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ORIGINAL PAPER

On the space of orthogonal multiplications in three and four dimensions and Cayley's nodal cubic

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Abstract The study of orthogonal multiplications is more than 100 years old and goes back to the works of Hurwitz and Radon. Yet, apart from the extensive literature on admissibility of domain and range dimensions near the Hurwitz-Radon range (in codimension ≤ 8), only sporadic and fragmentary results are known about full classification (in large codimension), more specifically, about the moduli space $\mathfrak{M}_m$ of orthogonal multiplications $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ (for various n), even for $m \leq 4$. In this paper we give an insight to the subtle geometries of $\mathfrak{M}_3$ and $\mathfrak{M}_4$. Orthogonal multiplications are intimately connected to quadratic eigenmaps between spheres via the Hopf-Whitehead construction. The 9-dimensional moduli space $\mathfrak{M}_3$ lies on the boundary of the 84-dimensional moduli of quadratic eigenmaps of S^5 into spheres. Similarly, the 36-dimensional moduli space $\mathfrak{M}_4$ is on the boundary of the 300-dimensional moduli of quadratic eigenmaps of S^7 into spheres. We will show that $\mathfrak{M}_3$ is the $SO(3) \times SO(3)$-orbit of a 3-dimensional convex body bounded by Cayley's nodal cubic surface with vertices in a real projective space $\mathbb{R}P^3$, the latter imbedded equivariantly and minimally in an 8-sphere of the space of quadratic spherical harmonics on S^3. For $\mathfrak{M}_4$, we show that it possesses two orthogonal 18-dimensional slices each of which is an $SO(4) \times SO(4)$-orbit of a 6-dimensional polytope $\mathcal{P} \subset \mathbb{R}^6$. This polytope itself is the convex hull of two orthogonal regular tetrahedra. The cor-

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responding orthogonal multiplications are explicitly constructible. Finally, we give an algebraic description of the 24-dimensional space of diagonalizable elements in $\mathfrak{M}_4$. The crucial fact in $\mathbb{R}^4$ is the splitting of the exterior product $\Lambda^2(\mathbb{R}^4)$ into self-dual and anti-self-dual components. The techniques employed here can be traced back to the work of Ziller and the author (Toth and Ziller 1999) in describing the 18-dimensional moduli for quartic spherical minimal immersions of S^3 into spheres. As a new feature, we point out the importance of multiplicity of the zeros of the polynomials that define the boundary $\partial\,\mathfrak{M}_m$ as a determinantal variety.

Keywords Orthogonal multiplication · Cayley's nodal cubic · Hopf-Whitehead construction · Moduli · Determinantal variety

Mathematics Subject Classification 52A05 · 52A38 · 52B11

1 Introduction and preliminaries

1.1 Orthogonal multiplications

A bilinear map $F : \mathbb{R}^\ell \times \mathbb{R}^m \to V$ into a Euclidean vector space V of dimension n is called an *orthogonal multiplication* if F is *normed*:

$$|F(x, y)| = |x| \cdot |y|, \quad x \in \mathbb{R}^\ell, \quad y \in \mathbb{R}^m,$$

where $|\cdot|$ stands for the Euclidean norms of the ambient vector spaces.
(Due to specific examples, it is of slight advantage to keep V arbitrary and only specify $V = \mathbb{R}^n$ if an orthonormal basis is a priori given in V.) Without loss of generality we will always assume that $1 \le \ell \le m$. In addition, restricting to the linear span of the image of F in V, we can also arrange (and will tacitly assume) that F is *full*, that is, the image of F is not contained in any proper linear subspace of V. With these the range dimension of an orthogonal multiplication $F : \mathbb{R}^\ell \times \mathbb{R}^m \to V$ satisfies $m \le n \le \ell m$, where $n = \dim V$. The lower bound is a direct consequence of normality. The upper bound follows from bilinearity, and is attained (among others) by the tensor product $F_\otimes : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^\ell \otimes \mathbb{R}^m$, $F_\otimes(x, y) = x \otimes y$, $x \in \mathbb{R}^\ell$, $y \in \mathbb{R}^m$. Two orthogonal multiplications $F : \mathbb{R}^\ell \times \mathbb{R}^m \to V$ and $F' : \mathbb{R}^\ell \times \mathbb{R}^m \to V'$ are called *range-equivalent* if $F' = U \cdot F$ for some (linear) isometry $U : V \to V'$.

The simplest examples of orthogonal multiplications are the real, complex, and quaternionic multiplications: $F_\cdot^{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F_\cdot^{\mathbb{R}}(x, y) = x \cdot y$, $x, y \in \mathbb{R}$; $F_\cdot^{\mathbb{C}} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $F_\cdot^{\mathbb{C}}(z, w) = z \cdot \bar{w}$, $z, w \in \mathbb{C}$; and $F_\cdot^{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, $F_\cdot^{\mathbb{H}}(p, q) = p \cdot \bar{q}$, $p, q \in \mathbb{H}$ (with quaternionic conjugation).
For a less trivial example, we let $F^\wedge : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \times \Lambda^2(\mathbb{R}^m) = \mathbb{R}^{m(m-1)/2+1}$ be defined by $F^\wedge(x, y) = (\langle x, y\rangle, x \wedge y)$, $x, y \in \mathbb{R}^m$. [See (Parker 1983, Remark 2, p. 371), and (Wu et al. 2015, H for $t = 0$).]

The study of *existence* of orthogonal multiplications $F : \mathbb{R}^\ell \times \mathbb{R}^m \to V$ (without fullness) goes back to Hurwitz (1898, 1923) and Radon (1922). For brevity, we say that F is *of type* (ℓ, m, n), and that (ℓ, m, n) is *admissible* if F (not necessarily full) exists.

In 1898 Hurwitz proved that (m, m, m) is admissible if and only if $m = 1, 2, 4, 8$. [$m = 8$ corresponds to multiplication of octonions (Cayley numbers).] For the existence of orthogonal multiplications of type (ℓ, m, m) Radon gave a full answer in 1922 as follows. Let $m = 2^p \cdot q$ with q odd. If $p = 4a + b$, $0 \leq b < 4$, then define (the Hurwitz-Radon function) $\rho(m) = 8a + 2^b$. Then (ℓ, m, m) is admissible if and only if $\ell \leq \rho(m)$. Equivalently, $\rho(m)$ is the largest ℓ such that (ℓ, m, m) is admissible. [Admissibility has an extensive literature; see the survey article of Shapiro (1984), and his monograph (Shapiro 2000).]

In view of the Hurwitz-Radon results it is natural to define the *codimension* of an orthogonal multiplication $F : \mathbb{R}^\ell \times \mathbb{R}^m \to V$ as $n - m \geq 0$, $n = \dim V$, and seek classification of orthogonal multiplications near the Hurwitz-Radon range, that is those of small codimension.

Codimension 1 orthogonal multiplications have been fully classified by Adem (1980, 1981). He showed that (up to isometries on the source and the range) an orthogonal multiplication of type $(\ell, m, m + 1)$ extends to an orthogonal multiplication of type $(\ell, m+1, m+1)$ if m is odd, and restricts to an orthogonal multiplication of type (ℓ, m, m) if m is even.

The role played by the parity of m can be illuminated here by noting that restricting the quaternionic multiplication $F_\cdot^{\mathbb{H}}$ to purely imaginary quaternions gives rise to a codimension 1 full orthogonal multiplication $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$, whereas, by Adem's result, *there is no full orthogonal multiplication* $F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^5$.

The full classification of codimension 2 orthogonal multiplications has been given by Gauchman and the author (Gauchman and Toth 1994,1996). They showed that a full orthogonal multiplication of type $(\ell, m, m+2)$ extends to an orthogonal multiplication of type $(\ell, m + 2, m + 2)$ if m is even. In addition, they also proved that, for m odd, the only possible types $(\ell, m, m + 2)$ are $\ell = 3$ and $m = 4r + 1$, $r \geq 1$ (and the corresponding orthogonal multiplications can be explicitly constructed from the quaternionic vector space multiplication $\mathbb{H} \times \mathbb{H}^r \to \mathbb{H}^r$ by restriction).

As an application relevant to our study here, we see that *there is no full orthogonal multiplication* $F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^6$. Indeed, if F existed then it could be extended to an orthogonal multiplication of type $(4, 6, 6)$ which contradicts to Hurwitz-Radon's $\rho(6) = 2$.

Apart from the extensive work on admissibility, there are only sporadic results of *complete* classification of orthogonal multiplications in specific domain and range dimensions. Parker (1983) gave an *algebraic* classification of orthogonal multiplications of type $(2, 2, n)$ and $(3, 3, n)$, and, most recently, in their studies of quadratic self-eigenmaps of S^7, Wu-Xiong-Zhao (2015) proved that, up to isometries on the domain and the range, there is only a 1-parameter family of full orthogonal multiplications $F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^7$. Note the specific example $F^\wedge : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^7$ ($m = 4$) above.

1.2 Connection to eigenmaps: the Hopf-Whitehead construction

The Hopf-Whitehead construction associates to an orthogonal multiplication $F : \mathbb{R}^\ell \times \mathbb{R}^m \to V$ the *quadratic polynomial map* $f_F : \mathbb{R}^{\ell+m} \to \mathbb{R} \times V$ defined by

$$f_F(x, y) = (|x|^2 - |y|^2, 2\,F(x, y)), \quad x \in \mathbb{R}^\ell, \quad y \in \mathbb{R}^m. \tag{1}$$

By normality, f_F is automatically *spherical*, that is, its restriction to the unit sphere $S^{\ell+m-1} \subset \mathbb{R}^{\ell+m}$ maps to the unit sphere $S_{\mathbb{R}\times V}$ of $\mathbb{R} \times V$ and we obtain $f_F : S^{\ell+m-1} \to S_{\mathbb{R}\times V}$. Moreover, by bilinearity, all but the first component of f_F (with respect to a basis in V, say) are *harmonic*, and the first component is harmonic if and only if $\ell = m$.

In general, a spherical map $f : S^m \to S_V$ into the unit sphere of a Euclidean vector space S_V of V, defined by harmonic k-homogeneous polynomials (or, by restriction, spherical harmonics of order k) is said to be a k-eigenmap (between the respective spheres). Eigenmaps furnish important examples of harmonic maps (of constant energy density) between spheres. By (1), the Hopf-Whitehead construction associates to an orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^m \to V$ a quadratic eigenmap $f_F : S^{2m-1} \to S_{\mathbb{R}\times V}$. Since our main motivation and interest are in quadratic eigenmaps between spheres, from now on we will assume that $\ell = m$.

Examples of the Hopf-Whitehead construction include the complex square $f_{F^{\mathbb{R}}} : S^1 \to S^1$, the (classical) *Hopf map* $f_{F^{\mathbb{C}}} : S^3 \to S^2$, and the *quaternionic Hopf map* $f_{F^{\mathbb{H}}} : S^7 \to S^4$. Moreover, we have $f_{F_\otimes} : S^{2m-1} \to S^{m^2}$, and $f_{F^\wedge} : S^{2m-1} \to S^{m(m-1)/2+1}$; in particular, for $m = 4$, we obtain $f_{F_\otimes} : S^7 \to S^{16}$, and $f_{F^\wedge} : S^7 \to S^7$.

The classification of all k-eigenmaps and orthogonal multiplications of type (m, m, n) (for various n) are long standing and difficult problems; see Eells-Lemaire's reports (Eells and Lemaire 1978, 1980; Toth 2002).

The *space of components* of an orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^m \to V$ is the linear subspace $V_F = \{\alpha \cdot F \mid \alpha \in V^*\}$ of the space of bilinear forms $(\mathbb{R}^m \otimes \mathbb{R}^m)^* = (\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \cong \mathbb{R}^m \otimes \mathbb{R}^m$. [We will identify $\mathbb{R}^m$ with its dual $(\mathbb{R}^m)^*$ via the standard basis $\{e_i\}_{i=1}^m \subset \mathbb{R}^m$.] Since F is full, precomposition with F gives a linear isomorphism $V^* \cong V_F$; in particular, we have $\dim V_F = \dim V = n$. Note that range-equivalent orthogonal multiplications have the same space of components.

1.3 Construction of the moduli $\mathfrak{M}_m$

Since an orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^m \to V$ is bilinear, there is a unique linear map $A : \mathbb{R}^m \otimes \mathbb{R}^m \to V$ such that $F = A \cdot F_\otimes$. Since F is full, A is onto. For $x, y \in \mathbb{R}^m$, we have

$$\begin{aligned} |F(x, y)|^2 - |x|^2|y|^2 &= |A(x \otimes y)|^2 - |x \otimes y|^2 \\ &= \left\langle (A^\top \cdot A - I)(x \otimes y), x \otimes y \right\rangle \\ &= \left\langle A^\top \cdot A - I, (x \otimes y)^2 \right\rangle, \end{aligned}$$

where the last scalar product is the one induced on the symmetric (tensor) square $S^2(\mathbb{R}^m \otimes \mathbb{R}^m)$. We obtain that normality of F is equivalent to

$$A^\top \cdot A - I \in \mathfrak{E}_m,$$

where

$$\mathfrak{E}_m = \left\{ (x \otimes y)^2 \mid x, y \in \mathbb{R}^n \right\}^\perp \subset S^2(\mathbb{R}^m \otimes \mathbb{R}^m). \tag{2}$$

(Once again, the orthogonal complement $\perp$ is understood with respect to the scalar product in $S^2(\mathbb{R}^m \otimes \mathbb{R}^m)$.) Since $A^\top \cdot A \geq 0$, (automatically) *positive semi-definite*, we obtain that, associating to F the symmetric endomorphism

$$\langle F \rangle = A^\top A - I \in S^2(\mathbb{R}^m \otimes \mathbb{R}^m),$$

gives rise to a *parametrization* of the range-equivalence classes of orthogonal multiplications $F : \mathbb{R}^m \times \mathbb{R}^m \to V$ (with various Euclidean vector spaces V) by the *convex body*

$$\mathfrak{M}_m = \{C \in \mathfrak{E}_m \mid C + I \geq 0\}. \tag{3}$$

(Injectivity of the parametrization on the range-equivalence classes follows easily by using polar decomposition.) The inverse of the parametrization is given by $\mathfrak{M}_m \ni C \mapsto (C + I)^{1/2} \cdot F_\otimes$ (made full by restriction).

By construction, the origin of $\mathfrak{E}_m$ corresponds to $F_\otimes$, and it is an interior point of $\mathfrak{M}_m$.

Since $\sum_{i,j=1}^m (e_i \otimes e_j)^2 = I$, by (2), the symmetric endomorphisms in $\mathfrak{E}_m$ are orthogonal to the identity I. Therefore they are traceless, or equivalently, the sum of their eigenvalues is zero. By (3), all the eigenvalues are ≥ -1, hence they are bounded; in particular, $\mathfrak{M}_m$ is *compact*. (For more details, see Toth (1987).)

The compact convex body $\mathfrak{M}_m$ is said to be the *moduli space* for orthogonal multiplications of $\mathbb{R}^m (\times \mathbb{R}^m)$.

By the generalized Sylvester's criterion, $C + I \geq 0$ (in (3)) if and only if *all* principal minors of $C + I$ are non-negative. (The original Sylvester's criterion applies for positive definite matrices and requires positivity of the upper left principal minors only.)

