Solutions Manual[1]

Chapter 0.

0.1.1. Let $A = \{1\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$.

0.1.2. Let $A = \{1\}$, and $B = C = \{\{1\}\}$.

0.4.3. Let A, B, resp. C be the sets of passwords that do not contain a, b, and c, resp. We have $|A| = |B| = |C| = 2^8$, $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $|A \cap B \cap C| = 0$. By the Principle of Inclusion-Exclusion, we have $|A \cup B \cup C| = 3 \cdot 2^8 - 3 \cdot 1 + 0$. Hence the number of passwords sought is $3^8 - (3 \cdot 2^8 - 3)$.

0.4.4. The smallest sum is $1 + 2 + \cdots + m = m(m+1)/2$, the largest is $(n - m + 1) + (n - m + 2) + \cdots + n = n(n+1)/2 - (n-m)(n-m+1)/2$. Thus the number sought is $n(n+1)/2 - m(m+1)/2 - (n-m)(n-m+1)/2 + 1$.

Chapter 1.

1.1.2. $a < b$ means that $b = a + c$ for some $c \in \mathbb{N}$. If $c = 1$ then $b = \mathcal{S}(a) < \mathcal{S}(a)$, a contradiction. If $c \neq 1$ then $c = \mathcal{S}(d)$ for some $d \in \mathbb{N}$, and so $b = a + \mathcal{S}(d) = \mathcal{S}(a + d) < \mathcal{S}(a)$, a contradiction again.

1.3.1. $19 = 2 + 17 = 3 + 5 + 11$.

1.3.2. To show that 3 divides $p^2 - q^2 = (p - q)(p + q)$, write $p - q = 3k \pm 1$ and $p + q = 3l \pm 1$ with $k, l \in \mathbb{N}$ (otherwise the statement is clear). Deduce that $p = 3m \pm 1$ and $q = 3n \pm 1$ for some $m, n \in \mathbb{N}$. To show that $8 = 2^3$ divides $p^2 - q^2$, write $p = 2k + 1$ and $q = 2l + 1$.

1.3.3. Factor: $2^{18} - 64 = 2^{18} - 2^6 = 2^6(2^{12} - 1) = 2^6(2^6 + 1)(2^3 + 1)(2^3 - 1)$. This gives $2^{18} - 64 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$.

1.3.4. We have $400 = 2^4 \cdot 5^2$, so that $n \in \mathbb{N}$, $n \leq 400$, is relatively prime to 400 if it is not a multiple of 2 or 5. Using the Principle of Inclusion-Exclusion, the number of multiples of 2, 5 and 10 is this range are $400/2 = 200$, $400/5 = 80$, $400/10 = 40$, respectively. Hence the number sought is $400 - 200 - 80 + 40 = 160$.

1.4.1. The general induction step $n \Rightarrow n + 1$ amounts to

$$\frac{n+1}{2n}\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2(n+1)}.$$

1.4.2. The general induction step $n \Rightarrow n + 1$, after canceling the common factors gives

$$\frac{n(4n+1)}{5!} + \frac{n+1}{3!} = \frac{(n+4)(4n+5)}{5!}.$$

[1] Only the challenging and/or computational intensive problems are treated here.

1.4.3. For the general induction step $n \Rightarrow n+1$, we need to show

$$\frac{1}{2} - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} \geq \frac{1}{2}.$$

Canceling and rearranging, the inequality follows.
1.4.4. Assume $\sqrt{a}+\sqrt{b} \in \mathbb{Q}$, $0 < a, b \in \mathbb{Q}$. Then $(\sqrt{a}+\sqrt{b})^2 = a + b + 2\sqrt{ab} \in \mathbb{Q}$. Hence $\sqrt{ab} \in \mathbb{Q}$ (so that $ab = n^2$ for some $n \in \mathbb{N}$). Finally, $\sqrt{a}(\sqrt{a}+\sqrt{b}) = a+\sqrt{ab} \in \mathbb{Q}$, and therefore $\sqrt{a} \in \mathbb{Q}$.
1.4.5. Let $q = a/b \in \mathbb{Q}$, $0 < a < b$, $a, b \in \mathbb{N}$, and assume that $\sqrt[3]{1-q^3} = c/d \in \mathbb{Q}$, $c, d \in \mathbb{N}$. Substituting $q = a/b$, we obtain $1 - (a/b)^3 = (c/d)^3$. Eliminating the denominators, we obtain $(bd)^3 = (ad)^3 + (bc)^3$. This is impossible by Fermat's Last Theorem in the exponent 3. The generalization to arbitrary exponents follows the same lines using Wiles' resolution of the Fermat problem.

Chapter 2.

2.1.1. Split into two cases: $x \geq 0$ and $x < 0$, and see that all $x \in \mathbb{R}$ are solutions.
2.1.2. The inequality obviously holds for $x \leq 0$. If $0 < x \leq 1$ then $x^2 \leq x$ and the inequality gives $0 \leq -x^2$ with no solutions. If $x > 1$ then $x^2 > x$ and the inequality gives $0 \leq x(x-2)$. In this case, we have $x > 2$.
2.1.3. For $i = 1, 2, \ldots, n$, the triangle inequality gives

$$|r-r_i|+|r-r_{2n-i+1}| \geq |r-r_i|+|r_{2n-i+1}-r| \geq |(r-r_i)+(r_{2n-i+1}-r)| = r_{2n-i+1}-r_i$$

with equality if and only if $r_i \leq r \leq r_{2n-i+1}$. Thus, we have

$$|r-r_1|+|r-r_2|+\cdots+|r-r_{2n}| \geq (r_{2n}-r_1)+(r_{2n-1}-r_2)+\cdots+(r_{n+1}-r_n)$$

with equality if and only if $r_n \leq r \leq r_{n+1}$.
2.1.4. Since $\sqrt{100} = 10$, this is the same question as which is bigger $\sqrt{101}$ or $10+1/20$. Squaring, we obtain $101 < (10+1/20)^2$.
2.1.5. We have

$$\sqrt{a+b+2\sqrt{ab}} = \sqrt{\sqrt{a}^2 + 2\sqrt{a}\sqrt{b} + \sqrt{b}^2} = \sqrt{(\sqrt{a}+\sqrt{b})^2} = \sqrt{a}+\sqrt{b}.$$

2.1.6. We have $2 \leq \sqrt{a}+\sqrt{b} \leq 20$. Since $\sqrt{a}+\sqrt{b}$ must be a square, the possible values are $4, 9, 16$. It is easy to see that both a and b must also be squares: $a = c^2$ and $b = d^2$, $c, d \in \mathbb{N}$. Since $c + d = 4, 9, 16$, we obtain $4 + 9 + 16$ possible values of c, d and hence the same for a, b.
2.1.7. $\sqrt{2} = \sqrt[4]{4} < \sqrt[3]{3}$.

2.1.8. 4.

2.1.9. For (a), in the general induction step $n \Rightarrow n+1$ the induction hypothesis gives $2^{2n} - 1 = 3k$ for some $k \in \mathbb{N}$. Using this, we have $2^{2(n+1)} - 1 = 4 \cdot 2^{2n} - 1 = 4(3k+1) - 1 = 12k + 3 = 3(4k+1)$. (b) is similar.

2.1.10. Since m and n are relatively prime, we have $1 = k \cdot m + l \cdot n$ for some $k, l \in \mathbb{Z}$. We have $a = (a^m)^k \cdot (a^l)^n = (b^n)^k \cdot (a^l)^n = (b^k a^l)^n$. Now, $b^k a^l \in \mathbb{Q}$, but, since its nth power is an integer, it is a natural number $u = b^k a^l \in \mathbb{N}$. With this, we also have $b = u^m$.

2.1.11. The general induction step $n \Rightarrow n+1$ amounts to showing

$$(a+b)(a^n + b^n) \leq 2(a^{n+1} + b^{n+1}).$$

Multiplying out, we obtain

$$ab^n + a^n b \leq a^{n+1} + b^{n+1}.$$

This factors as

$$0 \leq (a-b)(a^n - b^n) = (a-b)^2(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

2.1.13. $n = 1, 2, \ldots, 11$.

2.2.1. (a) $3/11$; (b) $879\,(187/333) = 292,894/333$; (c) $92,259,159,322/99,900,000$.

2.2.2. $1/64$.

2.2.3. Use Bernoulli's inequality.

2.3.3. Use induction to show that the sequence $(a_n)_{n \in \mathbb{N}_0}$ is strictly increasing and bounded above by τ. Hence, by the Monotone Convergence Theorem, the sequence converges. Now, let $n \to \infty$ in the inductive definition.

Chapter 3.

3.1.1. $(a_n - b_n)_{n \in \mathbb{N}}$ is a null sequence since $\lim_{n\to\infty}(a_n - b_n)^2 = \lim_{n\to\infty}((a_n + b_n)^2 - 4a_n b_n) = 2^2 - 4 = 0$. Thus $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$.

3.1.2. (a) Expand $(\sqrt{a_{n+1}} - \sqrt{a_n})^2 \geq 0$. (b) Let $a_n = 1/n$ if n is odd, and $a_n = 1/n^2$ if n is even.

