indeed, using the infinite geometric series formula, we calculate

$$\begin{aligned} \int_0^\infty \frac{\sin(ux)}{e^{2\pi x}-1}\,dx &= \int_0^\infty \sin(ux)\frac{e^{-2\pi x}}{1-e^{-2\pi x}}\,dx = \int_0^\infty \sin(ux)\sum_{k=1}^\infty e^{-2\pi kx}\,dx \\ &= \sum_{k=1}^\infty \int_0^\infty \sin(ux)e^{-2\pi kx}\,dx = \sum_{k=1}^\infty \frac{u}{(2\pi k)^2+u^2} = \frac{1}{2\pi}\sum_{k=1}^\infty \frac{u/(2\pi)}{k^2+(u/(2\pi))^2} \end{aligned}$$

where, in the last but one step, we also used the formula

$$\int_0^\infty \sin(ux)e^{-ax}\,dx = \frac{u}{a^2+u^2}, \quad a > 0,$$

via integration by parts (twice). For the last infinite sum, we now use the expansion of the hyperbolic cotangent function in Exercise 4(c) at the end of Section 4.8:

$$\frac{\pi\coth(\pi t)}{2} - \frac{1}{2t} = \sum_{k=1}^\infty \frac{t}{k^2+t^2}.$$

After scaling, and simplifying, we obtain

$$\int_0^\infty \frac{\sin(ux)}{e^{2\pi x}-1}\,dx = \frac{\coth(u/2)}{4} - \frac{1}{2u}.$$

The first equality of the example follows. For the second, we use the definition of the hyperbolic cotangent function:

$$\coth\left(\frac{u}{2}\right) = \frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}} = \frac{e^u+1}{e^u-1} = 1 + \frac{2}{e^u-1},$$

and the second equality also follows.

We now turn to the proof of the second Binet formula. We begin with the observation that the arctangent function in the integrand in this formula appeared in an improper integral in Example 4.7.1. We write this, for $t, x > 0$, as

$$\arctan\left(\frac{x}{t}\right) = \frac{\pi}{2} - \arctan\left(\frac{t}{x}\right) = \int_0^\infty e^{-tv/x}\frac{\sin(v)}{v}\,dv = \int_0^\infty \sin(ux)\frac{e^{-tu}}{u}\,du,$$

where, in the last step, we performed the substitution $v = ux$. We substitute this into the improper integral $I(t)$, say, in the second Binet formula, and begin to calculate as

$$\begin{aligned} I(t) &= 2\int_0^\infty \frac{\arctan(x/t)}{e^{2\pi x}-1}\,dx = 2\int_0^\infty \frac{1}{e^{2\pi x}-1}\int_0^\infty \sin(ux)\frac{e^{-tu}}{u}\,du\,dx \\ &= 2\int_0^\infty \int_0^\infty \frac{\sin(ux)}{e^{2\pi x}-1}\,dx\frac{e^{-tu}}{u}\,du \\ &= \int_0^\infty \left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u-1}\right)\frac{e^{-tu}}{u}\,du, \end{aligned}$$

where we used Fubini's theorem to interchange the improper integrals (Section 4.5). This, however, is the improper integral in the first Binet formula. Hence the second Binet formula follows from the first.

To close this section, we note yet another formula discovered by **Kummer**. It is usually written as

$$\ln\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) = -\frac{1}{2}\ln(2\sin(\pi a)) + \frac{1}{2}(\gamma+\ln(2\pi))(1-2a) + \frac{1}{\pi}\sum_{n=1}^\infty \frac{\ln(n)}{n}\sin(2n\pi a), \quad 0 < a < 1.$$

We will give a proof of Kummer's formula in Section 4.20 as a more or less direct consequence of the Hurwitz formula.

Exercises

1. Show that

$$\frac{\Gamma'(t)}{\Gamma(t)} = \ln(t) - \frac{1}{2t} - 2\int_0^\infty \frac{x\,dx}{(t^2+x^2)(e^{2\pi x}-1)}, \quad t > 0.$$

2. For $m \in \mathbb{N}$, approximate the expression in parentheses in the improper integral of the first Binet formula by $\sum_{\ell=1}^m x^{2\ell-1} B_{2\ell}/(2\ell)!$. (See the (second) remark after the statement of the first Binet formula.) Use the definition of the gamma function and its basic properties to show that the corresponding improper integral is

$$\int_0^\infty \sum_{\ell=1}^m \frac{x^{2\ell-1}}{(2\ell)!} B_{2\ell} \frac{e^{-tx}}{x}\,dx = \sum_{\ell=1}^m \frac{B_{2\ell}}{2\ell(2\ell-1)} \frac{1}{t^{2\ell-1}}.$$

in agreement with the Euler-Maclaurin formula for $\Gamma(t+1)$ in Example 4.15.8.

4.17 The Riemann Zeta Function

We introduce the **Riemann zeta function** $\zeta : (1,\infty) \to \mathbb{R}$ by the **Dirichlet series**:[114]

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad 1 < s \in \mathbb{R}.$$

Note that the sum converges for the stated parameter values since

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} < 1 + \int_1^\infty \frac{dx}{x^s} = 1 + \frac{1}{s-1}, \quad 1 < s \in \mathbb{R}.$$

The connection of the zeta function to number theory is given by the following famous formula of Euler:[115]

$$\zeta(s) = \prod_{p\in\Pi} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad 1 < s \in \mathbb{R},$$

[114] A Dirichlet series is a series of the form $\sum_{n=1}^\infty a_n/n^s$, where $(a_n)_{n\in\mathbb{N}}$ is a sequence. Clearly, if this sequence is bounded then the series converges absolutely for $s > 1$.

[115] This is the starting point of Riemann's original paper *Ueber die Anzahl der Primzahlen unter einer gegebener Grösse*, Monatsberichte der Berliner Akademie, Novermber, 1859.

where Π denotes the set[116] of all prime numbers.

For the proof of Euler's formula, we use the fundamental theorem of arithmetic which asserts that any natural number $2 \le n \in \mathbb{N}$ can be written as $n = \prod_{p\in\Pi} p^{a_p}$, where $a_p \in \mathbb{N}_0$ is zero except for finitely many primes $p \in \Pi$.
To derive Euler's formula, for fixed $q \in \Pi$ and $m \in \mathbb{N}$, we let

$$N_{q,m} = \left\{ 2 \le n \in \mathbb{N} \mid n = \prod_{p\in\Pi} p^{a_p},\ p \le q,\ a_p \le m \right\}.$$

We then have the finite sum

$$\sum_{n\in N_{q,m}} \frac{1}{n^s} = \prod_{\substack{p\le q\\ p\in\Pi}} \sum_{a=0}^{m} \frac{1}{p^{as}} = \prod_{\substack{p\le q\\ p\in\Pi}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{ms}}\right).$$

Letting $m \to \infty$, the infinite geometric series formula gives

$$\sum_{n\in N_q} \frac{1}{n^s} = \prod_{\substack{p\le q\\ p\in\Pi}} \sum_{a=0}^{\infty} \frac{1}{p^{as}} = \prod_{\substack{p\le q\\ p\in\Pi}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $N_q = \cup_{m\in\mathbb{N}} N_{q,m}$, the set of all natural numbers $2 \le n \in \mathbb{N}$ that can be written as a product of primes $p \le q$. Finally, since the natural numbers $2 \le n \in \mathbb{N}$ for which $n \le q$ automatically belong to N_q, we have

$$\left| \zeta(s) - \prod_{\substack{p\le q\\ p\in\Pi}} \left(1 - \frac{1}{p^s}\right)^{-1} \right| \le \frac{1}{(q+1)^s} + \frac{1}{(q+2)^s} + \cdots$$

where the limit of the last sum as $q \to \infty$ is zero due to the condition $s > 1$. Euler's formula follows.

It follows from the above that Euler's formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\Pi} \left(1 - \frac{1}{p^s}\right)^{-1}$$

is an analytic reformulation of the fundamental theorem of arithmentic.

[116]Since the product is absolutely convergent, the order in which the prime numbers are listed is irrelevant.

There are several consequences of this formula. First, for $s = 1$ Euler's formula specializes to

$$\prod_{p\in\Pi}\left(\frac{p}{p-1}\right) = \zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

showing the infinitude of primes.
Second, taking the natural logarithm, we obtain the equivalent form of the convergent series

$$\ln(\zeta(s)) = -\sum_{p\in\Pi}^{\infty}\ln\left(1-\frac{1}{p^s}\right), \quad 1 < s \in \mathbb{R}.$$

Using the Taylor series $\ln(1-x) = -\sum_{m=1}^{\infty} x^m/m$, $|x| < 1$, the right-hand side can be written as[117]

$$\ln(\zeta(s)) = \sum_{p\in\Pi}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mp^{ms}} = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\ln(n)}\frac{1}{n^s}, \quad 1 < s \in \mathbb{R},$$

where $\Lambda : (0,\infty) \to \mathbb{R}$ is the **von Mangoldt** function defined by

$$\Lambda(n) = \begin{cases} \ln(p) & \text{if } n = p^m,\ p \in \Pi,\ m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

We now introduce the **prime-counting function** $\pi : (0,\infty) \to \mathbb{N}_0$ by

$$\pi(x) = \sum_{\substack{p\leq x\\ p\in\Pi}} 1, \quad 0 < x \in \mathbb{R}.$$

[117] A comprehensive account of the number theoretical aspects of the Riemann zeta function is in Titmarsch, E.C., *The Theory of the Riemann Zeta Function*, 2nd edition, Oxford, 1986.

With this, for $1 < s \in \mathbb{R}$, we calculate

$$\begin{aligned}
\ln(\zeta(s)) &= -\sum_{n=2}^{\infty} (\pi(n) - \pi(n-1)) \ln\left(1 - \frac{1}{n^s}\right) \\
&= -\sum_{n=2}^{\infty} \pi(n) \left(\ln\left(1 - \frac{1}{n^s}\right) - \ln\left(1 - \frac{1}{(n+1)^s}\right)\right) \\
&= -\sum_{n=2}^{\infty} \pi(n) \left[\ln\left(1 - \frac{1}{x^s}\right)\right]_n^{n+1} \\
&= \sum_{n=2}^{\infty} \pi(n) \int_n^{n+1} \frac{s}{x(x^s-1)}\, dx \\
&= s \int_2^{\infty} \frac{\pi(x)}{x(x^s-1)}\, dx.
\end{aligned}$$

Note that the initial rearrangement of the series is permitted as it effects only consecutive terms, and the corresponding partial sums differ only by a null sequence since $\pi(n) \leq n$, $2 \leq n \in \mathbb{N}$, and[118]

$$\frac{1}{n^s} \leq -\ln\left(1 - \frac{1}{n^s}\right) \leq \frac{1}{n^s} \frac{1}{1 - 1/n^s}, \quad 1 < s \in \mathbb{R}.$$

In addition, we also employed the fundamental theorem of calculus to the effect

$$\left(\ln\left(1 + \frac{1}{x^s}\right)\right)' = \frac{sx^{-s-1}}{1 - x^{-s}} = \frac{s}{x(x^s-1)}.$$

Remark. The **prime number theorem** says that

$$\pi(x) \sim \frac{x}{\ln(x)}, \quad \text{as } x \to \infty.$$

Differentiating the log equivalent of Euler's formula (with respect to s), we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = (\ln(\zeta(s)))' = -\frac{d}{ds} \sum_{p \in \Pi} \ln\left(1 - p^{-s}\right) = \sum_{p \in \Pi} \frac{p^{-s} \ln(p)}{1 - p^{-s}} = \sum_{p \in \Pi} \sum_{m=1}^{\infty} \frac{\ln(p)}{p^{ms}},$$

[118]See the fundamental inequality for the natural logarithm in Section 10.3 in *Elements of Mathematics - History and Foundations.*

where the interchange of differentiation and the summation is allowed because the sum $\sum_{n=2}^{\infty} \ln(n)/(n^s - 1)$, $1 < s \in \mathbb{R}$, is uniformly convergent on closed intervals in $(1, \infty)$ (Proposition 1.3.10). The last double sum can be written as the Dirichlet sum

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad 1 < s \in \mathbb{R}.$$

There is an important link between the zeta and gamma functions which allows to express the zeta function in terms of an improper integral:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx, \quad 1 < s \in \mathbb{R}.$$

We call this the **zeta gamma relation**.
To derive this, we start with the Gamma function

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx = n^s \int_0^{\infty} u^{s-1} e^{-nu}\, du, \quad n \in \mathbb{N},$$

where we performed the substitution $x = nu$ with fixed $n \in \mathbb{N}$. Dividing and summing up (using the infinite geometric series formula), we obtain[119]

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} u^{s-1} e^{-nu}\, du = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{n=1}^{\infty} e^{-nu}\, du \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \frac{e^{-u}}{1 - e^{-u}}\, du = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1}}{e^u - 1}\, du, \quad 1 < s \in \mathbb{R}, \end{aligned}$$

where the interchange of the infinite sum and the improper integral is clearly allowed. The formula follows.

Remark. Although we will do this in a more general and systematic way shortly, this formula can be used to give a direct proof that the zeta function is analytic on $(1, \infty)$. See Exercise 1 at the end of this section.

The next example expresses the Bernoulli numbers in terms of an improper integral.

Example 4.17.1. We have[120]

$$\int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x} - 1}\, dx = (-1)^{n+1} \frac{B_{2n}}{4n}, \quad n \in \mathbb{N}.$$

[119] Here and below, we follow Riemann's original paper *Ueber die Anzahl der Primzahlen unter einer gegebener Grösse*, Monatsberichte der Berliner Akademie, Novermber, 1859.
[120] For a proof using contour integration on the complex plane, see Carda, Monatshefte für Math. und Phys. v. (1894) 321-324.

Indeed, the zeta gamma relation derived above specializes to

$$\zeta(2n) = \frac{1}{(2n-1)!}\int_0^\infty \frac{x^{2n-1}}{e^x-1}\,dx = \frac{(2\pi)^{2n}}{(2n-1)!}\int_0^\infty \frac{x^{2n-1}}{e^{2\pi x}-1}\,dx,$$

where we performed a linear change of variables. Euler's summation formula (Section 4.13) gives the value of $\zeta(2n)$ in terms of the Bernoulli number B_{2n}. The example follows.

We make a short detour here, and give a more transparent interpretation of the improper integral in the Euler-Maclaurin formula for the gamma function (Section 4.15) using the second Binet formula (Section 4.16). Recall that the integral in question is

$$\int_0^\infty \frac{\arctan(x/t)}{e^{2\pi x}-1}\,dx, \quad t>0.$$

We first use the finite geometric series formula as

$$\begin{aligned}\arctan(x) &= \int_0^x \frac{du}{1+u^2} = \int_0^x \sum_{\ell=1}^m (-1)^{\ell-1}u^{2(\ell-1)}\,du + (-1)^m\int_0^x \frac{u^{2m}}{1+u^2}\,du \\ &= \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{2\ell-1}x^{2\ell-1} + (-1)^m\int_0^x \frac{u^{2m}}{1+u^2}\,du.\end{aligned}$$

Replacing x by x/t, $t>0$, and changing the variable in the integral, we obtain

$$\arctan\left(\frac{x}{t}\right) = \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{2\ell-1}\frac{x^{2\ell-1}}{t^{2\ell-1}} + \frac{(-1)^m}{t^{2m-1}}\int_0^x \frac{u^{2m}}{t^2+u^2}\,du.$$

We now substitute this into the improper integral of the second Binet formula and calculate

$$\begin{aligned}\int_0^\infty \frac{\arctan(x/t)}{e^{2\pi x}-1}\,dx &= \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{2\ell-1}\frac{1}{t^{2\ell-1}}\int_0^\infty \frac{x^{2\ell-1}}{e^{2\pi x}-1}\,dx \\ &\qquad + \frac{(-1)^m}{t^{2m-1}}\int_0^\infty\int_0^x \frac{u^{2m}}{t^2+u^2}\,du\frac{dx}{e^{2\pi x}-1} \\ &= \sum_{\ell=1}^m \frac{B_{2\ell}}{4\ell(2\ell-1)}\frac{1}{t^{2\ell-1}} + \frac{(-1)^m}{t^{2m-1}}\int_0^\infty\int_0^x \frac{u^{2m}}{t^2+u^2}\,du\frac{dx}{e^{2\pi x}-1}.\end{aligned}$$

With this, the second Binet formula takes the form

$$\begin{aligned}\ln\Gamma(t+1) &= \ln\sqrt{2\pi} + (t+1/2)\ln(t) - t + \sum_{\ell=1}^m \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{t^{2\ell-1}} \\ &\qquad + 2\frac{(-1)^m}{t^{2m-1}}\int_0^\infty\int_0^x \frac{u^{2m}}{t^2+u^2}\,du\frac{dx}{e^{2\pi x}-1},\end{aligned}$$

where we used Example 4.17.1 above. We recognize here that the finite sum involving the Bernoulli numbers is the same as in the Euler-Maclaurin formula for the gamma function. Although the double integral here looks more complex than the improper integral in that formula; in fact, it is much easier to estimate. We have

$$\left|\int_0^\infty \int_0^x \frac{u^{2m}}{t^2+u^2}\,du \frac{dx}{e^{2\pi x}-1}\right| \le \frac{1}{t^2}\left|\int_0^\infty \int_0^x u^{2m}\,du \frac{dx}{e^{2\pi x}-1}\right|$$
$$= \frac{1}{2m+1}\frac{1}{t^2}\int_0^\infty \frac{x^{2m+1}}{e^{2\pi x}-1}\,dx = \frac{1}{t^2}\frac{|B_{2m+2}|}{4(m+1)(2m+1)},$$

where, one again, we used Example 4.17.1. Summarizing, we have

$$\left|\ln\Gamma(t+1) - \ln\sqrt{2\pi} - \left(t+\frac{1}{2}\right)\ln(t) + t - \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{t^{2\ell-1}}\right|$$
$$\le \frac{|B_{2m+2}|}{2(m+1)(2m+1)}\frac{1}{t^{2m+1}}.$$

The upper bound is the absolute value of the next ($m+1$st term) of the sum. Hence this estimate is equivalent to the statement that the expression

$$\ln\Gamma(t+1) - \ln\sqrt{2\pi} - \left(t+\frac{1}{2}\right)\ln(t) + t$$

is always **between** the sums

$$\sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{t^{2\ell-1}} \quad \text{and} \quad \sum_{\ell=1}^{m+1} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{t^{2\ell-1}}.$$

Finally, it is instructive to compare this with the estimate obtained in Section 4.15:

$$\left|\ln\Gamma(t+1) - \ln\sqrt{2\pi} - \left(t+\frac{1}{2}\right)\ln(t) + t - \sum_{\ell=1}^{m} \frac{B_{2\ell}}{2\ell(2\ell-1)}\frac{1}{t^{2\ell-1}}\right|$$
$$\le \frac{|B_{2m}|}{4\pi m}\left(\frac{1}{t^{2m}} - \frac{1}{(t+1/2)^{2m}}\right).$$

The ratio of the Euler-Maclaurin and Binet upper bounds above is

$$\frac{t}{m}\left(1 - \frac{1}{(1+1/(2t))^{2m}}\right)\left|\frac{B_{2m}}{B_{2m+2}}\right|\frac{(m+1)(2m+1)}{2\pi}$$
$$\sim \pi\frac{t}{m}\left(1 - \frac{1}{(1+1/(2t))^{2m}}\right) \quad \text{as } m\to\infty,$$

where we used the asymptotics for the Bernoulli numbers in Section 4.13. On the other hand, for fixed $m \in \mathbb{N}$, we have

$$\lim_{t\to\infty} \frac{t}{m}\left(1 - \frac{1}{(1+1/(2t))^{2m}}\right) = 1.$$

Both estimates give excellent approximations; for example, for $t = 10$ and $m = 1, 2, 3, 4, 5$, the Euler-Maclaurin and Binet estimates are given in pairs below:

$$\begin{array}{ll} m=1 & [1.233059838 \cdot 10^{-5}, 2.777777778 \cdot 10^{-6}] \\ m=2 & [2.351481461 \cdot 10^{-8}, 7.936507936 \cdot 10^{-9}] \\ m=3 & [1.602820401 \cdot 10^{-10}, 5.952380952 \cdot 10^{-11}] \\ m=4 & [2.143025539 \cdot 10^{-12}, 8.417508418 \cdot 10^{-13}] \\ m=5 & [4.655122283 \cdot 10^{-14}, 1.917526918 \cdot 10^{-14}]. \end{array}$$

We now return to the main line. The Dirichlet sum defining the zeta function diverges for $s \le 1$. The simplest idea to extend it to $0 < s < 1$ is to consider the **alternating zeta function**[121] $\zeta_a : (0, \infty) \to \mathbb{R}$ defined by

$$\zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \quad 0 < s \in \mathbb{R}.$$

This sum converges[122] for all $s > 0$ by the alternating series test. Alternatively, this can be seen directly as

$$\zeta_a(s) = \sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s}\right) = \sum_{n=1}^{\infty}\int_{2n-1}^{2n} \frac{s}{x^{s+1}}\,dx \le s\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{s+1}}$$

which converges for $s > 0$.
Moreover, for $s > 1$, we have

$$\zeta(s) - \zeta_a(s) = \sum_{n=1}^{\infty} \frac{1-(-1)^{n+1}}{n^s} = \sum_{n=1}^{\infty}\frac{2}{(2n)^s} = \frac{1}{2^{s-1}}\zeta(s),$$

and hence

$$\zeta(s) = \frac{1}{1-2^{1-s}}\zeta_a(s), \quad s > 1.$$

[121] Also called the Dirichlet eta function. Note that this function was studied by Euler in 1749.