By convexity, any ray in $\mathfrak{E}_m$ emanating from the origin has its (unique) intersection with the boundary of $\mathfrak{M}_m$ at exactly the point C for which $\det(C + I)$ vanishes on the ray the *first time*. It follows that the boundary $\partial \mathfrak{M}_m$ is a *determinantal variety* in the sense that it is contained in the zero-set $\{X \in \mathfrak{E}_m \mid \det(X + I) = 0\}$. In fact, $\operatorname{int} \mathfrak{M}_m$ is the largest connected (convex) subset in $\mathfrak{E}_m$ which contains the origin and whose boundary is contained in this zero-set.

Returning to Sylvester's criterion, it also follows that to determine whether $C \in \mathfrak{E}_m$ is a boundary point of $\mathfrak{M}_m$, we need only to show $\det(C+I) = 0$ and $\det(tC+I) > 0$ for $t \in [0, 1)$.

Given an orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^m \to V$ with $F = A \cdot F_\otimes$, the linear map $A : \mathbb{R}^m \otimes \mathbb{R}^m \to V$ is onto (as F is assumed to be full). Identifying $\mathbb{R}^m \otimes \mathbb{R}^m$ with its dual, we obtain

$$(\langle F \rangle + I)(\mathbb{R}^m \otimes \mathbb{R}^m) = V_F. \tag{4}$$

In particular, $\operatorname{rank}(\langle F \rangle + I) = \dim V_F = \dim V = n$.

Moreover, (4) gives another important fact: Within $\mathfrak{M}_m$, we have

$$\begin{aligned} \langle F \rangle &= \lambda_1 \langle F_1 \rangle + \cdots + \lambda_k \langle F_k \rangle, \lambda_1 + \cdots + \lambda_k = 1, \quad 0 < \lambda_i < 1, \ i = 1, \ldots, k \\ &\Rightarrow V_F = V_{F_1} + \cdots V_{F_k}. \end{aligned} \tag{5}$$

(In Toth and Ziller (1999) this is called the 'Connecting Lemma'.) This motivates to introduce a *natural stratification* on $\mathfrak{M}_m$ in which an *open stratum* consists of those range-equivalence classes of orthogonal multiplications that share the same space of components. By (5), the (open) strata are *convex*; in particular, the affine span of an open stratum intersected with $\mathfrak{M}_m$ gives the *closure* of that stratum. The interior of $\mathfrak{M}_m$ is an open stratum with the largest space of components $\mathbb{R}^m \otimes \mathbb{R}^m = \operatorname{span}\{x_i \cdot y_j \mid i, j = 1, \ldots, m\}$. At the (relative) boundary points of an open stratum the dimension of the space of components, or equivalently, the range dimension of the respective orthogonal multiplications, *decreases*.

By (4), the range dimension of a full orthogonal multiplication corresponding to $C \in \mathfrak{M}_m$ is $\operatorname{rank}(C + I)$. We also saw that, for $C \in \partial\,\mathfrak{M}_m$, the ray $t \mapsto tC + I$ consists of *positive definite* endomorphisms for $t \in [0, 1)$, and its determinant vanishes (first time) at $t = 1$. Now, a simple consideration of the eigenvalues shows that the *multiplicity of* $t = 1$ *as a root of the polynomial* $t \mapsto \det(tC + I)$, $t \in \mathbb{R}$, (of degree m^2) is equal to $m^2 - \operatorname{rank}(C + I)$, the corank of $C + I$. (Based on this, we also define the corank of a full orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ as $m^2 - n$.) Since this polynomial is usually explicitly computable, this provides a quick and efficient way to find the range dimension of the orthogonal multiplication that corresponds to C without determining the orthogonal multiplication itself.

A more precise algebraic description of $\mathfrak{E}_m$ can be given as follows. Given $C \in S^2(\mathbb{R}^m \otimes \mathbb{R}^m)$, expanding, we have

$$\langle C, (x \otimes y)^2 \rangle = \langle C(x \otimes y), x \otimes y \rangle = \sum_{i,j,k,l=1}^{m} c_{ijkl} x_i y_j x_k y_l, \tag{6}$$

where $x = \sum_{i=1}^m x_i e_i$, $y = \sum_{i=1}^m y_j e_j$, and $c_{ijkl} = \langle C(e_i \otimes e_j), e_k \otimes e_l \rangle$, $1 \le i, j, k, l \le m$. Now, the sum in (6) is zero for all $x, y \in \mathbb{R}^m$ if and only if the coefficients c_{ijkl} are *skew-symmetric* with respect to $i \leftrightarrow k$ and $j \leftrightarrow l$. We obtain

$$\mathfrak{E}_m = \Lambda^2(\mathbb{R}^m) \otimes \Lambda^2(\mathbb{R}^m) \cong so(m) \otimes so(m). \tag{7}$$

Here the *exterior product* $\Lambda^2(\mathbb{R}^m)$ of 2-*vectors* in $\mathbb{R}^m$ is identified with the Lie algebra $so(m)$ of skew-symmetric $m \times m$-matrices via the isomorphism that associates to the 2-vector $e_i \wedge e_j$, $1 \le i < j \le m$, the skew-symmetric matrix with (ij)-entry $+1$, (ji)-entry -1, and zeros elsewhere.

By (7), we have

$$\dim \mathfrak{M}_m = \dim \mathfrak{E}_m = \frac{m^2(m-1)^2}{4}.$$

Examples For the examples in Sect. 1.1, we have

$$\begin{aligned}
\langle F_\cdot^{\mathbb{C}} \rangle &= -e_1 \wedge e_2 \otimes e_1 \wedge e_2 \\
\langle F_\cdot^{\mathbb{H}} \rangle &= (e_1 \wedge e_2 - e_3 \wedge e_4) \otimes (e_1 \wedge e_2 - e_3 \wedge e_4) \\
&\quad + (e_1 \wedge e_3 + e_2 \wedge e_4) \otimes (e_1 \wedge e_3 + e_2 \wedge e_4) \\
&\quad + (e_1 \wedge e_4 - e_2 \wedge e_3) \otimes (e_1 \wedge e_4 - e_2 \wedge e_3),
\end{aligned}$$

$$\langle F^{\wedge}\rangle = \sum_{1\le i<j\le 4} (e_i \wedge e_j) \otimes (e_i \wedge e_j) \quad (m = 4).$$

Precomposition of orthogonal multiplications of $\mathbb{R}^m$ with (pairs of) orthogonal transformations of $\mathbb{R}^m$ gives rise to an orthogonal $O(m) \times O(m)$-action on $\mathfrak{E}_m$. Via the isomorphism in (7), this orthogonal $O(m) \times O(m)$-module structure on $\mathfrak{E}_m$ is the *adjoint action* $\mathrm{Ad} \otimes \mathrm{Ad}$ on $so(m) \otimes so(m)$ given by (pairs of) conjugations. By definition, this action preserves the moduli $\mathfrak{M}_m$ within $\mathfrak{E}_m$.

The first moduli $\mathfrak{M}_1$ is the singleton consisting of $\langle F_{\cdot}^{\mathbb{R}}\rangle$. The second moduli $\mathfrak{M}_2$ is a line segment with endpoints $\langle F_{\cdot}^{\mathbb{C}}\rangle$ and $\langle F_{\cdot}^{\prime\mathbb{C}}\rangle$, where $F_{\cdot}^{\prime\mathbb{C}}(z, w) = z \cdot w$, $z, w \in \mathbb{C}$ (no conjugation). The midpoint of $\mathfrak{M}_2$ is the origin $\langle F_{\otimes}\rangle$ $(m = 2)$.

Remark Given an orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^m \to V$ we can construct another $\tilde{F} : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to V \times \mathbb{R}^{2m+1}$ by

$$\begin{aligned}&\tilde{F}(x, x_{m+1}, y, y_{m+1})\\ &\quad = (F(x, y), x_{m+1} \cdot y, y_{m+1} \cdot x, x_{m+1} y_{m+1}), x, y \in \mathbb{R}^m, x_{m+1}, y_{m+1} \in \mathbb{R}.\end{aligned}$$

The correspondence $\langle F\rangle \mapsto \langle \tilde{F}\rangle$ gives rise to a linear imbedding $\mathfrak{M}_m \to \mathfrak{M}_{m+1}$ onto a linear slice of $\mathfrak{M}_{m+1}$, and it is equivariant with respect to the natural inclusion $O(m) \times O(m) \to O(m + 1) \times O(m + 1)$.

The connection between orthogonal multiplications and quadratic eigenmaps exists also on the level of the moduli as follows. Denote by $\mathcal{H}_m^k$, $m \ge 2$, the linear space of spherical harmonics of order k on S^m. As noted above, a map $f : S^m \to S_V$ into the unit sphere S_V of a Euclidean vector space V is a k-eigenmap if its space of components $V_f = \{\alpha \cdot f \mid \alpha \in V^*\}$ is contained in $\mathcal{H}_m^k$. The set of range-equivalence classes of full k-eigenmaps of S^m can be parametrized by the compact convex body $\mathcal{L}_m^k = \{C \in \mathcal{E}_m^k \mid C + I \ge 0\}$, where $\mathcal{E}_m^k$ is a certain linear subspace of the symmetric square $S_0^2(\mathcal{H}_m^k)$ of tracefree symmetric endomorphisms of $\mathcal{H}_m^k$. (For more details, see Toth (2002).) Precomposition by isometries in $O(m + 1)$ gives rise to an irreducible $O(m+1)$-module structure on $\mathcal{H}_m^k$. Then $\mathcal{E}_m^k$ is an $O(m+1)$-submodule with respect to the induced $O(m + 1)$-module structure on the symmetric square, and the moduli $\mathcal{L}_m^k$ is $O(m + 1)$-invariant. We have

$$\dim \mathcal{L}_m^k = \dim \mathcal{E}_m^k = \binom{\binom{m+k}{m} - \binom{m+k-2}{m} + 1}{2} - \binom{m + 2k}{m}.$$

Now, for $k = 2$, the Hopf-Whitehead construction $F \mapsto f_F$ in (1) on the respective sets of range-equivalence classes gives rise to an imbedding $\mathfrak{M}_m \to \mathcal{L}_{2m-1}^2$ which is equivariant with respect to the inclusion $O(m) \times O(m) \to O(2m)$. The image of the moduli $\mathfrak{M}_m$ is the intersection of its affine span with the boundary of $\mathcal{L}_{2m-1}^2$.

For example, for $m = 3$, the 9-dimensional moduli $\mathfrak{M}_3$ is on the boundary of the 84-dimensional moduli $\mathcal{L}_5^2$, and, for $m = 4$, the 36-dimensional moduli $\mathfrak{M}_4$ is on the boundary of the 300-dimensional moduli $\mathcal{L}_7^2$.

2 Statement of results

2.1 The moduli $\mathfrak{M}_3$

Parker (1983) gave a full *algebraic* classification of orthogonal multiplications of type $(3, 3, n)$ but did not realize the elegant geometry of $\mathfrak{M}_3$. To get to this, we first define

$$\Theta_0 = \{\alpha \in [-1, 1]^3 \mid Q(\alpha) \geq 0\}.$$

where $Q : \mathbb{R}^3 \to \mathbb{R}$ is Cayley's cubic polynomial given by

$$Q(\alpha) = 1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 + 2\alpha_1\alpha_2\alpha_3, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3. \tag{8}$$

Up to the scaling factor -2, *Cayley's nodal cubic surface* (in $\mathbb{R}^3 \subset \mathbb{R}P^3$) is defined by $Q(\alpha) = 0$; see Cayley (1869). This is an important example in classical surface theory (Hunt 1996). We will call the convex body Θ_0 *Cayley's tetrahedron*.
Clearly Θ_0 contains the regular tetrahedron $\Delta_0 \subset [-1, 1]^3$ with vertices the alternate vertices of the cube $[-1, 1]^3$ and one vertex at $(1, 1, 1)$. The 1-skeleton comprised by the six edges of Δ_0 is on the boundary of Θ_0, and the (smooth open) faces of Δ_0 are 'inflated' to the 'sides' of Θ_0.

Since the maximum inflation rate (from Δ_0 to Θ_0) is $3/2$, and it occurs at the centroids of the faces of Δ_0, the *Minkowski measure of symmetry* of Θ_0 (Grünbaum 1963) is equal to $3/(3/2) = 2$.

Cayley's tetrahedron Θ_0 is 'regular' in the sense that it inherits the symmetry group of Δ_0, the symmetric group $\mathcal{S}_4$ on four letters.

(Θ_0 is a good example of a convex body whose extremal set is *not closed*; in fact, the only non-extremal points of Θ_0 comprise the six *open* edges.)

We have the following:

Theorem A *Parametrize $\mathfrak{E}_3$ by the space of 3×3-matrices $M(3, 3)$ through the linear isomorphism $C : M(3, 3) \to \mathfrak{E}_3$ defined by*

$$C(\mathcal{X}) = \sum_{i,j=1}^{3} x_{ij} E_i \otimes E_j \in \mathfrak{E}_3, \quad \mathcal{X} = [x_{ij}]_{i,j=1}^3 \in M(3, 3),$$

where the basis $\{E_i\}_{i=1}^3 \subset \Lambda^2(\mathbb{R}^3) \cong \Lambda^1(\mathbb{R}^3) = \mathbb{R}^3$ corresponds to the standard basis in $\mathbb{R}^3$ via the Hodge $*$ operator on the exterior algebra $\Lambda^*(\mathbb{R}^3)$. Restricting C to the linear subspace $D(3, 3) = \mathbb{R}^3 \subset M(3, 3)$ of diagonal matrices, we denote the image of C, the linear subspace of 'diagonal elements,' by

$$\mathfrak{D}_3 = \{C(\alpha) \mid \alpha \in \mathbb{R}^3\} \subset \mathfrak{E}_3,$$

where

$$C(\alpha) = C(\mathrm{diag}\,(\alpha_1, \alpha_2, \alpha_3)) = \sum_{i=1}^{3} \alpha_i E_i \otimes E_i, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3.$$

Then the intersection

$$\Theta = \mathfrak{M}_3 \cap \mathfrak{D}_3$$

is a convex body which, under this parametrization, corresponds to Cayley's tetrahedron $\Theta_0 \subset [-1, 1]^3$*:*

$$C(\Theta_0) = \Theta.$$

With respect to the $SO(3) \times SO(3)$*-module structure of* $\mathfrak{E}_3$*, we have*

$$\mathfrak{M}_3 = (SO(3) \times SO(3))\,\Theta.$$

Under this action, the vertices (the ordinary double points) of Θ *are contained in a single* $SO(3) \times SO(3)$*-orbit which is a projective space* $\mathbb{R}P^3$ *equivariantly and minimally imbedded into an 8-sphere* S^8 *of* $\mathfrak{E}_3$*. Up to isometries and scaling,* $\mathbb{R}P^3$ *is the image of the standard minimal immersion of* S^3 *into the (ambient sphere of the) space of quadratic spherical harmonics* $\mathcal{H}_3^2$ *on* S^3*.*
Finally, the vertices of Θ *correspond to orthogonal multiplications with range dimension 4, the (open) edges to range dimension 7, the (inflated open) sides to range dimension 8, and the interior to range dimension 9.*

Remark The crux in the proof of Theorem A is the formula

$$\det\,(C(\alpha) + I) = \prod_{i=1}^{3}(1 - \alpha_i^2) \cdot Q(\alpha), \quad \alpha \in \mathbb{R}^3, \tag{9}$$

where Q is Cayley's cubic given in (8).

The last statement of theorem A implies that, modulo isometries on the source and the range, there is a 9-dimensional set of orthogonal multiplications with range dimension 9, a 2-dimensional set with range dimension 8, a 1-dimensional set with range dimension 7, and a finite set with range dimension 4. Actually, it is easy to see that the lowest range dimension 4 corresponds to quaternionic multiplication restricted to *imaginary* quaternions, and it is unique up to isometries on the source and the range.

