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ORIGINAL PAPER

Minimal simplices inscribed in a convex body

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Abstract Measuring how far a convex body $\mathcal{K}$ (of dimension n) with a base point $O \in \operatorname{int}\mathcal{K}$ is from an inscribed simplex $\Delta \ni O$ in "minimal" position, the interior point O can display regular or singular behavior. If O is a regular point then the $n+1$ chords emanating from the vertices of Δ and meeting at O are affine diameters, chords ending in pairs of parallel hyperplanes supporting $\mathcal{K}$. At a singular point O the minimal simplex Δ degenerates. In general, singular points tend to cluster near the boundary of $\mathcal{K}$. As connection to a number of difficult and unsolved problems about affine diameters shows, regular points are elusive, often non-existent. The first result of this paper uses Klee's fundamental inequality for the critical ratio and the dimension of the critical set to obtain a general existence for regular points in a convex body with large distortion (Theorem A). This, in various specific settings, gives information about the structure of the set of regular and singular points (Theorem B). At the other extreme when regular points are in abundance, a detailed study of examples leads to the conjecture that the simplices are the only convex bodies with no singular points. The second and main result of this paper is to prove this conjecture in two different settings, when (1) $\mathcal{K}$ has a flat point on its boundary, or (2) $\mathcal{K}$ has n isolated extremal points (Theorem C).

Keywords Convex body · Critical ratio · Critical set · Distortion · Simplex

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1 Preliminaries and statement of results

Let $\mathbb{E}^n$ be an n-dimensional Euclidean vector space ($n \geq 2$) with distance function d, and $\mathcal{K} \subset \mathbb{E}^n$ a *convex body*, a *compact* convex set in $\mathbb{E}^n$ with *non-empty interior*. (It is well known that all convex bodies in $\mathbb{E}^n$ have dimension n [5,19].) Given a point $O \in \operatorname{int}\mathcal{K}$ and

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a hyperplane $\mathcal{H} \subset \mathbb{E}^n$ passing through O, there are exactly two hyperplanes $\mathcal{H}'$ and $\mathcal{H}''$ parallel to $\mathcal{H}$ and both *supporting* $\mathcal{K}$. Letting $M(\mathcal{H}, O) \leq 1$ denote the ratio that $\mathcal{H}$ divides the distance between $\mathcal{H}'$ and $\mathcal{H}''$, and taking the infimum over all $\mathcal{H}$, we arrive at the function $\mu : \operatorname{int}\mathcal{K} \to \mathbb{R}$, $\mu(O) = \inf_{\mathcal{H}\ni O} M(\mathcal{H}, O)$. The *Minkowski measure of symmetry* is then defined as $\mu^* = \sup_{O\,\in\,\operatorname{int}\mathcal{K}} \mu(O)$. (For the general theory of measures of symmetry, see [4].) By definition, $\mu \leq 1$ so that $\mu^* \leq 1$, and the upper bound $\mu^* = 1$ is attained (at an interior point O^* of $\mathcal{K}$) if and only if $\mathcal{K}$ is symmetric (with respect to O^*).

In 1897 H. Minkowski proved that $\mu(g) \geq 1/n$, where g is the *centroid* of $\mathcal{K}$. (See [13,15] or [14], and [3].) Minkowski's proof was in dimensions $n = 2, 3$, and was subsequently extended by J. Radon in 1916 to any dimensions $n \geq 2$ [17]. In particular, we have $\mu^* \geq 1/n$. (Note that a simple application of Helly's theorem actually gives this latter inequality directly [1, 11.7.6].) A straightforward exercise then shows that $\mu^* = 1/n$ holds for simplices, and a more delicate argument gives the converse: $\mu^* = 1/n$ if and only if $\mathcal{K}$ is an n-simplex. (See also the discussion below).

Instead of enclosing $\mathcal{K}$ between two parallel supporting hyperplanes, one can also consider *chords* of $\mathcal{K}$ passing through $O \in \operatorname{int}\mathcal{K}$. For $C \in \partial\mathcal{K}$, let $\Lambda(C, O)$ denote the ratio into which O divides the chord of $\mathcal{K}$ starting at C passing through O and ending up at the *opposite* $C^o \in \partial\mathcal{K}$ of C (with respect to O).

This defines the *distortion function* $\Lambda : \partial\mathcal{K} \times \operatorname{int}\mathcal{K} \to \mathbb{R}$:

$$\Lambda(C, O) = \frac{d(C, O)}{d(C^o, O)}, \quad C \in \partial\mathcal{K}, \ \ O \in \operatorname{int}\mathcal{K}.$$

Clearly, $(C^o)^o = C$ and $\Lambda(C^o, O) = 1/\Lambda(C, O)$, $C \in \partial\mathcal{K}$. We let

$$\ell(O) = \sup_{C\in\partial\mathcal{K}} \Lambda(C, O), \quad O \in \operatorname{int}\mathcal{K},$$

noting that the supremum is attained by continuity of the distortion function on the compact boundary of $\mathcal{K}$ (Lemma 1 in [23]). Finally, we let

$$\ell^* = \inf_{O\in\operatorname{int}\mathcal{K}} \ell(O)$$

noting again that the infimum is attained since $\ell(O) \to \infty$ as O tends to any boundary point of $\mathcal{K}$.

A simple geometric argument gives $\mu = 1/\ell$, in particular, $\ell^* = 1/\mu^*$.

The functions ℓ (and μ) are *quasi-convex* on $\operatorname{int}\mathcal{K}$; in fact, the level-sets are the members of Hammer's exhaustion of $\mathcal{K}$ into a monotonic family $\{\mathcal{K}_t\}_{\ell^*\leq t\leq 1}$ ($\mathcal{K}_1 = \mathcal{K}$) of convex subsets [7]. For $\ell^* < t \leq 1$, $\mathcal{K}_t$ is a convex body, and the *critical set* $\mathcal{K}^* = \mathcal{K}_{\ell^*} = \{O^* \in \operatorname{int}\mathcal{K} \mid \ell(O^*) = \ell^*\}$ is a compact convex set.

Remark Klee [10] used the ratio $\rho(C, O) = d(C, O)/d(C, C^o)$, related to our distortion via $1/\rho = 1 + 1/\Lambda$. The corresponding extremal value r^* with $1/r^* = 1 + 1/\ell^*$ is called the *critical ratio* of $\mathcal{K}$. Neumann [16] and some of the subsequent authors used $\max(d(C, O), d(C^o, O))$ instead of $d(C, O)$ in the definition of ρ. Since our main interest is in the maxima on $\partial\mathcal{K}$, the two definitions give the same results. The principal reason to use the distortion function is that our formulas become simpler and more transparent. Note also that, instead of ℓ we will actually use the ratio $1/(1+\ell)$. Although the latter is just a rescaling of Minkowski's μ, it has a slight advantage being *concave* on $\operatorname{int}\mathcal{K}$ (Corollary to Proposition 1 in [23]), and thereby automatically quasi-convex. (Note that concavity of $1/(1+\ell)$ implies that ℓ is convex. Finally, apart from being quasi-convex, μ is *not* convex/concave even for $\mathcal{K}$ a metric ball.)

By the above, we have

$$1 \leq \ell^* \leq n, \tag{1}$$

with the lower bound attained if and only if $\mathcal{K}$ is (centrally) symmetric (with respect to then unique O^* in $\mathcal{K}^*$), and the upper bound is attained if and only if $\mathcal{K}$ is a simplex.