[122] Not absolutely, but uniformly on closed subintervals of $(0, \infty)$. It is a basic fact (that we will not show here) that the sum of the series is an analytic function.

This formula allows us to extend the zeta function $\zeta(s)$ to $s > 0$, $s \neq 1$.

As an immediate byproduct, we have

$$\lim_{s\to 1}(s-1)\zeta(s) = \lim_{s\to 1}\frac{(s-1)\zeta_a(s)}{1-2^{1-s}} = \frac{\zeta_a(1)}{\ln(2)} = 1,$$

since $\zeta_a(1) = \sum_{n=1}^{\infty}(-1)^{n+1}/n = \ln(2)$. One we show that the zeta function is analytic away from 1, this indicates that it has a first order pole at $s = 1$ with residue 1.

In analogy with the zeta function, we have the integral formula

$$\zeta_a(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx, \quad 0 < s \in \mathbb{R}.$$

Indeed, we have

$$\begin{aligned}
\zeta_a(s) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} = \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\int_0^\infty (-1)^{n+1}u^{s-1}e^{-nu}\,du \\
&= \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-nu}\,du = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{e^{-u}}{1+e^{-u}}\,du \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \frac{u^{s-1}}{e^u+1}\,du, \quad 1 < s \in \mathbb{R},
\end{aligned}$$

where the interchange of the infinite sum and the improper integral is clearly allowed. The formula follows.

Remark. As in the case of the zeta function, this formula can be used to give a direct proof that the alternating zeta function is analytic on $(0, \infty)$. See Exercise 2 at the end of this section. In particular, the formula connecting ζ and ζ_a above provides an **analytic continuation** of the zeta function from $s > 1$ to $s > 0$.

The zeta and gamma relation gives us a hint that there may be other integral formulas for the zeta function with more extended domains. The first and simplest attempt is to use the first Euler-Maclaurin fomula of the previous section to this effect

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{k^s} &= \int_1^n \frac{dx}{x^s} - s\int_1^n \frac{P_1(x)}{x^{s+1}}\,dx + \frac{1}{2}\left(\frac{1}{n^s}+1\right) \\
&= -\frac{1}{s-1}\left(\frac{1}{n^{s-1}}-1\right) + \frac{1}{2}\left(\frac{1}{n^s}+1\right) - s\int_1^n \frac{P_1(x)}{x^{s+1}}\,dx, \quad 2 \le n \in \mathbb{N}.
\end{aligned}$$

Keeping $1 < s \in \mathbb{R}$, we let $n \to \infty$, and obtain

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{n^s} = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{P_1(x)}{x^{s+1}}\,dx.$$

The improper integral (absolutely) converges for $s > 0$ (and conditionally for $s > -1$). We will prove below (in a more general setting) the simple fact that the improper integral on the right-hand side above defines an analytic function on $(0, \infty)$. Hence, the entire right-hand side is an analytic continuation of the zeta function from $(1, \infty)$ to $(0, \infty)$ across a simple pole at 1 with residue 1. (The same fact has been indicated for the previous extension of the zeta function (using ζ_a) for $0 < s < 1$.)

An interesting byproduct is the following:

Example 4.17.2. [123] We have

$$\lim_{s\to 1^+} \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{s^n} \right) = \lim_{s\to 1^+} \left(\zeta(s) - \frac{1}{s-1} \right) = \gamma.$$

Indeed, we have

$$\lim_{s\to 1^+} \left(\zeta(s) - \frac{1}{s-1} \right) = \frac{1}{2} - \int_1^\infty \frac{P_1(x)}{x^2}\, dx = \gamma,$$

where, in the last equality, we used Example 4.15.2. By the infinite geometric series formula, the example follows.

Returning to the Euler-Maclaurin summation formula, we now perform integration by parts on the last improper integral before the example as

$$\int_1^\infty \frac{P_1(x)}{x^{s+1}}\, dx = \frac{1}{2} \int_1^\infty \frac{P_2'(x)}{x^{s+1}}\, dx = -\frac{1}{12} + \frac{s+1}{2} \int_1^\infty \frac{P_2(x)}{x^{s+2}}\, dx,$$

since the boundary terms

$$\left[\frac{P_2(x)}{x^{s+1}} \right]_1^\infty = -P_2(1) = -B_2 = -\frac{1}{6}.$$

Substituting, we obtain

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)}{2} \int_1^\infty \frac{P_2(x)}{x^{s+2}}\, dx.$$

Since the improper integral converges for $-1 < s \in \mathbb{R}$, this gives an extension of $\zeta(s)$ to $s > -1$. It also shows $\zeta(0) = -1/2 = B_1$.

[123] For an elementary and direct proof, see Sondow, J., *An antisymmetric formula for Euler's constant*, Math. Magazine, Vol. 71, No.3 (Jun. 1998) 219-220.

Performing integration by parts can be inductively applied to the improper integral.[124] In general, we obtain

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} - B_1 + \frac{s}{2!}B_2 + \frac{s(s+1)}{3!}B_3 + \cdots + \frac{s(s+1)(s+2)\cdots(s+m-1)}{(m+1)!}B_{m+1} \\ &- \frac{s(s+1)(s+2)\cdots(s+m)}{(m+1)!}\int_1^\infty \frac{P_{m+1}(x)}{x^{s+m+1}}\,dx, \quad s > -m,\ m \in \mathbb{N}.\end{aligned}$$

This extends the zeta function to all real numbers except at 1, where it has a simple pole with residue 1.

We now show that these extensions (for various $m \in \mathbb{N}_0$) are analytic, and hence they define successive **analytic continuations** of the zeta function.
Since, the coefficients of the Bernoulli numbers (as well as the integral) are polynomials in s, it is enough to prove that the improper integral on the right-hand side is analytic for $s > -m$.
To do this we give an estimate of the growth rate of the higher derivatives of the integral in s. The nth derivative $n \in \mathbb{N}$ is

$$\frac{d^n}{ds^n}\int_1^\infty \frac{P_{m+1}(x)}{x^{s+m+1}}\,dx = (-1)^n \int_1^\infty \frac{P_{m+1}(x)}{x^{s+m+1}}\,(\ln(x))^n\,dx,$$

where the differentiation can be interchanged with the improper integral by Proposition 4.4.2. For $n \in \mathbb{N}_0$, we estimate as

$$\left|\int_1^\infty \frac{P_{m+1}(x)}{x^{s+m+1}}\,(\ln(x))^n\,dx\right| \leq \sup_{x\in[0,1]}|B_{m+1}(x)|\int_1^\infty \frac{(\ln(x))^n}{x^{s+m+1}}\,dx = K\frac{n!}{(s+m)^{n+1}},$$

where K stands for the supremum, and we used Example 4.3.1. The claimed analiticity now follows (Section 2.4).

As a byproduct of the expansion of the zeta function above, we also obtain

$$\zeta(-m) = (-1)^m\frac{B_{m+1}}{m+1}, \quad m \in \mathbb{N}_0.$$

Indeed, substituting $s = -m$ in the formula above, the initial sum of $m+1$ terms cancel due to the inductive formula for the Bernoulli numbers (Section 4.12), and the fact that all odd Bernoulli numbers are zero except $B_1 = -1/2$. A notable consequence of this is that the zeta function vanishes on all negative even integers. These are called the **trivial zeros** of the zeta function.

[124] Alternatively, we can also use the Euler-Maclaurin summation formulas of the previous section. Our direct method is of slight technical convenience as it involves derivatives of all order, not just the odd ones.

A more compact expression can be obtained from the first Euler-Maclaurin formula by making a more precise estimate of the improper integral error term; indeed, the following formula was Riemann's principal observations about the zeta function

$$\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(t^{-(s+1)/2} + t^{s/2-1}\right)\psi(t)\,dt, \quad 1 < s \in \mathbb{R},$$

where

$$\psi(t) = \sum_{n=1}^\infty e^{-\pi n^2 t}, \quad 0 < t \in \mathbb{R}.$$

The importance of this is that the right-hand side is invariant under the transformation $s \leftrightarrow 1-s$. This allows to define the zeta function for $s < 0$ by a single formula. This formula, to be proved below, is the so called **functional equation**:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \quad s \in \mathbb{R}\setminus\{1\}.$$

History. The functional equation was originally conjectured by Euler, and was given two proofs in Riemann's famous paper *Ueber die Anzahl der Primzahlen unter einer gegebener Grösse*, Monatsberichte der Berliner Akademie, Novermber, 1859, written on the occasion of his admission to the Prussian Academy of Sciences in 1859. He also proved analytic continuation of the zeta function to the whole complex plane except a simple pole at 1.

Note that, evaluating the functional equation on negative odd integers $s = -2n+1$, $n \in \mathbb{N}$, and using our earlier result, we obtain

$$\zeta(-2n+1) = -\frac{B_{2n}}{2n} = (-1)^n 2^{-2n+1}\pi^{-2n}(2n-1)!\zeta(2n).$$

As a byproduct, we recover Euler's summation formula in Section 4.13:

$$\zeta(2n) = \sum_{k=1}^\infty \frac{1}{k^{2n}} = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}, \quad n \in \mathbb{N}.$$

Remark. No simple formula is known for $\zeta(2n+1)$, $n \in \mathbb{N}$. For $n = 1$, in 1979, R. Apery used the series representation for the so-called **Catalan number**

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^\infty \frac{(-1)^n}{n^3\binom{2n}{n}},$$

to prove that it is irrational. More recently, Zudilin[125] proved that one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. A general result of Ball and Rivoal[126] asserts that the number of irrationals in the set $\{\zeta(2k+1) \mid k = 1, \ldots, n\}$, $n \in \mathbb{N}$, is $\geq 1/(2(1+\ln(2)))$ for large n.

For an elementary proof of the functional equation, we need some preparations.[127]

We make a minor modification of technical convenience in that we consider a continuously differentiable function $f : [-m, n] \to \mathbb{R}$, $2 \leq m, n \in \mathbb{N}$, and write

$$\sum_{k=-m}^{n} f(k) = \int_{-m}^{n} f(x)\, dx + \frac{1}{2}\left(f(n) + f(-m)\right) + \int_{-m}^{n} P_1(x) f'(x)\, dx, \quad 2 \leq m, n \in \mathbb{N}.$$

We now extend $f : (-\infty, \infty) \to \mathbb{R}$, and let $m, n \to \infty$. We have

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x)\, dx + \int_{-\infty}^{\infty} P_1(x) f'(x)\, dx,$$

where we assume that the infinite sum, and the improper integrals exist (in particular, $\lim_{n\to\pm\infty} f(x) = 0$). We now substitute the (Fourier) expansion of P_1 derived in Section 4.14 into the last integral and obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f(n) &= \int_{-\infty}^{\infty} f(x)\, dx - \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} f'(x)\, dx \\ &= \int_{-\infty}^{\infty} f(x)\, dx - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} f'(x) \sin(2\pi n x)\, dx,\end{aligned}$$

where, once again, we assume that the intechange of the improper integral and the infinite sum is legitimate. Finally, we perform integration by parts, and arrive at the formula

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{\infty}^{\infty} f(x)\, dx + 2\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(x) \cos(2\pi n x)\, dx,$$

or equivalently

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \cos(2\pi n x)\, dx.$$

---

[125] See Zudilin, W., *One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational*, Russ. Math. Surv. 56 (2001) 193-206.

[126] See Ball, K. and Rivoal, T., *Irrationaité d'une infinite de la fonction zeta aux entiers impairs*, Invent. Math. 146 (2001) 193-207.

[127] Note that there are no less than seven methods of proof of the functional equation in Titmarsch, E.C. *The Theory of the Riemann Zeta Function*, 2nd edition, Oxford, 1986.

This is (the real form of) the **Poisson summation formula**. The assumptions on f for this to be valid hold for **Schwarz functions**, continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ up to any order such that, for any $c \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\lim_{x\to\pm\infty} \frac{|f^{(n)}(x)|}{|x|^c} = 0.$$

History. The Poisson summation formula was discovered by the French mathematician and physicist Siméon Denis Poisson (1781 – 1840).

We apply the Poisson summation formula for the parametric (Schwarz) function $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ defined by

$$f(x,t) = e^{-\pi x^2 t}, \quad x \in \mathbb{R},\ 0 < t \in \mathbb{R}.$$

We calculate[128]

$$\int_{-\infty}^{\infty} e^{-\pi x^2 t} \cos(2\pi n x)\, dx = 2 \int_0^{\infty} e^{-\pi x^2 t} \cos(2\pi n x)\, dx$$
$$= \frac{2}{\sqrt{\pi t}} \int_0^{\infty} e^{-u^2} \cos\left(\frac{2\sqrt{\pi} n}{\sqrt{t}} u\right) du = \frac{1}{\sqrt{t}} e^{-\frac{\pi n^2}{t}}.$$

Substituting this into the Poisson formula, we obtain

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}}, \quad t > 0.$$

We now begin with the proof of the functional equation for the zeta function.[129] Substituting $a = s/2 - 1$, $0 < s \in \mathbb{R}$, $n = n^2\pi$, $n \in \mathbb{N}$, and $c = 1$ into the formula in the remark after Example 4.6.3, we obtain

$$\int_0^{\infty} t^{s/2-1} e^{-\pi n^2 t}\, dt = \frac{\Gamma(s/2)}{\pi^{s/2} n^s}, \quad s > 0,\ n \in \mathbb{N}.$$

We now impose $s > 1$, sum up with respect to $n \in \mathbb{N}$ and rearrange as

$$\frac{\Gamma(s/2)}{\pi^{s/2}} \zeta(s) \;=\; \frac{\Gamma(s/2)}{\pi^{s/2}} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} t^{s/2-1} e^{-\pi n^2 t}\, dt = \int_0^{\infty} t^{s/2-1} \sum_{n=1}^{\infty} e^{-\pi n^2 t}\, dt,$$

[128]For the last equality, letting $I(s) = \int_0^\infty e^{-u^2} \cos(su)\, du$, $s \in \mathbb{R}$, differentiating and integrating by parts, we obtain $I'(s) = -(s/2)I(s)$, $I(0) = \sqrt{\pi}/2$, and hence $I(s) = (\sqrt{\pi}/2)e^{-s^2/4}$, $s \in \mathbb{R}$.

[129]Once again, here we follow Riemann's original paper *Ueber die Anzahl der Primzahlen unter einer gegebener Grösse*, Monatsberichte der Berliner Akademie, Novermber, 1859. This includes his notations except for the Gamma function which he denoted by $\Pi(s) = \Gamma(s-1)$. See also Titmarsch, E.C., *The Theory of the Riemann Zeta Function*, 2nd edition, Oxford, 1986.

where the interchange of the summation with the improper integral is allowed by absolute convergence. It is convenient to introduce the shorthand notation $\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$, $t > 0$. With this, we have so far

$$\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s) = \int_0^\infty t^{s/2-1}\psi(t)\,dt = \int_0^1 t^{s/2-1}\psi(t)\,dt + \int_1^\infty t^{s/2-1}\psi(t)\,dt,$$

where we split the integral for future purposes. In terms of ψ, the Poisson formula is equivalent to

$$2\psi(t) + 1 = \frac{1}{\sqrt{t}}\left(2\psi(1/t) + 1\right), \quad t > 0.$$

We use this to replace the integrand in the first integral on the right-hand side as

$$\begin{aligned}\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s) &= \int_0^1 t^{s/2-1}\psi(t)\,dt + \int_1^\infty t^{s/2-1}\psi(t)\,dt \\ &= \int_0^1 t^{s/2-3/2}\psi\left(\frac{1}{t}\right)\,dt + \frac{1}{2}\int_0^1 t^{s/2-1}\left(\frac{1}{\sqrt{t}} - 1\right) + \int_1^\infty t^{s/2-1}\psi(t)\,dt \\ &= \int_1^\infty t^{-s/2-1/2}\psi(t)\,dt + \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty t^{s/2-1}\psi(t)\,dt,\end{aligned}$$

where, in the last step, we changed the variable t to $1/t$. Simplifying, we obtain

$$\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s) = -\frac{1}{s(1-s)} + \int_1^\infty \left(t^{-s/2-1/2} + t^{s/2-1}\right)\psi(t)\,dt.$$

The crux is that the right-hand side is invariant under the change $1 - s \mapsto s$. This gives

$$\frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s) = \frac{\Gamma(1/2 - s/2)}{\pi^{1/2-s/2}}\zeta(1-s).$$

By Euler's reflection formula $\Gamma(s/2)\Gamma(1-s/2) = \pi/\sin(\pi s/2)$ in Proposition 4.9.1, this can be written as

$$\zeta(s) = \Gamma(1/2 - s/2)\Gamma(1 - s/2)\pi^{s-3/2}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).$$

Finally, using the Legendre duplication formula with $t = 1/2 - s/2$ in Proposition 4.10.1, this simplifies to the following

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

The functional equation for the zeta function follows.

We close this chapter by presenting another proof of the functional equation for the zeta function due to Hardy.[130] (We will give yet another proof as a special case of Hermite's formula in Section 4.20.) We start by recalling from Section 4.14 the Fourier expansion

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi.$$

Splitting the sum into odd-even parts, for $0 < x < \pi$, we calculate

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(n(2x))}{n} = \frac{1}{2}\frac{\pi - 2x}{2}.$$

This gives

$$\sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1} = \frac{\pi - x}{2} - \frac{\pi - 2x}{4} = \frac{\pi}{4}, \quad 0 < x < \pi.$$

It is straightforward to extend this using periodicity to the expansion[131]

$$\sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1} = (-1)^m \frac{\pi}{4}, \quad m\pi < x < (m+1)\pi, \ m \in \mathbb{Z}.$$

We now multiply both sides by x^{s-1}, $0 < s < 1$, and integrate over $(0, \infty)$ with respect to x. We will do this for each side separately. For the left-hand side, we have

$$\begin{aligned}\int_0^{\infty} x^{s-1} \sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1}\, dx &= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{\sin((2n+1)x)}{x^{1-s}}\, dx \\ &= \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^{\infty} \frac{\sin((2n+1)x)}{x^{1-s}}\, dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{s+1}} \Gamma(s) \sin\left(\frac{\pi s}{2}\right),\end{aligned}$$

where we used the (scaled) formula

$$\int_0^{\infty} \frac{\sin(ax)}{x^{s-1}} = a^{-s}\Gamma(s) \sin\left(\frac{\pi s}{2}\right), \quad a > 0,$$

[130] See Hardy, G.H. *A new proof of the functional equation for the zeta function*, Mat. Tidsskrift, B (1922) 71-73. The proof uses analiticity of the alternatig zeta function ζ_a on $(0, \infty)$ as in Exercise 2 at the end of this section.