Given $\alpha \in \Theta_0$, an orthogonal multiplication $F^\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \to V$ representing $C(\alpha) \in \Theta \subset \mathfrak{M}_3$ can be explicitly constructed by the formula

$$F^\alpha = (C(\alpha) + I)^{1/2} \cdot F_\otimes, \quad F_\otimes : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3.$$

Here F^α is not necessarily full as its range is $\mathbb{R}^3 \otimes \mathbb{R}^3$. To obtain fullness, restriction to the linear span $V \subset \mathbb{R}^3 \otimes \mathbb{R}^3$ of the image of F^α is necessary.

2.2 The moduli $\mathfrak{M}_4$

I. The Splitting of $\mathfrak{E}_4$

Let $m = 4$. The Hodge $*$ operator on the exterior algebra $\Lambda^*(\mathbb{R}^4)$ restricts to a symmetric *endomorphism* $* : \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4)$ with $** = I$. Hence the eigenvalues of $*$ on $\Lambda^2(\mathbb{R}^4)$ are ± 1, and the corresponding (3-dimensional) eigenspaces $\Lambda^2_\pm$ of *self-dual* and *anti-self-dual* 2-vectors give rise to the orthogonal splitting

$$\Lambda^2(\mathbb{R}^4) = \Lambda^2_- \oplus \Lambda^2_+. \tag{10}$$

Via the $O(4)$-module structure on $\Lambda^2(\mathbb{R}^4)$ the diagonal matrix $\gamma = \mathrm{diag}(1, 1, 1, -1) \in O(4)$, $\gamma^2 = I$, interchanges the eigenspaces: $\gamma : \Lambda^2_\pm \leftrightarrow \Lambda^2_\mp$.

For the $SO(4)$-module structure of $\Lambda^2(\mathbb{R}^4)$, first note that, under the identification $\mathbb{C}^2 = \mathbb{R}^4$ by $(z, w) \mapsto (x, y, u, v)$, $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$, the special unitary group $SU(2)$ becomes a subgroup of $SO(4)$. Moreover, the orthogonal matrix γ above conjugates $SU(2)$ to another subgroup $SU(2)' = \gamma SU(2)\gamma \subset SO(4)$, and we have the almost product structure $SO(4) = SU(2) \cdot SU(2)'$ with $SU(2)$ and $SU(2)'$ both normal in $SO(4)$, and $SU(2) \cap SU(2)' = \{\pm I\}$.

Theorem B *The ± 1-eigenspaces of the Hodge $*$ operator (applied to each factor of the tensor product) give rise to the splitting*

$$\mathfrak{E}_4 \cong \Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4) = \mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{-,+} \oplus \mathfrak{E}_4^{+,-} \oplus \mathfrak{E}_4^{+,+}$$

where

$$\mathfrak{E}_4^{-,-} = \Lambda^2_- \otimes \Lambda^2_-, \quad \mathfrak{E}_4^{-,+} = \Lambda^2_- \otimes \Lambda^2_+, \quad \mathfrak{E}_4^{+,-} = \Lambda^2_+ \otimes \Lambda^2_-, \quad \mathfrak{E}_4^{+,+} = \Lambda^2_+ \otimes \Lambda^2_+. \tag{11}$$

All components $\mathfrak{E}_4^{\pm,\pm}$ are $SO(4) \times SO(4)$-submodules of $\mathfrak{E}_4$. More precisely, with respect to the almost direct product $SU(2) \cdot SU(2)' = SO(4)$ (applied to each factor), they are fixed-point sets:

$$\mathfrak{E}_4^{-,-} = \mathfrak{E}_4^{SU(2)' \times SU(2)'}, \quad \mathfrak{E}_4^{-,+} = \mathfrak{E}_4^{SU(2)' \times SU(2)},$$
$$\mathfrak{E}_4^{+,-} = \mathfrak{E}_4^{SU(2) \times SU(2)'}, \quad \mathfrak{E}_4^{+,+} = \mathfrak{E}_4^{SU(2) \times SU(2)}$$

with respective complementary (irreducible) module structures:

$$(SU(2) \times SU(2)) \cdot \mathfrak{E}_4^{-,-} = \mathfrak{E}_4^{-,-}, \quad (SU(2) \times SU(2)') \cdot \mathfrak{E}_4^{-,+} = \mathfrak{E}_4^{-,+},$$
$$(SU(2)' \times SU(2)) \cdot \mathfrak{E}_4^{+,-} = \mathfrak{E}_4^{+,-}, \quad (SU(2)' \times SU(2)') \cdot \mathfrak{E}_4^{+,+} = \mathfrak{E}_4^{+,+}.$$

The group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(I, I), (I, \gamma), (\gamma, I), (\gamma, \gamma)\} \subset O(4) \times O(4)$ acts simply transitively and orthogonally among the components with γ interchanging the respective signs:

$$(I, \gamma) : \mathfrak{E}_4^{\pm,\pm} \leftrightarrow \mathfrak{E}_4^{\pm,\mp}, \quad (\gamma, I) : \mathfrak{E}_4^{\pm,\pm} \leftrightarrow \mathfrak{E}_4^{\mp,\pm}.$$

II. The equivariant moduli $\mathfrak{M}_4^{\pm,\pm}$
We define the (linear) slices

$$\mathfrak{M}_4^{-,-} = \mathfrak{M}_4 \cap \mathfrak{E}_4^{-,-} = \mathfrak{M}_4^{SU(2)'\times SU(2)'}, \quad \mathfrak{M}_4^{-,+} = \mathfrak{M}_4 \cap \mathfrak{E}_4^{-,+} = \mathfrak{M}_4^{SU(2)'\times SU(2)},$$
$$\mathfrak{M}_4^{+,-} = \mathfrak{M}_4 \cap \mathfrak{E}_4^{+,-} = \mathfrak{M}_4^{SU(2)\times SU(2)'}, \quad \mathfrak{M}_4^{+,+} = \mathfrak{M}_4 \cap \mathfrak{E}_4^{+,+} = \mathfrak{M}_4^{SU(2)\times SU(2)}.$$

In constructing the moduli $\mathfrak{M}_4$ we factored out the isometries on the ranges so that $\mathfrak{M}_4^{\pm,\pm}$ parametrizes the orthogonal multiplications $F : \mathbb{R}^4\times\mathbb{R}^4 \to V$ that are *equivariant* with respect to the group $SU(2)/SU(2)' \times SU(2)/SU(2)'$ that fixes $\mathfrak{M}_4^{\pm,\pm}$ (where / indicates the respective choices). For this reason we call these *equivariant moduli*.

By Theorem B, the equivariant moduli are mutually equivalent via linear isometries provided by the elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$(I, \gamma) : \mathfrak{M}_4^{\pm,\pm} \leftrightarrow \mathfrak{M}_4^{\pm,\mp}, \quad (\gamma, I) : \mathfrak{M}_4^{\pm,\pm} \leftrightarrow \mathfrak{M}_4^{\mp,\pm}. \tag{12}$$

Finally, note that, by convexity, we have

$$\left[\mathfrak{M}_4^{-,-}, \mathfrak{M}_4^{-,+}, \mathfrak{M}_4^{+,-}, \mathfrak{M}_4^{+,+}\right] \subset \mathfrak{M}_4,$$

where the square brackets indicate convex hull. As we will see below, the inclusion is *proper*, a phenomenon that we call 'bulging.' (The importance of this has also been observed by Ziller and the author in (Toth and Ziller 1999, p. 88) for the moduli of quartic spherical minimal immersions of S^3.)

The next result gives a complete geometric description of the equivariant slices $\mathfrak{M}_4^{\pm,\pm}$ of $\mathfrak{M}_4$. By (12), we need only to discuss $\mathfrak{M}_4^{-,-}$. Note the apparent similarity with Theorem A.

Theorem C *Parametrize $\mathfrak{E}_4^{-,-}$ by $M(3,3)$ through the linear isomorphism $C^{-,-} : M(3,3) \to \mathfrak{E}_4^{-,-}$ defined by*

$$C^{-,-}(\mathcal{X}) = \sum_{i,j=1}^{3} x_{ij} E_i \otimes E_j \in \mathfrak{E}_4^{-,-}, \quad \mathcal{X} = [x_{ij}]_{i,j=1}^3 \in M(3,3),$$

where $\{E_i\}_{i=1}^3 \subset \Lambda_-^2$ is the canonical basis:

$$E_1 = e_1 \wedge e_2 - e_3 \wedge e_4, \quad E_2 = e_1 \wedge e_3 + e_2 \wedge e_4, \quad E_3 = e_1 \wedge e_4 - e_2 \wedge e_3. \tag{13}$$

Restricting $C^{-,-}$ to the linear subspace $D(3,3) = \mathbb{R}^3 \subset M(3,3)$ of diagonal matrices, we denote the image of $C^{-,-}$, the linear subspace of 'diagonal elements,' by

$$\mathfrak{D}_4^{-,-} = \left\{C^{-,-}(\alpha) \,|\, \alpha \in \mathbb{R}^3\right\} \subset \mathfrak{E}_4^{-,-},$$

where

$$C^{-,-}(\alpha) = C^{-,-}(\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)) = \sum_{i=1}^{3} \alpha_i E_i \otimes E_i, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3.$$

Then the intersection

$$\Delta^{-,-} = \mathfrak{M}_4^{-,-} \cap \mathfrak{D}_4^{-,-}$$

is a tetrahedron which, under this parametrization, corresponds to the regular tetrahedron Δ_0 *inscribed in the cube* $[-1, 1]^3 \subset \mathbb{R}^3$ *with vertices the alternative vertices of the cube and with one vertex at* $(1, 1, 1)$*:*

$$C^{-,-}(\Delta_0) = \Delta^{-,-}.$$

With respect to the $SU(2) \times SU(2)$*-module structure on* $\mathfrak{E}_4^{-,-}$*, the equivariant moduli* $\mathfrak{M}_4^{-,-}$ *is the* $SU(2) \times SU(2)$*-orbit of* $\Delta^{-,-}$*:*

$$\mathfrak{M}_4^{-,-} = (SU(2) \times SU(2))\, \Delta^{-,-}.$$

Moreover, under this action the vertices of $\Delta^{-,-}$ *are contained in a single* $SU(2) \times SU(2)$*-orbit which is a projective space* $\mathbb{R}P^3$ *equivariantly and minimally imbedded into an 8-sphere* S^8 *of* $\mathfrak{E}_4^{-,-}$*. Up to isometries and scaling,* $\mathbb{R}P^3$ *is the image of the standard minimal immersion of* S^3 *into the (ambient sphere of the) space of quadratic spherical harmonics on* S^3*.*

The crux in the proof of Theorem C is the formula

$$\det\,(C^{-,-}(\alpha) + I) = R(\alpha)^4, \quad \alpha \in \mathbb{R}^3, \tag{14}$$

where $R : \mathbb{R}^3 \to \mathbb{R}$ is the 'tetrahedral' quartic polynomial given by

$$\begin{aligned} R(\alpha) &= (1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)^2 + 8\alpha_1\alpha_2\alpha_3 - 4((\alpha_1\alpha_2)^2 + (\alpha_2\alpha_3)^2 + (\alpha_3\alpha_1)^2) \\ &= (1 + \alpha_1 + \alpha_2 + \alpha_3)(1 + \alpha_1 - \alpha_2 - \alpha_3)(1 - \alpha_1 + \alpha_2 - \alpha_3)(1 - \alpha_1 - \alpha_2 + \alpha_3). \end{aligned} \tag{15}$$

The first expression of R is for analogy with Cayley's cubic in (8). The second expression shows that R is tetrahedral in the sense that it vanishes precisely on (the plane extensions of) the faces of Δ_0.

The passage from $\mathfrak{M}_4^{-,-}$ to the rest of the equivariant moduli $\mathfrak{M}_4^{\pm,\pm}$ is effected by the non-trivial elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. For future purposes we let $C^{\pm,\pm} : M(3, 3) \to \mathfrak{E}_4^{\pm,\pm}$ be defined as

$$C^{-,+} = (I, \gamma)\, C^{-,-}, \quad C^{+,-} = (\gamma, I)\, C^{-,-}, \quad C^{+,+} = (\gamma, \gamma)\, C^{-,-},$$

so that $C^{\pm,\pm}$ parametrizes $\mathfrak{E}_4^{\pm,\pm}$ by $M(3, 3)$.

In addition, we let

$$\mathfrak{D}_4^{-,+} = (I, \gamma)\mathfrak{D}_4^{-,-}, \quad \mathfrak{D}_4^{+,-} = (\gamma, I)\mathfrak{D}_4^{-,-}, \quad \mathfrak{D}_4^{+,+} = (\gamma, \gamma)\mathfrak{D}_4^{-,-},$$

be the respective linear subspaces of 'diagonal elements' with tetrahedra

$$\Delta^{\pm,\pm} = C^{\pm,\pm}(\Delta_0) = \mathfrak{M}_4^{\pm,\pm} \cap \mathfrak{D}_4^{\pm,\pm}.$$

Given $\alpha \in \Delta_0$, an orthogonal multiplication $F^\alpha : \mathbb{R}^4 \times \mathbb{R}^4 \to V$ representing $C^{-,-}(\alpha) \in \Delta^{-,-}$ can be explicity constructed as

$$F^\alpha = \left(C^{-,-}(\alpha) + I\right)^{1/2} \cdot F_\otimes, \quad F_\otimes : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4 \otimes \mathbb{R}^4.$$

Here $V \subset \mathbb{R}^4 \otimes \mathbb{R}^4$ is the linear span of the image of F^α.

Example By (13), the second example in Sect. 1.3 gives

$$\langle F_\cdot^{\mathbb{H}} \rangle = \langle F^{(1,1,1)} \rangle = C^{-,-}(1, 1, 1) = \sum_{i=1}^{3} E_i \otimes E_i \in \mathfrak{M}_4^{-,-}.$$

Recall from Sect. 1.3 that the possible *range dimensions* of full orthogonal multiplications parametrized by $\Delta^{-,-}$ (and therefore, using Theorem C, by the entire $\mathfrak{M}_4^{-,-}$) can be obtained by calculating the possible multiplicities of the root $t = 1$ of the degree 16 polynomial $t \mapsto \det(tC^{-,-}(\alpha) + I)$, $t \in \mathbb{R}$, where $\alpha \in \Delta_0$. On the other hand, this determinant is given in (14) as $t \mapsto R(t\alpha)^4$, $t \in \mathbb{R}$, where R is the tetrahedral quartic. We immediately see that the possible coranks and hence also the possible range dimensions are multiples of 4, that is, they are 4, 8, 12, 16.

Constructing explicit examples at the cardinal points [vertex (such as $(1, 1, 1)$ above), midpoint of an edge, centroid of a face] of Δ_0 we obtain that all these range dimensions are realized:

Corollary *The range dimensions of orthogonal multiplications parametrized by the equivariant moduli $\mathfrak{M}_4^{\pm,\pm}$ are* 4, 8, 12, 16. *The $SU(2) \times SU(2)$-orbit of the vertices of $\Delta^{-,-}$ corresponds to full orthogonal multiplications with range dimension 4, the (open) edges to range dimension 8, the (open) faces to range dimension 12, and the (relative) interior of $\Delta^{-,-}$ to range dimension 16.*

III. The pairing of $\mathfrak{M}_4^{-,-}$ and $\mathfrak{M}_4^{+,+}$

For the next step, it is natural to pair $\mathfrak{E}_4^{-,-}$ with $\mathfrak{E}_4^{+,+}$ as their acting groups $SU(2) \times SU(2)$ and $SU(2)' \times SU(2)'$ overlap only in the subgroup $\{\pm I\} \times \{\pm I\}$ (acting trivially on $\mathfrak{M}_4$).