3.1.3. We have

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} (a_0 + nd) b_0 r^n = a_0 b_0 \sum_{n=0}^{\infty} r^n + d b_0 \sum_{n=1}^{\infty} n r^n = \frac{a_0 b_0}{1-r} + \frac{d b_0 r}{(1-r)^2},$$

where we used the Infinite Geometric Series Formula and Example 3.1.5.

3.1.4. $a_{n+1} - a_n = -t(a_n - a_{n-1})$, $n \in \mathbb{N}$. Use induction to show that $a_n - a_{n-1} = (-t)^{n-1}(a_1 - a_0)$, $n \in \mathbb{N}$. Finally, write $a_n - a_0 = \sum_{k=1}^{n}(a_k - a_{k-1})$, and use the Infinite Geometric Series Formula.

3.1.5. Let $A \subset \{1, 2, \dots, 2n\}$ have the stated property, and denote $a = \min A$ and $b = \max A$. Then, we have $1 \le a \le n$ and $b = 2n - a + 1$. All the elements of A are contained in the set $\{a, a+1, \dots, b\}$. This latter set has $b-a-1$ elements, so that the number of choices of A with $a = \min A$ (and $\max A = b$) is $2^{b-a-1} = 2^{2(n-a)}$. Hence the total number of subsets with the stated property is $\sum_{a=1}^{n} 2^{2(n-a)}$. A simple application of the Finite Geometric Series Formula gives $\sum_{a=1}^{n} 2^{2(n-a)} = \sum_{c=0}^{n-1} 2^{2c} = (4^n - 1)/3$.

3.1.6. (v) Fix m and proceed with induction with respect to n.

3.1.7. Split the number of ways to two cases according to whether the sum starts with 1 or with 2. These give the inductive formula $F_n = F_{n-1} + F_{n-2}$, $2 \le n \in \mathbb{N}$.

3.1.8. Let B_n, $n \in \mathbb{N}$, be the number of n-digit binary integers with no consecutive zeros. Any such integer must start with 1. If the second digit is 1 then, deleting the first digit, it follows that the number of such binary integers is B_{n-1}. If the second digit is 0 then, by assumption, the third must be 1. Deleting the first two digits, it follows that the number of such binary integers is B_{n-2}. We obtain $B_n = B_{n-1} + B_{n-2}$. This is the inductive formula for the Fibonacci sequence, and so $B_n = F_n$, $n \in \mathbb{N}$.

3.1.9. τ and $-1/\tau$ are the two solutions of the quadratic equation $x^2 = x + 1$. Use induction to show that, for these solutions, we also have $x^n = F_n x + F_{n-1}$, $n \in \mathbb{N}$. For the general induction step $n \Rightarrow n+1$, we calculate

$$x^{n+1} = x^n \cdot x = F_n x^2 + F_{n-1}x = F_n(x+1) + F_{n-1}x = (F_n + F_{n-1})x + F_n = F_{n+1}x + F_n.$$

Therefore we have

$$\tau^n = F_n \tau + F_{n-1} \quad \text{and} \quad (-1/\tau)^n = F_n(-1/\tau) + F_{n-1}.$$

Subtracting, the Binet formula follows.

3.1.10. Fix m and proceed with induction with respect to n.

3.1.11. For (a), we use induction with respect to $n \in \mathbb{N}$. For the general induction step $n \Rightarrow n+1$, we use the first identity in the previous exercise as $F_{m(n+1)} = F_{mn+m} = F_{(mn-1)+m+1} = F_{mn}F_{m+1} + F_{mn-1}F_m$. Thus, if $F_m | F_{mn}$ then $F_m | F_{m(n+1)}$, and the induction is complete. For (b), we use induction again $n \Rightarrow n+1$, as $\gcd(F_{n+1}, F_{n+2}) = \gcd(F_{n+1}, F_{n+1} + F_n) = \gcd(F_{n+1}, F_n) = \gcd(F_n, F_{n+1}) = 1$, and the induction is complete. For (c), we let $m \le n$, $m, n \in \mathbb{N}$. By the division algorithm, we have $n = mq + r$, $0 \le r < m$, $q \in \mathbb{N}$, $r \in \mathbb{N}_0$. We first show $\gcd(F_m, F_{mq+1}) = 1$. Indeed, by (a), we have $F_m | F_{mq}$, so that, using (b), we obtain $1 \le \gcd(F_m, F_{mq+1}) \le \gcd(F_{mq}, F_{mq+1}) = 1$. Turning to the main line, first note that, by (a), we may assume $1 \le r$. Using the inductive formula in the previous exercise and (b) (and the identities of the greatest common divisor), we calculate

$$\begin{aligned}\gcd(F_m, F_n) &= \gcd(F_m, F_{mq+r}) = \gcd(F_m, F_{mq+(r-1)+1}) \\ &= \gcd(F_m, F_{mq+1}F_r + F_{mq}F_{r-1}) = \gcd(F_m, F_{mq+1}F_r) = \gcd(F_m, F_r).\end{aligned}$$

This patterns the Euclidean algorithm. Using this pattern, after finitely many steps we arrive at

$$\gcd(F_m, F_n) = \cdots = \gcd(F_k, 0) = F_k,$$

where F_k is the **last** non-zero remainder. Hence $k = \gcd(m, n)$, and (c) follows.

3.1.12. We calculate

$$\begin{aligned} S_n &= 1 + 11 + 111 + \cdots + \overbrace{11\ldots 1}^{n} = \frac{1}{9}(9 + 99 + \cdots + \overbrace{99\ldots 9}^{n}) \\ &= \frac{1}{9}((10-1) + (10^2 - 1) + \cdots + (10^n - 1)) \\ &= \frac{1}{9}(10 + 10^2 + \cdots + 10^n - n) \\ &= \frac{1}{9}\left(\frac{10(10^n - 1)}{10 - 1} - n\right) = \frac{10^{n+1} - 10 - 9n}{81}. \end{aligned}$$

3.2.1. 16^{4x^2}.

3.2.2. For the general induction step $n \Rightarrow n+1$, we need to derive the lower and upper bounds

$$2\left(\sqrt{n+2} - \sqrt{n+1}\right) < \frac{1}{\sqrt{n+1}} \quad \text{and} \quad 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}.$$

Simplifying and rearranging, and squaring, the inequalities follow.

3.3.1. Rewrite the logarithms into powers in the exponent abc.

3.3.2. Let $m = [\log_n(x)] = \log_n[x] \in \mathbb{N}_0$. By definition of the greatest integer, we have $m \leq \log_n(x) < m+1$. Equivalently, $n^m \leq x < n^{m+1}$. Moreover, we also have $m = \log_n[x] \in \mathbb{N}_0$; that is, $n^m = [x]$. This gives $n^m \leq x < n^m + 1$, $m \in \mathbb{N}_0$. Since this is more restrictive than the previous, this is the solution.

Chapter 4.

4.2.1. For $x \in \mathbb{R} \setminus \mathbb{Q}$ irrational, and $0 < \epsilon \in \mathbb{R}$, let $0 < \delta \in \mathbb{R}$ be the (positive) distance of x to the closest a/b, $\gcd(a, b) = 1$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, such that $b \leq 1/\epsilon$.

4.3.2. Let $g(x) = f(x) + x$, $x \in \mathbb{R}$, where f is from Example 4.3.2.

Chapter 5.

5.1.1. Let $F \in \ell_1$ such that $d(O,F)/d(O,C) = r$, and L the intersection of the line segment $[C,F]$ and the line through K parallel to the line extension of $[O,D]$. The triangles $\triangle[O,C,L]$, $\triangle[O,L,F]$, and $\triangle[O,C,F]$ remain similar for all C satisfying the given ratio. Hence the set of L constructed as above will be on a half-line ℓ with end-point O. Finally, consider the half-line ℓ' through K parallel to ℓ, and with end-point $H \in \ell_1$. Using the second given ratio, it follows that H remains constant for all C, D, and K. Thus, the points K will stay on this half-line ℓ'.

5.2.1. The triangle consists of 10 points:

$$T = \{(0,0),(2,0),(4,0),(6,0),(1,1),(3,1),(5,1),(2,2),(4,2),(3,3)\}.$$

There are $\binom{10}{3} = 120$ ways to select three points from T. We need to deduct the collinear triples. There are 3 on the sides of T of number $3 \cdot \binom{4}{3} = 12$, and 3 through the center $(3,1)$ of number $3 \cdot \binom{3}{3} = 3$. So the total number of non-degenerate triangles is $120 - 12 - 3 = 105$.

5.2.2. We may assume $d \neq 0 \neq e$. We have $a_n = a_0 + nd$ and $b_n = b_0 + ne$, $n \in \mathbb{N}_0$. These give $(a_n - a_0)/d = (b_n - b_0)/e$, and hence the equation of the line is $(x - a_0)/d = (y - b_0)/e$.