[131] This is actually the Fourier series of the saltus function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (-1)^m \pi/4$ if $m\pi < x < (m+1)\pi$ and $f(x) = 0$ if $x = m\pi$, $m \in \mathbb{Z}$.

at the end of Section 4.6. The infinite sum can be written in terms of the zeta function since

$$\zeta(s+1) = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{s+1}} + \frac{1}{2^{s+1}} \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{s+1}} + \frac{1}{2^{s+1}} \zeta(s+1),$$

so that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{s+1}} = \left(1 - \frac{1}{2^{s+1}}\right) \zeta(s+1), \quad 0 < s < 1.$$

With this, the left hand-side is

$$\int_0^{\infty} x^{s-1} \sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1}\, dx = \left(1 - \frac{1}{2^{s+1}}\right) \zeta(s+1)\Gamma(s) \sin\left(\frac{\pi s}{2}\right)$$

The right-hand side can be written as

$$\frac{\pi}{4} \sum_{m=0}^{\infty} (-1)^m \int_{m\pi}^{(m+1)\pi} x^{s-1}\, dx.$$

We first note that the series here converges[132] for $s < 1$, and gives an analytic function. Assuming $s < 0$, we calculate

$$\begin{aligned}
\frac{\pi}{4} \sum_{m=0}^{\infty} (-1)^m \int_{m\pi}^{(m+1)\pi} x^{s-1}\, dx &= \frac{\pi}{4s} \sum_{m=0}^{\infty} (-1)^m \left[x^s\right]_{m\pi}^{(m+1)\pi} \\
&= \frac{\pi^{s+1}}{4s} \sum_{m=0}^{\infty} (-1)^m \left((m+1)^s - m^s\right) = \frac{\pi^{s+1}}{2s} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{-s}} \\
&= \frac{\pi^{s+1}}{2s} \zeta_a(-s) = \frac{\pi^{s+1}}{2s} \left(1 - 2^{s+1}\right) \zeta(-s).
\end{aligned}$$

where we used the alternating zeta function, and its relation for the zeta function. Since the left-hand side and the right-hand side overlap for $0 < s < 1$, by the unicity of analytic functions, they are equal everywhere on their domain of analiticity. We obtain

$$\left(1 - \frac{1}{2^{s+1}}\right) \zeta(s+1)\Gamma(s) \sin\left(\frac{\pi s}{2}\right) = \frac{\pi^{s+1}}{2s} \left(1 - 2^{s+1}\right) \zeta(-s).$$

Rearranging, we have

$$\zeta(s+1)\Gamma(s+1) \sin\left(\frac{\pi s}{2}\right) = -2^s \pi^{s+1} \zeta(-s),$$

[132] Albeit not absolutely.

where we used $s\Gamma(s) = \Gamma(s+1)$. Finally, replacing s by its opposite $-s$, we arrive at

$$\zeta(1-s)\Gamma(1-s)\left(\frac{\pi s}{2}\right) = 2^{-s}\pi^{s-1}\zeta(s).$$

This is the functional equation for the zeta function.

Exercises

1. Use the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx, \quad 1 < s \in \mathbb{R}.$$

in the text to prove that the zeta function is analytic on $(1, \infty)$ by the following steps. (a) Notice first that it is enough to prove that the improper integral is analytic in $s > 1$ (as the quotient of analytic functions is analytic, and the gamma function is analytic; see Section 4.6.). (b) For $n \in \mathbb{N}$, derive the formula

$$\frac{d^n}{ds^n}\left(\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx\right) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}(\ln(x))^n\, dx, \quad s > 1.$$

(c) Use the estimate $e^x - 1 = 2e^{x/2}\sinh(x/2) \geq 2xe^{x/2}$, $x \geq 0$, and the method at the beginning of Section 4.6 ($t = s - 1$) to estimate

$$\begin{aligned}\left|\int_0^\infty \frac{x^{s-1}}{e^x - 1}(\ln(x))^n\, dx\right| &\leq \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, |\ln(x)|^n\, dx \\ &\leq \frac{1}{2}\int_0^\infty x^{s-2}e^{-x/2}\, |\ln(x)|^n\, dx \leq \frac{1}{2}\frac{n!}{(s-1)^{n+1}} + K\, 4^n\, n!,\ s > 1,\ n \in \mathbb{N},\end{aligned}$$

where $x^{b-2} \leq Ke^{x/4}$, $x \geq 1$, and $1 < s < b$. (d) Finally, use condition on the growth rate of the Taylor coefficients (Section 2.4) to conclude that ζ is analytic on $(1, \infty)$.

2. Make the necessary changes in Exercise 1 in the use of the integral formula

$$\zeta_a(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1}\, dx, \quad 1 < s \in \mathbb{R}.$$

to prove that the alternating zeta function is analytic on $(0, \infty)$.

3. Show that

$$\ln(n) = \sum_{d|n} \Lambda(d), \quad n \in \mathbb{N}.$$

4. Show that

$$\frac{1}{\zeta(s)} = \prod_{p\in\Pi}\left(1-\frac{1}{p^m}\right) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s},$$

where μ is the Möbius function given by $\mu(1)=1$, $\mu(n)=(-1)^k$ if n is the product of k distinct primes, and zero otherwise.

4.18 The Digamma and Polygamma Functions

The **digamma function** $\Psi:\mathbb{R}\setminus(-\mathbb{N}_0)\to\mathbb{R}$ is defined as the logarithmic derivative of the Gamma function

$$\Psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t\in\mathbb{R}\setminus(-\mathbb{N}_0).$$

In section 4.10, using the Weierstrass representation of the Gamma function, we actually obtained an absolutely convergent series representation of the digamma function[133]

$$\Psi(t) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{t+n-1}\right) = -\gamma + \sum_{n=1}^{\infty}\frac{t-1}{n(t+n-1)}$$

as well as its derivative

$$\Psi'(t) = \sum_{n=1}^{\infty}\frac{1}{(t+n-1)^2}.$$

An immediate consequence of the last formula is that the digamma function Ψ is **strictly increasing** everywhere. Note that these series are uniformly convergent on closed subintervals of $\mathbb{R}\setminus(-\mathbb{N}_0)$. In addition, we have

$$\Psi''(t) = -2\sum_{n=1}^{\infty}\frac{1}{(t+n-1)^3},$$

showing that the digamma function is strictly concave **on** $(0,\infty)$.
Some specific values of Ψ and Ψ' are readily obtained (with increasing complexity)

[133] In Section 4.10, we resricted the domain to $(0,\infty)$; however, these representations are clearly valid for $\mathbb{R}\setminus(-\mathbb{N}_0)$.

as follows:

$$\begin{aligned}
\Psi(1) &= -\gamma \\
\Psi(2) &= -\gamma + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \gamma \\
\Psi(1/2) &= -\gamma - 2 \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = -\gamma - 2 \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right) = -\gamma - 2\ln(2) \\
\Psi'(1) &= \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \\
\Psi'(2) &= \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \zeta(2) - 1 = \frac{\pi^2}{6} - 1 \\
\Psi'(1/2) &= 4 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 4 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2} = 3\zeta(2) = \frac{\pi^2}{2}.
\end{aligned}$$

The series representation of Ψ above is revealing. It shows that, at every non-positive integer $-m$, $m \in \mathbb{N}_0$, the digamma function Ψ has a simple pole; in fact, for $|t+m| < 1$, the function $\Psi(t)$ is equal to $-1/(t+m)$ plus an absolutely convergent series. It follows that the graph of Ψ consists of strictly increasing branches between consecutive non-positive integers, whereas it has a single strictly increasing branch over the non-positive integers. Hence Ψ has a simple zero x_0 on the positive axis with $1 < x_0 < 2$ (by the intermediate value theorem as $\Psi(1) = -\gamma < 0 < 1 - \gamma = \Psi(2)$), and, for every $m \in \mathbb{N}$, it has a simple zero $x_m \in (-m, -m+1)$. The sequence $(x_m)_{m \in \mathbb{N}_0}$ of zeros of the digamma function is of importance as the Gamma function takes local extrema at exactly on these points. (This follows as they are critical points of the logaritmically convex Γ.)

History. The zeros of the digamma function can be approximated to any precision. Setting the number of digits to 25, say, a standard computer algebra system gives

$$\begin{aligned}
x_0 &\approx 1.461632144968362341262660 \\
x_1 &\approx -.5040830082644554092582693 \\
x_2 &\approx -1.573498473162390458778286 \\
x_3 &\approx -2.610720868444144650001538 \\
x_4 &\approx -3.635293366436901097839182 \\
x_5 &\approx -4.653237761743142441714598 \\
x_6 &\approx -5.667162441556885535849474.
\end{aligned}$$

In 1881 Hermite showed that

$$\lim_{m\to\infty} (\ln(m))^2 \left(x_m + m - \frac{1}{\ln(m)} \right) = 0.$$

A better approximation is provided by the asymptotic relation

$$x_m \sim -m + \frac{1}{\pi} \arctan\left(\frac{\pi}{\ln(m) + 1/(8m)} \right), \quad m \to \infty.$$

The zeros also satisfy the following[134]

$$\sum_{n=0}^{\infty} \frac{1}{x_m^2} \;=\; \gamma^2 + \frac{\pi^2}{2},$$

and similar formulas for $\sum_{n=0}^{\infty} 1/x_m^3$ and $\sum_{n=0}^{\infty} 1/x_m^4$.

As we will see in this section, the study of the digamma function provides important information about the properties of the gamma function. The following result, due to Gautschi,[135] illustrates this point.

Proposition 4.18.1. *We have*

$$\frac{2}{\frac{1}{\Gamma(t)} + \frac{1}{\Gamma(1/t)}} \geq 1, \quad t > 0,$$

and equality is attained at $t = 1$*.*

Remark. The expression on the left-hand side of the Gautschi inequality is the **harmonic mean** of $\Gamma(t)$ and $\Gamma(1/t)$, $t > 0$. It is well-known[136] that the harmonic mean is dominated by the geometric mean, which, in turn, is dominated by the arithmetic mean. It follows that we also have

$$\Gamma(t)\Gamma\left(\frac{1}{t}\right) \geq 1 \quad \text{and} \quad \Gamma(t) + \Gamma\left(\frac{1}{t}\right) \geq 2, \quad t > 0.$$

Proof of Proposition 4.19.1. By the discussion above, $x_0 \in (1, 2)$ is the only critical point of the (logarithmically convex and positive) gamma funtion on $(0, \infty)$. Hence Γ on $(0, \infty)$ attains its (unique) absolute minimum at x_0. The expression on the left-hand side of the inequality to be proven is then strictly increasing on (x_0, ∞).

[134]See Mezö, I. and Hoffman, M., *Zeros of the digamma function and its Barnes G-function analogue*, Integral Transforms and Special Functions, 28 (11) (2017) 846-858.

[135]See Gautschi, W., *A harmonic mean inequality for the gamma function*, SIAM J. Math. Anal., Vol. 5.No. 2 (1974) 278-281. We follow this original proof.

[136]See *Elements of Mathematics - History and Foundations*; Section 9.5.

Therefore, it is enough to show that the inequality holds on the interval $(1, x_0]$ (as the expression on the left-hand side is invariant under the substitution $t \to 1/t$, $t > 0$). It is technically convenient to use the variable $s = \ln(t)$, $0 < t \in \mathbb{R}$, with $t = e^s$, $s \in \mathbb{R}$, and also to introduce the function $g : \mathbb{R} \to \mathbb{R}$ by $g(s) = 1/\Gamma(e^s)$, $s \in \mathbb{R}$. With this, the inequality to be proven rewrites as

$$g(s) + g(-s) < 2g(0) = 2, \quad s \in (0, \ln(x_0)]$$

Expanding g into Taylor series with the Lagrange form of the remainder (Section 2.3), we obtain

$$\begin{aligned} g(s) &= g(0) + sg'(0) + \frac{s^2}{2} g''(u), \quad 0 < u < s; \\ g(-s) &= g(0) - sg'(0) + \frac{s^2}{2} g''(v), \quad -s < v < 0. \end{aligned}$$

Adding, we arrive at

$$g(s) + g(-s) - 2g(0) = \frac{s^2}{2} \left(g''(u) + g''(v) \right), \quad -s < v < 0 < u < s.$$

Thus, it remains to show that

$$g''(u) + g''(v) < 0, \quad \ln(1/x_0) \leq -s < v < 0 < u < s \leq \ln(x_0).$$

To calculate g'' we begin

$$g'(s) = \frac{d}{ds} \frac{1}{\Gamma(e^s)} = -e^s \frac{\Gamma'(e^s)}{\Gamma^2(e^s)} = -t \frac{\Psi(t)}{\Gamma(t)},$$

where, in the last equality, we reverted to the variable t. Differentiating again, and using $dt/ds = (d/ds)(e^s) = e^s = t$, we have

$$\begin{aligned} g''(s) &= -t \frac{d}{dt} \left(t \frac{\Psi(t)}{\Gamma(t)} \right) = -t \frac{\Psi(t)}{\Gamma(t)} - t^2 \frac{\Psi'(t)\Gamma(t) - \Psi(t)\Gamma'(t)}{\Gamma^2(t)} \\ &= -\frac{1}{\Gamma(t)} \left(t\Psi(t) + t^2 \Psi'(t) - t^2 \Psi^2(t) \right). \end{aligned}$$

Next, we claim that both $t\Psi(t)$ and $t^2\Psi'(t)$ are strictly increasing on $(1/2, x_0]$. (Since $1/2 < 1/x_0 < 1 < x_0$, this covers the interval $[1/x_0, x_0]$.) For the first expression, we use the infinite series representation above

$$t\Psi(t) = -\gamma t + \sum_{n=1}^{\infty} \frac{t(t-1)}{n(t+n-1)} = -1 + (1-\gamma)t + \sum_{n=1}^{\infty} \frac{t(t-1)}{(n+1)(t+n)},$$

where we split off the first term and moved the summation index up by one. Now, it is elementary to observe that the expression $t(t-1)/(t+n)$ has positive derivative for $t > n(\sqrt{1+1/n}-1) = 1/(\sqrt{1+1/n}+1)$, and hence, for $t > 1/2$.
This shows that $t\Psi(t)$ is strictly increasing on $(1/2, \infty)$; and we also note that $t\Psi(t) < 0$ for $0 < t < x_0$ (since $\Psi(x_0) = 0$).
The second expression $t^2\Psi'(t)$ is strictly increasing on $(0, \infty)$ as it is obvious by the expansion above

$$t^2\Psi'(t) = \sum_{n=1}^{\infty} \left(\frac{t}{t+n-1}\right)^2,$$

since the same holds for the generic term $t/(t+n-1) = 1 - 1/(t+n-1)$, $t > 0$. We also note that $t^2\Psi'(t) > 0$ for $t > 0$.
We now derive two estimates on the expression $t\Psi(t) + t^2\Psi'(t) - t^2\Psi^2(t)$. The first estimate is for $1 < t < x_0$. Using the monotonicity properties above, we have

$$t\Psi(t) + t^2\Psi'(t) - t^2\Psi^2(t) \geq \Psi(1) + \Psi'(1) - \Psi^2(1) = -\gamma + \frac{\pi^2}{6} - \gamma^2 = 0.7345404792\ldots$$

The second estimate is for $1/2 < t < x_0$. Once again, using the monotonicity properties above, we have

$$\begin{aligned} t\Psi(t) + t^2\Psi'(t) - t^2\Psi^2(t) &> \frac{1}{2}\Psi\left(\frac{1}{2}\right) + \frac{1}{4}\Psi'\left(\frac{1}{2}\right) - \frac{1}{4}\Psi^2\left(\frac{1}{2}\right) \\ &= -\frac{1}{2}(\gamma + 2\ln(2)) + \frac{\pi^2}{8} - \frac{1}{4}(\gamma + 2\ln(2))^2 = -0.7118973685\ldots \end{aligned}$$

In terms of the variable $s = \ln(t)$, and the intermediate points in the Taylor remainders u, v with $\ln(1/x_0) < -s < v < 0 < u < s < \ln(x_0)$, these give

$$g''(u)\Gamma(e^u) \leq -0.7345404792\ldots \quad \text{and} \quad g''(v)\Gamma(e^v) \leq 0.7118973685\ldots$$

in particular, $g''(u) < 0$. Adding, we obtain

$$g''(u)\Gamma(e^u) + g''(v)\Gamma(e^v) < 0.$$

Now, if $g''(v) \leq 0$ then $g''(u) + g''(v) < 0$ and we are done. Thus, we may assume that $g''(v) > 0$. Then, we have

$$0 > g''(u)\Gamma(e^u) + g''(v)\Gamma(e^v) > g''(u)\Gamma(e^u) + g''(v)\Gamma(e^u) = (g''(u) + g''(v))\,\Gamma(e^u),$$

where $\Gamma(e^v) > \Gamma(e^u)$ since $1/x_0 < e^v < 1 < e^u < x_0$. Hence, $g''(u) + g''(v) < 0$ follows again.
The proof of the proposition is complete.

Remark. There is a proliferation of variants of the Gautschi inequality; see, for example Alzer, H., *Gamma function inequalities*, Numer. Algor. 49 (2008) 53-84, and the references therein. Some of the estimates on the ratio of gamma functions discussed at the end of Section 4.10 also involve the digamma functions; for example, Gautschi's original estimate

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left((1-s)\Psi(x+1)\right), \quad x>0,\ 0<, s<1,$$

as well as the sharper Kershaw inequalities[137]

$$\exp\left((1-s)\Psi\left(x+\sqrt{s}\right)\right) < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left((1-s)\Psi\left(x+\frac{1}{2}(s+1)\right)\right),\ x>0,\ 0<s<1.$$

See also the survey article of Qi, F., *Bounds for the ratio of two gamma functions*, Journal of Inequalities and Applications, Vol. 2010.

Returning to the main line, the series representation of the digamma function above gives

$$\Psi(t+1)-\Psi(t)=\sum_{n=1}^{\infty}\left(\frac{1}{t+n-1}-\frac{1}{t+n}\right)=\frac{1}{t};$$

that is

$$\Psi(t+1)=\Psi(t)+\frac{1}{t}, \quad t\in\mathbb{R}\setminus(-\mathbb{N}_0).$$

Iterating, we obtain the **inductive formula for the digamma function**

$$\Psi(t+m)=\Psi(t)+\frac{1}{t}+\frac{1}{t+1}+\cdots+\frac{1}{t+m-1}, \quad t\in\mathbb{R}\setminus(-\mathbb{N}_0),\ m\in\mathbb{N}.$$

In particular ($t=1$ and m shifted down):

$$\Psi(m)=-\gamma+H_{m-1}, \quad 2\leq m\in\mathbb{N},$$

where $H_\ell=\sum_{k=1}^{\ell}1/k$, $m\in\mathbb{N}$, is the ℓth harmonic number.

Taking the logarithmic derivative of both sides of Euler's reflection formula (Proposition 4.9.1), a straightforward computation gives the **reflection formula for the digamma function**

$$\Psi(1-t)=\Psi(t)+\pi\cot(\pi t), \quad t\in\mathbb{R}\setminus\mathbb{Z}.$$

[137]See Kershaw, D., *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. 41 (1983) 607-611.