Theorem D *We have*

$$\begin{aligned}\mathfrak{M}_4 \cap \left(\mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{+,+}\right) &= (SO(4) \times SO(4))\,[\Delta^{-,-}, \Delta^{+,+}] \\ &= \left[(SU(2) \times SU(2))\,\Delta^{-,-}, (SU(2)' \times SU(2)')\,\Delta^{+,+}\right]\end{aligned}$$

$$= \left[\mathfrak{M}_4^{-,-}, \mathfrak{M}_4^{+,+}\right], \tag{16}$$

where the square brackets indicate convex hull.

Since (I, γ) (and (γ, I)) restrict to linear isometries $\mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{+,+} \leftrightarrow \mathfrak{E}_4^{-,+} \oplus \mathfrak{E}_4^{+,-}$ and leave the moduli invariant, for the *complementary* configuration we also have

$$\begin{aligned}\mathfrak{M}_4 \cap \left(\mathfrak{E}_4^{-,+} \oplus \mathfrak{E}_4^{+,-}\right) &= (SO(4) \times SO(4))\,[\Delta^{-,+}, \Delta^{+,-}] \\ &= \left[(SU(2) \times SU(2)')\,\Delta^{-,+}, (SU(2)' \times SU(2))\,\Delta^{+,-}\right] \\ &= \left[\mathfrak{M}_4^{-,+}, \mathfrak{M}_4^{+,-}\right].\end{aligned}$$

Under the restriction $C^{-,-} \times C^{+,+} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{D}_4^{-,-} \oplus \mathfrak{D}_4^{+,+}$, $D(3,3) = \mathbb{R}^3$, the 6-dimensional polytope $\Pi = [\Delta^{-,-}, \Delta^{+,+}]$ in (16) corresponds to the convex hull $\Pi_0 = [\Delta_0, \Delta_0] \subset \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ of two regular tetrahedra Δ_0 contained in the two copies of $\mathbb{R}^3$. The key point in the proof of Theorem D is the splitting of the determinant

$$\det\left(C^{-,-}(\alpha) + C^{+,+}(\beta) + I\right) = \prod_{\substack{\sigma_1\sigma_2\sigma_3=\tau_1\tau_2\tau_3=1 \\ \sigma_1,\sigma_2,\sigma_3,\tau_1,\tau_2,\tau_3\in\{\pm1\}}} \left(1 + \sum_{i=1}^{3}(\sigma_i\alpha_i + \tau_i\beta_i)\right) \tag{17}$$

into 16 linear factors, each vanishing on a hyperplane in $\mathbb{R}^6$ that cuts out a pair of (2-)faces from the two copies of Δ_0.

The same description holds for the affine (actually linear) copy Π; it is the convex hull of 16 half-spaces of $\mathfrak{D}_4^{-,-} \oplus \mathfrak{D}_4^{+,+}$ whose boundary hyperplanes cut out a pair of faces from $\Delta^{\pm,\pm}$. (For future comparison, we note that the diagonal $\mathbb{R}^3_\Delta = \{(\alpha,\alpha) \mid \alpha \in \mathbb{R}^3\} \subset \mathbb{R}^3 \times \mathbb{R}^3$ cuts out from Π_0 the regular tetrahedron Δ_0 scaled by $1/2$. The anti-diagonal $\mathbb{R}^3_{\Delta'} = \{(\alpha,-\alpha) \mid \alpha \in \mathbb{R}^3\} \subset \mathbb{R}^3 \times \mathbb{R}^3$ cuts out from Π_0 a *rhombic dodecahedron*, a zonohedron with 12 rhombic faces, and vertices $(\pm1/2, \pm1/2, \pm1/2)$, $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$.)
The number N_i of the i-faces, $i = 0, \ldots, 5$, of Π_0 (and also Π) are tabulated as follows:

N_0	N_1	N_2	N_3	N_4	N_5
8	28	56	68	48	16

The symmetry group of $\Pi = [\Delta^{-,-}, \Delta^{+,+}]$ is generated by $\mathcal{S}_4 \times \mathcal{S}_4$ (with $\mathcal{S}_4$, the symmetric group on 4 letters) and $\mathbb{Z}_2 = \{(I, I), (\gamma, \gamma)\}$.
As before, examples of orthogonal multiplications parametrized by points in Π can be constructed explicitly. Modulo this symmetry group, the following table summarizes those that correspond to the cardinal points of Π, their parameters, and the corresponding ranks/range dimensions:

Point	$(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3)$	Rank
V	$(1, 1, 1; 0, 0, 0)$	4
E/I	$(1, 0, 0; 0, 0, 0)$	8
E/II	$(1/2, 1/2, 1/2; 1/2, 1/2, 1/2)$	7
F/I	$(-1/3, -1/3, -1/3; 0, 0, 0)$	12
F/II	$(1/3, 1/3, 1/3; 2/3, 0, 0)$	10
C_3/I	$(1/4, 1/4, 1/4; -1/4, -1/4, -1/4)$	13
C_3/II	$(1/2, 0, 0; 1/2, 0, 0, 0)$	12
C_4	$(2/5, 0, 0; -1/5, -1/5, -1/5)$	14
C_5	$(-1/6, -1/6, -1/6; -1/6, -1/6, -1/6)$	15

The following notations are used: V= vertex of any of the two $\Delta^{\pm,\pm}$; E/I = midpoint of an edge of any of the two $\Delta^{\pm,\pm}$; E/II = midpoint of an edge of Π connecting two vertices in separate $\Delta^{\pm,\pm}$; F/I = centroid of a face of any of the two $\Delta^{\pm,\pm}$; F/II = centroid of a face of Π, the convex hull of a vertex and an edge on separate $\Delta^{\pm,\pm}$; C_3/I = centroid of a 3-face of Π, the convex hull of a vertex and a face on separate $\Delta^{\pm,\pm}$; C_3/II = centroid of a 3-face of Π, the convex hull of two edges on separate $\Delta^{\pm,\pm}$; C_4 = centroid of a 4-face of Π, the convex hull of an edge and a face on separate $\Delta^{\pm,\pm}$; C_5 = centroid of a 5-face of Π, the convex hull of two faces on separate $\Delta^{\pm,\pm}$.

Remark Under E/II one recognizes the Wu-Xiong-Zhao example (from Sect. 1.3):

$$\langle F^\wedge \rangle \in [\mathfrak{M}_4^{-,-}, \mathfrak{M}_4^{+,+}]$$

which corresponds to the midpoint of two vertices of $\Delta^{-,-}$ and $\Delta^{+,+}$ parametrized by $(1/2)(1, 1, 1, 0, 0, 0) + (1/2)(0, 0, 0, 1, 1, 1)$.

The table above gives the following:

Corollary *The range dimensions of orthogonal multiplications parametrized by the intersection* $\mathfrak{M}_4 \cap \left(\mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{+,+}\right)$ *are* $4, 7-8, 10, 12-16$.

IV. Diagonalizable elements in $\mathfrak{M}_4$

The entire space $\mathfrak{E}_4$ is parametrized by the 4-fold product $M(3,3)^4$ through the linear isomorphism $C^{-,-} \times C^{+,+} \times C^{-,+} \times C^{+,-} : M(3,3)^4 \to \mathfrak{E}_4$. Restriction in each component $M(3,3)$ to $\mathbb{R}^3 = D(3,3) \subset M(3,3)$ gives the parametrization of the linear subspace of 'diagonal elements'

$$\mathfrak{D}_4 = \mathfrak{D}_4^{-,-} \oplus \mathfrak{D}_4^{+,+} \oplus \mathfrak{D}_4^{-,+} \oplus \mathfrak{D}_4^{+,-} \subset \mathfrak{E}_4$$

by $\mathbb{R}^{12} = (\mathbb{R}^3)^4$.

To shed some light on the complexity of the entire moduli $\mathfrak{M}_4$ we introduce the following definition. An endomorphism $C \in \mathfrak{E}_4$ is called *diagonalizable* if there exist $U, V \in O(4)$ such that $(U, V)C \in \mathfrak{D}_4$. Since $\mathfrak{D}_4$ is 12-dimensional, it follows that the space $(O(4) \times O(4))\mathfrak{D}_4$ of diagonalizable elements is 24 dimensional within

the 36-dimensional $\mathfrak{E}_4$. This situation prevails in the moduli $\mathfrak{M}_4$; the space $(O(4) \times O(4))(\mathfrak{M}_4 \cap \mathfrak{D}_4)$ of diagonalizable elements is also 24 dimensional.

As a simple application of Theorem D, it follows that *any* endomorphism in $\mathfrak{E}_4$ is the sum of two diagonalizable elements. (As a quick proof, project the endomorphism to $\mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{+,+}$ and to $\mathfrak{E}_4^{-,+} \oplus \mathfrak{E}_4^{+,-}$. Apply Theorem D to the projections (after scaling), and conclude that the projections are diagonalizable.)

In the rest of this paper we give a detailed algebraic description of the space of diagonalizable elements. This amounts to study the intersection

$$\Gamma = \mathfrak{M}_4 \cap \mathfrak{D}_4$$

defined by

$$C^{-,-}(\alpha) + C^{+,+}(\beta) + C^{-,+}(\mu) + C^{+,-}(\nu) + I \geq 0,\ \alpha, \beta, \mu, \nu \in \mathbb{R}^3. \tag{18}$$

By Theorems B-C, we have the linear slices

$$\Gamma \cap \mathfrak{E}_4^{-,-} = \Gamma^{SU(2)' \times SU(2)'} = \Delta^{-,-}, \quad \Gamma \cap \mathfrak{E}_4^{-,+} = \Gamma^{SU(2)' \times SU(2)} = \Delta^{-,+},$$
$$\Gamma \cap \mathfrak{E}_4^{+,-} = \Gamma^{SU(2) \times SU(2)'} = \Delta^{+,-}, \quad \Gamma \cap \mathfrak{E}_4^{+,+} = \Gamma^{SU(2) \times SU(2)} = \Delta^{+,+}.$$

In addition, by Theorem D, we have

$$\Gamma \cap (\mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{+,+}) = [\Delta^{-,-}, \Delta^{+,+}] \quad \text{and} \quad \Gamma \cap (\mathfrak{E}_4^{-,+} \oplus \mathfrak{E}_4^{+,-}) = [\Delta^{-,+}, \Delta^{+,-}].$$

An additional complexity of $\mathfrak{M}_4$ is 'bulging,' that is *proper* inclusion in

$$[\Delta^{-,-}, \Delta^{+,+}, \Delta^{-,+}, \Delta^{+,-}] \subsetneqq \Gamma.$$

Finally, a technical problem concerning the boundary of Γ is that it is a determinantal variety described by a degree 16 polynomial in 12 variables. Although it splits into 4 degree 4 factors, each factor contains over 200 monomials.

To give a technically manageable description of this variety we introduce some notations. Let

$$\Sigma = \left\{ \sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{\pm 1\}^3 \mid \sigma_1 \sigma_2 \sigma_3 = 1 \right\}.$$

Geometrically, Σ is the set of vertices of the regular simplex Δ_0, and it is also the group whose elements are the diagonals of the three half-turns about the coordinate axes (plus the identity). Σ acts linearly on $\mathbb{R}^3$ as $\sigma \cdot \alpha = (\sigma_1 \alpha_1, \sigma_2 \alpha_2, \sigma_3 \alpha_3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$.

With these, (15) can be written in the compact form

$$R(\alpha) = \prod_{\sigma \in \Sigma} (1 + S(\sigma \cdot \alpha)), \quad \alpha \in \mathbb{R}^3,$$

where $S(\alpha) = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha \in \mathbb{R}^3$ (the first elementary symmetric polynomial). Clearly, R is Σ-invariant.

As another example, (17) reduces to

$$\det\left(C^{-,-}(\alpha) + C^{+,+}(\beta) + I\right) = R^{\Sigma}(\alpha, \beta) = \prod_{\sigma \in \Sigma} R(\alpha + \sigma \cdot \beta), \tag{19}$$

where R^{Σ} is a convenient notation.
The passage $R^4(\alpha) \mapsto R^{\Sigma}(\alpha, \beta)$ from (14) to (19) is revealing. It suggests that to obtain a compact form for the determinant of the positive semi-definite matrix in (18) one should mimic this process. We first need to calculate the determinant of the 'hybrid' $C^{-,-}(\alpha) + C^{-,+}(\beta) + I$, $\alpha, \beta \in \mathbb{R}^3$.

To do this, we let $T : \mathbb{R}^6 \to \mathbb{R}$ be the biquadratic form given by

$$T(\alpha, \beta) = \sum_{i,j=1}^{3} \alpha_i^2 \beta_j^2 - 2 \sum_{i=1}^{3} \alpha_i^2 \beta_i^2, \quad \alpha, \beta \in \mathbb{R}^3.$$

In the variables α_i^2, β_i^2, $i = 1, 2, 3$, T is a quadratic form with matrix entries $+1$ for $1 \le i \ne j \le 3$, and -1 for $1 \le i = j \le 3$. In particular, T is automatically Σ-invariant in each variable separately. Moreover, T is symmetric, that is, we have $T(\alpha, \beta) = T(\beta, \alpha)$, $\alpha, \beta \in \mathbb{R}^3$.
Now a somewhat tedious calculation shows

$$\det(C^{-,-}(\alpha) + C^{-,+}(\beta) + I) = G(\alpha, \beta)^4, \quad \alpha, \beta \in \mathbb{R}^3,$$

where

$$G(\alpha, \beta) = R(\alpha) + R(\beta) + 2T(\alpha, \beta) - 1, \quad \alpha, \beta \in \mathbb{R}^3. \tag{20}$$

(Clearly, $G(\alpha, 0) = G(0, \alpha) = R(\alpha)$, $\alpha \in \mathbb{R}^3$.)

In complete analogy with (14) and (19), we now have

$$\begin{aligned} &\det\left(C^{-,-}(\alpha) + C^{+,+}(\beta) + C^{-,+}(\mu) + C^{+,-}(\nu) + I\right) \\ &\quad = G^{\Sigma}(\alpha, \beta, \mu, \nu) = \prod_{\sigma \in \Sigma} G^{\sigma}(\alpha, \beta, \mu, \nu), \quad \alpha, \beta, \mu, \nu \in \mathbb{R}^3, \end{aligned} \tag{21}$$

where the 4 factors G^{σ}, $\sigma \in \Sigma$, in the determinant G^{Σ} are given by

$$\begin{aligned} G^{\sigma}(\alpha, \beta, \mu, \nu) &= G(\alpha + \sigma \cdot \beta, \mu + \sigma \cdot \nu) \\ &= R(\alpha + \sigma \cdot \beta) + R(\mu + \sigma \cdot \nu) + 2T(\alpha + \sigma \cdot \beta, \mu + \sigma \cdot \nu) - 1. \end{aligned} \tag{22}$$

Remark 1 The validity of the formulas in (20)–(22) can be checked using a computer algebra system. (As noted above, this amounts to match 800+ monomials of degree 16 in 12 variables.) Note also the special cases in (19) with $\mu = \nu = 0$, and in (14) with $\beta = \mu = \nu = 0$.