5.2.4. The set A can be defined as the set of points $(x,y) \in [0,1] \times [0,1]$ such that $(1-r)x + ry = r(1-r)$ holds for some $r \in [0,1]$. Define $p(r,x,y) = (1-r)x + ry - r(1-r) = r^2 - (1+x-y)r + x$, $(x,y) \in [0,1] \times [0,1]$, $r \in \mathbb{R}$. We have $p(0,x,y) = x \geq 0$ and $p(1,x,y) = y \geq 0$. For a given $(x,y) \in [0,1] \times [0,1]$, $p(r,x,y)$ is a quadratic polynomial in r, and therefore it attains a zero in $r \in [0,1]$ if and only if $p((1+x-y)/2, x, y) \leq 0$. This gives $4x \leq (1+x-y)^2$. Substitute $x = u^2$, $y = v^2$, $u, v \in [0,1]$, factor, and obtain $u + v \leq 1$.

5.4.1. For the side lengths a, b, c of the right-triangle, we have $a = b/q$ and $c = bq$. After simplification, the triangle inequalities give $q^2 < 1+q$, $1 < q+q^2$, and $q < 1+q^2$. Since the roots of the polynomial $x^2 - x - 1$ are τ and $-1/\tau$, the first inequality $q^2 - q - 1 < 0$ gives $-1/\tau < q < \tau$. The second inequality can be rewritten as $0 < (-q)^2 - (-q) - 1$, and therefore it gives $-q < -1/\tau$ or $-q > \tau$, or equivalently $q > 1/\tau$ or $q < -\tau$. The second alternative is not realized. Finally, the last inequality is automatic.

5.5.1. 2.

5.5.2. The line segment $[O,C]$ splits the triangle $\triangle[A,B,C]$ into two isosceles sub-triangles. Now apply the pons asinorum to both sub-triangles along with the fact that the sum of the interior angle measures in a triangle is equal to π. The proof of the central angle theorem is similar.

5.5.3. (a) Consider first the line through P and O that meets the circle in the diagonal points $A_0, B_0 \in S$. Clearly, $\mathfrak{p}_S(P) = d(O,A_0) \cdot d(O,B_0)$. Assuming $A \in [B,P]$

and $A_0 \in [B_0, P]$, use Thales' theorem to show that the triangles $\triangle[P, A, A_0]$ and $\triangle[P, B, B_0]$ are similar. Finally, use Birkhoff's Postulate of Similarity. (c) There is a unique point Q on the line segment $[O_1, O_2]$ such that $\mathfrak{p}_{S_1}(Q) = \mathfrak{p}_{S_2}(Q)$. The radical line is perpendicular to $[O_1, O_2]$ and passes through Q.

5.5.4. By scaling, we may assume $d(A, C') = 1$. Let $x = d(A', C')$. The equilateral triangle with vertices the midpoints of the sides of the original triangle $\triangle[A, B, C]$ is congruent to $\triangle[A', B', C']$, in particular, its (common) side length is also x. Therefore, the side length of the original triangle $\triangle[A, B, C]$ must be $2x$. Applying the intersecting chord theorem (Exercise 5.5.3) to the line through the points A and A', we obtain $d(A, C')d(A, A') = x^2$. This gives $1 + x = x^2$. Since $x > 0$ it must be the golden number τ.

5.5.5. By Thales' theorem this set is the open disk with diameter $[A, B]$ and, for non-degeneracy, the diameter is removed.

5.5.6. We may assume that $A = (0, 0)$ and $B = (1, 0)$. The condition $d(P, A)^2 = q^2 \cdot d(P, B)^2$, $P = (x, y) \in \mathbb{R}^2$, can be written as $x^2 + y^2 = q^2((x-1)^2 + y^2)$. Expanding and simplifying, we obtain $(x + q/(1 - q^2))^2 + y^2 = q^2(2 - q^2)/(1 - q^2)$, the equation of a circle.

5.5.7. Let B_1 be the foot of the altitude from B of the triangle $\triangle[A, B, D]$, and C_1 the foot of the altitude from C of the triangle $\triangle[A, C, D]$. The points B_1 and C_1 are on the angular bisector from A. Using Birkhoff's Postulate of Similarity for various similar triangles, we obtain

$$\frac{d(A, B)}{d(A, C)} = \frac{d(B, B_1)}{d(C, C_1)} = \frac{d(B, D)}{d(C, D)}.$$

5.5.8. Let ℓ be the common perpendicular bisector of the three chords. Let x and y be the heights of the two disjoint circular domains whose boundaries are the circular arcs surmounted on the first two parallel chords. Using the power of the (three) intersection points of this line with the chords, by Exercise 5.5.3, we obtain $x(y + d) = (a/2)^2$, $(x + d/2)(y + d/2) = (c/2)^2$, $(x + d)y = (b/2)^2$. Add the first and the third equations and compare it with the second.

5.5.9. Let O_1, O_2, O_3 be the centers of the three circles of radius r, and O the center of the circle of radius R. Due to the tangency conditions, O is the center of the equilateral triangle $\triangle[O_1, O_2, O_3]$ with side length $2r$. Hence $d(O, O_i) = (2/3)(2r)\sqrt{3}/2 = 2r\sqrt{3}/3$, $i = 1, 2, 3$. (a) With the tangency condition of the outer circle, we have $R = r + 2r\sqrt{3}/3$. We obtain $R/r = 1 + 2\sqrt{3}/3$. (b) Once again the tangency condition for the sides of the outer triangle give the side length as $2r + 2r\sqrt{3} = 2r(1 + \sqrt{3})$ so that the perimeter is $6r(1 + \sqrt{3})$.

5.5.10. Let $0 < r \in \mathbb{R}$ be the radius of the small circle. The tangency conditions give $2(1 + r) = 2\sqrt{2}$, and so $r = \sqrt{2} - 1$.

5.5.11. Let $d = d(A, O)$, and P and Q the points of tangency of the two tangent lines from A to the circle such that $B \in [A, P]$ and $C \in [A, Q]$. Let R be the point of tangency of the third tangent line to the circle. We have $d(A, P) = d(A, Q) = \sqrt{d^2 - r^2}$. By tangency again, we have $d(B, P) = d(B, R)$ and $d(C, Q) = d(C, R)$. Hence the perimeter is equal to $2\sqrt{d^2 - r^2}$.

5.5.12. Splitting the sides at the points of tangency, the two legs of the trapezoid have length $(a + c)/2$. Since the height is $2r$, by the Pythagorean theorem, we have $4r^2 + (a - c)^2/4 = (a + c)^2/4$. Hence, $r = \sqrt{ac}/2$.

5.7.1. Let $d(A, B) = a$, $d(B, C) = b$, $d(B, E') = u$, $d(C, E'') = v$. By the angle trisection condition we have $d(A, E') = 2u$ and $d(A, E'') = 2(a-v)$. The Pythagorean theorem now gives $a^2 + u^2 = 4u^2$ and $b^2 + (a - v)^2 = 4(a - v)^2$. Hence $a = \sqrt{3}u$ and $b = \sqrt{3}(\sqrt{3}u - v)$.

5.7.2. It is clear that the overlap is a rhombus. Let x be its side length. The Pythagorean theorem gives $x^2 = (a - x)^2 + b^2$. Hence $x = (a^2 + b^2)/(2a)$.

5.7.3. The two line segments connecting the center of each smaller circle and the center of the big circle are hypotenuses of right triangles with vertical and horizontal sides. Since these hypotenuses go through the points of tangency, their lengths are $3 + 1 = 4$ and $3 + 2 = 5$. Since all three circles touch the left vertical side of the rectangle, the lengths of the horizontal sides of the right triangles are $3 - 1 = 2$ and $3 - 2 = 1$. The Pythagorean theorem gives the vertical sides as $\sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$ and $\sqrt{5^2 - 1^2} = \sqrt{24} = 2\sqrt{6}$. Adding these up plus the vertical contribution of the small circles to the height, we obtain $1 + 2\sqrt{3} + 2\sqrt{6} + 2 = 3 + 2\sqrt{3} + 2\sqrt{6}$.

5.7.4. Let $O_m = (2m, 0)$, $O_0 = 0$, and $P \in S_n$ the point of tangency of ℓ with S_n. Let $Q \in S_1$ be the foot of the altitude from O_1 to the chord $[A_n, B_n]$ of S_1. Since the triangles $\triangle[0, O_1, Q]$ and $\triangle[0, O_n, P]$ are similar (right-)triangles, we have $2n/2 = 1/d(O_1, Q)$. This gives $d(O_1, Q) = 1/n$. Finally, the Pythagorean theorem applied to the right-triangle $\triangle[O_1, A_n, Q]$ gives $d(A_n, B_n)/2 = d(A_n, Q) = \sqrt{1^2 - d(O_1, Q)^2}$. With these, we obtain $d(A_n, B_n) = 2\sqrt{1 - 1/n^2}$.

5.7.5. After simplification, the Pythagorean equation reduces to $F_n^2 + F_{n-1}^2 = F_{2n-1}^2$. This is a special case of Exercise 3.1.10.

5.7.7. For the side lengths a, b, c of the right-triangle, we have $a = b - d$ and $c = b + d$. The Pythagorean equation $(b - d)^2 + b^2 = (b + d)^2$ simplifies to $b = 4d$. With this we obtain $a = 3d$ and $c = 5d$.

5.7.8. Let h denote the length of the altitude. By assumption, we have $h = c/2$. The altitude line splits the triangle into two similar triangles; in particular $a/h = c/b$, or equivalently, $2ab = 2hc = c^2$. The Pythagorean theorem then gives $a^2 + b^2 = c^2 = 2ab$. This rewrites as $(a - b)^2 = 0$, so that $a = b$.