Remark Following the previous remark, the upper estimate was first proved by Neumann [16] for $n = 2$, and by Süss [21] and Hammer [6] for general $n \geq 2$. Characterization of the simplex with $\ell^* = n$ was stated by Süss [21] and proved by Klee [10]. The statement that $\ell^* = 1$ characterizes the symmetric convex bodies is clear.

As noted above, the critical set $\mathcal{K}^*$ is a closed *convex* set [7]. (For a simple and direct proof of this, see [20, 5.2].) In the planar case ($n = 2$) $\mathcal{K}^*$ reduces to a single point [16], and the fact that for $n \geq 3$ it may be non-degenerate has first been recognized by Hammer and Sobczyk [8]. (A simple example is furnished by a vertical cylinder on an equilateral triangle [8].)

A significant improvement of the non-trivial upper estimate in (1) is Klee's [10] fundamental inequality:

$$\ell^* + \dim \mathcal{K}^* \leq n. \tag{2}$$

In his paper Klee actually proved much more (also needed here later). Letting $\mathcal{M}(O) = \{C \in \partial\mathcal{K} \mid \Lambda(C, O) = \ell(O)\}$, $O \in \operatorname{int}\mathcal{K}$, the sets $\mathcal{M}(O^*)$, with O^* traversing the *relative interior* of $\mathcal{K}^*$, stay the same, say, $\mathcal{M}^*$. Moreover, if $\ell^* + \dim \mathcal{K}^* \geq n - 1$ then $\mathcal{M}^*$ consists of at least $\lceil \ell^* \rceil + 1$ elements ($\lceil x \rceil$ is the *ceiling* of $x \in \mathbb{R}$), and we have

$$\operatorname{int}\mathcal{K}^* \subset \operatorname{int}[\mathcal{K}^* \cup \mathcal{M}^*], \tag{3}$$

where the square brackets mean *convex hull*.

In the study moduli spaces of spherical minimal immersions, in [22,23] the author introduced a sequence of measures of symmetry $\{\sigma_m\}_{m\geq 1}$ associated to a convex body $\mathcal{K} \subset \mathbb{E}^n$ (of dimension n) with a specified interior point $O \in \operatorname{int}\mathcal{K}$. The mth measure of symmetry σ_m is defined as follows. First, an *m-configuration of* $\mathcal{K}$ with respect to O is a *multi-set* $\{C_0, \ldots, C_m\} \subset \partial\mathcal{K}$ (with repetition allowed) such that the convex hull $[C_0, \ldots, C_m]$ contains O. Then

$$\sigma_m(\mathcal{K}, O) = \inf_{\{C_0,\ldots,C_m\}\in\mathfrak{C}_m(\mathcal{K},O)} \sum_{i=0}^{m} \frac{1}{1 + \Lambda(C_i, O)}, \tag{4}$$

where $\mathfrak{C}_m(\mathcal{K}, O)$ denotes the set of all m-configurations of $\mathcal{K}$ (with respect to O).

Algebraically, σ_m is an 'm-average' of the rescaled distortion giving the Minkowski function μ above, and, as we will see below, geometrically $\sigma_m(\mathcal{K}, O)$ measures how far the m-dimensional slices of $\mathcal{K}$ across O are from an m-simplex.

Compactness of $\mathcal{K}$ implies that the infimum in (4) is attained, but, as examples show, minimal m-configurations are by no means unique. Since a 1-configuration of $\mathcal{K}$ is an antipodal pair of points, we have $\sigma_1(\mathcal{K}, O) = 1$.

The functions $\sigma_m(\mathcal{K}, .)$ are continuous on $\operatorname{int}\mathcal{K}$ and extend continuously to $\partial\mathcal{K}$ (Theorem D/(b) in [23]):

$$\lim_{d(O,\partial\mathcal{K})\to 0} \sigma_m(\mathcal{K}, O) = 1. \tag{5}$$

For the range of σ_m, we have

$$1 \leq \sigma_m(\mathcal{K}, O) \leq \frac{m+1}{2}. \tag{6}$$

The lower bound is attained if and only if $\mathcal{K}$ has an m-dimensional simplicial slice across O, that is, there exists an m-dimensional affine subspace $\mathcal{E} \subset \mathbb{E}^n$ containing O such that $\mathcal{K} \cap \mathcal{E}$ is an m-simplex. Assuming $m \geq 2$, the upper bound is attained if and only if $\mathcal{K}$ is (centrally) symmetric at O. (For the details here and below, see [22,23]).

The sequence $\{\sigma_m\}_{m \geq 1}$ is *sub-arithmetic*, that is, for $k, m \geq 1$, we (obviously) have

$$\sigma_{m+k}(\mathcal{K}, O) \leq \sigma_m(\mathcal{K}, O) + \frac{k}{1 + \ell(O)}. \tag{7}$$

Equality holds for $m = n$ and $k \geq 1$, that is, the sequence $\{\sigma_m(\mathcal{K}, O)\}_{m \geq 1}$ is *arithmetic* from the nth term onwards with difference $1/(1+\ell(O))$. (This is a consequence of Carathéodory's theorem; see [19] or [1, 11.1.8.6]; for a recent proof and generalizations of the latter, see [2].)

The sequence $\{\sigma_m\}_{m \geq 1}$ is *super-additive*

$$\sigma_{m+k} - \sigma_m \geq \sigma_k - \sigma_1, \quad \sigma_1 = 1, \quad k, m \geq 1.$$

In particular, $\{\sigma_m(\mathcal{K}, O)\}_{m \geq 1}$ starts with an initial string of 1's, and afterwards it is *increasing*. The *length* l of this string (the dimension of a maximal simplicial slice of $\mathcal{K}$ across O) is $\leq [\ell(O)]$ ($[x]$ is the greatest integer of $x \in \mathbb{R}$). (This is an easy consequence of (7) (for $m = l, k = n - l$), and using the trivial lower bound $(n+1)/(1+\ell)$ for σ_n).

An n-configuration $\{C_0, \ldots, C_n\} \in \mathfrak{C}(\mathcal{K}, O)$ is called *simplicial* if $[C_0, \ldots, C_n]$ is an n-simplex and O is contained in its interior. (In what follows, whenever convenient, the subscript n will be suppressed.) The set of simplicial configurations is denoted by $\Delta(\mathcal{K}, O) \subset \mathfrak{C}(\mathcal{K}, O)$. In the definition (4) of $\sigma(\mathcal{K}, O)$ (for $m = n$) the infimum can be restricted to $\Delta(\mathcal{K}, O)$, but a minimizing sequence of simplicial configurations may not subconverge in $\Delta(\mathcal{K}, O)$. If *every* minimizing sequence subconverges within $\Delta(\mathcal{K}, O)$ then we call the point O *regular*. By (7) (for $m = n - 1$ and $k = 1$), O is regular if and only if

$$\sigma(\mathcal{K}, O) < \sigma_{n-1}(\mathcal{K}, O) + \frac{1}{1 + \ell(O)}. \tag{8}$$

By continuity of the measures of symmetry and ℓ, the set $\mathcal{R} \subset \operatorname{int} \mathcal{K}$ of regular points, the regular set, is *open*. An interior point O at which a minimizing sequence *degenerates* is called *singular*, and the set of singular points is denoted by $\mathcal{S}$. By the above, O is singular if and only if equality holds in (8), and the singular set $\mathcal{S}$ is relatively closed in $\operatorname{int} \mathcal{K}$.