Remark 2 Cayley's cubic Q, the tetrahedral quartic R, and the biquadratic form T are not independent. We have the identity

$$Q(2\alpha) = 2R(\alpha) + 2T(\alpha;\alpha) - 1, \quad \alpha \in \mathbb{R}^3. \tag{23}$$

(Note that Q is also Σ-invariant.) This (used twice) gives an equivalent form of G in which the principal (degree 4) part is clearly recognizable:

$$G(\alpha,\beta) = Q(2\alpha)/2 + Q(2\beta)/2 - T(\alpha,\alpha) - T(\beta,\beta) + 2T(\alpha,\beta), \quad \alpha,\beta \in \mathbb{R}^3.$$

In particular, we have

$$G(\alpha/2,\alpha/2) = Q(\alpha) \quad \text{and} \quad G(\alpha/2,-\alpha/2) = 1 - |\alpha|^2,$$

where $|\alpha|^2 = \alpha_1^2 - \alpha_2^2 - \alpha_3^2$, $\alpha \in \mathbb{R}^3$.

G^Σ in (22) is Σ-invariant in each variable separately (since R and T are). In addition, G^Σ is symmetric with respect to the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ whose non-trivial elements act as double transpositions on the 4 variables. Finally, G^Σ is invariant with respect to single transpositions in the first two and the last two variables. (These 2 transpositions and $\mathbb{Z}_2 \times \mathbb{Z}_2$ generate the symmetry group of G^Σ, an order 8 permutation subgroup of $\mathcal{S}_4$.)

For $0 \neq (\alpha,\beta) \in \mathbb{R}^6$, we denote by $\tau(\alpha,\beta) > 0$ the first (positive) root of the quartic polynomial $t \mapsto G(t\alpha, t\beta)$, $t \in \mathbb{R}$. (A simple compactness argument shows that $\tau(\alpha,\beta)$ exists.) We will show that $\tau(\alpha,\beta)$ is the unique number $\tau > 0$ satisfying

$$G(\tau\alpha,\tau\beta) = 0 \quad \text{and} \quad Q(\tau\alpha) + Q(\tau\beta) \geq 1.$$

In particular, $\tau(\alpha,\beta)\alpha$ and $\tau(\alpha,\beta)\beta$ both belong to Cayley's tetrahedron Θ_0.

Theorem E *Under the parametrization $C^{-,-} \times C^{+,+} \times C^{-,+} \times C^{+,-} : \mathbb{R}^{12} \to \mathfrak{D}_4$, $\mathbb{R}^3 = D(3,3)$, the intersection $\Gamma = \mathfrak{M}_4 \cap \mathfrak{D}_4$ corresponds to the convex body*

$$\Gamma_0 = \left\{ (\alpha,\beta,\mu,\nu) \in \Omega_0 \mid \min_{\sigma\in\Sigma} \tau(\alpha + \sigma\cdot\beta, \mu + \sigma\cdot\nu) \geq 1 \right\}, \tag{24}$$

where $\Omega_0 \subset \mathbb{R}^{12}$ is the 12-cube, the common intersection of 24 half-spaces, given by the inequalities

$$-1 \leq (\alpha_i + \sigma_i\beta_i) \pm (\mu_i + \sigma_i\nu_i) \leq 1,\ \alpha,\beta,\mu,\nu \in \mathbb{R}^3, \quad \sigma_i \in \{\pm 1\}, \quad i = 1,2,3. \tag{25}$$

For $(\alpha,\beta,\mu,\nu) \in \Gamma_0$, the corank $16-n$ of a full orthogonal multiplication $F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^n$, $4 \leq n \leq 16$, corresponding to $C = C^{-,-}(\alpha) + C^{+,+}(\beta) + C^{-,+}(\mu) + C^{+,-}(\nu) \in \Gamma$ is given by $16 - n = \sum_{\sigma\in\Sigma} c_\sigma$, where $c_\sigma \in \{0,1,2,3\}$, $\sigma \in \Sigma$, is the multiplicity of the first positive root $t = 1$ of the quartic polynomial $t \mapsto G(t(\alpha + \sigma\cdot\beta), t(\mu + \sigma\cdot\nu))$, $t \in \mathbb{R}$. (The multiplicity c_σ is zero if there is no

root on $[0, 1]$.*) If* $c_\sigma \geq 1$ *then* $Q(\alpha + \sigma \cdot \beta) + Q(\mu + \sigma \cdot \nu) \geq 1$, *and* $\alpha + \sigma \cdot \beta$ *and* $\mu + \sigma \cdot \nu$ *both belong to Cayley's tetrahedron* Θ_0. *We have* $c_\sigma = 1$ *if and only if* $G(\alpha + \sigma \cdot \beta, \mu + \sigma \cdot \nu) = 0$ *and* $Q(\alpha + \sigma \cdot \beta) + Q(\mu + \sigma \cdot \nu) > 1$. *Moreover,* $c_\sigma = 2$ *if and only if* $G(\alpha + \sigma \cdot \beta, \mu + \sigma \cdot \nu) = 0$, $Q(\alpha + \sigma \cdot \beta) + Q(\mu + \sigma \cdot \nu) = 1$ *and* $|\alpha + \sigma \cdot \beta|^2 + |\mu + \sigma \cdot \nu|^2 < 3$. *Finally,* $c_\sigma = 3$ *if and only if* $G(\alpha + \sigma \cdot \beta, \mu + \sigma \cdot \nu) = 0$, $Q(\alpha + \sigma \cdot \beta) + Q(\mu + \sigma \cdot \nu) = 1$, *and* $|\alpha + \sigma \cdot \beta|^2 + |\mu + \sigma \cdot \nu|^2 = 3$. *In this case* $\alpha + \sigma \cdot \in \Sigma$ *(vertex of* Θ_0*) and* $\mu + \sigma \cdot \nu = 0$, *or* $\mu + \sigma \cdot \nu \in \Sigma$ *(vertex of* Θ_0*) and* $\alpha + \sigma \cdot \beta = 0$.

Although a *geometric* description of the full boundary is not feasible, it is possible to obtain a variety of explicitly constructible examples of orthogonal multiplications corresponding to specific boundary points. In the following table we chose the mid-points of the line segments connecting the cardinal points of $\Pi = [\Delta^{-,-}, \Delta^{+,+}]$ and those of $[\Delta^{-+}, \Delta^{+,-}]$ (listed in the previous table). The results, along with the corresponding ranks (range dimensions), are tabulated as follows:

	V	E/I	E/II	F/I	F/II	C_3/I	C_3/II	C_4	C_5
V	8	12	11	12[n_1]	14	10[2]	14	12[5/3]	13[3/2]
E/I		12	14	12[n_2]	12	14[n_3]	14	14[n_4]	15[n_2]
E/II			11	15[n_1]	14	13[$\sqrt{2}$]	14	13[5/3]	13[3/2]
F/I				12[3/2]	14[3/2]	15[2]	14[n_2]	14[15/8]	15[2]
F/II					14	13[2]	14	14[5/3]	15[3/2]
C_3/I						13[$\sqrt{2}$]	15[$\sqrt{2}$]	14[10/7]	13[3/5]
C_3/II							14	14[5/3]	15[n_2]
C_4								14[n_5]	13[15/8]
C_5									15[3/2]

The lower triangular block is not filled for transparency. Bulging with bulging ratio is indicated with square brackets after the range dimension, where $n_1 = (1+\sqrt{13})/4$, $n_2 = (-1+\sqrt{13})/4$, $n_3 = (\sqrt{5}-1)/2$, $n_4 = (-3+\sqrt{29})/2$, $n_5 = (-15+5\sqrt{17})/4$.

It is important to note that the only additional rank (or range dimension) not listed previously is 11. In addition, since the range dimension 9 is missing throughout it is natural to have the following:

Conjecture *There is no full orthogonal multiplication* $F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^9$.

V. The fixed points of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathfrak{C}_4$

As an application of Theorem E, we can determine the fixed point sets of the various non-trivial elements in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting on Γ.

Consider the idempotent pair $(1/2)((I, I) \pm (\gamma, \gamma))$ in the group ring $\mathbb{Q}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ acting on $\mathfrak{E}_4$, and, by convexity, on $\mathfrak{M}_4$. Their images are the (18-dimensional orthogonal) ± 1-eigenspaces of (γ, γ) on $\mathfrak{E}_4$. The $+1$-eigenspace is the fixed-point set $\mathfrak{E}_4^{(\gamma,\gamma)}$.

Similarly, the images of the idempotent pairs $(1/2)((I, I) \pm (I, \gamma))$ and $(1/2)((I, I) \pm (\gamma, I)$ are the (18-dimensional orthogonal) ± 1-eigenspaces of (I, γ) and (γ, I).

These fixed point sets

$$\mathfrak{E}_4^{(I,\gamma)},\ \mathfrak{E}_4^{(\gamma,I)},\ \mathfrak{E}_4^{(\gamma,\gamma)} \subset \mathfrak{E}_4$$

form a 'bouquet,' that is they span $\mathfrak{E}_4$ and *any two* intersect in the common 9-dimensional linear subspace $\mathfrak{E}_4^{\mathbb{Z}_2\times\mathbb{Z}_2}$.

Restricting to the linear subspace $\mathfrak{D}_4$ of diagonal elements we obtain a bouquet of 6 dimensional linear subspaces

$$\mathfrak{D}_4^{(I,\gamma)},\ \mathfrak{D}_4^{(\gamma,I)},\ \mathfrak{D}_4^{(\gamma,\gamma)} \subset \mathfrak{D}_4$$

with 3-dimensional common intersection $\mathfrak{D}_4^{\mathbb{Z}_2\times\mathbb{Z}_2}$.

For brevity, we consider only the fixed point set $\Gamma^{(\gamma,\gamma)} = \mathfrak{M}_4 \cap \mathfrak{D}_4^{(\gamma,\gamma)}$ along with $\Gamma^{\mathbb{Z}_2\times\mathbb{Z}_2} = \mathfrak{M}_4 \cap \mathfrak{D}_4^{\mathbb{Z}_2\times\mathbb{Z}_2}$.

Among $C^{\pm,\pm}$ in (18), (γ,γ) acts as a double transposition $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$. Therefore $\Gamma^{(\gamma,\gamma)}$ is given by

$$C^{-,-}(\alpha/2) + C^{+,+}(\alpha/2) + C^{-,+}(\mu/2) + C^{+,-}(\mu/2) + I \geq 0,\ \alpha,\mu \in \mathbb{R}^3, \quad (26)$$

where the $1/2$ factors are inserted to preserve convexity (and for technical convenience).

Within this and with yet another $1/2$ scaling, $\Gamma^{\mathbb{Z}_2\times\mathbb{Z}_2}$ is given by

$$C^{-,-}(\alpha/4) + C^{+,+}(\alpha/4) + C^{-,+}(\alpha/4) + C^{+,-}(\alpha/4) + I \geq 0,\ \alpha,\beta \in \mathbb{R}^3.$$

Under the parametrization in (26), $\Gamma^{(\gamma,\gamma)}$ corresponds to the convex body $\Gamma_0^{(\gamma,\gamma)} \subset \mathbb{R}^6$. We now use (20)–(22) and the subsequent Remark 2 to specify G^σ, $\sigma \in \Sigma$, in our setting. If $\sigma \neq (1,1,1)$ fixes the ith coordinate (as a half-turn) then we have $G^\sigma(\alpha/2,\alpha/2,\mu/2,\mu/2) = (1-(\alpha_i+\mu_i)^2)(1-(\alpha_i-\mu_i)^2)$. If $\sigma = (1,1,1)$ then we have $G^{(1,1,1)}(\alpha/2,\alpha/2,\mu/2,\mu/2) = G(\alpha,\mu)$. Thus, by expanding G, we obtain that $\Gamma_0^{(\gamma,\gamma)}$ is given by

$$-1 \leq \alpha_i \pm \mu_i \leq 1, \quad i = 1,2,3, \quad (27)$$

$$\begin{aligned}(1-\alpha_1^2-\alpha_2^2-\alpha_3^2-\mu_1^2-\mu_2^2-\mu_3^2)^2 &+ 8(\alpha_1\alpha_2\alpha_3+\mu_1\mu_2\mu_3)\\ -4((\alpha_1\alpha_2)^2+(\alpha_2\alpha_3)^2+(\alpha_3\alpha_1)^2+(\mu_1\mu_2)^2+(\mu_2\mu_3)^2+(\mu_3\mu_1)^2)&\\ -4((\alpha_1\mu_1)^2+(\alpha_2\mu_2)^2+(\alpha_3\mu_3)^2) &\geq 0. \quad (28)\end{aligned}$$

The 12 inequalities in (27) define a 6-dimensional cube. Second, unlike its simpler cousin in (14)–(15), the left-hand side of (28) does not split into simpler factors. Setting $\alpha = 0$ or $\mu = 0$, however, it does, so that each of the two coordinate spaces $\mathbb{R}^3$ in $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ cuts out the regular tetrahedron Δ_0 from $\Gamma_0^{(\gamma,\gamma)}$. Moreover, by (23) and the subsequent formulas in Remark 2, the diagonal $\mathbb{R}^3_\Delta \subset \mathbb{R}^3 \times \mathbb{R}^3$ cuts out from $\Gamma_0^{(\gamma,\gamma)}$ Cayley's tetrahedron $(1/2)\Theta_0$ scaled by $1/2$, and the anti-diagonal $\mathbb{R}^3_{\Delta'} \subset \mathbb{R}^3 \times \mathbb{R}^3$ cuts out from $\Gamma_0^{(\gamma,\gamma)}$ the unit ball $\mathcal{B}_0$ scaled by $1/2$. Intuitively, $\Gamma_0^{(\gamma,\gamma)}$ can be considered as a 'hybrid' between the regular tetrahedron Δ_0 and Cayley's tetrahedron Θ_0.

Remark 1 The reappearance of Θ_0, the key component in the geometry of $\mathfrak{M}_3$, is not surprising. In fact, the fixed point set $(\mathbb{R}^4 \times \mathbb{R}^4)^{(\gamma,\gamma)}$ is $\mathbb{R}^3 \times \mathbb{R}^3$, where $\mathbb{R}^3 \subset \mathbb{R}^4$ is the linear subspace orthogonal to $(0, 0, 0, 1) \in \mathbb{R}^4$. Thus, acting on orthogonal multiplications $F : \mathbb{R}^4 \times \mathbb{R}^4 \to V$, the fixed point set of (γ, γ) corresponds to restriction to $\mathbb{R}^3 \times \mathbb{R}^3$.

Remark 2 By (21)–(22) (with $\beta = \nu = 0$), we see that, up to the scaling factor $1/2$, (27)–(28) also determine the hybrid intersection $\mathfrak{M}_4 \cap (\mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{-,+})$.