In a similar vein, taking the logarithmic derivative of both sides of the Legendre duplication formula, we obtain

$$\Psi(t) + \Psi(t+1/2) = \frac{2^{2t-1}}{\Gamma(2t)} \left(\frac{\Gamma(2t)}{2^{2t-1}}\right)' = 2\left(\Psi(2t) - \ln(2)\right)$$

since $(2^{2t-1})' = \left(e^{\ln(2)\cdot(2t-1)}\right)' = \ln(2)\cdot 2^{2t}$. Rearranging, we arrive at the **duplication formula for the digamma function**:

$$\Psi(2t) = \frac{1}{2}\left(\Psi(t) + \Psi(t+1/2)\right) + \ln(2).$$

In particular, substituting $t = 1/2$ and using $\Psi(1) = -\gamma$, we get

$$\Psi(1/2) = -\gamma - 2\ln(2).$$

Moreover, substituting $t = m + 1/2$, $m \in \mathbb{N}$, we obtain

$$\Psi(2m+1) = \frac{1}{2}\left(\Psi(m+1/2) + \Psi(m+1)\right) + \ln(2).$$

Using $\Psi(m+1) = -\gamma + H_m$ and $\Psi(2m+1) = -\gamma + H_{2m}$, after rearranging, we have

$$\Psi(m+1/2) = -\gamma - 2\ln(2) + 2\sum_{k=1}^{m} \frac{1}{2k-1}, \quad n \in \mathbb{N},$$

since $2H_{2m} - H_m = 2\sum_{k=1}^{m} 1/(2k-1)$.

The **Legendre-Gauss multiplication theorem for the digamma function** can be obtained from that of the gamma function by logarithmic differentiation in a straightforward way:

$$m\Psi(t) = m\ln(m) + \sum_{k=0}^{m-1} \Psi\left(\frac{t+k}{m}\right), \quad m \in \mathbb{N},\ t \in \mathbb{R} \setminus (\mathbb{N}_0).$$

Example 4.18.1. We have

$$\zeta'(0) = -\frac{\ln(2\pi)}{2}.$$

To derive this, we begin by the version of the functional equation of the zeta function above (after using the invariance $1-s \mapsto s$; see Section 4.17), which, by merging the powers of π, we write as

$$\zeta(1-s) = \pi^{-s+1/2}\frac{\Gamma(s/2)}{\Gamma(1/2-s/2)}\zeta(s).$$

Using the functional equation of the gamma function:

$$\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)=\frac{2}{1-s}\Gamma\left(\frac{3}{2}-\frac{s}{2}\right),$$

this becomes

$$-\zeta(1-s)=\pi^{-s+1/2}\frac{\Gamma(s/2)}{2\Gamma(3/2-s/2)}(s-1)\zeta(s).$$

We now perform logarithmic differentiation on each factor (and multiply through -1), and obtain

$$\frac{\zeta'(1-s)}{\zeta(1-s)}=\ln\pi-\frac{1}{2}\Psi\left(\frac{s}{2}\right)-\frac{1}{2}\Psi\left(\frac{3-s}{2}\right)+\frac{(d/ds)((s-1)\zeta(s))}{(s-1)\zeta(s)},$$

where we used the fact that the digamma function is the logarithmic derivative of the gamma function. We now evaluate this on $s=1$. We have $\zeta(0)=-1/2$ (Section 4.17), $\Psi(1/2)=-\gamma-2\ln(2)$, and $\Psi(1)=-\gamma$. For the last term, we first note that $\lim_{s\to 1}(s-1)\zeta(s)=1$ (Section 4.17), and then

$$\lim_{s\to 1}\frac{(s-1)\zeta(s)-1}{s-1}=\lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)=\gamma.$$

(See Example 4.17.2.) Putting everything together, the example follows.

We now discuss integral representations of the digamma function. We introduce the parametric integral $I:(0,\infty)\to\mathbb{R}$ by

$$I(t)=\int_0^1\left(\frac{t}{1-x}+\frac{1-x^t}{1-x}\frac{1}{\ln(x)}\right)dx=\int_0^1\frac{t\ln(x)+1-x^t}{(1-x)\ln(x)}\,dx,\quad t>0.$$

We first note that the integrand has removable discontinuities at the end-points, and hence the integral exists. Indeed, this is obvious at $x=0$. For the end-point $x=1$, this follows from the equivalent form

$$I(t)=t\int_0^1\left(\frac{1}{1-x}+\frac{1}{\ln(x)}\right)dx+\int_0^1\left(\frac{1-x^t}{1-x}-t\right)\frac{1}{\ln(x)}\,dx,$$

and the fundamental estimate of the natural logarithm[138]

$$\frac{x}{1+x}\le\ln(1+x)\le x,\quad -1<x\in\mathbb{R}.$$

[138]See *Elements of Mathematics - History and Foundations*, Section 10.3.

(The range of the first integrand is $[0, 1]$ with removable discontinuities at the end-points.)
Moreover, we have

$$I(0) = \lim_{t \to 0^+} I(t) = 0.$$

(The interchange of the limit and the integration is allowed by the Arzelà bounded convergence theorem; Section 3.2.)
Differentiating, we have

$$\begin{aligned} I'(t) &= \frac{d}{dt} \int_0^1 \left(\frac{t}{1-x} + \frac{1-x^t}{1-x} \frac{1}{\ln(x)} \right) dx \\ &= \int_0^1 \frac{d}{dt} \left(\frac{t}{1-x} + \frac{1-x^t}{1-x} \frac{1}{\ln(x)} \right) dx \\ &= \int_0^1 \frac{1-x^t}{1-x} \, dx, \quad t > 0. \end{aligned}$$

The last integral here is Euler's **harmonic number** H_t**,** $0 < t \in \mathbb{R}$. The name comes form the fact that, for $t = n \in \mathbb{N}$, by the geometric series formula, this integral reduces to the harmonic number $H_n = \sum_{k=1}^n 1/k$; see history insert in Section 4.15. Note that the interchange of the differentiation with the integration is allowed as the conditions of the corollary to Proposition 4.4.1 are satisfied. (The integrand has removable discontinuity at $x = 1$.) Note also that

$$I'(0) = \lim_{t \to 0^+} I'(t) = 0.$$

(Once again, here the interchange of the limit and the integration is allowed by the Arzelà bounded convergence theorem; Section 3.2.)
Finally, differentiating again, for $t > 0$, we obtain

$$\begin{aligned} I''(t) &= \frac{d}{dt} \int_0^1 \frac{1-x^t}{1-x} \, dx = -\int_0^1 \frac{x^t \ln(x)}{1-x} \, dx \\ &= -\int_0^1 \sum_{n=0}^{\infty} x^{n+t} \cdot \ln(x) \, dx = -\sum_{n=0}^{\infty} \int_0^1 x^{n+t} \cdot \ln(x) \, dx \\ &= -\sum_{n=0}^{\infty} \int_0^1 \frac{d}{dt} x^{n+t} \, dx = -\sum_{n=0}^{\infty} \frac{d}{dt} \int_0^1 x^{n+t} \, dx \\ &= -\sum_{n=0}^{\infty} \frac{d}{dt} \left(\frac{1}{n+t+1} \right) = \sum_{n=0}^{\infty} \frac{1}{(n+t+1)^2} = \Psi'(t+1). \end{aligned}$$

where, one again, the interchanges of the infinite sum and differentiation with the integration are allowed. Integrating both sides, we obtain

$$I'(t) = \Psi(t+1) + C, \quad t > 0.$$

Substituting $t = 0$, and using $I'(0) = 0$ and $\Psi(1) = -\gamma$, we obtain $C = \gamma$. Thus, up to this point, we have

$$I'(t) = \Psi(t+1) + \gamma, \quad t > 0.$$

Since $\Psi(t+1) = (d/dt) \ln \Gamma(t+1)$, another integration gives

$$I(t) = \ln \Gamma(t+1) + \gamma t + C, \quad t > 0.$$

Finally, substituting $t = 0$, we have $0 = I(0) = \ln \Gamma(1) + C = C$, and hence

$$I(t) = \ln \Gamma(t+1) + \gamma t, \quad t > 0.$$

Summarizing, we arrive at the integral formula

$$\int_0^1 \left(\frac{t}{1-x} + \frac{1-x^t}{1-x} \frac{1}{\ln(x)} \right) dx = \ln \Gamma(t+1) + \gamma t, \quad t > 0.$$

In particular, for $t = 1$, this specializes to

$$\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln(x)} \right) dx = \gamma.$$

Substituting this back to the previous expression for $\ln \Gamma(t+1)$, we obtain

$$\ln \Gamma(t+1) = \int_0^1 \left(\frac{x^t - 1}{x - 1} - t \right) \frac{dx}{\ln(x)}, \quad t > 0.$$

We interrupt the computation by the following formula of Euler:

Example 4.18.2. We have[139]

$$\ln B(t,s) = \ln \left(\frac{t+s}{ts} \right) + \int_0^1 \frac{(1-x^t)(1-x^s)}{(1-x)\ln(x)} \, dx, \quad t, s > 0.$$

Indeed, using the formula just derived, we calculate

$$\begin{aligned}
\ln B(t,s) &= = \ln \left(\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)} \right) = \ln \left(\frac{t+s}{ts} \right) + \ln \left(\frac{\Gamma(t+1)\Gamma(s+1)}{\Gamma(t+s+1)} \right) \\
&= \ln \left(\frac{t+s}{ts} \right) + \int_0^1 \left(\frac{1-x^t}{1-x} - t + \frac{1-x^s}{1-x} - s - \frac{1-x^{t+s}}{1-x} + (t+s) \right) \frac{dx}{\ln(x)} \\
&= \ln \left(\frac{t+s}{ts} \right) + \int_0^1 \frac{(1-x^t)(1-x^s)}{(1-x)\ln(x)} \, dx.
\end{aligned}$$

The example follows.

[139] See Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020; Exercise 36, p. 262.

Returning to our computation, as a byproduct of the computation above ($I'(t)$), we obtain the integral formula

$$\Psi(t+1) = -\gamma + \int_0^1 \frac{1-x^t}{1-x}\,dx.$$

Substituting $x = e^{-u}$, this transforms into

$$\Psi(t+1) = -\gamma + \int_0^\infty \frac{1-e^{-tu}}{e^u-1}\,du.$$

On the other hand, making the same substitution into the formula for γ in the example above, we obtain

$$\int_0^\infty \left(\frac{1}{e^u-1} - \frac{1}{ue^u}\right) du = \gamma.$$

Substituting this value of γ into the previous formula and shifting the argument of Ψ, we arrive at

$$\Psi(t) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-tx}}{1-e^{-x}}\right) dx, \quad t > 0,$$

where we also reverted to the previous variable. This is the **Gauss representation of Ψ**.
The substitution $x = \ln(1+u)$ for the integral corresponding to the second term gives

$$\int_0^\infty \frac{e^{-tx}}{1-e^{-x}}\,dx = \int_0^\infty \frac{du}{u(1+u)^t}, \quad t > 0,$$

and we arrive at the **Dirichlet form of the digamma function**[140]

$$\Psi(t) = \int_0^\infty \left(e^{-x} - \frac{1}{(1+x)^t}\right)\frac{dx}{x}, \quad t > 0.$$

Remark. Note the special case ($t = 1$) that gives yet another expression for the Euler-Mascheroni constant:

$$\gamma = \int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}.$$

Example 4.18.3. We have[141]

$$\int_0^1 \frac{x^{t-1}}{1+x}\,dx = \frac{1}{2}\Psi\left(\frac{t+1}{2}\right) - \frac{1}{2}\Psi\left(\frac{t}{2}\right).$$

[140] See ibid.
[141] See ibid; Exercise 30, p. 262.

Using the initial integral representation of Ψ above, we calculate

$$\Psi\left(\frac{t+1}{2}\right)-\Psi\left(\frac{t}{2}\right)=-\gamma+\int_0^1\frac{1-x^{\frac{t+1}{2}-1}}{1-x}\,dx+\gamma-\int_0^1\frac{1-x^{\frac{t}{2}-1}}{1-x}\,dx$$
$$=\int_0^1\frac{x^{\frac{t}{2}-1}-x^{\frac{t+1}{2}-1}}{1-x}\,dx=\int_0^1\frac{x^{\frac{t}{2}-1}}{1+x^{\frac{1}{2}}}\,dx=\int_0^1\frac{u^{t-1}}{1+u}\,du,$$

where, in the last equality, we used the substitution $x=u^2$. The example follows.

To complete the circle, we now show that the Gauss representation above and the Stirling formula imply the first Binet formula (Section 4.16) for the gamma function. We start with the derivative

$$I'(t)=\int_0^\infty\left(\frac{1}{u}-\frac{1}{2}-\frac{e^{-u}}{1-e^{-u}}\right)e^{-tu}\,du$$

of the improper integral in the Binet formula, and compare it with the Gauss representation above

$$\Psi(t+1)=\int_0^\infty\left(\frac{e^{-u}}{u}-\frac{e^{-(t+1)u}}{1-e^{-u}}\right)du,$$

where we moved up the value of the parameter t to $t+1$. Taking the difference, and rearranging, we obtain

$$I'(t)-\Psi(t+1)=-\int_0^\infty\frac{e^{-u}-e^{tu}}{u}\,du-\frac{1}{2}\int_0^\infty e^{-tu}\,du=-\ln(t)-\frac{1}{2t},$$

where we used Exercise 1 at the end of Section 4.4. Now, recalling that $\Psi(t+1)=(\ln\Gamma(t+1))'$ and integrating the right-hand side, we get

$$\ln\Gamma(t+1)=\left(t+\frac{1}{2}\right)\ln(t)-t+\int_0^\infty\left(\frac{1}{2}-\frac{1}{u}+\frac{1}{e^u-1}\right)\frac{e^{-tu}}{u}\,du+C.$$

This is the first Binet formula for the gamma function, up to the value of the constant C. We now let $t\to\infty$, and note that the improper integral tends to zero. Finally, we use the Stirling formula for the gamma function (Section 4.10):

$$\lim_{t\to\infty}\frac{\Gamma(t+1)}{t^{t+1/2}e^{-t}}=\sqrt{2\pi},$$

to conclude that $C=\ln\sqrt{2\pi}$.

As a byproduct of the computations above, we obtained the integral formula

$$\int_0^\infty\left(\frac{1}{2}-\frac{1}{x}+\frac{1}{e^x-1}\right)e^{-tx}\,dx=-\Psi(t+1)+\ln(t)+\frac{1}{2t}=-\Psi(t)+\ln(t)-\frac{1}{2t},\quad t>0,$$

interesting in its own right. For example, for $x > 0$, the trivial inequality $1 + x < e^x$ gives $1/x > 1/(e^x - 1)$, and hence

$$\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right) e^{-tx}\, dx < \frac{1}{2}\int_0^\infty e^{-tx}\, dx = \frac{1}{2t}, \quad t > 0.$$

On the other hand, for $0 < x < 2$, the trivial inequality $e^x < 1 + x + x^2/(2 - x)$ gives $1/(e^x - 1) > 1/x - 1/2$, $x > 0$ (since this obviously holds for $x \geq 2$). Hence

$$\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right) e^{-tx}\, dx > 0, \quad t > 0.$$

Combining these with the integral representation of the digamma function above, we arrive at the **fundamental estimate of the digamma function**

$$\ln t - \frac{1}{t} < \Psi(t) < \ln t - \frac{1}{2t}, \quad t > 0.$$

This is due to Horst Alzer.[142] The sharper estimate

$$\ln\left(t + \frac{1}{2}\right) - \frac{1}{t} < \Psi(t) < \ln\left(t + e^{-\gamma}\right) - \frac{1}{t}, \quad t > 0.$$

was proved by Elezovic, Giordano and Pecaric.[143]

Next we reexamine Example 4.1.6 from a point of view of the beta and digamma functions:

Example 4.18.4. We have

$$\int_0^{\pi/2} \ln(\sin(x))\, dx = -\frac{\pi}{2}\ln 2.$$

The crux here is that the integral can be obtained from the Wallis integral (Section 4.11)

$$W(t) = \int_0^{\pi/2} \sin^t(x)\, dx = \frac{1}{2}B\left(\frac{t+1}{2}, \frac{1}{2}\right), \quad t > -1,$$

by differentiation

$$W'(0) = \frac{d}{dt}\int_0^{\pi/2} \sin^t(x)\, dx\bigg|_{t=0} = \int_0^{\pi/2} \frac{d}{dt}\sin^t(x)\bigg|_{t=0} dx = \int_0^{\pi/2} \ln(\sin(x))\, dx,$$

[142] See Alzer, H., *On some inequalities for the gamma and psi functions*, Math. Comp. 66 (217) (1997) 373-389.

[143] See Elezovic, N., Giordano, C. and Pecaric, J., *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. 2 (2000) 239-252; and for the constants being the best possible, Qi, F. and Guo. B.-N., *Sharp inequalities for the psi function and harmonic numbers*, arXiv:0902.2524.

where the interchange of differentiation and the improper integral is allowed by the remark following Proposition 4.4.2. (See also the higher derivative of the gamma function in Section 4.6.) We now calculate

$$\begin{aligned}\frac{d}{dt}B\left(\frac{t+1}{2},\frac{1}{2}\right)\Bigg|_{t=0} &= \Gamma\left(\frac{1}{2}\right)\frac{d}{dt}\frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)}\Bigg|_{t=0} = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\left(\Gamma'\left(\frac{1}{2}\right)\Gamma(1)-\Gamma\left(\frac{1}{2}\right)\Gamma'(1)\right)\\ &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)^2\left(\Psi\left(\frac{1}{2}\right)-\Gamma'(1)\right) = \frac{\pi}{2}\left(-\gamma-2\ln 2+\gamma\right) = -\pi\ln 2.\end{aligned}$$

The example follows.

The values of the digamma function are known for all positive rational numbers in $(0,1)$ via the **Gauss digamma theorem**:

Proposition 4.18.2. *Let $p < q$, $p, q \in \mathbb{N}$. Then we have*

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \ln(2q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + \sum_{k=1}^{q-1}\cos\left(2p\pi\frac{k}{q}\right)\ln\sin\left(\pi\frac{k}{q}\right).$$

We will give a proof of this in Section 4.19. One of the stunning features of this formula is that it expresses the value of the digamma function on rational numbers by using only elementary functions. The first few values are as follows

$$\begin{aligned}\Psi\left(\frac{1}{2}\right) &= -\gamma - 2\ln 2\\ \Psi\left(\frac{1}{3}\right) &= -\gamma - \frac{3\ln 3}{2} - \frac{\pi\sqrt{3}}{6}\\ \Psi\left(\frac{1}{4}\right) &= -\gamma - 3\ln 2 - \frac{\pi}{2}\\ \Psi\left(\frac{1}{6}\right) &= -\gamma - 2\ln 2 - \frac{3\ln 3}{2} - \frac{\pi\sqrt{3}}{2}.\end{aligned}$$

We now introduce the **polygamma functions** $\Psi_n : \mathbb{R}\setminus(-\mathbb{N}_0) \to \mathbb{R}$ as the higher derivatives $\Psi_n = \Psi^{(n)}$, $n \in \mathbb{N}$. By repeated differentiation of the series representation of the digamma function at the beginning of this section, we obtain the expansion

$$\Psi_n(t) = (-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{1}{(t+k-1)^{n+1}}, \quad n \in \mathbb{N}, \quad t \in \mathbb{R}\setminus(-\mathbb{N}_0).$$

Substitution ($t = 1$) yields

$$\Psi_n(1) = (-1)^{n+1} n! \zeta(n+1), \quad n \in \mathbb{N}.$$

Differentiating the inductive formula of the digamma function at the beginning of this section, we obtain the **inductive formula for the polygamma functions** as

$$\Psi_n(t+1) = \Psi_n(t) + (-1)^n \frac{n!}{t^{n+1}}, \quad n \in \mathbb{N}, \quad t \in \mathbb{R} \setminus (-\mathbb{N}_0)$$

Differentiating the duplication formula for the digamma function n times, we obtain

$$2^{n+1} \Psi_n(2t) = \Psi_n(t) + \Psi_n(t+1/2), \quad n \in \mathbb{N}.$$

In particular, substituting $t = 1/2$ and using $\Psi_n(1) = (-1)^{n+1} n! \zeta(n+1)$, we get

$$\Psi_n(1/2) = (-1)^{n+1} \left(2^{n+1} - 1\right) n! \zeta(n+1), \quad n \in \mathbb{N}.$$

Moreover, substituting $t = m + 1/2$, $m \in \mathbb{N}$, gives

$$2^{n+1} \Psi_n(2m+1) = \Psi_n(m+1/2) + \Psi_n(m+1), \quad m, n \in \mathbb{N}.$$

We now use

$$\Psi_n(m+1) = (-1)^{n+1} n! \sum_{k=1}^{\infty} \frac{1}{(m+k)^{n+1}} = (-1)^{n+1} n! \left(\zeta(n+1) - \sum_{k=1}^{m} \frac{1}{k^{n+1}} \right),$$

to obtain

$$\Psi_n(m+1/2) = (-1)^{n+1} n! \left((1 - 2^{n+1}) \zeta(n+1) + 2^{n+1} \sum_{k=1}^{2m} \frac{1}{k^{n+1}} - \sum_{k=1}^{m} \frac{1}{k^{n+1}} \right), \ m, n \in \mathbb{N}.$$

(Note that this also holds for $m = 0$ with the finite sums absent.)