Let $O \in \mathcal{R}$ be a regular point. Since a *minimal* configuration $\{C_0, \ldots, C_n\}$ is simplicial, $\Lambda(., O)$ attains *local maximum* at each C_i, $0 \leq i \leq n$. In addition (Lemma 2.1 in [22]), using suitable translations along line segments in the boundary of $\mathcal{K}$, the configuration points can be replaced by *extremal points* (in the sense of convex geometry [19]). (Recall that a boundary point of $\mathcal{K}$ is extremal if it is not contained in the interior of a line segment on the boundary of $\mathcal{K}$.) Moreover, from the study of local extrema of $\Lambda(., O)$ (Section 7 in [22], and also (3.2) in [10]), it also follows that each chord $[C_i, C_i^o]$, $0 \leq i \leq n$, is an *affine diameter* in the sense that there are parallel hyperplanes passing through C_i and C_i^o both supporting $\mathcal{K}$. Summarizing, we see that at each regular point $n + 1$ (affinely independent) affine diameters meet. Regularity aside, to characterize the set of these points in a convex body is a difficult and unsolved problem; in particular, it is not known whether the centroid of a convex body has this property or not [4,11,12]. (For a thorough overview of results on affine diameters, see the survey article [20].)

As can be expected, regular points are elusive. Our first result is the following:

Theorem A *Let $\mathcal{K} \subset \mathbb{E}^n$ be a convex body. Assume that $\ell^* > n - 1$. Then the critical set $\mathcal{K}^*$ consists of a single regular point $O^* \in \mathcal{R}$ and*

$$\sigma(\mathcal{K}, O^*) = \frac{n+1}{1+\ell^*}. \tag{9}$$

By (1), no assumption is needed for the planar case ($n = 2$):

Corollary *Let $\mathcal{K}$ be a planar convex body. Then the critical set $\mathcal{K}^*$ is a singleton O^* and we have*

$$\sigma(\mathcal{K}, O^*) = \frac{3}{1+\ell^*}.$$

Remark Let G be a compact (not necessarily connected) Lie group acting on $\mathbb{E}^n$ linearly with no nonzero fixed points, and assume that $\mathcal{K}$ is G-invariant. Then the critical set $\mathcal{K}^*$ is also G-invariant. The *centroid* of $\mathcal{K}^*$ is left fixed by G, therefore, by assumption, it must be the origin. It follows that $0 \in \mathcal{K}^*$ and (9) (obviously) holds with O^* replaced by 0. (A minimal configuration can be selected from the G-invariant set $\mathcal{M}^* \subset \partial\mathcal{K}$).

The examples of a double regular tetrahedron or a vertical cylinder on an equilateral triangle show that the lower bound in Theorem A is sharp in the sense that there are convex bodies with $\ell^* = n - 1$ and *no regular points*. A closer inspection of the proof of Theorem A (in Section 2) shows that if, for a convex body $\mathcal{K}$, we have $\ell^* = n - 1$ *and* the critical set $\mathcal{K}^*$ consists of a single point then this critical point O^* is singular if and only if $\mathcal{K}$ has an $(n-1)$-dimensional simplicial slice across O^*.

This motivates the following definition. Let $\mathcal{K} \subset \mathbb{E}^n$ be a convex body and $1 \le k \le n$. We say that $\mathcal{K}$ is *simplicial in codimension k* if $\mathcal{K}$ possesses an $(n-k)$-dimensional simplicial slice across *any* $O \in \operatorname{int}\mathcal{K}$. By the above, $\mathcal{K}$ is simplicial in codimension k if and only if $\sigma_{n-k} = 1$ *identically* on $\operatorname{int}\mathcal{K}$. Clearly, a convex body $\mathcal{K}$ is automatically simplicial in codimension $n-1$, and $\mathcal{K}$ is simplicial of codimension 0 if and only if it is a simplex. (As a simple example, the n-dimensional cube $\mathcal{C}_n$ is not simplicial in codimension 1, although $\sigma_{n-1} = 1$ away from the inscribed cross-polytope.)

Let $\mathcal{K}$ be simplicial in codimension k. Since $\sigma_{n-k} = 1$, by (7), we have

$$\frac{n+1}{1+\ell^*} \le \sigma(\mathcal{K}, O^*) \le 1 + \frac{k}{1+\ell^*}, \quad O^* \in \mathcal{K}^*. \tag{10}$$

In particular, $\ell^* \ge n - k$ and, by Klee's inequality (2), $\dim \mathcal{K}^* \le k$.

Theorem B *Let $\mathcal{K} \subset \mathbb{E}^n$ be a codimension 1 simplicial convex body. Then $\ell^* \ge n-1$ and equality holds if and only if $\mathcal{K}^* \subset \mathcal{S}$. The measure of symmetry $\sigma(\mathcal{K}, .)$ as a function on the interior of $\mathcal{K}$ is concave. Given $O \in \operatorname{int}\mathcal{K}$, for any $C \in \mathcal{M}(O)$, the line segment $[O, C^o)$ intersects $\mathcal{R}$ and $\mathcal{S}$ in intervals (one of which may be empty). In particular, $\mathcal{S} \cup \partial\mathcal{K}$ is path-connected and $\mathcal{R}$ is (topologically) $(n-1)$-connected.*

Remark The examples below and Example 2 in Section 4 illustrate Theorem B. Note also that for the n-dimensional cube $\mathcal{C}_n$, $n \ge 3$, $\sigma(\mathcal{C}_n, .)$ is *not* concave [25].

For $n = 2$ the situation is much simpler. We summarize our findings as follows.

Corollary *Let $\mathcal{K}$ be a planar convex body ($n = 2$). Then $1 \le \ell^* \le 2$. If $\ell^* = 1$ then $\mathcal{K}$ is symmetric with all interior points singular. If $1 < \ell^* < 2$ then the regular set $\mathcal{R}$ is non-empty and simply connected, and $\mathcal{S} \cup \partial\mathcal{K}$ is connected. If $\ell^* = 2$ then $\mathcal{K}$ is a triangle with all interior points regular.*

Example 1 The unit half-disk $\mathcal{K} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ y \geq 0\}$ is a simple but important example. As noted by Hammer [6], $\ell^* = \sqrt{2}$ with the (unique) critical point at $O^* = (0, \sqrt{2} - 1)$, and with the *centroid* of $\mathcal{K}$ at $(0, 4/3\pi)$, *different* from O^*. By the corollary above, O^* is a regular point. Moreover, letting $C_\pm = (\pm 1, 0)$ and $C_0 = (0, 1)$, in Example 1 of Section 4 we will show that the regular set $\mathcal{R}$ is the interior of the triangle $\Delta = [C_-, C_0, C_+]$. Using this we will determine $\sigma(\mathcal{K}, .)$ explicitly. In particular, we will see that the *maximum* of $\sigma(\mathcal{K}, .)$ on $\operatorname{int}\mathcal{K}$ is attained at yet *another* point, $(0, 1/2)$.