Corollary *Under the parametrization above, the intersection*

$$\Gamma^{(\gamma,\gamma)} = \mathfrak{M}_4 \cap \mathfrak{D}_4^{(\gamma,\gamma)}$$

corresponds to the hybrid convex body $\Gamma_0^{(\gamma,\gamma)} \subset \mathbb{R}^6$. *The tetrahedral intersections* Δ_0 *of* $\Gamma_0^{(\gamma,\gamma)}$ *with the two coordinate spaces* $\mathbb{R}^3$ *in* $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ *correspond in* $\Gamma^{(\gamma,\gamma)}$ *to the arithmetic means of the tetrahedra*

$$(1/2)((I, I) + (\gamma, \gamma))\Delta^{-,-} \quad \text{and} \quad (1/2)((I, \gamma) + (\gamma, I))\Delta^{-,-}.$$

The range dimensions of the full orthogonal multiplications parametrized by the cardinal points of these tetrahedra are listed in the first table under E/II *(vertex),* C_3/II *(midpoint of an edge),* C_5 *(centroid of a face).*

The intersection of $\Gamma_0^{(\gamma,\gamma)}$ *with the diagonal* $\mathbb{R}^3_\Delta$, *the scaled Cayley's tetrahedron* $(1/2)\Theta_0$, *corresponds to* $\Gamma^{\mathbb{Z}_2\times\mathbb{Z}_2}$. *The range dimensions of the full orthogonal multiplications that correspond to the cardinal points of* $(1/2)\Theta_0$ *are 11 (vertex), 14 (midpoint of an edge), and 15 (centroid of an inflated face).*

The intersection of $\Gamma_0^{(\gamma,\gamma)}$ *with the anti-diagonal* $\mathbb{R}^3_\Delta$, *corresponds to the scaled unit ball* $(1/2)\mathcal{B}_0$. *The range dimension of the full orthogonal multiplications parametrized by the spherical boundary of* $(1/2)\mathcal{B}_0$ *is constant 15.*

3 Proofs

We will follow a somewhat unusual order B, C, A, D, E, since the proof of Theorem A is conceptually similar (and simpler) than the proof of Theorem C.

3.1 Adjoint representation of $O(4)$: Proof of Theorem B

Theorem B follows from elementary facts on the adjoint representation of the orthogonal group $O(4)$ on its Lie algebra $so(4)$. Recall from Sect. 1.3 the identification $so(4) \cong \Lambda^2(\mathbb{R}^4)$.

First, we parametrize $SU(2)$ by

$$A(z, w) = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}, \quad |z|^2 + |w|^2 = 1, \ z, w \in \mathbb{C}. \tag{29}$$

Note that $SU(2)$ is identified with the group $S^3 \subset \mathbb{H}$ of unit quaternions by $(z, w) \mapsto z + \mathrm{j}w$, $z, w \in \mathbb{C}$. Under this identification, multiplication by $A(z, w)$ corresponds to quaternionic multiplication by $z + \mathrm{j}w$.
As in Sect. 2.2, under the identification $\mathbb{C}^2 = \mathbb{R}^4$ by $(z, w) \mapsto (x, y, u, v)$, $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$, $x, y, u, v \in \mathbb{R}$, the special unitary group $SU(2)$ becomes a subgroup of $SO(4)$. Moreover, we have the almost product structure $SO(4) = SU(2) \cdot SU(2)'$ with $SU(2)$ and $SU(2)' = \gamma SU(2)\gamma$, $\gamma = \mathrm{diag}\,(1, 1, 1, -1) \in O(4)$, both normal in $SO(4)$, and $SU(2) \cap SU(2)' = \{\pm I\}$.

Now a simple computation shows that the eigenspace Λ^2_- *coincides* with $\Lambda^2(\mathbb{R}^4)^{SU(2)'}$, the fixed point set of $SU(2)'$ on $\Lambda^2(\mathbb{R}^4)$. In addition, it is invariant under the action of $SU(2)$, and, in fact, it is a (real) irreducible $SU(2)$-module (by restriction). Similarly, $\Lambda^2_+ = \Lambda^2(\mathbb{R}^4)^{SU(2)}$ is an irreducible $SU(2)'$-module.

With these, we obtain

$$\Lambda^2(\mathbb{R}^4) = \Lambda^2_- \oplus \Lambda^2_+ = \Lambda^2(\mathbb{R}^4)^{SU(2)'} \oplus \Lambda^2(\mathbb{R}^4)^{SU(2)}, \tag{30}$$

as an orthogonal direct sum of irreducible $SU(2)$- and $SU(2)'$-modules. Finally, as stated in Sect. 2.2, $\gamma \in O(4)$ interchanges the eigenspaces: $\gamma : \Lambda^2_\pm \leftrightarrow \Lambda^2_\mp$.

Applying (30) to each component of $\mathfrak{E}_4 = \Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4)$, Theorem B follows. Since γ *generates* $O(4)$ over $SO(4)$, as a byproduct, we also see that $\Lambda^2(\mathbb{R}^4)$ is *irreducible* as an $O(4)$-module.

We will need an explicit formula for the action of $SU(2)$ on Λ^2_-. We claim that, with respect to the canonical basis $\{E_i\}_{i=1}^3 \subset \Lambda^2_-$ in (13), the adjoint action $Ad(A(z, w))$, $A(z, w) \in SU(2)$, on $\Lambda^2_- = \mathbb{R}^3$ is by (left) multiplication by the matrix $U(z, w) \in SO(3)$ given as

$$Ad(A(z, w)) \sim U(z, w) = \begin{bmatrix} |z|^2 - |w|^2 & 2\Im(z\bar{w}) & 2\Re(z\bar{w}) \\ 2\Im(zw) & \Re(z^2 + w^2) & -\Im(z^2 - w^2) \\ -2\Re(zw) & \Im(z^2 + w^2) & \Re(z^2 - w^2) \end{bmatrix}. \tag{31}$$

To derive this is a straightforward computation noting that, under the identifications $\Lambda^2(\mathbb{R}^4) = so(4)$ and $\mathbb{R}^4 = \mathbb{C}^2$, we have $E_1 = A(-\mathrm{i}, 0)$, $E_2 = A(0, -1)$, $E_3 = A(0, -\mathrm{i})$. Note also that this action of $SU(2)$ is *not effective*, since $Ad(A(-z, -w)) = Ad(A(z, w))$; this explains why we have $SO(3) \cong SU(2)/\{\pm I\}$.

Remark Not unexpectedly (Schur's orthogonality relations applied to the complex $SU(2)$-module W_2) the nine matrix entries in (31) form an L^2-orthonormal basis of the space of quadratic spherical harmonics $\mathcal{H}^2_3$ over S^3. One also finds that the entries of the first row comprise the components of the Hopf map $f_{F^{\mathbb{C}}_{\cdot}} : S^3 \to S^2$.

3.2 Proof of Theorem C

Recall that, using the canonical basis in (13), we parametrize the ambient 9-dimensional linear space $\mathfrak{E}_4^{-,-} = \Lambda^2_- \otimes \Lambda^2_-$ by 3×3-matrices $\mathcal{X} = [x_{ij}]_{i,j=1}^3 \in$

$M(3,3)$ as

$$C^{-,-}(\mathcal{X}) = \sum_{i,j=1}^{3} x_{ij} E_i \otimes E_j. \tag{32}$$

This is the tensor (Kronecker) product of 4×4 skew-symmetric matrices, a 16×16 symmetric matrix. By definition of the moduli, $C^{-,-}(\mathcal{X}) \in \mathfrak{M}_4^{-,-}$ if and only if $C^{-,-}(\mathcal{X}) + I \geq 0$. We have the following:

Proposition *Given $\mathcal{X} \in M(3,3)$, with row vectors $X_i \in \mathbb{R}^3$, $i = 1,2,3$, we have $C^{-,-}(\mathcal{X}) \in \mathfrak{M}_4^{-,-}$ if and only if the following inequalities hold:*

(I) $1 - |X_i|^2 \geq 0$, $i = 1,2,3$;
(II) $1 - \|\mathcal{X}\|^2 + 2\det(\mathcal{X}) \geq 0$;
(III) $(1 - \|\mathcal{X}\|^2)^2 + 8\det(\mathcal{X}) - 4\|\mathrm{adj}\,(\mathcal{X})\|^2 \geq 0$,

where $\|\cdot\|$ is the 'entrywise' L^2-norm of matrices and adj *is the adjoint matrix.*

Proof The (relative) interior of the equivariant moduli $\mathfrak{M}_4^{-,-}$ is given by

$$\operatorname{int} \mathfrak{M}_4^{-,-} = \{C \in \mathfrak{E}_4^{-,-} \mid C + I > 0\}.$$

To prove the proposition it is enough to show that this interior is characterized by (I)-(III) with *strict* inequalities.

According to Sylvester's criterion, positive definiteness of $C(\mathcal{X}) + I$, $\mathcal{X} \in M(3,3)$, is equivalent to all principal upper left minors to be positive. Now a technical calculation (which may be facilitated by the use of a computer algebra system) shows that these 16 principal minors are as follows:

$$1, 1, 1, 1, P, P^2, P^3, P^4, P^3Q, P^2Q^2, PQ^3, Q^4, Q^3R, Q^2R^2, QR^3, R^4, \tag{33}$$

where $P = 1 - |X_1|^2$, and Q and R are the left-hand sides of (II) and (III), respectively. (The notation is justified by restriction to $\mathbb{R}^3 = D(3,3) \subset M(3,3)$ whereas Q and R become Cayley's cubic and the tetrahedral quartic, respectively.)

To complete the proof we will show that $1 - |X_1|^2 > 0$ and $Q > 0$ imply $1 - |X_2|^2 > 0$ and $1 - |X_3|^2 > 0$. To do this, we replace $Q > 0$ by the weaker inequality

$$|X_1|^2 + |X_2|^2 + |X_3|^2 < 1 + 2|X_1||X_2||X_3|.$$

Factoring this, we obtain

$$(|X_2| - |X_3|)^2 < (1 - |X_1|)(1 + |X_1| - 2|X_2||X_3|).$$

Since $|X_1| < 1$, we must have $2|X_2||X_3| < 1 + |X_1| < 2$. Thus $|X_2| < 1$ or $|X_3| < 1$. Assuming the first (say), we factor again

$$(|X_1| - |X_2|)^2 < (1 - |X_3|)(1 + |X_3| - 2|X_1||X_2|).$$

Now, if $|X_3| > 1$ were to hold then we would get $1 + |X_3| < 2|X_1||X_2| < 2$, a contradiction. The proposition follows.

As an initial geometric insight to the inequalities (I)-(III), assume that the rows X_i, $i = 1, 2, 3$, of $\mathcal{X} \in M(3, 3)$ are of maximal (unit) length. Then $\|\mathcal{X}\|^2 = 3$ and (II) implies that $\mathcal{X} \in SO(3)$. (The signed volume of the parallelepiped spanned by the unit vectors X_i, $i = 1, 2, 3$, must be 1.) Changing the signs of two of the three rows of $\mathcal{X}$ in all possible ways, we obtain three additional matrices which, along with $\mathcal{X}$ form the vertices of a regular tetrahedron in $\mathfrak{E}_4^{-,-} = \mathbb{R}^9$ with center at the origin and inscribed in a cube of edge length 2. Now a simple computation shows that equality holds in (III) on the *faces* of the tetrahedron. We obtain that the solid regular tetrahedron with the ascribed vertices satisfies (I)–(III). We will see below that the entire solution set of (I)–(III) will be an $SU(2) \times SU(2)$-orbit of such tetrahedra. □

Remark 1 The last part of the proof of the proposition can be simplified by using the well-known refinement of Sylvester's criterion: A matrix is positive semi-definite if and only if *all* its principal minors are non-negative.
Remark 2 The regularity of the sequence in (33) suggests that there may be a shorter and more insightful calculation of the principal minors rather than the 'brute force' method used here.
Remark 3 From the discussion on the general properties of the moduli $\mathfrak{M}_4$ it follows that $C^{-,-}(\mathcal{X}) \in \mathfrak{M}_4^{-,-}$ if and only if (III) holds for $t \cdot \mathcal{X}$, $t \in [0, 1]$, (in place of $\mathcal{X}$) with strict inequality for $t \in [0, 1)$. In particular, we will see shortly that (II) is redundant in the sense that (I) and (III) imply (II).

We now derive an explicit formula for the action of $SU(2) \times SU(2)$ on $\mathfrak{E}_4^{-,-}$. As in (32), let $C^{-,-}(\mathcal{X}) \in \mathfrak{E}_4^{-,-}$, $\mathcal{X} \in M(3, 3)$, and $A(z, w), A(z', w') \in SU(2)$. Using (31), we have

$$Ad(A(z, w)) \otimes Ad(A(z', w')) \cdot C^{-,-}(\mathcal{X}) = C^{-,-}(U(z, w)^\top \cdot \mathcal{X} \cdot U(z', w')). \quad (34)$$

Indeed, we calculate

$$\begin{aligned}
&Ad(A(z, w)) \otimes Ad(A(z', w')) \cdot C^{-,-}(\mathcal{X}) \\
&\quad = \sum_{i,j=1}^{3} x_{ij}\, Ad(A(z, w)) \cdot E_i \otimes Ad(A(z', w')) \cdot E_j \\
&\quad = \sum_{i,j=1}^{3} x_{ij} \sum_{k=1}^{3} U(z, w)_{ik} E_k \otimes \sum_{l=1}^{3} U(z', w')_{jl} E_l \\
&\quad = \sum_{k,l=1}^{3} \sum_{i,j=1}^{3} U(z, w)^\top_{ki}\, x_{ij}\, U(z', w')_{jl}\, E_k \otimes E_l \\
&\quad = \sum_{k,l=1}^{3} (U(z, w)^\top \cdot \mathcal{X} \cdot U(z', w'))_{kl}\, E_k \otimes E_l \\
&\quad = C^{-,-}(U(z, w)^\top \cdot \mathcal{X} \cdot U(z', w')).
\end{aligned}$$

Now, given $\mathcal{X} \in M(3,3)$, recall from the *singular value decomposition theorem* that there exist $U, U' \in O(3)$ such that $\mathcal{X} = U \cdot \mathrm{diag}\,(\alpha_1, \alpha_2, \alpha_3) \cdot U'^{\top}$, where the diagonal entries, the so-called *singular values*, satisfy $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0$. Since the ambient space is odd dimensional, inserting negative signs if needed, we can attain that $U, U' \in SO(3)$ at the expense of getting negative diagonal entries in the diagonal matrix. Changing the notations accordingly, we then have $U^{\top} \cdot \mathcal{X} \cdot U' = \mathrm{diag}\,(\alpha_1, \alpha_2, \alpha_3)$ with $U, U' \in SO(3)$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Summarizing, and using the notations of Theorem C, we obtain that every $SU(2) \times SU(2)$-orbit in $\mathfrak{E}_4^{-,-}$ intersects the linear subspace $\mathfrak{D}_4^{-,-}$ of 'diagonal elements.'

We now bring in the moduli $\mathfrak{M}_4^{-,-}$. The conditions (I)–(III) that guarantee $C^{-,-}(\alpha) \in \mathfrak{M}_4^{-,-}$, $\alpha \in \mathbb{R}^3$, reduce to the following:
$(I)_0$ $P_i(\alpha) \leq 0$, $i = 1, 2, 3$;
$(II)_0$ $Q(\alpha) \geq 0$;
$(III)_0$ $R(\alpha) \geq 0$,
where $P_i(\alpha) = 1 - \alpha_i^2$, $\alpha \in \mathbb{R}^3$, $i = 1, 2, 3$, Q is Cayley's cubic given in (8), and R is the tetrahedral quartic given in (15). (Note that $(I)_0$ constrains $\alpha \in \mathbb{R}^3$ to the cube $[-1, 1]^3$.)