The **Legendre-Gauss multiplication theorem for the polygamma functions** can be obtained from that of the digamma function by differentiation in a straightforward way:

$$m^{n+1} \Psi_n(t) = \sum_{k=0}^{m-1} \Psi_n\left(\frac{t+k}{m}\right), \quad m, n \in \mathbb{N}, \ t \in \mathbb{R} \setminus (\mathbb{N}_0).$$

Differentiating both sides of the Gauss representation of the digamma function immediately gives the following integral representation of the polygamma functions

$$\Psi_n(t) = (-1)^{n+1}\int_0^\infty \frac{x^n e^{-tx}}{1-e^{-x}}\,dx = -\int_0^1 \frac{x^{t-1}}{1-t}(\ln(x))^n\,dx, \quad t>0,\ n\in\mathbb{N}.$$

Using the zeta gamma relation, in Example 4.17.1 we gave a formula expressing the Bernoulli numbers as improper integrals. There is yet another (somewhat more subtle) formula of this kind that we are now able to derive as follows.[144]

Example 4.18.5. We have

$$\int_0^\infty \frac{x^{2n-1}}{\sinh(x)}\,dx = (-1)^{n+1}\frac{(2^{2n}-1)\pi^{2n}}{2n}B_{2n}, \quad n\in\mathbb{N}.$$

To derive this, we first differentiate n times both sides of the formula

$$\Psi(t+1) = -\gamma + \int_0^\infty \frac{1-e^{-tx}}{e^x-1}\,dx,$$

and obtain

$$\begin{aligned}\Psi_n(t+1) &= \Psi^{(n)}(t+1) = (-1)^{n+1}\int_0^\infty \frac{x^n e^{-tx}}{e^x-1}\,dx = (-1)^{n+1}2^{n+1}\int_0^\infty \frac{x^n e^{-2tx}}{e^{2x}-1}\,dx\\ &= (-1)^{n+1}2^{n+1}\int_0^\infty \frac{x^n e^{-(2t+1)x}}{e^x-e^{-x}}\,dx = (-1)^{n+1}2^n\int_0^\infty \frac{x^n e^{-(2t+1)x}}{\sinh(x)}\,dx.\end{aligned}$$

At $t=-1/2$, this gives

$$\Psi_n(1/2) = (-1)^{n+1}2^n\int_0^\infty \frac{x^n}{\sinh(x)}\,dx, \quad n\in\mathbb{N}.$$

Combining this with our previous expression of $\Psi_n(1/2)$, we arrive at

$$\int_0^\infty \frac{x^n}{\sinh(x)}\,dx = \frac{2^{n+1}-1}{2^n}\,n!\zeta(n+1), \quad n\in\mathbb{N}.$$

Replacing n by $2n-1$, we obtain

$$\int_0^\infty \frac{x^{2n-1}}{\sinh(x)}\,dx = \frac{2^{2n}-1}{2^{2n-1}}\,(2n-1)!\zeta(2n).$$

Finally, Euler's summation formula finishes the proof.

[144] See also Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020; Exercise, p. 126.

The higher order derivatives of the digamma function at $t = 1$ obtained above can be written as Taylor coefficients

$$\frac{\Psi_n(1)}{n!} = \frac{\Psi^{(n)}(1)}{n!} = (-1)^{n+1}\zeta(n+1), \quad n \in \mathbb{N}.$$

These give the Taylor expansion

$$\Psi(1+t) = -\gamma + \sum_{k=2}^{\infty}(-1)^k \zeta(k)\, t^{k-1}, \quad |t| < 1,$$

or, inserting the geometric series formula:

$$\Psi(1+t) = -\frac{1}{1+t} - (\gamma - 1) + \sum_{k=2}^{\infty}(-1)^k \left(\zeta(k) - 1\right)\, t^{k-1}, \quad |t| < 1.$$

Remark. As an interesting byproduct, the Taylor expansion of Ψ above gives the expansion of the cotangent function at $t = 0$. Indeed, using Euler's reflection formula along with the inductive formula for Ψ as in the beginning of this section, we calculate

$$\begin{aligned}
\pi \cot(\pi t) &= \frac{1}{t} + \Psi(1-t) - \Psi(1+t) = \frac{1}{t} - \sum_{k=2}^{\infty} \zeta(k) t^{k-1} - \sum_{k=1}^{\infty} (-1)^k \zeta(k) t^{k-1} \\
&= \frac{1}{t} - \sum_{k=2}^{\infty} \left(1 + (-1)^k\right) \zeta(k) t^{k-1} = \frac{1}{t} - 2\sum_{k=1}^{\infty} \zeta(2k) t^{2k-1} \\
&= \frac{1}{t} + \sum_{k=1}^{\infty} (-1)^k \frac{(2\pi)^{2k}}{(2k)!} B_{2k} t^{2k-1},
\end{aligned}$$

where, in the last step, we used Euler's summation formula.

One of the advantages of the Taylor expansion of Ψ is that it leads directly (by integration) to the series expansion of the Gamma function:

$$\ln(\Gamma(1+t)) = -\gamma t + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k}\, t^k, \quad |t| < 1,$$

or equivalently

$$\ln(\Gamma(1+t)) = -\ln(1+t) - (\gamma - 1)t + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k} t^k, \quad |t| < 1,$$

Note that, substituting $t = 1$ into the first expansion, we obtain the formula

$$\gamma = \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k}$$

due to Euler. (The alternating series converges.) Using the substitution $t = 1/2$ and rearranging, we obtain

$$\gamma = \ln\left(\frac{4}{\pi}\right) + 2\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k \cdot 2^k}.$$

The series that involve the zeta function on integer values as part of the coefficients converge slowly. To remedy this, we may consider the more symmetric form

$$\frac{1}{2}\ln\frac{\Gamma(1+t)}{\Gamma(1-t)} = -\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) - (\gamma - 1)t - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{2k+1} t^{2k+1}.$$

We also observe that the Euler reflection formula (Proposition 4.9.1) can be written as

$$\Gamma(1+t)\Gamma(1-t) = \frac{\pi t}{\sin(\pi t)}.$$

Taking the natural logarithm, and using it in the formula above, we arrive at the faster converging formula due to Legendre:

$$\ln\Gamma(1+t) = \frac{1}{2}\ln\left(\frac{\pi t}{\sin(\pi t)}\right) - \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) - (\gamma - 1)t - \sum_{k=1}^{\infty} \frac{(\zeta(2k+1) - 1)}{2k+1} t^{2k+1}$$

History. The symmetrized formula for $t = 1/2$ specializes to

$$\gamma = 1 - \ln\left(\frac{3}{2}\right) - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{4^k(2k+1)}.$$

In 1887 Stieltjes calculated $\zeta(k)$, $k = 2, \ldots, 70$ up to 32 decimal precisison (extending previous computations of Legendre for $k = 2, \ldots, 35$ up to 16 digits). He then used this formula to calculate γ up to 32 decimal digits.

In the remainder of this section we give various estimates on the polygamma functions.

We start with the trivial estimate $1 + x < e^x$, $x > 0$, of the natural exponential function. We write this as $x/(1-e^{-x}) < 1+x$, $x>0$. Multiplying through by $x^{n-1}e^{-tx}$, $n\in\mathbb{N}$, and integrating, we obtain

$$\int_0^\infty \frac{x^n e^{-tx}}{1-e^{-x}}\,dx < \int_0^\infty x^{n-1}e^{-tx}\,dx + \int_0^\infty x^n e^{-tx}\,dx, \quad t>0.$$

By Example 4.6.1, and the integral formula for the polygamma functions, this gives

$$(-1)^{n+1}\Psi_n(t) < \frac{(n-1)!}{t^n} + \frac{n!}{t^{n+1}}, \quad t>0,\ n\in\mathbb{N}.$$

On the other hand, since $\tanh(x) < x$, $x>0$, we have $1 < (x/2)\coth(x/2)$, $x>0$. We

$$\frac{x}{2}\coth\left(\frac{x}{2}\right) = \frac{x}{2}\,\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{x}{2}\left(\frac{2}{1-e^{-x}}-1\right) = \frac{x}{1-e^{-x}} - \frac{x}{2},$$

so that we obtain

$$1+\frac{x}{2} < \frac{x}{1-e^{-x}}, \quad x>0.$$

As before this gives

$$\frac{(n-1)!}{t^n} + \frac{n!}{2t^{n+1}} < (-1)^{n+1}\Psi_n(t), \quad t>0,\ n\in\mathbb{N}.$$

Combining these, we obtain the **fundamental estimate for the polygamma functions**

$$\frac{(n-1)!}{t^n} + \frac{n!}{2t^{n+1}} < (-1)^{n+1}\Psi_n(t) < \frac{(n-1)!}{t^n} + \frac{n!}{t^{n+1}}, \quad t>0,\ n\in\mathbb{N}.$$

A sharper upper bound can be obtained by using the inequality

$$\frac{x}{e^x-1} < 1-\frac{x}{2}+\frac{x^2}{12}, \quad x>0.$$

Although the right-hand side is the first three terms of the series expansion of the left-hand side in terms of the Bernoulli numbers (Section 4.12), an elementary proof of this inequality can be obtained by using $\sum_{k=1}^{6} x^k/k! < e^x-1$, $x>0$, and by noting that

$$\left(1-\frac{x}{2}+\frac{x^2}{12}\right)\sum_{k=1}^{6}\frac{x^k}{k!} = x + \frac{x^5}{6!} + \frac{x^6}{2\cdot 6!} + \frac{x^8}{12\cdot 6!}, \quad x>0.$$

Now, the inequality just proved can be written as

$$\frac{x}{1-e^{-x}} < 1+\frac{x}{2}+\frac{x^2}{12}, \quad x>0.$$

The procedure above then gives

$$(-1)^{n+1}\Psi_n(t) < \frac{(n-1)!}{t^n} + \frac{n!}{2t^{n+1}} + \frac{(n+1)!}{12t^{n+2}}, \quad t > 0,\ n \in \mathbb{N}.$$

Remark. For $n = 1$, an alternative derivation[145] of the inequalities

$$\frac{1}{t} + \frac{1}{2t^2} < \Psi_1(t) < \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3}, \quad t > 0,$$

can be given as follows. Define $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \Psi_1(t), \quad t > 0.$$

Then, by the inductive formula for Ψ_1, we have $\Psi_1(t) - \Psi_1(t+1) = 1/t^2$, so that

$$\begin{aligned} f(t+1) - f(t) &= \frac{1}{t+1} + \frac{1}{2(t+1)^2} + \frac{1}{6(t+1)^3} - \frac{1}{t} - \frac{1}{2t^2} - \frac{1}{6t^3} + \frac{1}{t^2} \\ &= -\frac{1}{6(t+1)^3 t^3} < 0. \end{aligned}$$

Hence, inductively,

$$f(t) > f(t+1) > f(t+2) > \cdots > \lim_{t\to\infty} f(t) = 0,$$

where we used that $\lim_{t\to\infty} \Psi_1(t) = 0$. The upper bound follows. The proof for the lower bound is similar (with the cubic term absent).

As an application of these inequalities, and complementing Proposition 4.18.1, we now prove that the digamma function satisfies the following harmonic mean inequality due to Horst Alzer and Graham Jameson:[146]

Proposition 4.18.3. *We have*

$$\frac{2}{\frac{1}{\Psi(t)} + \frac{1}{\Psi(1/t)}} \geq -\gamma,$$

where equality holds if and only if $t = 1$.

[145] This method is used in Guo, B.-N. and Qi, F., *Refinements of lower bounds for polygamma functions*, Proc. Amer. Math. Soc., Vol. 141, No. 3 (March 2013) 1007-1015; see the proof of Lemma 1 for the weaker upper bound $\Psi_1(t) < e^{1/t} - 1$, $t > 0$. The extension of this method for $2 \leq n \in \mathbb{N}$ is possible but leads to combinatorial complexity.

[146] See Alzer, J. and Jameson, G., *A harmonic mean inequality for the digamma function and related results*, Rend. Sem. Mat. Univ. Padova, Vol. 137 (2017) 203-209.

Recall that, as the series expansion of the digamma function Ψ shows, it is strictly increasing (everywhere) as well as **strictly concave** on $(0,\infty)$. In addition, as noted in the proof of Proposition 4.18.1 above, $t^2\Psi'(t)$ is strictly increasing on $(0,\infty)$. Since

$$\frac{d}{dt}\Psi\left(\frac{1}{t}\right) = -\frac{1}{t^2}\Psi'\left(\frac{1}{t}\right), \quad t>0,$$

we see that $\Psi(1/t)$ is **strictly convex** on $(0,\infty)$.
For the proof of Proposition 4.18.3, we need three lemmas.

Lemma 1. *We have*

$$\Psi(t)+\Psi\left(\frac{1}{t}\right) \le -2\gamma.$$

The upper bound is attained if and only if $t=1$.

Proof. We define the function $F:(0,\infty)\to\mathbb{R}$ by $F(t)=\Psi(t)+\Psi(1/t)$, $t>0$. For future reference, note that $F(1)=-2\gamma$ and $F'(1)=0$. We now claim that F is **strictly concave**.To show this, we differentiate as

$$\begin{aligned} t^4F''(t) &= 2t\Psi'\left(\frac{1}{t}\right)+\Psi''\left(\frac{1}{t}\right)+t^4\Psi''(t) \\ &= 2t\left(\Psi'\left(1+\frac{1}{t}\right)+t^2\right)+\Psi''\left(1+\frac{1}{t}\right)-2t^3+t^4\Psi''(t) \\ &= 2t\Psi'\left(1+\frac{1}{t}\right)+\Psi''\left(1+\frac{1}{t}\right)+t^4\Psi''(t), \end{aligned}$$

where in the second step we used the inductive formulas for the polygamma functions $\Psi_1=\Psi'$ and Ψ_2''. Finally, we bring in the the fundamental estimate for Ψ'' and that of Ψ' with the refined upper bound as above, and calculate

$$\begin{aligned} t^4F''(t) &= 2t\left(\frac{1}{1+1/t}+\frac{1}{2(1+1/t)^2}+\frac{1}{6(1+1/t)^3}\right) \\ &\quad -\frac{1}{(1+1/t)^2}-\frac{1}{(1+1/t)^3}-t^4\left(\frac{1}{t^2}+\frac{1}{t^3}\right) \\ &= -\frac{t}{3(t+1)^3}\left(3t^4+2t^3+9t^2+9t+3\right)<0, \quad t>0. \end{aligned}$$

The claim follows.
Since F is strictly concave, F' is strictly decreasing. Since $F'(1)=0$, it follows that F' is positive on $(0,1)$ and negative on $(1,\infty)$. Hence, F attains its unique absolute

maximum at $t = 1$ with value -2γ. Reverting to the definition of F, we obtain the estimate

$$\Psi(t) + \Psi\left(\frac{1}{t}\right) \leq -2\gamma,$$

with strict inequality for $t \neq 1$.

Lemma 2. *For $0 < s < 1$, we have*

$$\Psi(1+s)\Psi(1-s) < \gamma^2,$$

and the upper bound is sharp.

PROOF. For $0 < x_0 - 1 \leq s < 1$ the inequality is obvious since then $(0 <)\, 1 - s < x_0 \leq 1 + s$, and hence $\Psi(1-s) \leq 0 \leq \Psi(1+s)$. Thus, we may assume that $0 < s < x_0 - 1\,(< 1/2)$. We now use the Taylor series expansion of Ψ above as

$$-\Psi(1+s) = \gamma + \sum_{k=2}^{\infty}(-1)^{k+1}\zeta(k)s^{k-1}.$$

(Note that s is within the radius of convergence 1.) Since the series is alternating, we obtain

$$0 < -\Psi(1+s) \leq \gamma - \zeta(2)s + \zeta(3)s^2,$$

and

$$0 < -\Psi(1-s) \leq \gamma + \zeta(2)s + \zeta(3)\sum_{k=2}^{\infty}s^k = \gamma + \zeta(2)s + \zeta(3)\frac{s^2}{1-s} \leq \gamma + \zeta(2)s + 2\zeta(3)s^2.$$

Combining these, we obtain

$$\Psi(1-s)\Psi(1+s) \leq \gamma^2 - (\zeta(2)^2 - 3\gamma\zeta(3))s^2 - \zeta(2)\zeta(3)s^3 + 2\zeta(3)^2s^4.$$

Now, numerical evaluation shows that $\zeta(3) < \pi^4/(108\gamma)$ so that the coefficient of the quadratic term (s^2) is negative. Since $\zeta(2)\zeta(3) > \zeta(3)^2 > 2\zeta(3)^2 s$, the sum of the last two terms is also negative. The lemma follows. Note finally that $\Psi(1)^2 = \gamma^2$ so that the estimate is sharp.

Lemma 3. *We have*

$$\Psi(t)\Psi\left(\frac{1}{t}\right) \leq \gamma^2.$$

The upper bound is attained if and only if $t = 1$.

PROOF. We may assume that $t > 1$. Moreover, if $t \geq x_0$ then $\Psi(1/t) < 0 \leq \Psi(t)$, so that we may actually assume $1 < t \leq x_0$. Letting $t = 1+s$, we have $0 < s < x_0 - 1 < 1$ and $1 - s < 1/(1+s) = 1/t$. Hence, by Lemma 2:

$$\Psi(t)\Psi\left(\frac{1}{t}\right) < \Psi(1+s)\Psi(1-s) < \gamma^2,$$

as $\Psi(t) = \Psi(1+s) < 0$. The lemma follows.

PROOF OF PROPOSITION 4.18.3. Combining Lemmas 1 and 3, for $t > 0$, $t \neq 1$, we calculate

$$2\frac{\Psi(t)\Psi(1/t)}{\Psi(t) + \Psi(1/t)} > 2\frac{\gamma^2}{\Psi(t) + \Psi(1/t)} > 2\frac{\gamma^2}{-2\gamma^2} = -\gamma.$$

The proposition follows.