Example 2 An illustrative example for the case $1 < \ell^* < 2$ is a $(2m + 1)$-sided regular polygon $\mathcal{P}_{2m+1} \subset \mathbb{R}^2, m \geq 2$. (For definiteness, we may assume that $\mathcal{P}_{2m+1}$ is inscribed in the unit circle of $\mathbb{R}^2$.) Clearly, the critical set $\mathcal{P}^*_{2m+1}$ consists of a single point O^*, the centroid of $\mathcal{K}$, and consequently $\ell^* = \sec(\frac{\pi}{2m+1})$. (The respective dihedral group G leaves only the origin fixed; see the remark after Theorem A.) As shown in [26], the regular set is the interior of the *star-polygon* $\left\{\frac{2m+1}{m}\right\}$ of $\mathcal{P}_{2m+1}$. Note that the distance of O^* to the singular set is $\left(1 - 2\cos\left(\frac{2m\pi}{2m+1}\right)\right)^{-1}$ which decreases to $1/3$ as $m \to \infty$. It follows that the open disk with center O^* and radius $1/3$ is contained in the regular set for all $m \geq 2$. (Note that the limit of $\mathcal{P}_{2m+1}$ as $m \to \infty$ is the disk all of whose interior points are singular.)

Based on this, one may expect a positive lower bound for $d(O^*, \mathcal{S})$ for certain classes of convex bodies with $O^* \in \mathcal{R}$. In Example 2 of Section 4, we show however that, for 3-dimensional cones $\mathcal{K}_m, m \geq 1$, with base $\mathcal{P}_{2m+1}$ and sharing the same vertex V, the critical set $\mathcal{K}^*_m$ consists of a single regular point O^* and $\lim_{m\to\infty} d(O^*, \mathcal{S}_m) = 0$, where $\mathcal{S}_m$ is the singular set of $\mathcal{K}_m$.

Simplices do not have singular points, in fact, for any interior point of a simplex, the unique minimal configuration is comprised of the vertices of the simplex. In [24] we *conjectured* that the converse was also true, that is, a convex body all of whose interior points are regular is a simplex. Our second and main result resolves this conjecture under two *different* assumptions:

Theorem C *Let $\mathcal{K} \subset \mathbb{E}^n$ be a convex body with all its interior points regular. Assume that one of the following conditions hold:*

(I) *There is a flat point on $\partial\mathcal{K}$, that is a point $C \in \partial\mathcal{K}$ with a hyperplane $\mathcal{H} \subset \mathbb{E}^n$ supporting $\mathcal{K}$ such that C is contained in the (non-empty) interior of $\partial\mathcal{K} \cap \mathcal{H}$ in $\mathcal{H}$.*
(II) *Assume that $\mathcal{K}$ has (at least) n isolated extremal points on its boundary. Then $\mathcal{K}$ is an n-simplex.*

Theorem C can be paraphrased as an existence result for singular points for non-simplicial convex bodies. In particular, we have the following:

Corollary *Let $\mathcal{K} \subset \mathbb{E}^n$ be a convex polytope of dimension n which is not a simplex. Then the singular set of $\mathcal{K}$ is non-empty.*

The proof of Theorem C is long and technical and will be given in Sect. 3. The assumptions (I) and (II) are very different and so are the respective proofs. Accordingly, we will split the proof of Theorem C into two parts. Note that Part II has been announced and discussed in [25] with an inductive proof outlined. We will give here a complete, simplified, and different proof of the general induction step.

Remark In the other extreme, the interior of any *symmetric* convex body consists of singular points only (Theorem A in [26]). It is natural to ask about the converse, that is, whether a convex body all of whose interior points are singular is symmetric. The examples noted above (a double regular tetrahedron or a vertical cylinder on an equilateral triangle) immediately show that this is false for $n \geq 3$. Surprisingly, as a byproduct of Corollary to Theorem B above shows, this is true for $n = 2$: A planar convex body is symmetric if and only if all of its interior points are singular.

We close this section with the following illustration of Part II of Theorem C:

Example 3 Let $\Delta = [C_1, \ldots, C_{n-1}] \subset \mathbb{R}^{n-2}$, $n \geq 3$, be an $(n-2)$-simplex with vertices $C_1, \ldots, C_{n-1}$, and assume that the origin 0 is an interior point of Δ. Let $\Gamma \subset \mathbb{R}^2$ be the circle with center $(0, 1)$ and radius 1. We consider Δ and Γ imbedded in $\mathbb{R}^n$ via $\mathbb{R}^{n-2} \times \mathbb{R}^2 = \mathbb{R}^n$. The convex hull $\mathcal{K} = [\Delta, \Gamma] \subset \mathbb{R}^n$ is a codimension 1 simplicial convex body in $\mathbb{R}^n$. In fact, for any $C \in \Gamma \setminus \{0\}$, the hyperplane containing $\mathbb{R}^{n-2}$ and C intersects $\mathcal{K}$ in the $(n-1)$-simplex $[\Delta, C]$. The set of extremal points $\mathcal{K}^0$ splits into 'discrete' and 'continuous' parts: the *isolated* extremal points $C_1, \ldots, C_{n-1}$ and the subset $\Gamma \setminus \{0\}$. (In particular, $\mathcal{K}^0$ is not closed.) By Theorem C/(II), the singular set $\mathcal{S}$ of $\mathcal{K}$ is non-empty. Finally, note that in the proof of Theorem C/(II) of Sect. 3, $\mathcal{K}$ satisfies condition $\mathcal{P}_{n-1}$ (but not $\mathcal{P}_n$ as there are not enough isolated extremal points).

2 Theorems A–B: existence of regular points

To prove Theorem A we let $\ell^* > n-1$. By (2), we have $\dim \mathcal{K}^* = 0$. Since $\mathcal{K}^*$ is convex, it must consist of a single point, say, O^*. The inclusion in (3) then reduces to $O^* \in [\mathcal{M}^*]$, where $\mathcal{M}^* = \mathcal{M}(O^*)$. Hence there exists $\{C_0, \ldots, C_n\} \subset \mathcal{M}^*$ such that $O^* \in [C_0, \ldots, C_n]$. (Since $\mathcal{M}^*$ consists of at least $\lceil \ell^* \rceil + 1 > n$ points, $C_0, \ldots, C_n$ can be chosen mutually distinct. The fact that no more than $n+1$ of them are needed follows from Carathéodory's theorem [19].) With this, we arrive at an n-configuration $\{C_0, \ldots, C_n\} \in \mathfrak{C}(\mathcal{K}, O^*)$. By definition, we then have

$$\sigma(\mathcal{K}, O^*) \leq \frac{n+1}{1+\ell^*}.$$

On the other hand, since $\ell^* = \max_{\partial\mathcal{K}} \Lambda(., O^*)$, the opposite inequality obviously holds. Finally, to show that O^* is regular, we estimate

$$\sigma(\mathcal{K}, O^*) = \frac{n}{1+\ell^*} + \frac{1}{1+\ell^*} < 1 + \frac{1}{1+\ell^*} \leq \sigma_{n-1}(\mathcal{K}, O^*) + \frac{1}{1+\ell^*},$$

where we used (6). Theorem A follows.