5.8.1. Let $\rho_{l'}(A) = A'$ and $\rho_{l''}(A) = A''$. Since reflection in a line is distance preserving, for any choice of $B \in \ell'$ and $C \in \ell''$, the perimeter of the triangle $\triangle[A, B, C]$ is equal to the length of the open polygonal path consisting of the line segments $[A', B]$, $[B, C]$, and $[C, A'']$. The shortest path between A' and A'' is realized by the length of the (single) line segment $[A', A'']$. Since our angle is acute, this line segment intersects the half-lines ℓ', resp. ℓ'', at B_0, resp. C_0. By construction, the triangle $\triangle[A, B_0, C_0]$ has the least perimeter.

5.9.1. Start with a constructible circle with a constructible point on its perimeter. Construct a square and a regular hexagon inscribed into the circle with one vertex being the given point. Since the perpendicular bisector of two constructible points is constructible, Archimedes' duplication gives a constructible octagon, and a constructible dodecagon.

Chapter 6.

6.1.1. $p(x-1) = x^3 - 3x^2$.

6.1.2. Assume that $|x| + |y| + |z|$ is the smallest **natural** number that x, y, z is a solution. Since x is even, $x = 2w$ say, we have $8w^3 = 2y^3 + 4z^3$, or $y^3 = 2(-z)^3 + 4w^3$. Observe that $(y, -z, w)$ is another solution which has less absolute value sum. Conclude that $(0, 0, 0)$ is the only solution.

6.1.3. We factor as $ax^2 + bx + b - a = a(x^2 - 1) + b(x + 1) = (x + 1)(ax + b - a)$. The first root is -1, the second is $1 - b/a$, Since a does not divide b, the second root is not an integer.

6.2.1. Since $p(x)$ is odd ($p(-x) = -p(x)$, $x \in \mathbb{R}$) it is enough to show that $p(n) \in \mathbb{Z}$ for $n \in \mathbb{N}$. We use induction with respect to $n \in \mathbb{N}_0$. For the general induction step $n \Rightarrow n + 1$, we calculate

$$\begin{aligned} p(n+1) &= \frac{(n+1)^5}{5} + \frac{(n+1)^3}{3} + \frac{7(n+1)}{15} \\ &= \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1}{5} + \frac{n^3 + 3n^2 + 3n + 1}{3} + \frac{7n+7}{15} \\ &= p(n) + (n^4 + 2n^3 + 2n^2 + n) + (n^2 + n) + \left(\frac{1}{5} + \frac{1}{3} + \frac{7}{15}\right). \end{aligned}$$

The constant is $p(1) = 1$.

6.2.2. $999^3 = (1000 - 1)^3 = 1000^3 - 3 \cdot 1000^2 + 3 \cdot 1000 - 1 = 997,002,999$.

6.2.3. Expand and rewrite as $(x^3 - 1)^2 + x^2(x - 1)^2 = 0$. Thus, $x = 1$ is the only solution.

6.2.4. Letting $u = x - a$ and $v = y + a$, the equation rewrites as $(u + v)^2 = uv$. Squaring and rearranging, we obtain $u^2 + uv + v^2 = 0$. Using the cubic identity, we have $u^3 - v^3 = (u - v)(u^2 + uv + v^2) = 0$. This gives $u^3 = v^3$, and hence $u = v$. The equation for u, v then implies $u = v = 0$. Reverting back to the original variables, we obtain that $x = a$ and $y = -a$ is the only solution.

6.3.3. For $p(x)$ constant, this follows from identity x in Exercise 6.3.2 above ($k = n$). In general, take repeated derivatives of the binomial expansion of the polynomial $(1-x)^n$ at $x = 1$.

6.3.5. $a > 1$ may be assumed (since otherwise we take the reciprocal of a). Show that $1 < a^{1/n} < 1 + a/n$ for $n \in \mathbb{N}$. Indeed, assume that $a^{1/n} \geq 1 + a/n$. Then, by the Binomial Formula, we have $a \geq (1 + a/n)^n \geq 1 + a$, a contradiction.

6.3.6. The first three cards must have ranks from the set $\{2, 3, 4, 5, 6\}$, so that the number of these arrangements is $\binom{5}{3}$ (with strictly increasing rank). The last three cards must have ranks from the set $\{8, 9, 10, J, Q, K, A\}$, so that the number of arrangements is $\binom{7}{3}$. As for the suites, the middle card can be any of the 4 suites. For each of the rest of the cards we can have 3 choices for the suite. Thus, the total number of arrangements is $4 \cdot 3^6 \cdot \binom{5}{3} \cdot \binom{7}{3} = 1,020,600$.

6.3.7. Given $k = 1, 2, \ldots, n$, there are $n - 1$ choices for a derangement to map k to $j \neq k$, $j = 1, 2, \ldots, n$. The number of derangements such that k is mapped to j but j is not mapped back to k is D_{n-1}. The number of derangements such that k is mapped to j and j is mapped back to k is D_{n-2}.

6.4.1. $x^3y^3 - x^3 - y^3 + 1 = (x^3 - 1)(y^3 - 1) = (x - 1)(x^2 + x + 1)(y - 1)(y^2 + y + 1)$.

6.4.2. Observing that $-a - 1$ is a solution, we obtain the factorization $(x + 1)(x + a)(x + a + 2)(x + 2a + 1) - a^2 = (x^2 + 2(a + 1)x + 2a)(x + a + 1)^2$.

6.5.1. The sum of the coefficients of $p(x)$ is zero. This means that 1 is a root, and hence $x - 1$ is a factor. We now perform synthetic division:

$$\begin{array}{c|ccccccc} & 1 & -1 & -3 & 2 & 3 & -1 & -1 \\ 1 & & 1 & 0 & -3 & -1 & 2 & 1 \\ \hline & 1 & 0 & -3 & -1 & 2 & 1 & 0 \end{array}$$

This gives the factorization

$$p(x) = (x - 1)(x^5 - 3x^3 - x^2 + 2x + 1).$$

The quotient still has the property that the sum of the coefficients is zero. Performing another synthetic division, we have

$$\begin{array}{c|cccccc} & 1 & 0 & -3 & -1 & 2 & 1 \\ 1 & & 1 & 1 & -2 & -3 & -1 \\ \hline & 1 & 1 & -2 & -3 & -1 & 0 \end{array}$$

This gives

$$p(x) = (x-1)^2(x^4+x^3-2x^2-3x-1).$$

This time the **alternating** sum of the coefficients is zero. This means that -1 is a root, and hence $x+1$ is a factor. Performing yet another synthetic division, we get

$$\begin{array}{r|rrrrr} & 1 & 1 & -2 & -3 & -1 \\ -1 & & -1 & 0 & 2 & 1 \\ \hline & 1 & 0 & -2 & -1 & 0 \end{array}$$

At this point, we have

$$p(x) = (x-1)^2(x+1)(x^3-2x-1).$$

Since the alternating sum of coefficients (including the zero coefficient of x^2) is still zero, we proceed with yet another synthetic division

$$\begin{array}{r|rrrr} & 1 & 0 & -2 & -1 \\ -1 & & -1 & 1 & 1 \\ \hline & 1 & -1 & -1 & 0 \end{array}$$

This gives

$$p(x) = (x-1)^2(x+1)^2(x^2-x-1).$$

Finally, the last quotient is quadratic, x^2-x-1, and the Quadratic Formula gives two irrational roots $(1\pm\sqrt{5})/2$, the golden number τ and its negative reciprocal $-1/\tau$. These, along with the roots $x=\pm 1$ of multiplicity 2, give all the roots of the sextic polynomial $p(x)$.

6.5.2. Polynomial division gives

$$\frac{n^2+15}{n+5} = n-5+\frac{40}{n+5}.$$

Hence, $n = 3, 5, 15, 35$.