Exercises

1. Use the representation of $\ln \Gamma$ as in Example 4.18.2 along with the technique in Example 4.18.3 to derive the following formula of Kummer[147]

$$\int_0^1 \frac{x^{t-1} - x^{s-1}}{(1+x)\ln(x)}\, dx = \ln\left(\frac{\Gamma\left(\frac{t+1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{t}{2}\right)}\right).$$

2. Use the Gauss representation of Ψ to obtain

$$\Psi(t) = -\int_0^1 \left(\frac{x^{t-1}}{1-x} + \frac{1}{\ln(x)}\right) dx, \quad t > 0.$$

3. Differentiate the formula $\Gamma' = \Gamma \cdot \Psi$ and evaluate it at 1 to obtain

$$\Gamma^{(n+1)}(1) = -\gamma\, \Gamma^{(n)}(1) + \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{(n-k)!} \zeta(k+1) \Gamma^{(n-k)}(1).$$

Use this inductively to derive an expression for

$$\Gamma^{(m)}(1) = \int_0^\infty e^{-x} (\ln(x))^m\, dx, \quad m = 2, 3, 4, \ldots,$$

[147] Ibid.

in terms of γ and the zeta function on positive integers. The first few cases are

$$\begin{aligned}
\int_0^\infty e^{-x}\ln(x)\,dx &= -\gamma\\
\int_0^\infty e^{-x}(\ln(x))^2\,dx &= \frac{\pi^2}{6}+\gamma^2\\
\int_0^\infty e^{-x}(\ln(x))^3\,dx &= -\frac{\pi^2}{2}\gamma-\gamma^3-2\zeta(3)\\
\int_0^\infty e^{-x}(\ln(x))^4\,dx &= \frac{3\pi^4}{20}+\pi^2\gamma^2+\gamma^4+8\gamma\zeta(3)\\
\int_0^\infty e^{-x}(\ln(x))^5\,dx &= -\frac{3\pi^4}{4}\gamma-\frac{5\pi^2}{3}\gamma^3-\gamma^5-\frac{10\pi^2}{3}\zeta(3)-20\gamma^2\zeta(3)-24\zeta(5).
\end{aligned}$$

4. Take the mth derivative, $m\in\mathbb{N}$, of both sides of the formula

$$\int_0^\infty x^{t-1}\sin(x)\,dx=\Gamma(t)\sin\left(\frac{\pi t}{2}\right)=\Gamma(t+1)\frac{\sin\left(\frac{\pi t}{2}\right)}{t},\quad 0\le t<1,$$

(see the remark at the end of 4.6) to obtain

$$\begin{aligned}
\int_0^\infty \frac{\sin(x)}{x}(\ln(x))^m\,dx &= \sum_{k=0}^{m}\binom{m}{k}\frac{d^k}{dt^k}\frac{\sin\left(\frac{\pi t}{2}\right)}{t}\Bigg|_{t=0}\Gamma^{(m-k)}(1)\\
&= \sum_{k=0}^{[m/2]}\frac{(-1)^k}{2k+1}\binom{m}{2k}\left(\frac{\pi}{2}\right)^{2k+1}\Gamma^{(m-2k)}(1).
\end{aligned}$$

Now, use the previous exercise to compile the following list:

$$\begin{aligned}
\int_0^\infty \frac{\sin(x)}{x}\ln(x)\,dx &= -\frac{\pi}{2}\gamma\\
\int_0^\infty \frac{\sin(x)}{x}(\ln(x))^2\,dx &= \frac{\pi^3}{24}+\frac{\pi}{2}\gamma^2\\
\int_0^\infty \frac{\sin(x)}{x}(\ln(x))^3\,dx &= -\frac{\pi^3}{8}\gamma-\frac{\pi}{2}\gamma^3-\pi\zeta(3)\\
\int_0^\infty \frac{\sin(x)}{x}(\ln(x))^4\,dx &= \frac{19\pi^5}{480}+\frac{\pi^3}{4}\gamma^2+\frac{\pi}{2}\gamma^4+4\pi\gamma\zeta(3)\\
\int_0^\infty \frac{\sin(x)}{x}(\ln(x))^5\,dx &= -\frac{19\pi^5}{96}\gamma-\frac{5\pi^3}{12}\gamma^3-\frac{\pi}{2}\gamma^5-\frac{5\pi^3}{6}\zeta(3)-10\pi\gamma^2\zeta(3)-12\pi\zeta(5).
\end{aligned}$$

4.19 The Hurwitz Zeta Function

The **Hurwitz zeta function** is defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\quad 1<s\in\mathbb{R},\ 0<a\in\mathbb{R}.$$

In what follows, we will tacitly assume that the 'parameter' a is a positive real number as powers with negative base (such as $a^s=e^{s\ln(a)}$) would lead to complex numbers and multiple valuedness.[148] Clearly, the Hurwitz zeta function is absolutely convergent for $s>1$, and uniformly convergent in s on any closed interval in $(1,\infty)$.
By definition, we have $\zeta(s,1)=\zeta(s)$, the Riemann zeta function.

The **functional equation of the Hurwitz zeta function** follows directly from the definition:[149]

$$\zeta(s,a+1)=\zeta(s,a)-\frac{1}{a^s},\quad 1<s\in\mathbb{R}.$$

By induction, for $n\in\mathbb{N}$, we obtain

$$\zeta(s,a)=\zeta(s,a+n)+\sum_{k=0}^{n-1}\frac{1}{(a+k)^s},\quad 1<s\in\mathbb{R}.$$

Remark. The expansion of the polygamma function Ψ_n in Section 4.18 reveals its relation to the Hurwitz zeta function as

$$\zeta(n+1,a)=\frac{(-1)^{n+1}}{n!}\Psi_n(a),\quad n\in\mathbb{N}.$$

Indeed, we have

$$\Psi_n(a)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(a+k)^{n+1}}=(-1)^{n+1}n!\zeta(n+1,a),$$

where we moved the index of the summation down by one.

History Adolf Hurwitz (1859 – 1919) was 22 years old when he invented the generalization of the Riemann zeta function named after him. A year later he already finished his habilitation and worked as a private dozent in Göttingen under the supervision of Felix Klein (1849 – 1925), and in contact with his mentor Weierstrass in Berlin and his friend Luigi Bianchi (1856 –1928) in Munich. Hans von Mangoldt (1854 – 1925) was also a private dozent in Göttingen at the time.

[148] Most authors, including Hurwitz himself, considered only $0<a\leq 1$.
[149] It is because of this that most authors restrict to $0<a\leq 1$.

As expected, the Hurwitz zeta function participates in several (mostly integral) formulas that are similar to the ones we developed in the previous sections. The first and most basic integral representation is the following:

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\, dx, \quad 1 < s \in \mathbb{R}.$$

For $a = 1$, this reduces to the zeta gamma relation. For this reason, we call this the **Hurwitz zeta gamma relation**.
To derive this, we calculate

$$\begin{aligned} \zeta(s,a)\Gamma(s) &= \sum_{n=0}^\infty \frac{1}{(n+a)^s} \int_0^\infty x^{s-1}e^{-x}\, dx = \sum_{n=0}^\infty \int_0^\infty x^{s-1}e^{-(n+a)x}\, dx \\ &= \int_0^\infty x^{s-1}e^{-ax} \sum_{n=0}^\infty e^{-nx}\, dx = \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\, dx, \end{aligned}$$

where, in the second step, we replaced x by $(n+a)x$ within the infinite sum. The integral formula follows.

As for the gamma and zeta functions, we could use this formula to show that the Hurwitz zeta function is analytic in $s > 1$. We defer the proof of analiticity to a later development; for a direct proof using this formula, see Exercise 1 at the end of this section.

We wish to extend the definition of the Hurwitz zeta function below $1 < s$. The simplest extension (to $-1 < s$) is the following.
Assuming $0 < a \leq 1$, a simple algebraic manipulation of the original definition gives

$$\zeta(s,a) = \zeta(s) - as\zeta(s+1) + \frac{1}{a^s} + \sum_{n=1}^\infty \frac{\left(1+\frac{a}{n}\right)^{-s} - \left(1 - s\frac{a}{n}\right)}{n^s}.$$

By Newton's binomial theorem, for fixed $n \in \mathbb{N}$, the numerator of the fraction under the infinite sum can be written as

$$\left(1+\frac{a}{n}\right)^{-s} - \left(1 - s\frac{a}{n}\right) = \frac{1}{n^2}\sum_{k=2}^\infty (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}\frac{a^k}{n^{k-2}}.$$

For fixed $n \in \mathbb{N}$, the power series (in a) here is absolutely convergent since $0 < a \leq 1$ and the radius of convergence is n; and actually the convergence is uniform on closed intervals in $(0,1]$ (Section 2.3.) Combining this fact with the denominator $n^s \cdot n^2 = n^{s+2}$, we see that the infinite series above (in n) converges for $s + 2 > 1$; that is, for $s > -1$. Since the zeta function is defined everywhere except at $s = 1$ where it has a

simple pole, this shows that the the formula above defines the Hurwitz zeta function for $s > -1$ except at $s = 1$.
As an immediate byproduct, for $s = 0$ (in the limiting sense), the formula above gives

$$\zeta(0, a) = -\frac{1}{2} - a \lim_{s\to 0} s\zeta(s + 1) + 1 = \frac{1}{2} - a \lim_{s\to 1}(s - 1)\zeta(s) = \frac{1}{2} - a,$$

where we used $\zeta(0) = -1/2$.

As for a more sophisticated extension, in 1930 Helmut Hasse derived the following series representation of the Hurwitz zeta function:

$$\zeta(s, a) = \frac{1}{s - 1} \sum_{n=0}^{\infty} \frac{1}{n + 1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a + k)^{1-s}.$$

The advantage of this formula is that the infinite series is uniformly convergent on any closed interval (see the remark below).
To derive Hasse's formula (for $s > 1$), first use a special case of Example 4.6.1:

$$(a + k)^{1-s} = \frac{1}{\Gamma(s - 1)} \int_0^{\infty} x^{s-2} e^{-(a+k)x}\, dx, \quad s > 1.$$

Using $\Gamma(s-1) = \Gamma(s)/(s-1)$, upon substitution and rearrangement of the right-hand side, we obtain

$$\begin{aligned}
&\frac{1}{s - 1} \sum_{n=0}^{\infty} \frac{1}{n + 1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a + k)^{1-s} \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{1}{n + 1} \int_0^{\infty} \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} (e^{-x})^k \right) x^{s-2} e^{-ax}\, dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{1}{n + 1} \int_0^{\infty} (1 - e^{-x})^n x^{s-2} e^{-ax}\, dx \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(1 - e^{-x})^n}{n + 1} x^{s-2} e^{-ax}\, dx \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}}\, dx = \zeta(s, a),
\end{aligned}$$

where we used

$$\sum_{n=1}^{\infty} (-1)^n \frac{(1 - e^{-x})^n}{n} = -\ln(e^{-x}) = x, \quad x \geq 0.$$

The Hasse formula follows.

Remark. Using powers of the forward difference operator Δ (Section 4.12), the finite sum in Hasse's formula can be concisely interpreted as

$$\Delta^n a^{1-s} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (a+k)^{1-s},$$

where $\Delta(a^{1-s}) = (a+1)^{1-s} - a^{1-s}$, $\Delta^2(a^{1-s}) = (a+2)^{1-s} - 2(a+1)^{1-s} + a^{1-s}$, etc. With this, Hasse's formula takes the form

$$\zeta(s,a) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^n (a^{1-s}) = \frac{1}{s-1} \frac{1}{\Delta} \ln(I+\Delta)(a^{1-s}).$$

A thoroughly elementary, albeit fairly long-winded, argument then proves that the infinite series in Hasse's formula converges uniformly on closed intervals, and that the Hurwitz zeta extension defined by this formula is analytic everywhere except having a simple pole at $s = 1$.[150]

We now begin to derive Euler-Maclaurin summation formulas for the Hurwitz zeta function. The treatment here parallels those in Section 4.17.
Using the first Euler-Maclaurin formula (with index starting at $k = 0$) in Section 4.16, we obtain

$$\sum_{k=0}^{n} \frac{1}{(k+a)^s} = \int_0^n \frac{dx}{(x+a)^s}\, dx - s \int_0^n \frac{P_1(x)}{(x+a)^{s+1}}\, dx + \frac{1}{2}\left(\frac{1}{(n+a)^s} + \frac{1}{a^s} \right), \ n \in \mathbb{N},$$

where we used $(d/dx)(x+a)^{-s} = -s(x+a)^{-s-1}$. The first integral calculates as

$$\int_0^n \frac{dx}{(x+a)^s}\, dx = \frac{1}{1-s}\left(\frac{1}{(n+a)^{s-1}} - \frac{1}{a^{s-1}} \right).$$

Substituting and rearranging, we obtain

$$\sum_{k=0}^{n} \frac{1}{(k+a)^s} = \frac{1}{2(n+a)^s} - \frac{1}{s-1}\frac{1}{(n+a)^{s-1}} + \frac{1}{s-1}\frac{1}{a^{s-1}} + \frac{1}{2a^s} - s\int_0^n \frac{P_1(x)}{(x+a)^{s+1}}\, dx.$$

Letting $n \to \infty$, we arrive at the **first Euler-Maclaurin formula for the Hurwitz zeta function**:

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \frac{1}{s-1}\frac{1}{a^{s-1}} + \frac{1}{2a^s} - s\int_0^{\infty} \frac{P_1(x)}{(x+a)^{s+1}}\, dx.$$

[150] See the original paper Hasse, H., *Ein Summierungsverfahren für die Riemannsche ζ-Reihe*, Mathematische Zeitschrift, 32 (1) (1930) 458-464; and, for a modern approach, Kanoussis, D.P., *A new proof of H. Hasse's global expression for the Riemann's zeta function*, https://reserachgate.net/publication/317823796, and the references therein.

The improper integral converges absolutely for $s > 0$, and conditionally for $s > -1$. As we will see below, it is also analytic for $s > 0$. This implies that $\zeta(s,a)$ is analytic for $s > 0$ with a simple pole at $s = 1$ of residue 1.

Example 4.19.1. For $-1 < s < 0$ and $0 < a \leq 1$, we have

$$\zeta(s,a) = -s \int_{-a}^{\infty} \frac{P_1(x)}{(x+a)^{s+1}}\, dx,$$

Indeed, we have

$$\begin{aligned}
\int_{-a}^{0} \frac{P_1(x)}{(x+a)^{s+1}}\, dx &= \int_{-a}^{0} \frac{x-[x]-1/2}{(x+a)^{s+1}}\, dx = \int_{-a}^{0} \frac{x+1/2}{(x+a)^{s+1}}\, dx \\
= \int_{0}^{a} \frac{u-a+1/2}{u^{s+1}}\, du &= \int_{0}^{a} \frac{du}{u^{s}} - \left(a - \frac{1}{2}\right) \int_{0}^{a} \frac{du}{u^{s+1}}\, du \\
&= -\frac{1}{s-1}\frac{1}{a^{s-1}} + \frac{1}{s}\left(a - \frac{1}{2}\right)\frac{1}{a^{s}},
\end{aligned}$$

where we used the substitution $x = u - a$. Incorporating this into the formula above, the example follows.

We now return to the first Euler-Maclaurin formula for the Hurwitz zeta function, and integrate by parts as

$$\int_{0}^{\infty} \frac{P_1(x)}{(x+a)^{s+1}}\, dx = \frac{1}{2}\int_{0}^{\infty} \frac{P_2'(x)}{(x+a)^{s+1}}\, dx = -\frac{1}{12a^{s+1}} + \frac{s+1}{2}\int_{0}^{\infty} \frac{P_2(x)}{(x+a)^{s+1}}\, dx,$$

where the contribution from the boundary term is

$$\left[\frac{P_2(x)}{(x+a)^{s+1}}\right]_0^{\infty} = -\frac{P_2(0)}{a^{s+1}} = -\frac{B_2}{a^{s+1}} = -\frac{1}{6a^{s+1}}.$$

Substituting, we obtain

$$\zeta(s,a) = \frac{1}{s-1}\frac{1}{a^{s-1}} + \frac{1}{2a^{s}} + \frac{s}{12a^{s+1}} - \frac{s(s+1)}{2}\int_{0}^{\infty} \frac{P_2(x)}{(x+a)^{s+2}}\, dx$$

Since the improper integral converges absolutely for $s > -1$ (and conditionally for $s > -2$) this gives an extension of the Hurwitz zeta function for $s > -1$. It also shows that $\zeta(0,a) = 1/2 - a$, a result that we obtained previously.
We recognize the Bernoulli numbers in the coefficients, and write

$$\zeta(s,a) = \frac{1}{s-1}\frac{1}{a^{s-1}} - \frac{B_1}{a^{s}} + \frac{s}{2!}\frac{B_2}{a^{s+1}} - \frac{s(s+1)}{2}\int_{0}^{\infty} \frac{P_2(x)}{(x+a)^{s+2}}\, dx.$$

For future purposes, note also that $\zeta(0,a) = 1/2 - a = -B_1(a)$, where $B_1(a)$ is the first Bernoulli polynomial in the indeterminate a.

We now perform integration by parts inductively, and arrive at the **general Euler-Maclaurin formula for the Hurwitz zeta function**:

$$\begin{aligned}\zeta(s,a) &= \frac{1}{(s-1)a^{s-1}} - \frac{B_1}{a^s} + \frac{s}{2!}\frac{B_2}{a^{s+1}} + \frac{s(s+1)}{3!}\frac{B_3}{a^{s+2}} + \cdots \\ &\qquad \cdots + \frac{s(s+1)(s+2)\cdots(s+m-1)}{(m+1)!}\frac{B_{m+1}}{a^{s+m}} \\ &- \frac{s(s+1)(s+2)\cdots(s+m)}{(m+1)!}\int_0^\infty \frac{P_{m+1}(x)}{(x+a)^{s+m+1}}\,dx, \quad s > -m,\ m \in \mathbb{N}.\end{aligned}$$

Before going any further, we derive the following important connection between the values of the Hurwitz zeta function on non-positive integers and the Bernoulli polynomials:

$$\zeta(-m,a) = -\frac{B_{m+1}(a)}{m+1}, \quad m \in \mathbb{N}_0.$$

Indeed, a direct substitution in the general Euler-Maclaurin formula for the Hurwitz zeta function gives

$$\begin{aligned}\zeta(m,a) &= -\frac{1}{m+1}\left(a^{m+1} + \frac{m+1}{1!}B_1 a^m + \frac{(m+1)m}{2!}B_2 a^{m-1}\right. \\ &\qquad \left. + \frac{(m+1)m(m-1)}{3!}B_3 a^{m-2} + \cdots + B_{m+1}\right) \\ &= -\frac{1}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}B_k a^{m-k+1} = -\frac{B_{m+1}(a)}{m+1},\end{aligned}$$

where the alternating signs disappear as every odd Bernoulli number vanishes except B_1, and, in the last equality, we used the expansion of the Bernoulli polynomials in terms of the Bernoulli numbers (Section 4.12). The stated formula follows.

As an obvious byproduct, for $s = -m$, $m \in \mathbb{N}_0$, the Hasse formula above gives

$$\zeta(-m,a) = -\frac{1}{m+1}\sum_{n=0}^{m+1}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(a+k)^{m+1} = -\frac{B_{m+1}(a)}{m+1}.$$

(The sum in k, for $s = -m$, interpreted as $\Delta^n(a^{m+1})$ is zero for $n > m+1$ since the forward difference operator Δ reduces the degree of polynomials by one; see the remark after the proof of Hasse's formula.) Hence

$$B_m(x) = \sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^m, \quad m \in \mathbb{N}_0.$$

where we replaced a with the generic variable x.

In view of this, the Hurwitz zeta function may be considered as a generalization of the Bernoulli polynomials to non-integer values.