If $\ell^* = n-1$ *and* $\mathcal{K}^* = \{O^*\}$, the argument above goes through (in the use of a minimal configuration $\{C_0, \ldots, C_n\} \subset \mathcal{M}^*$) and we obtain

$$\sigma(\mathcal{K}, O^*) = \frac{n+1}{1+\ell^*} \leq \sigma_{n-1}(\mathcal{K}, O^*) + \frac{1}{1+\ell^*}.$$

Equality holds ($O^* \in \mathcal{S}$) if and only if $\sigma_{n-1}(\mathcal{K}, O^*) = 1$, that is, if and only if $\mathcal{K}$ has a codimension 1 simplicial configuration across O^*.

To prove Theorem B we assume now that $\mathcal{K}$ is a codimension 1 simplicial convex body. The first statement of Theorem B follows directly from (10) with $k = 1$. For the second statement, concavity of $\sigma(\mathcal{K}, .)$, first note that $\sigma(\mathcal{K}, .)$ is always concave on the regular set $\mathcal{R}$ (Proposition 2 in [23]). Second, due to our assumption, on the singular set $\mathcal{S}$, $\sigma(\mathcal{K}, .)$ is equal to $1 + 1/(1+\ell)$ which, as a function on $\operatorname{int}\mathcal{K}$, is also concave (Corollary to Proposition

1 in [23] already cited). Now concavity of $\sigma(\mathcal{K}, .)$ follows by an elementary argument. (See Lemma 5 in [26].)

To prove the third statement of Theorem B, we first recall a consequence of the Comparison Lemma (Corollary 1 in [27]): Given $O \in \operatorname{int}\mathcal{K}$, if $C \in \mathcal{M}(O)$ then, for any $O' \in [O, C^o)$, we also have $C \in \mathcal{M}(O')$.

Let $C \in \mathcal{M}(O)$ as above, and parametrize the line segment $[O, C^o)$ as $\lambda \mapsto O_\lambda = (1-\lambda)O + \lambda C^o$, $0 \leq \lambda < 1$. Since $C \in \mathcal{M}(O_\lambda)$, we have

$$\ell(O_\lambda) = \Lambda(C, O_\lambda) = \frac{1}{1-\lambda}\Lambda(C, O) + \frac{\lambda}{1-\lambda} = \frac{1}{1-\lambda}\ell(O) + \frac{\lambda}{1-\lambda}.$$

We now observe that in the inequality

$$\sigma(\mathcal{K}, O_\lambda) \leq 1 + \frac{1}{1+\ell(O_\lambda)} = 1 + \frac{1-\lambda}{1+\ell(O)}, \tag{11}$$

the left-hand side is a concave function in $0 \leq \lambda < 1$, and the right-hand side is linear. By definition, equality holds for $\lambda \in [0, 1)$ if and only if O_λ is a singular point. Finally, by (5), for $\lambda \to 1$ both sides converge to 1. It follows that if equality holds in (11) for a *particular* $\lambda_0 \in [0, 1)$ then equality holds for *all* $\lambda \in [\lambda_0, 1)$. The third statement of Theorem B now follows. Finally, the last statement is an immediate topological consequence of the third.

3 Theorem C: a characterization of the simplex

In this section we prove Theorem C. We let $\mathcal{K} \subset \mathbb{E}^n$ be a convex body.

Part I. We assume that $\mathcal{K}$ has a flat point on its boundary: $O_0 \in \partial\mathcal{K}$. To prove Theorem C we first study the existence of regular points near O_0.

Lemma 1 *Let $\mathcal{K}$ be a convex body in $\mathbb{E}^n$ with a flat point O_0 on its boundary, and let $\{O_k\}_{k\geq 1} \subset \mathcal{R}$ such that* $\lim_{k\to\infty} O_k = O_0$. *Denote by $\mathcal{H}_0$ the unique hyperplane supporting $\mathcal{K}$ at O_0. Then $\mathcal{K}_0 = \mathcal{H}_0 \cap \partial\mathcal{K}$ is an $(n-1)$-simplex. In addition, if $\mathcal{K}_0 = [C_1, \ldots, C_n]$ then, for each $1 \leq i \leq n$, there exist parallel hyperplanes $\mathcal{H}_i$ and $\mathcal{H}_i'$ supporting $\mathcal{K}$ such that $[C_1, \ldots, \widehat{C_i}, \ldots, C_n] \subset \mathcal{H}_i$ and $C_i \in \mathcal{H}_i'$.*

Remark According to Example 2 of Sect. 1, the regular polygon $\mathcal{P}_{2m+1}$, $m \geq 2$, has no sequence of regular points converging to a flat boundary point and, clearly, there are no parallel supporting lines at the endpoints of any side.

As another example, the diagonals of a proper *trapezoid* $\mathcal{K} \subset \mathbb{R}^2$ split the trapezoid into four triangles and the regular set is the interior of the triangle with one side being the *longer* parallel side. Hence, a flat point on this longer side can be approximated by regular points, while the flat points on the other sides cannot be. This is also confirmed by Lemma 1 above as parallel supporting hyperplanes exist at the endpoints of the longer parallel side but not at the endpoints of the shorter side.

Proof of Lemma 1 By assumption, each point O_k, $k \geq 1$, is regular so that there exists a minimal *simplicial* configuration $\{C_{0,k}, \ldots, C_{n,k}\} \in \mathfrak{C}(\mathcal{K}, O_k)$. As noted in Section 1, we may assume that the configuration points $C_{i,k}$, $0 \leq i \leq n$, $k \geq 1$, are all *extremal* points of $\mathcal{K}$. In addition, since $\partial\mathcal{K}$ is compact, selecting a subsequence (if necessary) we may also assume that, for each $0 \leq i \leq n$, we have $C_{i,k} \to C_i \in \partial\mathcal{K}$ as $k \to \infty$. We now look for the possible location of each limit point $C_i \in \partial\mathcal{K}$, $0 \leq i \leq n$. Clearly, C_i cannot be in the relative interior of $\mathcal{K}_0$ since all configuration points are extremal points.

Consider first the case when $C_i \notin \mathcal{K}_0$. Since O_0 is in the interior of $\mathcal{K}_0$, for k large, the opposite $C_{i,k}^{o_k}$ of $C_{i,k}$ with respect to O_k must be in the relative interior of $\mathcal{K}_0$. For any of these, $\mathcal{H}_0$ is the unique supporting hyperplane of $\mathcal{K}$ at $C_{i,k}^{o_k}$. Since O_k is a regular point, the chord $[C_{i,k}, C_{i,k}^{o_k}]$ must be an *affine diameter*. Then, by definition, there exists a hyperplane $\mathcal{H}_0'$ supporting $\mathcal{K}$ at $C_{i,k}$ and parallel to $\mathcal{H}_0$. Clearly, $\mathcal{H}_0'$ depends only on $\mathcal{H}_0$ and $\mathcal{K}$.

Next, we prove that there cannot be any additional point $C_j \notin \mathcal{K}_0, 0 \leq j \neq i \leq n$. Indeed, if $C_i, C_j \notin \mathcal{K}_0$ then, by what was said above, for k large, the points $C_{i,k}$ and $C_{j,k}$ are both in $\mathcal{H}_0'$. We then slide $C_{j,k}$ to $C_{i,k}$ along the line segment connecting them and get a contradiction to the regularity of O_k. More precisely, we consider the 1-parameter family of multi-sets $t \mapsto \{C_{0,k}, \ldots, C_{i,k}, \ldots, \widehat{C_{j,k}}, (1-t)C_{j,k} + tC_{i,k}, \ldots, C_{n,k}\}, 0 \leq t \leq 1$. Since $\mathcal{H}_0$ and $\mathcal{H}_0'$ are parallel, $\Lambda(., O_k)$ evaluated on this family does not depend on t. The configuration condition that O_k is in the respective convex hull is valid at $t = 0$. Let $0 < t_0 \leq 1$ be the last parameter for which the configuration condition holds. If $t_0 < 1$ then the configuration at t_0 is (minimal but) not simplicial (as this is an open condition), a contradiction to the regularity of O_k. If $t_0 = 1$ then the (once again minimal) configuration has the point $C_{i,k}$ listed twice, also a contradiction to regularity.