6.5.4. The difference of cubes identity gives

$$\begin{aligned} x^{15}-1 = (x^5)^3-1 &= (x^5-1)((x^5)^2+x^5+1) \\ &= (x^5-1)(x^{10}+x^5+1) \\ &= (x-1)(x^4+x^3+x^2+x+1)(x^{10}+x^5+1). \end{aligned}$$

On the other hand, we also have

$$
\begin{aligned}
x^{15}-1 &= (x^3)^5-1 \\
&= (x^3-1)((x^3)^4+(x^3)^3+(x^3)^2+x^3+1) \\
&= (x^3-1)(x^{12}+x^9+x^6+x^3+1) \\
&= (x-1)(x^2+x+1)(x^{12}+x^9+x^6+x^3+1).
\end{aligned}
$$

Now, we note that $\operatorname{gcf}(x^2+x+1, x^4+x^3+x^2+x+1)=1$. Indeed, using the Euclidean Algorithm, we have

$$
\begin{aligned}
x^4+x^3+x^2+x+1 &= (x^2+x+1)x^2+x+1 \\
x^2+x+1 &= (x+1)x+1.
\end{aligned}
$$

Comparing the computations above, we obtain that x^2+x+1 is a factor of $x^{10}+x^5+1$. Using long division, we obtain

$$
\begin{array}{r|rrrrrrrrrrr}
 & & & x^8 & -x^7 & & +x^5 & -x^4 & +x^3 & & -x & +1 \\ \cline{2-12}
x^2+x+1 & x^{10} & & & & & +x^5 & & & & & +1 \\
 & -x^{10} & -x^9 & -x^8 \\ \cline{2-4}
 & & -x^9 & -x^8 \\
 & & x^9 & +x^8 & +x^7 \\ \cline{3-5}
 & & & & x^7 & & +x^5 \\
 & & & & -x^7 & -x^6 & -x^5 \\ \cline{5-7}
 & & & & & -x^6 \\
 & & & & & x^6 & +x^5 & +x^4 \\ \cline{6-8}
 & & & & & & x^5 & +x^4 \\
 & & & & & & -x^5 & -x^4 & -x^3 \\ \cline{7-9}
 & & & & & & & & -x^3 \\
 & & & & & & & & x^3 & +x^2 & +x \\ \cline{9-11}
 & & & & & & & & & x^2 & +x & +1 \\
 & & & & & & & & & -x^2 & -x & -1 \\ \cline{10-12}
 & & & & & & & & & & & 0
\end{array}
$$

Thus, we have

$$x^{10}+x^5+1=(x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1).$$

Remark. The complete factorization of the polynomial $x^8-x^7+x^5-x^4+x^3-x+1$ in the previous example is more involved. We give the details without proof.[2] The

[2]Since $x^{10}+x^5+1=(x^5)^2+x^5+1$, over the complex number field $\mathbb{C}$, the ten roots of this polynomial are the two sets of 5th roots of $(-1\pm \mathrm{i}\sqrt{3})/2$. Taking conjugate pairs, the quadratic factors can be recovered.

polynomial has no real root so that it splits into four irreducible quadratic factors.

First, the polynomial can be written as the product of two quartic polynomials as follows:

$$\begin{aligned}&x^8 - x^7 + x^5 - x^4 + x^3 - x + 1\\&\qquad = (x^4 - \tau x^3 + \tau x^2 - \tau x + 1)(x^4 + (1/\tau)x^3 - (1/\tau)x^2 + (1/\tau)x + 1),\end{aligned}$$

where $\tau = (1 + \sqrt{5})/2$ is the golden number.

Second, each quartic splits into a product of two quadratic polynomials as

$$\begin{aligned}&x^4 - \tau x^3 + \tau x^2 - \tau x + 1\\&\quad = \left(x^2 - \frac{1+\sqrt{5}+\sqrt{6}\sqrt{5-\sqrt{5}}}{4}x + 1\right)\left(x^2 - \frac{1+\sqrt{5}-\sqrt{6}\sqrt{5-\sqrt{5}}}{4}x + 1\right)\\&x^4 + (1/\tau)x^3 - (1/\tau)x^2 + (1/\tau)x + 1\\&\quad = \left(x^2 - \frac{1-\sqrt{5}+\sqrt{6}\sqrt{5-\sqrt{5}}}{4}x + 1\right)\left(x^2 - \frac{1-\sqrt{5}-\sqrt{6}\sqrt{5-\sqrt{5}}}{4}x + 1\right)\end{aligned}$$

6.6.1. There are no solutions beyond the obvious $x = 0, y = 1$ and $x = 1, y = 0$. First, notice that that $|x| \le 1$ and $|y| \le 1$. Clearly, $-1 < x, y < 0$ are impossible. Finally, for $0 < x, y < 1$, we can use the fact that $0 < x^4 < x^3$ and $0 < y^4 < y^3$.

6.6.2. Let x, y, z be the roots of the monic cubic polynomial $p(t) = (t - x)(t - y)(t - z)$. The Newton-Girard formulas give $s_1 = p_1 = 3$, $s_1^2 - 2s_2 = p_2 = 3$, $s_1^3 - 3s_2 s_1 + 3s_3 = p_3 = 3$. These give $s_1 = s_2 = 3$ and $s_3 = 1$. Thus, we have $p(t) = t^3 - 3t^2 + 3t - 1 = (t - 1)^3$. Hence $x = y = z = 1$ is the only solution.

6.6.4. Since D is symmetric with respect to r_1, r_2, r_3 as indeterminates, by the Fundamental Theorem of Symmetric Polynomials, it can be written as a polynomial in the elementary symmetric polynomials s_1, s_2, s_3 as indeterminates. Now use the Viète relations.

6.6.5. Since the roots are real and distinct, we have $4ac < b^2$. Moreover, the quadratic formula gives

$$-b < \pm\sqrt{b^2 - 4ac} < 2a - b.$$

The first inequality is equivalent to $\sqrt{b^2 - 4ac} < b$. This holds if and only if $b > 0$ and $4ac > 0$. Since $a > 0$, this gives $c > 0$. The second inequality immediately gives $b < 2a$, and, after squaring, it simplifies to $b < a + c$. Finally, $b < 2a$ is equivalent to $b^2 < 4a^2$, so that $4ac < b^2 < 4a^2$ gives $c < a$.

6.6.6. $r = 1 - c$ is a root for all $c \in \mathbb{R}$ so that we have the factorization $p(x) = (x + c - 1)(x^2 + x + c + 1)$. Thus the discriminant D of the quadratic factor must be negative. This gives $c > -3/4$.

6.7.1. By symmetry, we may assume $a_1 \leq a_2 \leq \cdots \leq a_n$, so that $s - a_1 \geq s - a_2 \geq \cdots \geq s - a_n$, and therefore $a_1/(s-a_1) \leq a_2/(s-a_2) \leq \cdots \leq a_n/(s-a_n)$. Now use the Chebyshev sum inequality for the last two sequences.

6.7.2. Use the AM-GM-inequality three times for each parentheses.

6.7.3. Use the AM-GM-inequality three times for each pair of terms on the left-hand side.

Chapter 7.

7.2.2. A simple comparison with the Cubic Formula shows that this is a root of the cubic $x^3 + 2x + 3$. This cubic has a single real root. On the other hand, -1 is clearly a root. It follows that this expression is equal to -1.

7.3.1. Divide by x^2 and set $t = x + 1/x$. Since $t^2 = x^2 + 1/x^2 + 2$, we obtain $p(x)/x^2 = q(t) = at^2 + bt + c - 2a = 0$. Assuming the discriminant $D = b^2 - 4a(c - 2a) \geq 0$, we have $q(t) = a(t - r_1)(t - r_2)$, where $r_1, r_2 = (-b \pm \sqrt{D})/(2a)$. Reverting back to the original polynomial, we obtain $p(x) = (x^2 - r_1 x + 1)(x^2 - r_2 x + 1)$.

7.4.1. Use the Rational Root Test to obtain 4 as a root. After factoring the other two roots are $-2 \pm \sqrt{3}$.

7.4.2. Use the Rational Root Test to obtain 2 as a root. The other two roots are complex conjugates.

7.5.1. First we dehomogenize as in Example 7.5.4. This amounts to setting $y = 1$. Using the Finite Geometric Series Formula, we have

$$x^{10} - 1 = (x^5)^2 - 1 = (x^5 - 1)(x^5 + 1) = (x-1)(x^4 + x^3 + x^2 + x + 1)(x+1)(x^4 - x^3 + x^2 - x + 1).$$

(Note the symmetry with respect to the substitution $x \mapsto -x$.) We now claim

$$x^4 + x^3 + x^2 + x + 1 = \left(x^2 + \tau x + 1\right)\left(x^2 - (1/\tau)x + 1\right),$$

where the coefficients of the two linear terms on the right-hand side are the golden number and its negative reciprocal. We now use the method of Exercise 7.3.1 and write

$$x^4 + x^3 + x^2 + x + 1 = x^2\left(x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\right).$$

For the expression in the parentheses on the right-hand side, we use the substitution $t = x + 1/x$. Squaring, we then have $t^2 = x^2 + 1/x^2 + 2$ so that

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = t^2 + t - 1.$$

The roots of the quadratic polynomial equation $t^2+t-1=0$ are $-(1\pm\sqrt{5})/2$. By the Factor Theorem, we then have

$$t^2+t-1=(t+\tau)\,(t-1/\tau)\,.$$

In terms of the original indeterminate x, this is equal to

$$\left(x+\frac{1}{x}+\tau\right)\left(x+\frac{1}{x}-1/\tau\right)=\frac{1}{x^2}\left(x^2+\tau x+1\right)\left(x^2-(1/\tau)x+1\right).$$

The claim above now follows.
Using this (with $\pm x$), we obtain

$$\begin{aligned} x^{10}-1 &= (x-1)(x^4+x^3+x^2+x+1)(x+1)(x^4-x^3+x^2-x+1)\\ &= (x-1)\left(x^2+\tau x+1\right)\left(x^2-(1/\tau)x+1\right)\\ &\quad\times(x+1)\left(x^2-\tau x+1\right)\left(x^2+(1/\tau)x+1\right). \end{aligned}$$

Finally, homogenizing, we arrive at

$$\begin{aligned} x^{10}-y^{10} &= (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)\\ &= (x-y)\left(x^2+\tau xy+y^2\right)\left(x^2-(1/\tau)xy+y^2\right)\\ &\quad\times(x+y)\left(x^2-\tau xy+y^2\right)\left(x^2+(1/\tau)xy+y^2\right). \end{aligned}$$

7.5.2. 8.