By (14)–(15) (as a special case of (20)–(22)), for $\alpha \in \mathbb{R}^3$, we have

$$\det\,(C^{-,-}(\alpha) + I) = R(\alpha)^4 = \prod_{\sigma \in \Sigma} (1 + \sigma_1\alpha_1 + \sigma_2\alpha_2 + \sigma_3\alpha_3)^4 \geq 0. \tag{35}$$

By a general property of the moduli discussed in Sect. 1.3, within $\mathfrak{D}_4^{-,-}$ this determinant vanishes on rays emanating from the origin the *first time* on the boundary of $\mathfrak{M}_4^{-,-} \cap \mathfrak{D}_4^{-,-}$. By convexity, we see that $C^{-,-}(\alpha) \in \mathfrak{M}_4^{-,-}$, $\alpha \in \mathbb{R}^3$, if and only if

$$1 + \sigma_1\alpha_1 + \sigma_2\alpha_2 + \sigma_3\alpha_3 \geq 0, \quad \sigma_1\sigma_2\sigma_3 = 1, \quad \sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}. \tag{36}$$

The 4 inequalities in (36) define the half-spaces containing the origin, whose boundary planes extend the 4 faces of the regular tetrahedon $\Delta_0 \subset [-1, 1]^3$ (inscribed in the cube $[-1, 1]^3$ with vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$). We conclude that $C^{-,-}(\alpha) \in \mathfrak{M}_4^{-,-}$ if and only if $\alpha \in \Delta_0$.

Remark In our present case ($m = 4$) the inequality in $(II)_0$ is *redundant*. As noted above, *restricted to the cube* $[-1, 1]^3$, it describes Cayley's tetrahedron Θ_0 containing Δ_0:

$$\Delta_0 \subset \Theta_0 \subset [-1, 1]^3.$$

Using the notations in Theorem C, we obtain that $\Delta^{-,-} = \mathfrak{M}_4^{-,-} \cap \mathfrak{D}_4^{-,-}$ is a regular tetrahedron (in its affine span) in $\mathfrak{D}_4^{-,-}$, and we have $C^{-,-}(\Delta_0) = \Delta^{-,-}$. Finally, since the action of $SU(2) \times SU(2)$ preserves $\mathfrak{M}_4^{-,-} \subset \mathfrak{E}_4^{-,-}$ and every $SU(2) \times SU(2)$-orbit contains a diagonal element, we see that the *entire equivariant moduli* $\mathfrak{M}_4^{-,-}$ is the $SU(2) \times SU(2)$-orbit of $\Delta^{-,-}$. The first statement of Theorem C follows.

As a byproduct, we also see that (I), $(III) \Rightarrow (II)$.

The action of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ carries the same structure to the other equivariant moduli:

$$\begin{aligned}\mathfrak{M}_4^{-,+} &= (SU(2) \times SU(2)')\,\Delta^{-,+},\\ \mathfrak{M}_4^{+,-} &= (SU(2)' \times SU(2))\,\Delta^{+,-},\\ \mathfrak{M}_4^{+,+} &= (SU(2)' \times SU(2)')\,\Delta^{+,+}.\end{aligned}$$

Indeed, for example, using (12) and Theorem C, we have

$$\begin{aligned}\mathfrak{M}_4^{+,+} &= (\gamma,\gamma)\mathfrak{M}_4^{-,-} = (\gamma,\gamma)(SU(2) \times SU(2))\,\Delta^{-,-}\\ &= ((\gamma SU(2)\gamma) \times (\gamma SU(2)\gamma))\,(\gamma,\gamma)\,\Delta^{-,-}\\ &= (SU(2)' \times SU(2)')\,\Delta^{+,+}.\end{aligned}$$

Recall from Sect. 2.2 that, for $\alpha \in \Delta_0$, a representative (full) orthogonal multiplication $F^\alpha : \mathbb{R}^4 \times \mathbb{R}^4 \to V$ of $C^{-,-}(\alpha) \in \mathfrak{M}_4^{-,-}$ is given by

$$F^\alpha = (C^{-,-}(\alpha) + I)^{1/2} \cdot F_\otimes,$$

where V is the linear span of the image of F^α.

As a specific example, we now determine the $SU(2) \times SU(2)$-orbit of $\langle F_\cdot^{\mathbb{H}}\rangle = \langle F^{(1,1,1)}\rangle = C^{-,-}(I)$. By (34) (with $\mathcal{X} = I$), the group $SU(2) \times SU(2)$ acts as $SO(3) \times SO(3)$ with isotropy subgroup $SO(3)_\Delta \subset SO(3) \times SO(3)$, the *diagonal*. It follows that the $SU(2)\times SU(2)$-orbit is $(SO(3)\times SO(3))/SO(3)_\Delta = SO(3) = \mathbb{R}P^3$, the real projective space. Moreover, the components of the corresponding orbit-map are the matrix elements in (31). As noted in the remark at the end of Sect. 3.1, they form an L^2-orthonormal basis in $\mathcal{H}_3^2$. By definition, this means that this orbit is the image of the degree 2 *standard minimal immersion* of S^3 into S^8.

The second statement and hence Theorem C follows.

The interior points of the tetrahedron $\Delta^{-,-}$ correspond to orthogonal multiplications with maximal range dimension 16. We just saw that the range dimension for the vertex (1, 1, 1) of $\Delta^{-,-}$ is 4 since it is the parameter of the quaternionic multiplication $F_\cdot^{\mathbb{H}}$. Thus, for all the vertices of $\Delta^{-,-}$, the range dimension is 4.

We now turn to the proof of the corollary and claim that the range dimension for points on the open faces is 12, and those of the open edges is 8. This can be done in two ways. First, due to the stratification of the moduli, this can be shown by exhibiting specific examples of orthogonal multiplications that correspond to (specific) points on an open face and an open edge of $\Delta^{-,-}$.

Now, as simple computation shows, the opposite of the vertex $\langle F^{(1,1,1)}\rangle = \langle F_\cdot^{\mathbb{H}}\rangle$, the centroid $\langle F^{(-1/3,-1/3,-1/3)}\rangle$ of the opposite face, has the following representation

$$F^{(-1/3,-1/3,-1/3)} = \frac{\sqrt{3}}{2}\begin{bmatrix} 1 & 1/3 & -1/3 & 1/3\\ 1/3 & 1 & 1/3 & -1/3\\ -1/3 & 1/3 & 1 & 1/3\\ 1/3 & -1/3 & 1/3 & 1\end{bmatrix} \cdot \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4\\ -x_2y_2 & x_2y_1 & x_3y_1 & x_4y_1\\ x_3y_3 & x_3y_4 & x_4y_2 & x_2y_3\\ -x_4y_4 & x_4y_3 & x_2y_4 & x_3y_2\end{bmatrix}.$$

The components of $F^{(-1/3,-1/3,-1/3)}$ are the entries of the product matrix. Since the rows of the coefficient matrix are linearly dependent, the range dimension of $F^{(-1/3,-1/3,-1/3)}$ is 12.

Moreover, $F^{(1,0,0)}$ corresponds to the midpoint of an edge of $\Delta^{-,-}$. This is range-equivalent to $(\gamma,\gamma)\cdot F^{\mathbb{C}}_{\otimes}$, where $F^{\mathbb{C}}_{\otimes}:\mathbb{C}\times\mathbb{C}\to\mathbb{C}^4$ is the complex tensor product given by $F^{\mathbb{C}}_{\otimes}(z_1,z_2,w_1,w_2)=(z_1\overline{w_1},z_1\overline{w_2},z_2\overline{w_1},z_2\overline{w_2})$. The range dimension is 8.

As noted above, the natural stratification of $\mathfrak{M}_4$ implies that the range dimensions above prevail in $\mathfrak{M}_4^{-,-}$. The corollary to Theorem C follows.

As a second proof, following Sect. 1.3, for $\alpha\in\partial\Delta_0$, we determine the multiplicity of the root $t=1$ of the polynomial $t\mapsto R(t\alpha)$, $t\in\mathbb{R}$. Although this can be seen directly from the factorization under (15) (or (35)), for the more complex cases that follow, it is instructive to give a few details.

First, since $C(\alpha)\in\partial\Delta^{-,-}$, we have $R(\alpha)=0$, and the multiplicity is at least one. Using this, we have

$$\begin{aligned}R(t\alpha)&=(1-t^2(\alpha_1^2+\alpha_2^2+\alpha_3^2))^2+8t^3\alpha_1\alpha_2\alpha_3\\&\quad-4t^4((\alpha_1\alpha_2)^2+(\alpha_2\alpha_3)^2+(\alpha_3\alpha_1)^2)\\&=1-t^4-2t^2(1-t^2)(\alpha_1^2+\alpha_2^2+\alpha_3^2)+8t^3(1-t)\alpha_1\alpha_2\alpha_3.\end{aligned}$$

Factoring out with $1-t$ (and substituting $t=1$), we obtain that $t=1$ is a root of multiplicity ≥ 2 if and only if (in addition to $R(\alpha)=0$) we have $Q(\alpha)=0$, where Q is Cayley's qubic. On the other hand, by geometry, the regular tetrahedron Δ_0 (defined by $R(\alpha)=0$) and Cayley's tetrahedron Θ_0 (defined by $Q(\alpha)=0$) intersect exactly at the 1-skeleton of Δ_0 comprised by the 6 edges. Going back to algebra, using this, another simple computation shows that the multiplicity is ≥ 3 if and only if $|\alpha|^2=3$ (vertex). Since R is of degree 4, and there must be a negative root (by compactness of $\Delta^{-,-}$), we see that the open edges correspond to corank $4\times 2=8$, and the vertices to corank $4\times 3=12$. The corollary follows again.

3.3 Proof of Theorem A

We postponed the proof of Theorem A up to this point as it is an easy modification of the proof of Theorem C.

Let $m=3$. We realize the $SO(3)\times SO(3)$-module structure on $\mathfrak{E}_3$ (by restriction) as follows. The Hodge * operator on the exterior algebra $\Lambda^*(\mathbb{R}^3)$ restricts to an $SO(3)$-isomorphism $\Lambda^2(\mathbb{R}^3)\to\Lambda^1(\mathbb{R}^3)=\mathbb{R}^3$. Under this isomorphism, the 2-vectors of the standard basis, $E_1=e_2\wedge e_3$, $E_2=e_3\wedge e_1$, $E_3=e_1\wedge e_2$, correspond to the respective basis elements e_1,e_2,e_3 in $\mathbb{R}^3$. We realize the standard action of $SO(3)$ on $\mathbb{R}^3$ as the lift to $SU(2)=S^3\subset\mathbb{H}$ on the linear space of *purely imaginary* quaternions $\mathbb{H}_0\subset\mathbb{H}$ by quaternionic adjoint $\mathrm{Ad}\,(q)$, $q\in S^3$. With the identifications $\mathbb{R}^4=\mathbb{C}^2$ and $\mathbb{C}^2\ni(z,w)\mapsto z+\mathrm{j}w\in\mathbb{H}$, $z,w\in\mathbb{C}$, made previously, we also have $\mathbb{R}^3\cong\mathbb{H}_0$ given by $(a,b,c)\mapsto \mathrm{i}a+\mathrm{j}b-\mathrm{k}c\in\mathbb{H}_0$, $a,b,c\in\mathbb{R}$. With these the (adjoint) action of a typical element $A(z,w)\in SU(2)$, $z,w\in\mathbb{C}$, on $\mathbb{R}^3$ is by left-multiplication by the matrix $U(z,w)\in SO(3)$ given explicitly in (31).

Parametrizing the linear space $\mathfrak{E}_3$ by $M(3,3)$ through the linear isomorphism $C : M(3,3) \to \mathfrak{E}_3$ the proof of (34) goes through with appropriate modifications. We obtain that $(U, U') \in SO(3) \times SO(3)$ acts on $M(3,3)$ as $\mathcal{X} \mapsto U^\top \cdot \mathcal{X} \cdot U'$, $\mathcal{X} \in M(3,3)$. Applying the singular value decomposition, we conclude, as before, that every $SO(3) \times SO(3)$-orbit contains a diagonal element $C(\alpha) \in \mathfrak{D}_3, \alpha \in \mathbb{R}^3$. Finally, a simple but somewhat tedious computation gives (9).

In perfect analogy with the case of $\mathfrak{M}_4^{-,-}$, using the fact that the determinant in (9) vanishes on rays emanating from the origin the first time on the boundary of the moduli, non-negativity of the right-hand side in (9) breaks up into factors. The first product gives $\mathfrak{M}_3 \subset [-1,1]^3$, and Cayley's cubic factor Q gives $(II)_0$. All except the last statement of Theorem A now follow by the same argument as the proof of Theorem C above.

Finally, it remains to determine $\operatorname{rank}(C(\alpha) + I)$ for $\alpha \in \partial\Theta_0$, or equivalently (for the corank), the possible multiplicities (depending on α) of $t = 1$ as a root of the (degree 9) polynomial $\det(tC(\alpha) + I)$, $t \in \mathbb{R}$. In view of (9), we need to determine the multiplicity of $t = 1$ as a root for the cubic $Q(t\alpha)$, $t \in \mathbb{R}$.

First, since $C(\alpha) \in \partial\Theta$, the multiplicity is at least one, and we have $Q(\alpha) = 0$. Using this, we have

$$\begin{aligned} Q(t\alpha) &= 1 - t^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 2t^3\alpha_1\alpha_2\alpha_3 \\ &= (t-1)(t^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 1) + t + 1). \end{aligned}$$

Canceling $t - 1$ (and substituting $t = 1$), we obtain that the multiplicity is ≥ 2 if and only if $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 3$ (vertex) and $\alpha_1\alpha_2\alpha_3 = 1$. Since the maximum multiplicity is 2, returning to (9) and accounting for more multiplicities from the product there if $\alpha_i = \pm 1$ (edges), $i = 1, 2, 3$, an easy case-by-case check gives the last statement of Theorem A.