Thus, we obtain that there may be *at most one* C_i, $0 \leq i \leq n$, with $C_i \notin \mathcal{K}_0$. If there is one, renumbering if necessary, we may assume this to be C_0, and let $I = \{1, \ldots, n\}$; otherwise, we let $I = \{0, \ldots, n\}$.

We then have $C_i \in \partial\mathcal{K}_0, i \in I$, and, using continuity of the distortion function, we have $\Lambda_{\mathcal{K}}(C_{i,k}, O_k) \to \Lambda_{\mathcal{K}_0}(C_i, O_0)$ as $k \to \infty$, where we indicated the dependence on the respective convex body by subscripts. (In the exceptional case of C_0, we have $\Lambda(C_{0,k}, O_k) \to \infty$, as $k \to \infty$.)

By the choice of the minimizing configurations, we obtain

$$\sigma(\mathcal{K}, O_k) = \sum_{i=0}^{n} \frac{1}{1 + \Lambda_{\mathcal{K}}(C_{i,k}, O_k)} \to \sum_{i \in I} \frac{1}{1 + \Lambda_{\mathcal{K}_0}(C_i, O_0)} = 1, \quad \text{as} \quad k \to \infty, \tag{12}$$

where the last equality is because of (5). From the study of the possible exceptional point it is clear that $\{C_i\}_{i \in I}$ is a configuration for O_0 in $\mathcal{K}_0$. Since $\mathcal{K}_0$ is $(n-1)$-dimensional, the only way the last equality in (12) can hold is that $I = \{1, \ldots, n\}$ and $\sigma_{n-1}(\mathcal{K}_0, O_0) = 1$, so that $\mathcal{K}_0$ is an $(n-1)$-simplex whose vertices are $C_1, \ldots, C_n$. The first statement of Lemma 1 follows.

We now define $\mathcal{V}$ as the set of those boundary points $C \in \partial\mathcal{K} \setminus \mathcal{K}_0$ at which there is a supporting hyperplane parallel to $\mathcal{H}_0$. By the proof of the first part of Lemma 1, is clear that $C_0 \in \mathcal{V}$ and the supporting hyperplane for any point in $\mathcal{V}$ must be $\mathcal{H}_0'$. Therefore we have

$$\mathcal{V} = \mathcal{K} \cap \mathcal{H}_0' = \partial\mathcal{K} \cap \mathcal{H}_0'.$$

For the second part of the proof of Lemma 1 as well as for the future we need the following:

Lemma 2 *Given $O \in \operatorname{int}\mathcal{K}$, assume that $\Lambda(., O)$ attains a local maximum at $C \in \partial\mathcal{K}$. Then*

$$C^o \in \mathcal{K}_0 \Rightarrow C \in \mathcal{V}. \tag{13}$$

Proof If $C^o \in \mathcal{K}_0$ then $C \notin \mathcal{K}_0$. Let $\mathcal{H}'$ be the hyperplane passing through C and parallel to $\mathcal{H}_0$. Since $\mathcal{H}_0$ supports $\mathcal{K}$ at C^o and $\Lambda(., O)$ attains a *local maximum* at C, it follows that $\mathcal{H}'$ supports $\mathcal{K}$. (See also the proposition in Section 7 of [23].) Thus $\mathcal{H}' = \mathcal{H}_0'$ and $C \in \mathcal{V}$.

We now return to the proof of Lemma 1. Since $C_{0,k}^o$ is in the (relative) interior of $\mathcal{K}_0$, for k large, we have $C_{0,k} \in \mathcal{V}$. Since O_k is regular, $C_{0,k}$ can be any point in $\mathcal{V}$, in particular, we can choose $C_{0,k} = C_0$ constant.

Recall from the first part of the proof that, for $0 \leq i \leq n$, $C_{i,k} \to C_i \in \mathcal{K}_0$, as $k \to \infty$, and $\mathcal{K}_0 = [C_1, \ldots, C_n]$. Clearly, for k large, $C_{i,k} \notin \mathcal{V}$. Hence, by Lemma 2, $C_{i,k}^{o_k} \notin \mathcal{K}_0$. Since $[C_{i,k}, C_{i,k}^{o_k}]$ is an affine diameter there exist parallel hyperplanes $\mathcal{H}_{i,k} \ni C_{i,k}^o$ and $\mathcal{H}'_{i,k} \ni C_{i,k}$ supporting $\mathcal{K}$.

Denote by $\delta_{i,k}$ the *dihedral angle* of the angular sector given by the hyperplanes $\mathcal{H}_0$ and $\mathcal{H}_{i,k}$ containing $\mathcal{K}$. Define $\delta'_{i,k}$ similarly (with $\mathcal{H}'_{i,k}$ in place of $\mathcal{H}_{i,k}$). Clearly, $0 < \delta_{i,k}, \delta'_{i,k} < \pi$. In addition, since $\mathcal{H}_{i,k}$ and $\mathcal{H}'_{i,k}$ are parallel, we also have $\delta_{i,k} + \delta'_{i,k} = \pi$. Selecting subsequences, we may assume that $\delta_{i,k} \to \delta_i$ and $\delta'_{i,k} \to \delta'_i$ as $k \to \infty$. Taking the respective limits, we obtain $\delta_i + \delta'_i = \pi$. By convexity, we also have $0 < \delta_i, \delta'_i < \pi$. Let $\mathcal{H}_i$ be the hyperplane containing $[C_1, \ldots, \widehat{C_i}, \ldots, C_n]$ and having dihedral angle δ_i with $\mathcal{H}_0$. By construction, $\mathcal{H}_i$ is the limit of the supporting hyperplanes $\mathcal{H}_{i,k}$, and so it must also support $\mathcal{K}$. Denote by $\mathcal{H}'_i$ the hyperplane containing C_i and parallel to $\mathcal{H}_i$. Again by construction, $\mathcal{H}'_i$ supports $\mathcal{K}$ at C_i. The second statement of Lemma 1 follows. □

Remark Since $\mathcal{K}$ is between the parallel supporting hyperplanes $\mathcal{H}_i$ and $\mathcal{H}'_i$, a simple comparison of distortions shows that, for k large, $\Lambda(C_i, O_k) \geq \Lambda(C_{i,k}, O_k)$. Hence, for large k, $\{C_0, \ldots, C_n\} \in \mathfrak{C}(\mathcal{K}, O_k)$ (with $C_0 \in \mathcal{V}$ arbitrary) is a *minimizing* configuration.

From now on, we will assume that each $\mathcal{H}_i$, $1 \leq i \leq n$, is *closest to* $\mathcal{K}$ in the sense that there is no supporting hyperplane between $\mathcal{H}_i$ and $\partial\mathcal{K} \setminus \mathcal{K}_0$.