We now return to the general Euler-Maclaurin formula above, and show that it implies analiticity of the Hurwitz zeta function $\zeta(s,a)$ in $s > -m$, $m \in \mathbb{N}$. The treatment here parallels the analogous result for the zeta function in Section 4.17. The nth derivative, $n \in \mathbb{N}$, of the improper integral in the general Euler-Maclaurin formula is

$$\frac{d^n}{ds^n}\int_0^\infty \frac{P_{m+1}(x)}{(x+a)^{s+m+1}}\,dx = (-1)^n \int_0^\infty \frac{P_{m+1}(x)}{(x+a)^{s+m+1}}\left(\ln(x+a)\right)^n\,dx$$
$$= (-1)^n \int_a^\infty \frac{P_{m+1}(u-a)}{u^{s+m+1}}\left(\ln(u)\right)^n\,du,$$

where the differentiation was interchanged with the improper integral by Proposition 4.4.2. For $n \in \mathbb{N}_0$, we estimate the last integral as

$$\left|\int_a^\infty \frac{P_{m+1}(u-a)}{u^{s+m+1}}\left(\ln(u)\right)^n\,du\right| \leq K\int_a^\infty \frac{|\ln(u)|^n}{u^{s+m+1}}\,du,$$

where $K = \sup_{u\in[0,1]}|B_{m+1}(u)|$ (Section 4.14). Now, if $a \geq 1$ then, as in the case of the zeta function, this is dominated by $\leq Kn!/(s+m)^{n+1}$. If $0 < a < 1$, then the last integral estimate needs to be augmented by

$$\int_a^1 \frac{(-\ln(u))^n}{u^{s+m+1}}\,du \leq (-\ln(a))^n \int_a^1 \frac{du}{u^{s+m+1}} = \frac{(-\ln(a))^n}{s+m}\left(\frac{1}{a^{s+m}}-1\right), \quad s > -m.$$

In either case the claimed analiticity now follows (Section 2.4). In particular, by unique continuation, the extension of the Hurwitz zeta function at the beginning of this section is the same as the one we just obtained.

The following result, due to Lerch,[151] is a generalization ($s = 1$) of Example 4.18.1:

Proposition 4.19.1. *We have*

$$\left.\frac{d\zeta(s,a)}{ds}\right|_{s=0} = \ln\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right).$$

Proof. We begin with differentiating both sides of the defining equality of the Hurwitz zeta function as

$$\frac{d^2\zeta(s,a)}{da^2} = s(s+1)\sum_{n=0}^\infty \frac{1}{(n+a)^{s+2}}.$$

[151] See Weil, A., *Elliptic Functions according to Eisenstein and Kronecker*, Springer-Verlag, New York, 1976, p. 60.

This gives

$$\frac{d^2}{da^2}\frac{d\zeta(s,a)}{ds}\bigg|_{s=0} = \frac{d}{ds}\frac{d^2\zeta(s,a)}{da^2}\bigg|_{s=0} = \sum_{n=0}^{\infty}\frac{1}{(n+a)^2} = \frac{d^2\ln\Gamma(a)}{da^2},$$

where the last equality is a property of the logarithmic derivative of the gamma function (Section 4.10). Hence

$$\frac{d^2}{da^2}\left(\frac{d\zeta(s,a)}{ds}\bigg|_{s=0} - \ln\Gamma(a)\right) = 0,$$

and we get

$$\frac{d\zeta(s,a)}{ds}\bigg|_{s=0} = \ln\Gamma(a) + Bs + C,$$

for some $B, C \in \mathbb{R}$.
On the other hand, differentiating the functional equation of the Hurwitz zeta function, we obtain

$$\frac{d\zeta(s,a+1)}{ds}\bigg|_{s=0} - \frac{d\zeta(s,a)}{ds}\bigg|_{s=0} = \ln(a).$$

Combining these, we obtain $B = 0$ since $\ln\Gamma(a+1) = \ln(a\Gamma(a)) = \ln(a) + \ln\Gamma(a)$. Finally, for $a = 1$, Example 4.18.1 gives $C = -\ln\sqrt{2\pi}$. The proposition follows.

Remark. In the course of the proof, we established the following

$$\frac{d^m\zeta(s,a)}{da^m} = (-1)^m s(s+1)\cdots(s+m-1)\zeta(s+m,a), \quad s \neq 1, 0, -1, \ldots, -m+1,\ m \in \mathbb{N}.$$

Another simple consequence of the Hurwitz zeta gamma relation is obtained by a simple rearrangement and various uses of the gamma function:

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{1}{\Gamma(s)}\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right)\frac{x^{s-1}}{e^{ax}}\,dx.$$

Indeed, we calculate the integral as

$$\begin{aligned}\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x-1}\right)\frac{x^{s-1}}{e^{ax}}\,dx &= \frac{1}{2}\int_0^\infty x^{s-1}e^{-ax}\,dx - \int_0^\infty x^{s-2}e^{-ax}\,dx \\ &+ \int_0^\infty \frac{x^{s-1}e^{-(a+1)x}}{1-e^{-x}}\,dx = \frac{\Gamma(s)}{2a^s} - \frac{\Gamma(s-1)}{2a^{s-1}} + \Gamma(s)\zeta(s,a+1) \\ &= \Gamma(s)\left(\frac{a^{-s}}{2} - \frac{a^{1-s}}{s-1} + \zeta(a,s) - a^{-s}\right) = \Gamma(s)\left(-\frac{a^{-s}}{2} - \frac{a^{1-s}}{s-1} + \zeta(a,s)\right),\end{aligned}$$

where we also used the functional equation for the Hurwitz zeta function. The stated integral formula now follows.

Note the special case ($a = 1$):

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{x^{s-1}}{e^x} \, dx.$$

As a consequence of integral formula just derived, we show that

$$\lim_{s \to 1} \left(\zeta(s, a) - \frac{1}{s-1} \right) = -\Psi(a), \quad a > 0.$$

To prove this, we calculate

$$\lim_{s \to 1} \left(\zeta(s, a) - \frac{1}{s-1} \right) = \lim_{s \to 1} \frac{a^{-s}}{2} + \lim_{s \to 1} \frac{a^{1-s} - 1}{s-1} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) e^{-ax} \, dx$$
$$= \frac{1}{2a} - \ln(a) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) e^{-ax} \, dx.$$

The last integral can be extracted from the proof of the first Binet formula via the Gauss representation (Section 4.18 before Example 4.18.4):

$$\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) e^{-ax} \, dx = -\Psi(a+1) + \ln(a) + \frac{1}{2a} = -\Psi(a) + \ln(a) - \frac{1}{2a}.$$

The stated limit relation follows.

The expression in parentheses of the improper integral in the left-hand side of the formula above can be approximated by $\sum_{\ell=1}^m x^{2\ell-1} B_{2\ell}/(2\ell)!$, $m \in \mathbb{N}$. (See the (second) remark after the statement of the first Binet formula.) Thus, we have

$$-\Psi(a) + \ln(a) - \frac{1}{2a} = \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) e^{-ax} \, dx$$
$$\approx \sum_{\ell=1}^m \frac{B_{2\ell}}{(2\ell)!} \int_0^\infty x^{2\ell-1} e^{-ax} \, dx = \sum_{\ell=1}^m \frac{B_{2\ell}}{(2\ell)! a^{2\ell}} \Gamma(2\ell) = \sum_{\ell=1}^\infty \frac{B_{2\ell}}{2\ell a^{2\ell}}$$

Rearranging, we obtain

$$\Psi(a) \approx \ln(a) - \frac{1}{2a} - \sum_{\ell=1}^m \frac{B_{2\ell}}{2\ell a^{2\ell}}, \quad a > 0$$

To get an estimate for the error term of the approximation, we can use the Euler-Maclaurin summation formula above (with m replaced by $2m$). We calculate as above

$$\begin{aligned}\Psi(a) &= -\lim_{s\to 1}\left(\zeta(s,a)-\frac{1}{s-1}\right) = -\lim_{s\to 1}\frac{a^{1-s}-1}{s-1}+\frac{B_1}{a}-\sum_{k=2}^{2m+1}\frac{B_k}{ka^k}+\int_0^\infty \frac{P_{m+1}(x)}{(x+a)^{2m+2}}\,dx \\ &= \ln(a)-\frac{1}{2a}-\sum_{k=1}^{m}\frac{B_{2\ell}}{2\ell a^{2\ell}}+\int_0^\infty \frac{P_{2m+1}(x)}{(x+a)^{2m+2}}\,dx.\end{aligned}$$

Thus, we have the error estimate

$$\left|\int_0^\infty \frac{P_{2m+1}(x)}{(x+a)^{2m+2}}\,dx\right| \le \sup_{x\in[0,1]}|B_{2m+1}(x)|\int_0^\infty \frac{dx}{(x+a)^{2m+2}} \le \frac{|B_{2m}|}{2\pi a^{2m+1}}.$$

(See Section 4.14.)
Note finally that, letting $m\to\infty$, the power series (in $1/a$) in the **asymptotic formula**

$$\Psi(a)\sim \ln(a)-\frac{1}{2a}-\sum_{\ell=1}^{\infty}\frac{B_{2\ell}}{2\ell a^{2\ell}},\quad a>0,$$

has radius of convergence 0 (by the asymptotics of the Bernoulli numbers), that is, the infinite sum does not converge for any $a>0$. Nevertheless, the partial sums approximate the digamma function with increasing accuracy as $a\to\infty$.

Returning to the main line, we now derive yet another another integral representation of the Hurwitz zeta function knows as the **Hermite formula**:

$$\zeta(s,a)=\frac{a^{-s}}{2}+\frac{a^{1-s}}{s-1}+2\int_0^\infty \frac{\sin(s\arctan(x/a))}{(a^2+x^2)^{s/2}(e^{2\pi x}-1)}\,dx.$$

To pass from the integral representation just derived to the Hermite formula, we need to show that the respective indefinite integrals match. We first note that, according to Example 4.16.1, we have

$$\frac{1}{2}\left(\frac{1}{2}-\frac{1}{x}+\frac{1}{e^x-1}\right)=\int_0^\infty \frac{\sin(xu)}{e^{2\pi u}-1}\,du.$$

Using this, we calculate

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{2}-\frac{1}{x}+\frac{1}{e^x-1}\right)\frac{x^{s-1}}{e^{ax}}\,dx=\frac{2}{\Gamma(s)}\int_0^\infty\left(\int_0^\infty\frac{\sin(xu)}{e^{2\pi u}-1}\,du\right)\frac{x^{s-1}}{e^{ax}}\,dx\\ &\quad=\frac{2}{\Gamma(s)}\int_0^\infty\int_0^\infty x^{s-1}e^{-ax}\sin(ux)\,dx\frac{du}{e^{2\pi u}-1}=2\int_0^\infty\frac{\sin(s\arctan(u/a))}{(a^2+u^2)^{s/2}(e^{2\pi u}-1)}\,du,\end{aligned}$$

where we used Example 4.6.3. The Hermite formula follows.

As a special case, the Hermite formula provides important integral representations for the polygamma functions Ψ_n, $n \in \mathbb{N}$. Substituting $s = n+1$, $n \in \mathbb{N}$, we have

$$\frac{(-1)^{n+1}}{n!}\Psi_n(a) = \zeta(n+1, a) = \frac{1}{2a^{n+1}} + \frac{1}{na^n} + 2\int_0^\infty \frac{\sin((n+1)\arctan(x/a))}{(a^2+x^2)^{(n+1)/2}(e^{2\pi x}-1)}\,dx$$

The crux here is that the trigonometric expression in the numerator of the integrand is a rational function (in both x and a), and it is expressible by the Chebyshev polynomials U_n, $n \in \mathbb{N}_0$ (Section 4.10). Using the definition, we have

$$\begin{aligned}\sin((n+1)\arctan(x/a)) &= U_n(\cos(\arctan(x/a))\sin(\arctan(x/a)) \\ &= U_n\left(\frac{a}{\sqrt{a^2+x^2}}\right)\frac{x}{\sqrt{a^2+x^2}}.\end{aligned}$$

Substituting, we arrive at

$$\frac{(-1)^{n+1}}{n!}\Psi_n(a) = \frac{1}{2a^{n+1}} + \frac{1}{na^n} + 2\int_0^\infty U_n\left(\frac{a}{\sqrt{a^2+x^2}}\right)\frac{x\,dx}{(a^2+x^2)^{n/2+1}(e^{2\pi x}-1)}.$$

For completeness, note that integrating both sides of the formula for $n = 1$, we obtain

$$\Psi(a) = -\frac{1}{2a} + \ln(a) - 2\int_0^\infty \frac{x\,dx}{(a^2+x^2)(e^{2\pi x}-1)}.$$

Remark. Using the integral equality in the proof of the Hermite formula, we also have

$$(-1)^{n+1}\Psi_n(a) = \frac{n!}{2a^{n+1}} + \frac{(n-1)!}{a^n} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x-1}\right)\frac{x^n}{e^{ax}}\,dx,$$

and, for completeness again

$$\Psi(a) = -\frac{1}{2a} + \ln(a) - \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x-1}\right)e^{-ax}\,dx.$$

We now return to the compact form of the first Euler-Maclaurin formula in Example 4.19.1, and expand the periodized Bernoulli polynomial $P_1(x)$ in the numerator of the integrand into Fourier series as in Section 4.14. For $-1 < s < 0$ and $0 < a \le 1$, we calculate

$$\begin{aligned}\zeta(s,a) &= -s\int_{-a}^\infty \frac{P_1(x)}{(x+a)^{s+1}}\,dx = -s\int_0^\infty \frac{P_1(x-a)}{x^{s+1}}\,dx \\ &= \frac{s}{\pi}\int_0^\infty \sum_{n=1}^\infty \frac{\sin(2n\pi(x-a))}{n}\frac{dx}{x^{s+1}} = \frac{s}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty \frac{\sin(2n\pi(x-a))}{x^{s+1}}\,dx\end{aligned}$$

We work out the improper integral using the formulas in the remark after Example 4.6.3 as

$$\begin{aligned}\int_0^\infty \frac{\sin(2n\pi(x-a))}{x^{s+1}}\,dx &= \cos(2n\pi a)\int_0^\infty \frac{\sin(2n\pi x)}{x^{s+1}}\,dx - \sin(2n\pi a)\int_0^\infty \frac{\cos(2n\pi x)}{x^{s+1}}\,dx\\
&= (2n\pi)^s\cos(2n\pi a)\int_0^\infty \frac{\sin(u)}{u^{s+1}}\,du - (2n\pi)^s\sin(2n\pi a)\int_0^\infty \frac{\cos(u)}{u^{s+1}}\,du\\
&= -(2n\pi)^s\Gamma(-s)\left(\cos(2n\pi a)\sin\left(\frac{\pi s}{2}\right) + \sin(2n\pi a)\cos\left(\frac{\pi s}{2}\right)\right)\\
&= (2\pi)^s\frac{\Gamma(1-s)}{s}\left(\sin\left(\frac{\pi s}{2}\right)\frac{\cos(2n\pi a)}{n^{-s}} + \cos\left(\frac{\pi s}{2}\right)\frac{\sin(2n\pi a)}{s^{-n}}\right)\end{aligned}$$

Substituting, for $s < 0$ and $0 < a \leq 1$, we obtain the **Hurwitz formula**[152]

$$\zeta(s,a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left(\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^\infty \frac{\cos(2n\pi a)}{n^{1-s}} + \cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^\infty \frac{\sin(2n\pi a)}{n^{1-s}}\right).$$

Note that, for $a = 1$, we recover the functional equation for the Riemann zeta function (Section 4.17):

$$\zeta(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s), s < 0.$$

Kummer's formula for the gamma function, noted in Section 4.16, is a direct consequence of the Hurwitz formula via Proposition 4.19.1.
Indeed, lettting $0 < a < 1$, performing the differentiation, we calculate

$$\begin{aligned}\ln\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) &= \left.\frac{d\zeta(s,a)}{ds}\right|_{s=0} = \left(-\frac{\Gamma'(1)}{\pi} + \frac{\Gamma(1)}{\pi}\ln(2\pi)\right)\sum_{n=1}^\infty \frac{\sin(2n\pi a)}{n}\\
&\quad + \frac{\Gamma(1)}{\pi}\left(\frac{\pi}{2}\sum_{n=1}^\infty \frac{\cos(2n\pi a)}{n} + \sum_{n=1}^\infty \frac{\sin(2n\pi a)}{n}\ln(n)\right)\\
&= (\gamma + \ln(2\pi))\left(\frac{1}{2} - a\right) - \frac{1}{2}\ln(2\sin(\pi a)) + \frac{1}{\pi}\sum_{n=1}^\infty \sin(2n\pi a)\frac{\ln(n)}{n}.\end{aligned}$$

[152]For other proofs, see Apostol, T., *Introduction to analytic number theory*, Springer, NY, 1976, p. 257; Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, pp. 268-269, and 3rd Edition, Dover, 2020; and Kanemitsu, S., Tanigawa, Y., Tsukada, H. and Yoshimoto, M., *Contributions to the theory of the Hurwitz zeta function*, Hardy-Ramanujan Journal, 30 (2007) 31-55; Section 4.

where we used the Fourier expansions

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\pi a)}{n} = \frac{1}{2} - a \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\cos(2n\pi a)}{n} = -\ln(2\sin(\pi a)), \quad 0 < a < 1.$$

(See Section 4.14 as well as Exercise 3 at the end of the section.)
We obtain **Kummer's formula**, usually written as

$$\ln\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) = -\frac{1}{2}\ln(2\sin(\pi a)) + \frac{1}{2}\left(\gamma + \ln(2\pi)\right)(1-2a) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\ln(n)}{n}\sin(2n\pi a), \quad 0 < a < 1.$$

It is interesting to observe that $\ln\Gamma$ is expressed in the right-hand side by elementary functions only.

Returning to the main line, we now show an important consequence of the Hurwitz formula for rational $a = p/q$, $p \le q$, $p, q \in \mathbb{N}$, as

$$\zeta\left(1-s, \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2\pi q)^s}\sum_{k=1}^{q}\cos\left(\frac{\pi s}{2} - 2k\pi\frac{p}{q}\right)\zeta\left(s, \frac{k}{q}\right),$$

with no restriction on s.
To derive this, we first replace s by $1-s$ in the Hurwitz formula, and join the two infinite sums:

$$\begin{aligned}\zeta(1-s, a) &= \frac{2\Gamma(s)}{(2\pi)^s}\left(\cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2n\pi a)}{n^s} + \sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2n\pi a)}{n^s}\right) \\ &= \frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\cos\left(\frac{\pi s}{2} - 2n\pi a\right)\frac{1}{n^s}, \quad s > 1.\end{aligned}$$

Evaluating this at $a = p/q$ as above, we obtain

$$\zeta\left(1-s, \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\cos\left(\frac{\pi s}{2} - 2n\pi\frac{p}{q}\right)\frac{1}{n^s}, \quad s > 1.$$

The crux is to write $n = mq + k$, and replace the infinite sum with variable $n \in \mathbb{N}$ by a double sum with $m \in \mathbb{N}_0$ and $k = 1, \ldots, q$. This gives

$$\begin{aligned}\zeta\left(1-s, \frac{p}{q}\right) &= \frac{2\Gamma(s)}{(2\pi)^s}\sum_{m=0}^{\infty}\sum_{k=1}^{q}\cos\left(\frac{\pi s}{2} - 2(mq+k)\pi\frac{p}{q}\right)\frac{1}{(mq+k)^s} \\ &= \frac{2\Gamma(s)}{(2\pi q)^s}\sum_{k=1}^{q}\cos\left(\frac{\pi s}{2} - 2k\pi\frac{p}{q}\right)\sum_{m=0}^{\infty}\frac{1}{(m+k/q)^s}.\end{aligned}$$

Since the last factor is $\zeta(s, k/q)$, the stated formula follows. This holds for $s > 1$ and, by analytic continuation, for all s.
For $p = q = 1$, this relation reduces (via Euler's reflection formula) to the functional equation of the zeta function.