Chapter 8.

8.1.1. Substituting $y=x^2/\sqrt{p}$ into $y^2+x(x+q/p)=0$, we obtain $x^4/p+x(x+q/p)=0$. Factoring x and simplifying, we obtain $x^3+px+q=0$.
8.2.1. This follows from the discussion on the reflective property of the parabola. For each tangent line, the midpoint M is on ℓ.
8.2.2. It is enough to prove this for the unit parabola given by $y=x^2$. A simple calculation gives

$$Q_1=\left(x_1,y_2+\frac{y_3-y_2}{x_3-x_2}(x_1-x_2)\right)\quad\text{and}\quad Q_2=\left(x_2,y_1+\frac{y_4-y_1}{x_4-x_1}(x_2-x_1)\right).$$

Taking slopes the condition of the two secants being parallel is equivalent to

$$\frac{y_3-y_2}{x_3-x_2}+\frac{y_4-y_1}{x_4-x_1}=\frac{y_2-y_1}{x_2-x_1}+\frac{y_4-y_3}{x_4-x_3}.$$

Substituting $y_1=x_1^2$, $y_2=x_2^2$, $y_3=x_3^2$, $y_4=x_4^2$, the equality holds.

8.2.3. It is enough to show this for the unit parabola given by $y = x^2$. Let $P_0 = (x_0, y_0)$ be a point of intersecting tangent lines. The equation of a line through P_0 with slope $m \in \mathbb{R}$ has the form $y = y_0 + m(x - x_0)$. Substituting $y = x^2$, we obtain $x^2 - mx + mx_0 - y_0 = 0$. A line is tangent to the parabola if and only if this quadratic equation has a unique solution; that is, its discriminant is zero. We have $D = m^2 - 4(mx_0 - y_0) = 0$. This gives us two slopes. The corresponding two tangent lines are perpendicular if and only if the product of their slopes is equal to -1. By the second Viète relation, this product is the constant term of this quadratic equation; that is $4y_0 = -1$. We obtain $y_0 = -1/4$, or equivalently, the point P_0 is on the directrix.

8.2.4. Given $m \in \mathbb{R}$, consider the pencil of parallel lines given by $y = mx + t$, $t \in \mathbb{R}$. Intersecting these parallel lines with the unit parabola $y = x^2$ amounts to solve the quadratic equation $x^2 - mx - t = 0$. Solutions exists if and only if the discriminant $D = m^2 + 4t \geq 0$; that is, $t \geq -m^2/4$. The midpoints of the first coordinates of the intersections is the constant $m/2$, the arithmetic mean of the roots. It follows that these midpoints fill the vertical half-line given by $(m/2, m^2/2 + t)$, $t \geq -m^2/4$.

8.2.5. We have $x_1 + x_2 = 0$ and $a(x_1^2 + x_2^2) + 2c = 0$. Solving, we obtain $x_1 = -x_2 = \pm\sqrt{-c/a}$.

8.3.1. The solution follows the method of Exercise 8.2.3 above. Let $P_0 = (x_0, y_0)$ be a point of intersecting tangent lines. The equation of a line through P_0 with slope $m \in \mathbb{R}$ has the form $y = y_0 + m(x - x_0)$. Substituting this into $x^2/a^2 + y^2/b^2 = 1$, and expanding, we obtain

$$\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) x^2 + \frac{2m(y_0 + mx_0)}{b^2} x + \frac{(y_0 + mx_0)^2}{b^2} - 1 = 0.$$

The vanising of the discriminant D, after simplification, gives

$$\frac{m^2}{b^2} - \frac{(y_0 + mx_0)^2}{a^2 b^2} + \frac{1}{a^2} = 0.$$

This is a quadratic equation in the slope m. The product of the two roots must be equal to -1. Using the second Viète relation, we obtain

$$\frac{\frac{1}{a^2} - \frac{y_0^2}{a^2 b^2}}{\frac{1}{b^2} - \frac{x_0^2}{a^2 b^2}} = -1.$$

This is equivalent to $x_0^2 + y_0^2 = a^2 + b^2$.

8.3.2. This is clearly true for circles. Now the unit cicle given by $x^2 + y^2 = 1$ can be transformed to the normal hyperbola by the transformation $(x, y) \mapsto (x/a, y/b)$,

$(x, y) \in \mathbb{R}^2$. This transformation preserves lines and ratios, so the same statement holds for ellipses.

8.4.1. This follows along the same lines as Exercise 8.2.3 with simple modifications.

8.4.2. First we reduce this to the rectangular hyperbola given by $y = 1/x$. We now follow the method of Exercise 8.2.4 as follows. Given $m \in \mathbb{R}$, consider the pencil of parallel lines given by $y = mx + t$, $t \in \mathbb{R}$. Clearly, chords exist only if $m < 0$. Intersecting these parallel lines with $y = 1/x$ amounts to solve the quadratic equation $mx^2 + tx - 1 = 0$. The discriminant $D = t^2 + 4m \geq 0$. This gives $t^2 \geq -4m$. Finally, the midpoints of the first coordinates of the intersections is $-t/(2m)$. With this, the midpoints can be parametrized by t as $(-t/(2m), t/2)$. Equivalently, the equation of the line is $y = -mx$.

8.4.4. Using the notation in the parametrization of the hyperbola in the main text, the parallelogram is given by the points O, P_t, Q_t, $Q_t - P_t$, where

$$P_t = (at, bt) \quad \text{and} \quad Q_t = \left(\frac{a}{2}\left(2t + \frac{1}{2t}\right), \frac{b}{2}\left(2t - \frac{1}{2t}\right)\right).$$

The line through O and $P_t - (Q_t - P_t) = 2P_t - Q_t$ is parallel to the other diagonal. Since

$$2P_t - Q_t = \left(\frac{a}{2}\left(2t - \frac{1}{2t}\right), \frac{b}{2}\left(2t + \frac{1}{2t}\right)\right),$$

the corresponding slope is

$$\frac{b\left(2t + \frac{1}{2t}\right)}{a\left(2t - \frac{1}{2t}\right)}.$$

On the other hand, the slope of the tangent line through Q_t is

$$\frac{\frac{1}{a}\left(2t + \frac{1}{2t}\right)}{\frac{1}{b}\left(2t - \frac{1}{2t}\right)}.$$

The two slopes are equal.

8.4.6. Clearly the three vertices cannot be on a single branch of the hyperbola. By symmetry, we may assume that one vertex is at $(-1, -1)$ and the other two are $(a, 1/a)$ and $(1/a, a)$, for some $0 < a \in \mathbb{R}$. The condition that the side lengths are equal is $2(1/a - a)^2 = (1/a + 1)^2 + (a + 1)^2$. Expanding and simplifying (in the use of the new variable $b = a + 1/a$) we obtain $a = 2 \pm \sqrt{3}$.

Chapter 9.

9.1.1. Using the identity[3] $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$, the numerator factors as $(x^2 + 1)(5x^2 + 4x + 1)$, so that the fraction becomes $x^2 + 1$.

[3] Sometimes termed as the Sophie Germain identity.

9.1.3. (c) $1/(1+x) = 1/(1-x^2) - x/(1-x^2)$ and $1/(x^4+x) = x^2/(x^6-1) - 1/(x^7-x)$.
9.2.1. (a) $-1/(x-2) + 5/(x+3) + 2/(x+1)$; (b) $1/(x-1) + 1/(x-1)^2 + 1/(x-1)^3$; (c) $(3x+2)/(x^2+1) + (x-2)/(x^2+x+1)$.
9.2.2. Using partial fraction decomposition, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} &= \sum_{k=1}^{n} \frac{k}{(k^2-k+1)(k^2+k+1)} \\
&= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k^2-k+1} - \frac{1}{k^2+k+1}\right) \\
&= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k(k-1)+1} - \frac{1}{(k+1)k+1}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{n^2+n+1}\right),
\end{aligned}$$

since the last sum is telescopic.
9.3.1. We have

$$\frac{1+2x-x^2}{1-x^2} = 1 + \frac{2x}{1-x^2}.$$

Hence, $y = 1$ is a horizontal asymptote, and $x = \pm 1$ are two vertical asymptotes.
9.4.1. xy/z.
9.4.2. $(x^{1/2} - y^{1/2})(x + x^{1/2}y^{1/2} + y)$.
9.4.3. First calculate $(1-\sqrt{2}+\sqrt{3})(1-\sqrt{2}-\sqrt{3}) = -2\sqrt{2}$. With this, we have $1/(1-\sqrt{2}+\sqrt{3}) = -\sqrt{2}(1-\sqrt{2}-\sqrt{3})/4$.
9.5.1. By Thales' Theorem, the triangle $\triangle[A,B,C]$ has right angle at the vertex C. As usual, we let $d(A,B) = c$, $d(B,C) = a$, $d(C,A) = b$, and $d(C,D) = h$. (a) Applying the Pythagorean theorem to the three right triangles $\triangle[A,B,C]$, $\triangle[B,D,C]$ and $\triangle[A,D,C]$, we obtain

$$\begin{aligned}
a^2 + b^2 &= c^2 = (x+y)^2 \\
a^2 &= y^2 + h^2 \\
b^2 &= x^2 + h^2.
\end{aligned}$$