Lemma 3 *Any affine diameter of* $\mathcal{K}$ *disjoint from* $\mathcal{K}_0$ *has endpoints on a pair* $\mathcal{H}_i$ *and* $\mathcal{H}'_i$, *for some* $i = 1, \ldots, n$, *or on a pair* $\mathcal{H}_i$ *and* $\mathcal{H}_j$, *for some distinct* $1 \leq i, j \leq n$.

Proof Let $[B, B'] \subset \mathcal{K}$ be an affine diameter disjoint from $\mathcal{K}_0$. Let $\mathcal{H}$ and $\mathcal{H}'$ be parallel hyperplanes supporting $\mathcal{K}$ with $B \in \mathcal{H}$ and $B' \in \mathcal{H}'$.

Assume that $B \notin \mathcal{H}_i$ for $1 \leq i \leq n$.

We fix $1 \leq i \leq n$. The hyperplane $\mathcal{F}_i = \langle B, C_1, \ldots, C_{i-1}, \widehat{C_i}, C_{i+1}, \ldots, C_n\rangle$ intersects $\mathcal{H}_i$ in $A = \langle C_1, \ldots, C_{i-1}, \widehat{C_i}, C_{i+1}, \ldots, C_n\rangle$. (Here and in what follows, the angular brackets mean affine span.) This hyperplane $\mathcal{F}_i$ is transversal to $\mathcal{H}$. (Otherwise, having B as a common point, they would be equal, $\mathcal{F}_i = \mathcal{H}$, and, due to the minimal choice of $\mathcal{H}_i$ above, we would also have $\mathcal{H} = \mathcal{H}_i$, contradicting to $B \notin \mathcal{H}_i$.) We now rotate $\mathcal{F}_i$ about A to $\mathcal{H}_i$ staying on one side of $\mathcal{K}_0$. We consider whether during this rotation the rotated hyperplanes stay transversal to $\mathcal{H}$. Assume not. Then, at one stage of the rotation, a rotated hyperplane is parallel to $\mathcal{H}$. Since this rotated hyperplane along with $\mathcal{H}$, $\mathcal{H}_0$, $\mathcal{H}_i$ and $\mathcal{H}'_i$ all contain a (translated) copy of A, the entire configuration can be understood via its intersection with the 2-dimensional $A^\perp$. Projecting C_i and B to $A^\perp$ (along A) we see that we must have $\mathcal{H} = \mathcal{H}'_i$ and so $B \in \mathcal{H}'_i$. In this case, we also have $\mathcal{H}' = \mathcal{H}_i$ (since they are parallel and both supporting) and so $B' \in \mathcal{H}_i$. We arrive at one of the stated scenarios: the affine diameter $[B', B]$ connects $\mathcal{H}_i$ and $\mathcal{H}'_i$.

In the second case, during the rotation, the rotated hyperplanes stay transversal to $\mathcal{H}$, in particular, $\mathcal{H}_i \cap \mathcal{H}$ intersect transversally.

It remains to consider this second case for all $1 \leq i \leq n$. The intersections $\mathcal{H}_i \cap \mathcal{H}$, $1 \leq i \leq n$, bound an $(n-1)$-simplex Δ in $\mathcal{H}$ and the rotation argument along with $B \notin \mathcal{H}_i$, $1 \leq i \leq n$, imply that B is in the interior of Δ.

Switching the roles of B and B', if $B' \notin \mathcal{H}_j$ for all $1 \leq j \leq n$, then, repeating the argument above and discarding the stated scenarios, we obtain that $\mathcal{H}_j \cap \mathcal{H}'$, $1 \leq j \leq n$, bound a simplex Δ' in $\mathcal{H}'$ with B' in its interior.

Since all the participating hyperplanes are supporting $\mathcal{K}$, it follows that the convex hull $[\Delta, \Delta']$ contains $\mathcal{K}$. This convex hull is a polytope which, in addition to its parallel simplicial cells Δ and Δ', has n other side cells supported by $\mathcal{H}_i$, $1 \leq i \leq n$. On the other hand, the configuration of the side cells intersected with the hyperplane $\mathcal{H}_0$ cuts out the simplex

$\mathcal{K}_0$ (Lemma 1). By assumption, $\mathcal{K}_0$ is disjoint from $[B, B']$. Since $\mathcal{H}_0$ is supporting $\mathcal{K}$, this implies that $\mathcal{H}_0 = \mathcal{H}$ or $\mathcal{H}_0 = \mathcal{H}'$. It follows that B or B' is in $\mathcal{K}_0$, a contradiction. The lemma follows. □

To finish the proof of Part I of Theorem C, from now on we assume that *all* interior points of $\mathcal{K}$ are regular (and $O_0 \in \partial\mathcal{K}$ is a flat point).

Lemma 4 *Let $B \in \partial\mathcal{K} \setminus \mathcal{K}_0$. Assume that $B \in \mathcal{H}_i$ for some $1 \le i \le n$. Then the intersection $\mathcal{K}_i = \mathcal{H}_i \cap \mathcal{K}$ is an $(n-1)$-simplex $[C, C_1, \ldots, \widehat{C_i}, \ldots, C_n]$ with $C \in \mathcal{V}$.*

Proof Due to its minimal choice made above, $\mathcal{H}_i$ is a supporting hyperplane of $\mathcal{K}$ at B. The n-simplex $[B, C_1, \ldots, \widehat{C_i}, \ldots, C_n] \subset \mathcal{H}_i$ must then be contained in the boundary of $\mathcal{K}$. Since *any* point in the relative interior of this simplex is a flat point and, by assumption, all interior points are regular, we can now apply Lemma 1. We obtain that $\mathcal{K}_i = \mathcal{H}_i \cap \mathcal{K}$ is an $(n-1)$-simplex. The intersection $\mathcal{K}_i \cap \mathcal{K}_0$ is the $(n-2)$-simplex $[C_1, \ldots, \widehat{C_i}, \ldots, C_n]$. We denote by C the missing vertex of $\mathcal{K}_i$. Applying the last statement of Lemma 1 to this situation (with $\mathcal{K}_i$ in place of $\mathcal{K}_0$), we see that $\mathcal{K}$ has a supporting hyperplane at C, parallel to $\mathcal{H}_0$. By convexity, this can only be $\mathcal{H}'_0$, so that $C \in \mathcal{V}$. The lemma follows. □

Lemma 5 *We have $\mathcal{K} = [\mathcal{K}_0, \mathcal{V}]$. For any $O \in \operatorname{int}\mathcal{K}$, the vertices $C_1, \ldots, C_n$ along with a point $C_0 \in \mathcal{V}$ form a minimal configuration with respect to O.*