3.4 Proof of Theorem D

We now return to $\mathfrak{M}_4$. As usual, we parametrize $\mathfrak{D}_4^{-,-} \oplus \mathfrak{D}_4^{+,+}$ by the restriction of the linear isomorphism $C^{-,-} \times C^{+,+} : M(3,3) \times M(3,3) \to \mathfrak{E}_4^{-,-} \oplus \mathfrak{E}_4^{+,+}$ to the linear subspace $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 = D(3,3) \times D(3,3)$. The intersection

$$\mathfrak{M}_4 \cap \left(\mathfrak{D}_4^{-,-} \oplus \mathfrak{D}_4^{+,+}\right)$$

is then defined by

$$C^{-,-}(\alpha) + C^{+,+}(\beta) + I \geq 0, \quad \alpha, \beta \in \mathbb{R}^3.$$

Specializing (20)–(22) to $\mu = \nu = 0$, as stated in (17), we obtain

$$\det\left(C^{-,-}(\alpha)+C^{+,+}(\beta)+I\right)=\prod_{\sigma\in\Sigma}R(\alpha+\sigma\cdot\beta)$$
$$=\prod_{\sigma,\tau\in\Sigma}\left(1+\sum_{i=1}^{3}(\sigma_i\alpha_i+\tau_i\beta_i)\right)\geq 0.$$

We now use (once again) the fact that, within $\mathfrak{D}_4^{-,-}\oplus\mathfrak{D}_4^{+,+}$, this determinant vanishes on rays emanating from the origin the first time on the boundary of $\mathfrak{M}_4\cap(\mathfrak{D}_4^{-,-}\oplus\mathfrak{D}_4^{+,+})$. Using convexity, we obtain that $C^{-,-}(\alpha)+C^{+,+}(\beta)\in\mathfrak{M}_4\cap(\mathfrak{D}_4^{-,-}\oplus\mathfrak{D}_4^{+,+})$, $\alpha,\beta\in\mathbb{R}^3$, if and only if

$$1+\sum_{i=1}^{3}(\sigma_i\alpha_i+\tau_i\beta_i)\geq 0, \tag{37}$$

with four choices in each group of signs satisfying

$$\sigma_1\sigma_2\sigma_3=1,\ \sigma_1,\sigma_2,\sigma_3\in\{\pm1\}\quad\text{and}\quad\tau_1\tau_2\tau_3=1,\ \tau_1,\tau_2,\tau_3\in\{\pm1\}.$$

In $\mathbb{R}^3\times\mathbb{R}^3=\mathbb{R}^6$, $\alpha,\beta\in\mathbb{R}^3$, for fixed $\sigma,\tau\in\Sigma$, (37) defines a half-space (containing the origin) with boundary hyperplane whose intersection with $\mathfrak{E}_4^{-,-}$ is a supporting hyperplane of $\Delta^{-,-}$ containing one of its faces, and similarly, its intersection with $\mathfrak{E}_4^{+,+}$ is supporting and containing a face of $\Delta^{+,+}$. Since there are exactly $4\times4=16$ hyperplanes with this property, we conclude

$$\mathfrak{M}_4\cap\left(\mathfrak{D}_4^{-,-}\oplus\mathfrak{D}_4^{+,+}\right)=[\Delta^{-,-},\Delta^{+,+}]. \tag{38}$$

Finally, given $C\in\mathfrak{M}_4\cap\left(\mathfrak{E}_4^{-,-}\oplus\mathfrak{E}_4^{+,+}\right)$, we have $C=C^{-,-}(\mathcal{X})+C^{+,+}(\mathcal{Y})$ with $\mathcal{X},\mathcal{Y}\in M(3,3)$. By Theorem C, $\mathcal{X}$ can be diagonalized with the action of $SU(2)\times SU(2)$ to a diagonal element $C^{-,-}(\alpha)$, $\alpha\in\mathbb{R}^3$, and similarly, $\mathcal{Y}$ can be diagonalized with the action of $SU(2)'\times SU(2)'$ to a diagonal element $C^{+,+}(\beta)$, $\beta\in\mathbb{R}^3$. It follows that C is in the $SO(4)\times SO(4)$-orbit of the sum $C^{-,-}(\alpha)+C^{+,+}(\beta)\in\mathfrak{M}_4\cap\left(\mathfrak{D}_4^{-,-}\oplus\mathfrak{D}_4^{+,+}\right)$. By (38), we obtain $C^{-,-}(\alpha)+C^{+,+}(\beta)\in[\Delta^{-,-},\Delta^{+,+}]$. Theorem D follows.

Remark Due to the complete factorization in (17) into linear factors, the polyhedral structure of $\Pi=[\Delta^{-,-},\Delta^{+,+}]$, and the table following Theorem D, the calculation of possible coranks via multiplicities of the root $t=1$ of the polynomial $t\mapsto\det(tC^{-,-}(\alpha)+tC^{+,+}(\beta)+I)$, $(\alpha,\beta)\in\partial\Pi$, $t\in\mathbb{R}$, can be dispensed with.

3.5 Proof of Theorem E

Recall the intersection $\Gamma=\mathfrak{M}_4\cap\mathfrak{D}_4$ given by (18), and the respective determinant in (20)–(22). Define $\Gamma_0\subset\mathbb{R}^{12}$ as the convex body corresponding to Γ:

$$(\alpha,\beta,\mu,\nu)\in\Gamma_0\Leftrightarrow C^{-,-}(\alpha)+C^{+,+}(\beta)+C^{-,+}(\mu)+C^{+,-}(\nu)\in\Gamma,$$

$$\Leftrightarrow C^{-,-}(\alpha) + C^{+,+}(\beta) + C^{-,+}(\mu) + C^{+,-}(\nu) + I \geq 0. \quad (39)$$

By the generalized Sylvester criterion, all principal minors of the positive semi-definite matrix in (39) are non-negative. Evaluating all 2×2 principal minors, we obtain

$$\begin{aligned}&(1 + (\alpha_i + \sigma_i\beta_i) + \tau_i(\mu_i + \sigma_i\nu_i))(1 - (\alpha_i + \sigma_i\beta_i) - \tau_i(\mu_i + \sigma_i\nu_i))\\ &\quad \geq 0, \ \ \sigma_i, \tau_i \in \{\pm 1\}, \ i = 1, 2, 3.\end{aligned}$$

These give (25). Note that the 24 half-spaces in (25) that define Ω_0 is a 12-cube since the unit normal vectors of the bounding hyperplanes form pairs of 12 orthonormal vectors in $\mathbb{R}^{12}$. We obtain $\Gamma_0 \subset \Omega_0$.

In view of (21)–(22) and (39), we have $(\alpha, \beta, \mu, \nu) \in \partial\Gamma_0$ if and only if $t = 1$ is the first positive root of the polynomial $t \mapsto \prod_{\sigma\in\Sigma} G(t(\alpha + \sigma \cdot \beta), t(\mu + \sigma \cdot \nu))$, $t \in \mathbb{R}$. By the definition before Theorem E, the first positive root of this polynomial is equal to $\min_{\sigma\in\Sigma} \tau(\alpha + \sigma \cdot \beta, \mu + \sigma \cdot \nu)$. Convexity of Γ_0 and $0 \in \operatorname{int} \Gamma_0$ now imply (24).

It remains to prove the last statement of Theorem E about the corank of diagonal elements in Γ. Using the parametrization in (39), as in Sect. 1.3, we let $(\alpha, \beta, \mu, \nu) \in \partial\Gamma_0$, and determine the possible multiplicities of the root $t = 1$ of the (degree 16) polynomial $t \mapsto G^{\Sigma}(t\alpha, t\beta, t\mu, t\nu)$, $t \in \mathbb{R}$. By (21), we have $G^{\Sigma} = \prod_{\sigma\in\Sigma} G^{\sigma}$, so that the corank of $C = C^{-,-}(\alpha) + C^{+,+}(\beta) + C^{-,+}(\mu) + C^{+,-}(\nu) \in \partial\Gamma$ is equal to $\sum_{\sigma\in\Sigma} c_\sigma$, where $c_\sigma \in \{1, 2, 3, 4\}$, $\sigma \in \Sigma$, is the multiplicity of the (possible) first root $t = 1$ of the quartic polynomial $t \mapsto G^{\sigma}(t\alpha, t\beta, t\mu, t\nu)$, $t \in \mathbb{R}$. (As usual the multiplicity is zero if this quartic is positive on $[0, 1]$. We will also see shortly that $c_\sigma \leq 3, \sigma \in \Sigma$.) Since $G^{\sigma}(t\alpha, t\beta, t\mu, t\nu) = G(t(\alpha+\sigma\cdot\beta), t(\mu+\sigma\cdot\nu))$, and since a linear change of variables does not change the multiplicity, it remains to determine the possible multiplicities of the quartic $t \mapsto G(t\alpha, t\beta), t \in \mathbb{R}$. By (25) just proved, for the new variables we have $-1 \leq \alpha_i \pm \beta_i \leq 1$, $i = 1, 2, 3$, in particular, $\alpha, \beta \in [-1, 1]^3$, and hence $|\alpha|^2 + |\beta|^2 \leq 3$.

Using (8) and (20) and homogeneity of the biquadratic form T, an easy calculation gives

$$\begin{aligned}G(t\alpha, t\beta) = {}& t^4 G(\alpha, \beta) + 4t^3(1 - t)\big(Q(\alpha) + Q(\beta) - 1\big)\\ &+ 2t^2(1 - t)^2\big(3 - |\alpha|^2 - |\beta|^2\big) + (1 + 3t)(1 - t)^3, \quad t \in \mathbb{R}. \quad (40)\end{aligned}$$

Without loss of generality we may assume that the multiplicity of $t = 1$ is at least 1, that is we have $G(\alpha, \beta) = 0$. Using this, (40) reduces to the following

$$\begin{aligned}&G(t\alpha, t\beta) = (1 - t)\\ &\quad \times \Big[4t^3\big(Q(\alpha) + Q(\beta) - 1\big) + 2t^2(1 - t)\big(3 - |\alpha|^2 - |\beta|^2\big) + (1 + 3t)(1 - t)^2\Big].\end{aligned}$$

Now, $t = 1$ is the first root of the quartic $t \mapsto G(t\alpha, t\beta)$, $t \in \mathbb{R}$, if and only if the cubic factor in the square brackets above is positive on $[0, 1)$. Since $|\alpha|^2 + \beta|^2 \leq 3$ and $(1 + 3t)(1 - t)^2 > 0$ for $0 \leq t < 1$, a simple analysis shows that this positivity

holds if and only if

$$Q(\alpha) + Q(\beta) \geq 1. \tag{41}$$

Recall now that we have $\alpha, \beta \in [-1, 1]^3$. Elementary calculus gives

$$\max_{\alpha \in [-1,1]^3} Q(\alpha) = 1 \tag{42}$$

with the maximum attained at the origin. Now (41)–(42) imply $Q(\alpha), Q(\beta) \geq 0$ so that α and β both belong to Cayley's tetrahedron Θ_0.

Note that, as a byproduct, the characterization of τ before Theorem E in Sect. 2.2 follows.

Returning to the main line, it is clear from (40) that $t = 1$ is a root of multiplicity at least 2 if and only if

$$Q(\alpha) + Q(\beta) = 1. \tag{43}$$

Assuming this we have

$$G(t\alpha, t\beta) = (1-t)^2\big(2t^2(3 - |\alpha|^2 - |\beta|^2) + (3t+1)(1-t)\big).$$

We obtain that $t = 1$ is a root of multiplicity at least 3 if and only if

$$|\alpha|^2 + |\beta|^2 = 3. \tag{44}$$

Note finally that in this case the multiplicity must be equal to 3 as the last root is $-1/3$.

For the last statement of Theorem E, assume that the multiplicity is equal to 3. Substituting (44) to (43) via (8), we obtain $\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 = 1$. Using this and (44) again to evaluate $R(2\alpha)$ and $R(2\beta)$ in (23), we obtain

$$G(\alpha, \beta) = 3 - T(\alpha, \alpha) - T(\beta, \beta) + 2T(\alpha, \beta) = 0.$$

Now, a simple computation gives

$$3 - T(\alpha, \alpha) - T(\beta, \beta) + 2T(\alpha, \beta) = 3 + x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_2x_3 + x_3x_1) = 0, \tag{45}$$

where

$$x_i = \alpha_i^2 - \beta_i^2 \in [-1, 1], \quad i = 1, 2, 3. \tag{46}$$

The crux is that the only points $(x_1, x_2, x_3) \in [-1, 1]^3$ satisfying (45) and (46) are $\pm(1, 1, 1)$. (Minimize the distance of (x_1, x_2, x_3) from the origin subject to (45).)

Thus, we have $\alpha_1^2 - \beta_1^2 = \alpha_2^2 - \beta_2^2 = \alpha_3^2 - \beta_3^2 = \pm 1$, $i = 1, 2, 3$. Summing up we obtain $|\alpha|^2 - |\beta|^2 = \pm 3$. This combined with (44) gives $|\alpha|^2 = 3$ (vertex of Θ_0) and $\beta = 0$, or $|\beta|^2 = 3$ (vertex of Θ_0) and $\alpha = 0$. The last statement of Theorem E follows.

3.6 Fixed points of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathfrak{D}_4$

Recall from Sect. 2.2 the linear subspace of all diagonal elements

$$\mathfrak{D}_4 = \mathfrak{D}_4^{-,-} \oplus \mathfrak{D}_4^{+,+} \oplus \mathfrak{D}_4^{-,+} \oplus \mathfrak{D}_4^{+,-} \subset \mathfrak{E}_4.$$

Each non-trivial element of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(I, I), (I, \gamma), (\gamma, I), (\gamma, \gamma)\} \subset O(4) \times O(4)$ acts on the set of components $\mathfrak{D}_4^{\pm,\pm}$ as a double transposition. As in Sect. 2.2, we have the bouquet of 6-dimensional linear subspaces

$$\mathfrak{D}_4^{(I,\gamma)},\ \mathfrak{D}_4^{(\gamma,I)},\ \mathfrak{D}_4^{(\gamma,\gamma)}$$

with common intersection $\mathfrak{D}_4^{\mathbb{Z}_2\times\mathbb{Z}_2}$.

The parametrization of $\mathfrak{D}_4^{(\gamma,\gamma)}$ by $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ in (26) is given by

$$\mathfrak{D}_4^{(\gamma,\gamma)} = \left\{ C^{-,-}(\alpha/2) + C^{+,+}(\alpha/2) + C^{-,+}(\mu/2) + C^{+,-}(\mu/2) \,|\, \alpha, \mu \in \mathbb{R}^3 \right\}.$$

Using (20)–(22), we obtain

$$\begin{aligned} &\det\left(C^{-,-}(\alpha/2) + C^{+,+}(\alpha/2) + C^{-,+}(\mu/2) + C^{+,-}(\mu/2) + I\right) \\ &\quad = (R(\alpha) + R(\mu) + 2T(\alpha;\mu) - 1) \prod_{i=1}^{3} \left(1 - (\alpha_i + \mu_i)^2\right) \left(1 - (\alpha_i - \mu_i)^2\right) \geq 0. \end{aligned}$$

Expanding the first factor, this gives (27)–(28).

For $\mu = 0$, $C^{-,-}(\alpha/2) + C^{+,+}(\alpha/2)$, $\alpha \in \Delta_0$, parametrize the arithmetic mean $(1/2)((I, I) + (\gamma, \gamma))\Delta^{-,-}$. In a similar vein, for $\alpha = 0$, $C^{-,+}(\mu/2) + C^{+,-}(\mu/2)$, $\mu \in \Delta_0$, parametrize $(1/2)((I, \gamma) + (\gamma, I))\Delta^{-,-}$.

The common intersection $\mathfrak{D}_4^{\mathbb{Z}_2\times\mathbb{Z}_2} \subset \mathfrak{D}_4^{(\gamma,\gamma)}$ is obtained by setting $\alpha = \mu$. With an additional scaling factor $1/2$, we have

$$\begin{aligned} &\det\left(C^{-,-}(\alpha/4) + C^{+,+}(\alpha/4) + C^{-,+}(\alpha/4) + C^{+,-}(\alpha/4) + I\right) \\ &\quad = Q(\alpha) \prod_{i=1}^{3} (1 - \alpha_i^2) \geq 0. \end{aligned}$$

We obtain that (up to the $1/2$ scaling) $\mathfrak{M}_4 \cap \mathfrak{D}_4^{\mathbb{Z}_2\times\mathbb{Z}_2}$ is Cayley's tetrahedron $\Theta_0 \subset [-1, 1]^3$.

In a similar vein, setting $\alpha = -\mu$, with an additional scaling factor $1/2$ we have

$$\begin{aligned} &\det\left(C^{-,-}(\alpha/4) + C^{+,+}(\alpha/4) + C^{-,+}(-\alpha/4) + C^{+,-}(-\alpha/4) + I\right) \\ &\quad = (1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2) \prod_{i=1}^{3} (1 - \alpha_i^2) \geq 0. \end{aligned}$$

We obtain that the anti-diagonal $\mathbb{R}^3_{\Delta'} \subset \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ defined by $\alpha = -\mu$ cuts out from Γ_0 the unit ball $\mathcal{B}_0$ scaled by $1/2$:

$$\Gamma_0 \cap \mathbb{R}^3_{\Delta'} = (1/2)\mathcal{B}_0.$$

Summarizing, we obtain

$$(1/2)[\mathcal{B}_0, \Theta_0] \subsetneqq \Gamma_0$$

with proper inclusion. Finally, a simple evaluation of the ranks of the respective positive semi-definite matrices on the boundary of $(1/2)\mathcal{B}_0$ gives the constant 15. The corollary follows.

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