We close this section by giving a proof of the **Gauss formula for the digamma function** stated in Proposition 4.19.2.
We need the following:

Lemma. *For $p = 1, \ldots, q-1$, we have*
(1) $\sum_{k=1}^{q} \sin\left(2p\pi\frac{k}{q}\right) = 0$*;*
(2) $\sum_{k=1}^{q} \cos\left(2p\pi\frac{k}{q}\right) = 0$*;*
(3) $\sum_{k=1}^{q} \sin\left(2p\pi\frac{k}{q}\right)\frac{k}{q} = -\frac{1}{2}\cot\left(\frac{p\pi}{q}\right)$*;*
(4) $\sum_{k=1}^{q} \cos\left(2p\pi\frac{k}{q}\right)\frac{k}{q} = \frac{1}{2}$*.*

Proof. All these are consequences of the Lagrange identities[153]

$$\begin{aligned}
\sum_{k=1}^{q} \sin(k\alpha) &= \frac{\cos(\alpha/2) - \cos((2q+1)\alpha/2)}{2\sin(\alpha/2)} \\
\sum_{k=1}^{q} \cos(k\alpha) &= \frac{-\sin(\alpha/2) + \sin((2q+1)\alpha/2)}{2\sin(\alpha/2)}.
\end{aligned}$$

For (1)-(2) we use the direct substitution $\alpha = 2p\pi/q$. For (3), we take the derivative of the Lagrange identity for cosine (with respect to α) as

$$\sum_{k=1}^{q} k\sin(k\alpha) = \frac{\sin((2q+1)\alpha/2)\cos(\alpha/2) - (2q+1)\cos((2q+1)\alpha/2)\sin(\alpha/2)}{4\sin^2(\alpha/2)},$$

and substitute $\alpha = 2p\pi/q$. Finally, for (4), we take the derivative of the Lagrange identity for sine (with respect to α) as

$$\sum_{k=1}^{q} k\cos(k\alpha) = \frac{\cos((2q+1)\alpha/2)\cos(\alpha/2) + (2q+1)\sin((2q+1)\alpha/2)\sin(\alpha/2) - 1}{4\sin^2(\alpha/2)},$$

and substitute $\alpha = 2p\pi/q$. The lemma follows.

[153] See *Elements of Mathematics - History and Foundations*, Exercise 11.3.16.

PROOF OF GAUSS' DIGAMMA THEOREM. We first replace s by $1-s$ (for technical convenience) in the Hurwitz formula for rational $a = p/2$ just derived above

$$(2q\pi)^{1-s}\zeta\left(s,\frac{p}{q}\right) = 2\Gamma(1-s)\sum_{k=1}^{q}\sin\left(\frac{\pi s}{2}+2p\pi\frac{k}{q}\right)\zeta\left(1-s,\frac{k}{q}\right),\ p<q,\ p,q\in\mathbb{N}.$$

In the first step we extract the residues at $s=1$ from both sides of this equality as

$$\lim_{s\to 1}(s-1)(2q\pi)^{1-s}\zeta\left(s,\frac{p}{q}\right) = \lim_{s\to 1}(s-1)\zeta\left(s,\frac{p}{q}\right) = 1,$$

and

$$2\lim_{s\to 1}(s-1)\Gamma(1-s)\sum_{k=1}^{q}\sin\left(\frac{\pi s}{2}+2p\pi\frac{k}{q}\right)\zeta\left(1-s,\frac{k}{q}\right)$$
$$= -2\sum_{k=1}^{q}\sin\left(\frac{\pi}{2}+2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right) = -2\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right).$$

Hence, the equality of the residues gives

$$1 = -2\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right).$$

On the other hand, since $\zeta(0,a) = 1/2 - a$, this can be written as

$$1 = -2\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\left(\frac{1}{2}-\frac{k}{q}\right) = -\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right) + 2\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\frac{k}{q}.$$

By the summation formulas in (2) and (4) of the lemma above, however, this always holds. Hence the equality of the residues gives nothing new.

In the second step, we return to our equation, deduct the poles with the calculated residues, and let $s\to 1$.[154] For the left-hand side, we obtain

$$\lim_{s\to 1}\left((2\pi q)^{1-s}\zeta\left(s,\frac{p}{q}\right) - \frac{1}{s-1}\right) = \lim_{s\to 1}\left(\zeta\left(s,\frac{p}{q}\right) - \frac{1}{s-1}\right)$$
$$+\lim_{s\to 1}\frac{(2\pi q)^{1-s}-1}{s-1}\lim_{s\to 1}(s-1)\zeta\left(s,\frac{p}{q}\right) = -\Psi\left(\frac{p}{q}\right) - \ln(2\pi q).$$

The corresponding limit of the right-hand side (suppressing the factor 2) is

$$\lim_{s\to 1}\left(\Gamma(1-s)\sum_{k=1}^{q}\sin\left(\frac{\pi s}{2}+2p\pi\frac{k}{q}\right)\zeta\left(1-s,\frac{k}{q}\right) + \frac{1}{s-1}\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)\right).$$

[154] This amounts to taking the constant term in the Laurent series expansions at $s=1$.

We rearrange this into two terms as

$$\lim_{s\to 1}\left(\Gamma(1-s)-\frac{1}{1-s}\right)\sum_{k=1}^{q}\sin\left(\frac{\pi s}{2}+2p\pi\frac{k}{q}\right)\zeta\left(1-s,\frac{k}{q}\right)$$
$$-\sum_{k=1}^{q}\lim_{s\to 1}\frac{1}{s-1}\left(\sin\left(\frac{\pi s}{2}+2p\pi\frac{k}{q}\right)\zeta\left(1-s,\frac{k}{q}\right)-\cos\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)\right).$$

Clearly, the first limit is

$$-\gamma\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)=-\gamma\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\left(\frac{1}{2}-\frac{k}{q}\right)=\frac{\gamma}{2},$$

where we used (2) and (4) of the lemma above.
The generic limit within the (finite) sum in the second term is the derivative

$$\begin{aligned}&\frac{d}{ds}\left(\sin\left(\frac{\pi s}{2}+2p\pi\frac{k}{q}\right)\zeta\left(1-s,\frac{k}{q}\right)\right)\Bigg|_{s=1}\\&\quad=\frac{\pi}{2}\cos\left(\frac{\pi}{2}+2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)-\sin\left(\frac{\pi}{2}+2p\pi\frac{k}{q}\right)\zeta'\left(0,\frac{k}{q}\right)\\&\quad=-\frac{\pi}{2}\sin\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)-\cos\left(2p\pi\frac{k}{q}\right)\zeta'\left(0,\frac{k}{q}\right).\end{aligned}$$

Putting everything together, so far we have

$$-\Psi\left(\frac{p}{q}\right)-\ln(2\pi q)=\gamma+\pi\sum_{k=1}^{q}\sin\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)+2\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\zeta'\left(0,\frac{k}{q}\right).$$

We calculate the first sum on the right-hand side as

$$\begin{aligned}&\sum_{k=1}^{q}\sin\left(2p\pi\frac{k}{q}\right)\zeta\left(0,\frac{k}{q}\right)=\sum_{k=1}^{q}\sin\left(2p\pi\frac{k}{q}\right)\left(\frac{1}{2}-\frac{k}{q}\right)\\&=\frac{1}{2}\sum_{k=1}^{q}\sin\left(2p\pi\frac{k}{q}\right)-\sum_{k=1}^{q}\sin\left(2p\pi\frac{k}{q}\right)\frac{k}{q}=\frac{1}{2}\cot\left(\frac{p\pi}{q}\right),\end{aligned}$$

where we used (1) and (3) of the lemma above. With this, we now have

$$-\Psi\left(\frac{p}{q}\right)-\ln(2\pi q)=\gamma+\frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right)+2\sum_{k=1}^{q}\cos\left(2p\pi\frac{k}{q}\right)\zeta'\left(0,\frac{k}{q}\right).$$

For the last sum, we use Proposition 4.19.1 as

$$\sum_{k=1}^{q} \cos\left(2p\pi\frac{k}{q}\right)\zeta'\left(0,\frac{k}{q}\right) = \sum_{k=1}^{q} \cos\left(2p\pi\frac{k}{q}\right)\ln\Gamma\left(\frac{k}{q}\right),$$

where the term $-\ln(\sqrt{2\pi})\sum_{k=1}^{q}\cos(2p\pi k/q)$ vanishes again by (2) of the lemma above. Substituting and rearranging, we arrive at

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \ln(2\pi q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) - 2\sum_{k=1}^{q-1}\cos\left(2p\pi\frac{k}{q}\right)\ln\Gamma\left(\frac{k}{q}\right).$$

(Note the change in the upper limit of the finite sum as the respective term vanishes.) As the final step, taking the logarithm of both sides of Euler's reflection formula (Proposition 4.9.1), and evaluating at k/q, we have

$$\ln\Gamma\left(\frac{k}{q}\right) + \ln\Gamma\left(\frac{q-k}{q}\right) = \ln\pi - \ln\sin\left(\pi\frac{k}{q}\right), \quad k = 1, \ldots, q-1.$$

Multiplying through by $\cos(2\pi pk/q) = \cos(2\pi p(q-k)/q)$ and summing up with respect to $k = 1, \ldots, q-1$, we obtain

$$2\sum_{k=1}^{q-1}\cos\left(2p\pi\frac{k}{q}\right)\ln\Gamma\left(\frac{k}{q}\right) = -\ln\pi - \sum_{k=1}^{q-1}\cos\left(2p\pi\frac{k}{q}\right)\ln\sin\left(\pi\frac{k}{q}\right),$$

Using this to replace the sum in our formula above, we finally arrive at the **Gauss digamma theorem**

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \ln(2q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + \sum_{k=1}^{q-1}\cos\left(2p\pi\frac{k}{q}\right)\ln\sin\left(\pi\frac{k}{q}\right), \; p < q, \; p, q \in \mathbb{N}.$$

Remark. This formula has various equivalent forms. A minor modification (using the trigonometric summation for cosine above) gives

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \ln q - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + \sum_{k=1}^{q-1}\cos\left(2p\pi\frac{k}{q}\right)\ln\left(2\sin\left(\pi\frac{k}{q}\right)\right).$$

A more interesting version can be obtained from the original Gauss digamma formula above by noticing that the terms in the (finite) sum stay the same under the symmetry

$k \to q-k$, $k = 1, \ldots, q-1$. This allows to pair up the respective terms. For q **odd**, the sum becomes

$$\sum_{k=1}^{q-1} \cos\left(2p\pi\frac{k}{q}\right) \ln\left(\sin\left(\pi\frac{k}{q}\right)\right) = 2\sum_{k=1}^{\frac{q-1}{2}} \cos\left(2p\pi\frac{k}{q}\right) \ln\left(\sin\left(\pi\frac{k}{q}\right)\right).$$

For q **even**, the middle term for $k = q/2$ vanishes and we have

$$\sum_{k=1}^{q-1} \cos\left(2p\pi\frac{k}{q}\right) \ln\left(\sin\left(\pi\frac{k}{q}\right)\right) = 2\sum_{k=1}^{\frac{q}{2}-1} \cos\left(2p\pi\frac{k}{q}\right) \ln\left(\sin\left(\pi\frac{k}{q}\right)\right).$$

The two cases can be combined, and we obtain

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \ln(2q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + 2\sum_{k=1}^{\left[\frac{q-1}{2}\right]} \cos\left(2p\pi\frac{k}{q}\right) \ln\sin\left(\pi\frac{k}{q}\right).$$

There is a proliferation of (finite) summation formulas of the digamma function on rational numbers. The next example is also due to Gauss;

Example 4.19.2. We have

$$\sum_{p=1}^{q} \Psi\left(\frac{p}{q}\right) = -q(\gamma + \ln q), \quad q \in \mathbb{N}.$$

Indeed, splitting off the top term $\Psi(1) = -\gamma$ and using the Gauss digamma theorem, we calculate

$$\begin{aligned}\sum_{p=1}^{q} \Psi\left(\frac{p}{q}\right) &= -\gamma + \sum_{p=1}^{q-1} \Psi\left(\frac{p}{q}\right) = -q\gamma - (q-1)\ln(2q) \\ &\quad - \frac{\pi}{2}\sum_{p=1}^{q-1} \cot\left(\frac{p\pi}{q}\right) + \sum_{k=1}^{q-1}\left(\sum_{p=1}^{q-1} \cos\left(2p\pi\frac{k}{q}\right)\right) \ln\sin\left(\pi\frac{k}{q}\right)\end{aligned}$$

By (2) of the lemma above, we have

$$\sum_{p=1}^{q-1} \cos\left(2p\pi\frac{k}{q}\right) = \sum_{p=1}^{q-1} \cos\left(2k\pi\frac{p}{q}\right) = -1;$$

and, by (1) and (3), of the same lemma

$$\sum_{p=1}^{q-1} \cot\left(\frac{p\pi}{q}\right) = -2\sum_{p=1}^{q-1}\sum_{k=1}^{q} \sin\left(2p\pi\frac{k}{q}\right)\frac{k}{q} = -2\sum_{k=1}^{q}\left(\sum_{p=1}^{q} \sin\left(2k\pi\frac{p}{q}\right)\right)\frac{k}{q} = 0.$$

Incorporating these, we have

$$\sum_{p=1}^{q} \Psi\left(\frac{p}{q}\right) = -q\gamma - (q-1)\ln(2q) - \sum_{k=1}^{q-1} \ln\sin\left(\pi\frac{k}{q}\right)$$

Finally, using the lemma in the proof of the Legendre-Gauss theorem (Proposition 4.10.2), we have

$$\sum_{k=1}^{q-1} \ln\sin\left(\pi\frac{k}{q}\right) = \ln\prod_{k=1}^{q-1}\sin\left(\pi\frac{k}{q}\right) = \ln\left(\frac{q}{2^{q-1}}\right) = \ln(q) - (q-1)\ln 2.$$

The example now follows.

Exercises.

1. Use the formula

$$\zeta(s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx, \quad s > 1,$$

in the text to prove that the Hurwitz zeta function is analytic in $s > 1$ by the following steps. (a) Notice first that it is enough to prove that the improper integral is analytic in $s > 1$ (as the quotient of analytic functions is analytic, and the gamma function is analytic; see Section 4.6.). (b) For $n \in \mathbb{N}$, derive the formula

$$\frac{d^n}{ds^n}\left(\int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx\right) = \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}(\ln(x))^n\,dx, \quad s > 1.$$

(c) Use the inequality $e^{-x} < 1 - x + x^2/2$, $x \geq 0$, and the method in Section 4.6, to estimate

$$\left|\int_0^1 \frac{x^{s-1}e^{-ax}}{1-e^{-x}}(\ln(x))^n\,dx\right| \leq \int_0^1 \frac{x^{s-2}e^{-ax}}{1-x/2}\left(-\ln(x)\right)^n\,dx$$
$$\leq 2\int_0^1 x^{s-2}e^{-ax}\left(-\ln(x)\right)^n\,dx \leq 2\int_0^1 x^{s-2}(-\ln(x))^n\,dx = 2\frac{n!}{(s-1)^{n+1}}, \quad n \in \mathbb{N}_0.$$

(d) Fix $b > 1$. Assuming $1 < s < b$ and $2 \leq n \in \mathbb{N}$, estimate

$$\int_1^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}(\ln(x))^n\,dx \leq \frac{1}{1-1/e}\int_1^\infty x^{s-1}e^{-ax}(\ln(x))^n\,dx$$
$$\leq \frac{K}{1-1/e}\int_1^\infty e^{-ax/2}(\ln(x))^n\,dx \leq \frac{K}{1-1/e}\frac{2^n n!}{a^n}\int_1^\infty\left(\frac{\ln(x)}{x}\right)^n\,dx$$
$$\leq \frac{K}{1-1/e}\frac{2^n n!}{a^n}\frac{n!}{(n-1)^{n+1}} \leq \frac{K}{1-1/e}\frac{2^{n+1}n!}{a^n},$$

where $x^{s-1} \le x^{b-1} \le Ke^{ax/2}$, $x \ge 1$, (with K depending on a), and $e^{ax/2} \ge (ax/2)^n/n!$, $n \in \mathbb{N}$. (e) Finally, use (c)-(d) and the condition on the growth rate of the Taylor coefficients (Section 2.4) to conclude that $\zeta(s,a)$ is analytic for $s > 1$.

2. Derive the limit relation

$$\lim_{s\to 1}\left(\zeta(s,a) - \frac{1}{s-1}\right) = -\Psi(a),$$

using the Hermite formula along with Exercise 1 at the end of Section 4.16.
Solution: Calculate

$$\begin{aligned}
\lim_{s\to 1}\left(\zeta(s,a) - \frac{1}{s-1}\right) &= \lim_{s\to 1}\frac{a^{-s}}{2} + \lim_{s\to 1}\frac{a^{1-s}-1}{s-1} + 2\int_0^\infty \frac{\sin(\arctan(x/a))}{(a^2+x^2)^{1/2}(e^{2\pi x}-1)}\,dx \\
&= \frac{1}{2a} - \ln(a)\lim_{s\to 1} a^{1-s} + 2\int_0^\infty \frac{\sin(\arctan(x/a))}{(a^2+x^2)^{1/2}(e^{2\pi x}-1)}\,dx \\
&= \frac{1}{2a} - \ln(a) + 2\int_0^\infty \frac{x\,dx}{(a^2+x^2)(e^{2\pi x}-1)}
\end{aligned}$$

For the last integral, Exercise 1 at the end of Section 4.16 gives

$$\Psi(a) = \frac{\Gamma'(a)}{\Gamma(a)} = \ln(a) - \frac{1}{2a} - 2\int_0^\infty \frac{x\,dx}{(a^2+x^2)(e^{2\pi x}-1)}.$$

3. Use the method of the proof of the fundamental equation for the Hurwitz zeta function for rational a to derive the multiplication theorem

$$q^s\zeta(s) = \sum_{p=1}^{q}\zeta\left(s, \frac{p}{q}\right), \quad p \le q,\ p, q \in \mathbb{N}.$$

4. Show that, applying the differentiation formula in Proposition 4.19.1 to the general Euler-Maclaurin formula for the Hurwitz zeta function, we obtain the Euler-Maclaurin formula for the gamma function in Section 4.15. Similarly, performing analogous computations for the Hermite formula, obtain the second Binet formula for the gamma function.

5. Use the method of the proof of the Gauss digamma theorem to derive the formula

$$\sum_{p=1}^{q-1}\Psi\left(\frac{p}{q}\right)\frac{p}{q} = -(q-1)\frac{\gamma}{2} - \frac{q\ln(q)}{2} - \frac{\pi}{2}\sum_{p=1}^{q-1}\cot\left(\frac{p\pi}{q}\right)\frac{p}{q}, \quad 2 \le q \in \mathbb{N}.$$

Further Reading

1. Andrews, G.E., Askey, R. and Roy, R., *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge Uniersity Press, 1999.

2. Artin, E., *The Gamma Function*, New York, Holt, Rinehart and Winston 1964, and Dover 2015.

3. Bourbaki, N., *Fonctions d'une variable réelle.* Éléments de mathématique. Springer, 1976.

4. Farrel, O.J. and Ross, B., *Solved problems in analysis*, Dover, 2013.

5. Hijab, O., *Introduction to Calculus and Classical Analysis*, Springer, 1997.

6. Katznelson, Y., *An Introduction to Harmonic Analysis*, 2nd ed. Dover, New York 1976.

7. Nielsen, N., *Handbuch der Theorie der Gamma-function*, Leipzig, 1906.

7. Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill, Inc. New York, 1976.

8. Titmarsch, E.C., *The Theory of the Riemann Zeta Function*, 2nd edition, Oxford, 1986.

9. Walker, P.L., *The Theory of Fourier Series and Integrals*, Wiley, 1986.

10. Whittaker, E.T. and Watson, G.N., *A Course in Modern Analysis*, 4th Edition, Cambridge, 1927, and 3rd Edition, Dover, 2020.

11. Zygmund, A., *Trigonometric Series*, 3rd ed. Cambridge University Press, 2002.