Subtracting the second and third equality from the first, and rearranging, we arrive at

$$d(C,D) = h = \sqrt{xy} = \sqrt{d(A,D)d(B,D)}.$$

This is the geometric interpretation of the **geometric mean**. Note that, since $d(O,A) = d(O,B) = (x+y)/2$, the AM-GM inequality $\sqrt{xy} \le (x+y)/2$ also follows with equality if and only if $x = y$. (b) Let $[D, E]$ be the altitude line of the triangle $\triangle[O, C, D]$ from the vertex D. We claim that $d(C, E)$ is the **harmonic mean** of $x = d(A, D)$ and $y = d(B, D)$. Indeed, letting $d(C, E) = u$, $d(O, E) = v$, as above, we get $d(D, E) = \sqrt{uv}$. The Pythagorean theorem aplied to the right triangle $\triangle[C, D, E]$ then gives

$$u^2 + uv = xy.$$

On the other hand, we have $u + v = (x + y)/2$. Combining these two equations, we obtain

$$d(C, E) = u = \frac{u^2 + uv}{u + v} = \frac{xy}{\frac{x+y}{2}} = \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$

For (c), the Pythagorean theorem applied to the triangle $\triangle[O, D, F]$ gives $d(D, F] = \sqrt{(x + y)^2 + (x - y)^2/2} = \sqrt{(x^2 + y^2)/2}$.

9.5.2. For $0 < x \in \mathbb{R}$, by the general AM-GM inequality, we have

$$\begin{aligned} x^m + \frac{1}{x^n} &= \frac{x^m}{n} + \cdots + \frac{x^m}{n} + \frac{1}{mx^n} + \cdots + \frac{1}{mx^n} \\ &\geq (m+n) \sqrt[m+n]{\left(\frac{x^m}{n}\right)^n \cdot \left(\frac{1}{mx^n}\right)^m} = \frac{m+n}{\sqrt[m+n]{m^m \cdot n^n}}. \end{aligned}$$

Equality holds if and only if $x^m/n = 1/(mx^n)$; that is, if and only if $x = \sqrt[m+n]{n/m}$.

9.6.1. n must be a multiple of 6.

9.6.2. $1 \leq x < 16$.

Chapter 10.

10.1.1. This follows from the general lower and upper estimate of $\exp(x)$ for $x = 1$. For $n = 1, 2, 3, 4$, we obtain $2 < e < 3$, $5/2 < e < 11/4$, $8/3 < e < 49/18$, $65/24 < e < 87/32$.

10.1.2. This is the Bernoulli inequality in disguise ($y = e^x$).

10.1.3. Use induction with respect to $n \in \mathbb{N}$.

10.3.1. As in the first solution of Example 3.2.6, we let $0 < b < a$, and $c = b/a$. (Note the switched roles of a and b.) Since $\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = a \lim_{n\to\infty} \sqrt[n]{1 + (b/a)^n}$, we need to show $\lim_{n\to\infty} \sqrt[n]{1 + c^n} = 1$ for $0 < c < 1$. Taking the natural logarithm (and using continuity), we have $\ln\left(\lim_{n\to\infty} \sqrt[n]{1 + c^n}\right) = \lim_{n\to\infty} \ln\left(\sqrt[n]{1 + c^n}\right) = \lim_{n\to\infty} \ln(1 + c^n)/n = 0$.

10.3.2. This is the AM-GM inequality in disguise.

10.3.7. $\sinh(\ln q) = (q-1/q)/2$ and $\cosh(\ln q) = (q+1/q)/2$; in particular $\sinh(\ln 2) = (2 - 1/2)/2 = 3/4$ and $\cosh(\ln 2) = (2 + 1/2)/2 = 5/4$.

10.4.1. First, assume that $x \in (0,1)$ is rational, and write $x = m/n$ with $0 < m < n$ and $m, n \in \mathbb{N}$. Use the general AM-GM inequality with $x_1 = \cdots = x_{n-m} = a^{x_0}$ and $x_{n-m+1} = \cdots = x_n = a^{x_1}$, and conclude that convexity holds in this case. For real $x \in [0,1]$ use (sequential) continuity. The geometric meaning of this inequality is that, for $a \neq 1$, the graph of the exponential function $y = a^x$ on the interval $[x_0, x_1]$ is below its secant line passing through the points (x_0, a^{x_0}) and (x_1, a^{x_1}).

10.5.1. Using continuity of the natural exponential and logarithmic functions and the Euler limit, we have

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x\to\infty}\left(\left(1+\frac{1}{x^2}\right)^{x^2}\right)^{1/x} = e^0 = 1.$$

10.5.2. Take the products of both sides of the inequality $(1+1/k)^k < e$ for $k = 1, 2, \ldots, n-1$, and obtain $n^{n-1}/(n-1)! < e^{n-1}$.

10.5.3. $(1+x/n)^n \leq e^x$, $-n < x$, with equality only at $x_0 = 0$. If n is odd then $(1+x/n)^n$ is negative for $x < -n$, so that the only solution is $x_0 = 0$. If n even then, for $x \leq -n$, the polynomial $(1+x/n)^n$ is decreasing (to zero), while e^x is increasing with a horizontal asymptote being the negative first axis. By the Intermediate Value Theorem, the graphs meet at a unique point $x_1 < -n$.

10.5.4. Let $1 < t = x/y \in \mathbb{R}$. Eliminating x, we obtain $e^{(t+1)y} = t$. This gives $y = \ln t/(t+1)$ and hence $x = t \cdot \ln t/(t+1)$. Thus, $t \in (1, \infty)$ parametrizes all solutions.

Chapter 11.

11.1.1. We have $d(B,E) = d(B,C)/\tan(5\pi/12)) = d(B,C)/(2+\sqrt{3})$. Hence $d(A,E) = d(A,B) - d(B,E) = d(B,C)(2 - 1/(2+\sqrt{3}) = d(B,C)\sqrt{3} = d(A,D)\sqrt{3}$. Thus, $\mu\angle AED = \pi/6$ so that $\mu\angle DEC = 5\pi/12$.

A more ad hoc approach is to define $E' \in [A,B]$ such that $d(D,C) = d(D,E')$ and verify that $E' = E$ by calculating angles.

11.2.1. By the Principle of Least Distance, the shortest path is along two tangential segments from $(0,0)$ and $(2a, 2b)$ to points of tangency at S with an intermediate circular path of $\mathbb{S}$ with end-points, the points of tangency. Letting $d = \sqrt{a^2+b^2}$, by the Pythagorean theorem, the common length of the tangential segments is $\sqrt{d^2-1}$ while the angle subtended by the circular arc connecting the two points of tangency is $\pi - 2\arccos(1/d)$. Thus the total length is $2\sqrt{d^2-1} + \pi - 2\arccos(1/d)$.

11.2.2. Let x be the shorter side length of the slimmer rectangle, and α the angle of tilt. We then have $x\sin\alpha + \cos\alpha = 1$ and $x + x\cos\alpha + \sin\alpha = 1$. These give $\alpha = \pi/6$ and $x = 2 - \sqrt{3}$.

11.3.1. This follows by inspection of the accompanying figure.
11.3.2. Let $a = \cos\alpha$, $b = \sin\alpha$, $c = \cos\beta$, $d = \sin\beta$, $\alpha, \beta \in \mathbb{R}$. Then $|ac + bd| = |\cos\alpha\cos\beta + \sin\alpha\sin\beta| = |\cos(\alpha - \beta)| \leq 1$.
11.3.3. By the Cauchy-Schwarz inequality, we have $c_{n+1} = a \cdot c_n + b \cdot \sqrt{1 - c_n^2} \leq \sqrt{a^2 + b^2} = 1$, $n \in \mathbb{N}_0$, and the first statement follows by induction. For the second, let $a = \cos\theta$, $b = \sin\theta$, $0 < \theta < \pi/2$, and define $0 \leq \theta_n < \pi/2$ with $c_n = \cos(\theta_n)$, $n \in \mathbb{N}_0$. Rewrite the inductive definition as $\cos(\theta_{n+1}) = \cos\theta \cdot \cos(\theta_n) + \sin\theta \cdot \sin(\theta_n) = \cos(\theta - \theta_n)$. Verify that $\theta \geq \theta_n$, that is, $a \leq c_n$, $n \in \mathbb{N}$, by induction. Finally, conclude that $\theta_{n+1} = \theta - \theta_n$, $n \in \mathbb{N}$.
11.3.4. The cubic formulas are reformulations of the triple angle formulas for sine and cosine. For quadruple angles, we have

$$\sin^4(\alpha) = \frac{3 - 4\cos(2\alpha) + \cos(4\alpha)}{8} \quad \text{and} \quad \cos^4(\alpha) = \frac{3 + 4\cos(2\alpha) + \cos(4\alpha)}{8}$$

11.3.5. Using the double angle formula $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$, these identities can be converted to powers of sine in the double angle 2α. With this, the identities follow from the half angle formula for sine and the previous exercise.
11.3.6. This follows form the definitions of arcsin and arccos.
11.3.7. We use here the identities $\sin(2\alpha) + \sin(2\beta) = 2\sin(\alpha + \beta)\cos(\alpha - \beta)$ and $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin(\alpha)\sin(\beta)$ both of which are consequences of the addition formulas for sine and cosine. (See also Exercise 11.3.15 below.) With these, we calculate

$$\begin{aligned}
&\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 2\sin(\alpha + \beta)\cos(\alpha - \beta) + 2\sin(\gamma)\cos(\gamma) \\
&\quad = 2\sin(\pi - \gamma)\cos(\alpha - \beta) + 2\sin(\gamma)\cos(\pi - (\alpha + \beta)) \\
&\quad = 2\sin(\gamma)\cos(\alpha - \beta) - 2\sin(\gamma)\cos(\alpha + \beta) \\
&\quad = 2\sin(\gamma)(\cos(\alpha - \beta) - \cos(\alpha + \beta)) = 4\sin(\gamma)\sin(\alpha)\sin(\beta).
\end{aligned}$$

11.3.8. We have $\alpha + \beta = \pi - \gamma$ so that $\tan(\alpha + \beta) = \tan(\pi - \gamma) = -\tan(\gamma)$. The addition formula for tangent gives

$$\frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} = -\tan(\gamma).$$

Multiplying out and rearranging, the identity follows.
11.3.9. Rewrite the cases $n = 3k + 1$, resp. $n = 3k + 2$, $k \in \mathbb{N}_0$, as $\pi/3 - k \cdot \pi/n = \pi/(3n)$, resp. $\pi/3 - k \cdot \pi/n = 2\pi/(3n)$.
11.3.10. This follows by direct substitution $x = \cos(\alpha)$, and using $T_n(\cos(\alpha)) = \cos(n\alpha)$ and $U_{n-1}(\cos(\alpha)) = \sin(n\alpha)/\sin(\alpha)$.

11.3.11. For $\alpha = 0$, we have $T_n(1) = T_n(\cos 0) = \cos 0 = 1$, and, for $\alpha = \pi$, we have $T_n(-1) = T_n(\cos\pi) = \cos(n\pi) = (-1)^n$, $n \in \mathbb{N}$. Similarly, we have $U_{n-1}(1) = \lim_{\alpha\to 0} U_{n-1}(\cos(\alpha)) = \lim_{\alpha\to 0}\sin(n\alpha)/\sin(\alpha) = n$ and $U_{n-1}(-1) = \lim_{\alpha\to\pi} U_{n-1}(\cos(\alpha)) = \lim_{\alpha\to\pi}\sin(n\alpha)/\sin(\alpha) = (-1)^n n$.

11.3.12. This is a direct consequence of the identity $2\cos(m\alpha)\cos(n\alpha) = \cos((m+n)\alpha) + \cos((m-n)\alpha)$ which itself follows from the addition formulas for cosine.

11.3.13. These follow again from the defining formula $T_n(\cos(\alpha)) = \cos(n\alpha)$, $n \in \mathbb{N}$. Restricted to $\alpha \in [0, \pi]$, we have $\cos(\alpha) \in [-1, 1]$, and the n roots are $\cos((2k+1)\pi/(2n))$, $k = 0, 1, \ldots, n-1$.

11.3.14. Straightforward computation using the addition formulas for cosine and sine.

11.4.1. We have $\alpha + \beta = \pi/2 - \gamma$ so that $\cot(\alpha+\beta) = \cot(\pi/2 - \gamma) = \tan(\gamma)$. The addition formula for cotangent gives

$$\frac{\cot(\alpha)\cot(\beta) - 1}{\cot(\alpha) + \cot(\beta)} = \frac{1}{\cot(\gamma)}.$$

Multiplying out and rearranging, the identity follows.

11.4.2. See Section 11.4.

11.4.3. Using the triple angle formulas for sine and cosine, we calculate

$$\tan(3\alpha) = \frac{\sin(3\alpha)}{\cos(3\alpha)} = \frac{3\sin(\alpha) - 4\sin^3(\alpha)}{4\cos^3(\alpha) - 3\cos(\alpha)}$$
$$\frac{3\tan(\alpha)\sec^2(\alpha) - 4\tan^3(\alpha)}{4 - 3\sec^2(\alpha)} = \frac{3\tan(\alpha) - \tan^3(\alpha)}{1 - 3\tan^2(\alpha)}.$$

The second formula follows form the first by taking reciprocals.

11.4.5. For (a), we consider $2\pi/3$ a double angle. Using the double angle formulas, we calculate

$$\cos\left(2\cdot\left(\frac{\pi}{3}\right)\right) = 2\cos^2\left(\frac{\pi}{3}\right) - 1 = 2\left(\frac{1}{2}\right)^2 - 1 = -\frac{1}{2}$$

and

$$\sin\left(2\cdot\left(\frac{\pi}{3}\right)\right) = 2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}.$$

For (b), we consider $3\pi/4$ a half angle. Using the half angle formulas we calculate

$$\cos\left(\frac{3\pi}{4}\right) = -\sqrt{\frac{1+\cos\left(\frac{3\pi}{2}\right)}{2}} = -\sqrt{\frac{1+0}{2}} = -\frac{\sqrt{2}}{2}.$$

Here we used the negative square root due to the fact the $3\pi/4$ falls into Quadrant II in which the cosine function is negative. We do the same for the sine function (with positive square root), and calculate

$$\sin\left(\frac{3\pi}{4}\right) = \sqrt{\frac{1-\cos\left(\frac{3\pi}{4}\right)}{2}} = \frac{\sqrt{2}}{2}.$$

For (c), we first write $5\pi/12 = \pi/4 + \pi/6$. Using the addition formulas, we calculate

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}-1}{2}\right)$$

and

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}+1}{2}\right).$$

11.4.6. We use the notations in the remark and figure following Example 5.5.2. We have $u = (-a+b+c)/2 = s-a$, where $s = (a+b+c)/2$. If O denotes the incenter, the center of the incircle of the triangle $\triangle[A, B, C]$, then $\triangle[A, P, O]$ has right angle at P. The definition of cotangent gives $\cot(\alpha/2) = (s-a)/r$. Applying this to all sides of the triangle $\triangle[A, B, C]$, the Law of Cotangents follows.

11.4.8. We have $\cos\alpha/\sin\alpha + \cos\gamma/\sin\gamma = 2\cos\beta/\sin\beta$. By the Law of Sines this reduces to $\cos\alpha/a + \cos\gamma/c = 2\cos\beta/b$. Now apply the Law of Cosines to each angle and simplify.

11.4.9. Since the sequence is geometric, we obtain $\cos^3(\alpha) = \sin^2(\alpha)$, or equivalently, $\cos^3(\alpha) + \cos^2(\alpha) - 1 = 0$. Hence, $x = \cos(\alpha)$ is the real root of the cubic $x^3 + x^2 - 1$. Thus, x is the negative of the real root of the cubic in Example 7.2.3 (c).

11.8.1. We denote the angles subtended by the chords $[A, B]$, $[B, C]$, $[C, D]$, $[D, A]$ by α, β, γ, δ. Since the angle sum in a triangle is equal to π, we have

$$\alpha + \beta + \gamma + \delta = \pi.$$

We now make use of the geometric meaning of the fractions in the law of sines as the diameter of the circumscribed circle. Since every sub-triangle of our quadrilateral has the same circumscribed circle of radius $R > 0$, say, we have

$$\frac{\sin(\alpha)}{d(A,B)} = \frac{\sin(\beta)}{d(B,C)} = \frac{\sin(\gamma)}{d(C,D)} = \frac{\sin(\delta)}{d(D,A)} = \frac{\sin(\alpha+\delta)}{d(B,D)} = \frac{\sin(\alpha+\beta)}{d(A,C)} = \frac{1}{2R}.$$

Substituting (and canceling $4R^2$), Ptolemy's equation rewrites as

$$\sin(\alpha)\cdot\sin(\gamma) + \sin(\beta)\cdot\sin(\delta) = \sin(\alpha+\beta)\cdot\sin(\alpha+\delta).$$

We now make use of the identity $2\sin(\alpha)\cdot\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta)$. Applying this (twice) to the left-hand side of Ptolemy's equation, we obtain

$$\begin{aligned}
&\sin(\alpha)\cdot\sin(\gamma) + \sin(\beta)\cdot\sin(\delta) \\
&\qquad = \cos(\alpha-\gamma) - \cos(\alpha+\gamma) + \cos(\beta-\delta) - \cos(\beta+\delta) \\
&\qquad\qquad = \cos(\alpha-\gamma) + \cos(\alpha-\delta),
\end{aligned}$$

since

$$\cos(\beta+\delta) = \cos(\pi-\alpha-\gamma) = -\cos(\alpha+\gamma).$$

Similarly, applying this identity to the right-hand side (again twice), we have

$$\begin{aligned}
&\sin(\alpha+\beta)\cdot\sin(\alpha+\delta) = \cos(\beta-\delta) - \cos(2\alpha+\beta+\delta) \\
&\qquad = \cos(\beta-\delta) - \cos(\alpha-\gamma+\pi) \\
&\qquad\qquad = \cos(\beta-\delta) + \cos(\alpha-\gamma).
\end{aligned}$$

Ptolemy's Theorem follows.