Proof Given $O \in \operatorname{int}\mathcal{K}$, by regularity, we may choose a minimizing configuration $\{B_0, \ldots, B_n\} \in \mathfrak{C}(\mathcal{K}, O)$ consisting of extremal points. Fix $0 \le i \le n$. If $B_i \in \mathcal{K}_0$ then, since B_i is extremal, it must be one of the vertices $\{C_1, \ldots, C_n\}$ of $\mathcal{K}_0$. If $B_i \notin \mathcal{K}_0$ but $B_i^o \in \mathcal{K}_0$ then, by Lemma 2, $B_i \in \mathcal{V}$. (Since O is regular, $\Lambda(., O)$ attains local maximum at B_i.) In the remaining case $[B_i, B_i^o]$ (with B_i^o with respect to O) is an affine diameter away from $\mathcal{K}_0$. We are in the position to apply Lemma 3. If $B_i \in \mathcal{H}_j$ for some $j = 1, \ldots, n$, then, by Lemma 4, B_i must be in the $(n-1)$-simplex $[C, C_1, \ldots, \widehat{C_j}, \ldots, C_n]$ with $C \in \mathcal{V}$. Since B_i is extremal, it must be one of the vertices of this simplex. Once again, we obtain that $B_i = C_k$, for some $k = 1, \ldots, n$, $k \ne j$, or $B_i \in \mathcal{V}$. Finally, if $B_i \in \mathcal{H}'_j$ and $B_i^o \in \mathcal{H}_j$ then B_i can be moved to C_j along the line segment $[B_i, C_j] \subset \mathcal{H}'_j$. This line segment is part of the boundary of $\mathcal{K}$ since $\mathcal{H}'_j$ is supporting $\mathcal{K}$. During this move the distortion $\Lambda(., O)$ does not decrease since $\mathcal{H}_j$ is parallel to $\mathcal{H}'_j$ and supports $\mathcal{K}$. In addition, the configuration condition stays intact since O is a regular point. We obtain that B_i can be moved to C_j retaining minimality.

Since a minimizing configuration for a regular point cannot contain multiple points, renumbering and making some moves if needed, we conclude that our minimizing configuration may be assumed to have the form $\{B_0, \ldots, B_k, C_{i_1}, \ldots, C_{i_l}\}$, $1 \le i_1 < \ldots < i_k \le n$, $k + l = n$, where $B_0, \ldots, B_k \in \mathcal{V}$. It remains to show that $k = 0$. Since $O \in [B_0, \ldots, B_k, C_{i_1}, \ldots, C_{i_l}]$, we have the convex linear combination

$$O = \sum_{i=0}^{k} \lambda_i B_i + \sum_{j=1}^{l} \lambda_{i_j} C_{i_j}, \quad \sum_{i=0}^{k} \lambda_i + \sum_{j=1}^{l} \lambda_{i_j} = 1, \ 0 \le \lambda_i, \lambda_{i_j} \le 1.$$

We now compress the first sum in the usual way letting $\mu_0 = \sum_{i=0}^{k} \lambda_i > 0$ and $C_0 = \frac{1}{\mu_0} \sum_{i=1}^{k} \lambda_i B_i \in \mathcal{V}$. We obtain

$$O = \mu_0 C_0 + \sum_{j=1}^{l} \lambda_{i_j} C_{i_j}. \tag{14}$$

This implies that the opposite of C_0 is in $\mathcal{K}_0$, in particular, $\Lambda(C_0, O) \geq \Lambda(B_i, O)$, $0 \leq i \leq k$. Thus, we have

$$\sigma(\mathcal{K}, O) = \sum_{i=0}^{k} \frac{1}{1+\Lambda(B_i, O)} + \sum_{j=1}^{l} \frac{1}{1+\Lambda(C_{i_j}, O)} \geq \frac{k+1}{1+\Lambda(C_0, O)} + \sum_{j=1}^{l} \frac{1}{1+\Lambda(C_{i_j}, O)}.$$

Again by (14), $\{C_0, \ldots, C_0, C_{i_1}, \ldots, C_{i_l}\}$ (with C_0 repeated k times) is a configuration with respect to O, so that the opposite inequality also holds. Since O is regular, $k = 0$ must hold. The lemma follows. □

We are now ready for the final step as follows:

Lemma 6 *$\mathcal{V}$ consists of a single point.*

Proof Let $C_0 \in \mathcal{V}$. If $\mathcal{V}$ consists of more than one point then the simplex $[C_0, \ldots, C_n]$ cannot be the whole $\mathcal{K}$. In particular, there is a point $O \in \operatorname{int}\mathcal{K}$ on the boundary of $[C_0, \ldots, C_n]$. Applying Lemma 5, there is $C_0' \in \mathcal{V}$ such that $\{C_0', C_1, \ldots, C_n\}$ is a minimal simplicial configuration with respect to O. Since the antipodal of C_0 with respect to O is on $\mathcal{K}_0$, we have $\Lambda(C_0, O) \geq \Lambda(C_0', O)$. By minimality, equality must hold. Thus, $\{C_0, C_1, \ldots, C_n\}$ is minimizing with respect to O. This is a contradiction to the regularity of O since it is on the boundary of the simplex $[C_0, C_1, \ldots, C_n]$. □

Combining Lemmas 5–6, we obtain that $\mathcal{K} = [C_0, C_1, \ldots, C_n]$. Part I of Theorem C follows.

Part II. We now change the setting and start the proof of Theorem C under the assumption in (II). Let $\mathcal{K}^0 \subset \partial\mathcal{K}$ denote the set of extremal points of $\mathcal{K}$. We call an extremal point $C \in \mathcal{K}^0$ *isolated* if C has an open neighborhood disjoint from $\mathcal{K}^0 \setminus \{C\}$. We begin with the following:

Proposition *Let $\mathcal{K} \subset \mathbb{E}^n$ be a convex body with all its interior points regular. Assume that $\mathcal{K}$ has (at least) two isolated extremal points $C_1, C_2 \in \mathcal{K}^0$. Then, for any plane $\mathcal{E} \subset \mathbb{E}^n$ containing $[C_1, C_2]$ and an interior point of $\mathcal{K}$, the intersection $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{E}$ is a triangle with $[C_1, C_2]$ as one side.*

This proposition is essentially Theorem 1.1 in [25] except we added here the assumption that the plane $\mathcal{E}$ contains an *interior* point of $\mathcal{K}$ (and consequently obtain a (non-degenerate) *triangular* intersection $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{E}$). The proof here is identical with that of Theorem 1.1.

We now turn to the main induction step of the proof. We assume that $\mathcal{K} \subset \mathbb{E}^n$ is a convex body with all its interior points regular. We let $C_1, \ldots, C_n$ be a fixed sequence of isolated extremal points. For $2 \leq m \leq n$, we let $\mathcal{P}_m$ denote the following statement:

For any $1 \leq i_1, \ldots, i_m \leq n$ distinct and any $O_0 \in \operatorname{int}\mathcal{K} \setminus \langle C_{i_1}, \ldots, C_{i_m}\rangle$, the set $\{C_{i_1}, \ldots, C_{i_m}\}$ is affinely independent, and the intersection $\mathcal{K} \cap \langle C_{i_1}, \ldots, C_{i_m}, O_0\rangle$ is an m-simplex with $[C_{i_1}, \ldots, C_{i_m}]$ as a side.

Note that $\mathcal{P}_2$ is our proposition above. Moreover, for reasons of dimension, $\mathcal{P}_n$ says that $\mathcal{K}$ is an n-simplex; Part II of Theorem C. Therefore we can use induction with respect to $m \geq 2$, with the initial step already accomplished.

For the general induction step $m-1 \Rightarrow m$, $3 \leq m \leq n$, we assume that $\mathcal{P}_{m-1}$ holds. Rearranging if necessary, we consider $C_1, \ldots, C_m$, and let $\mathcal{E} = \langle C_1, \ldots, C_m, O_0\rangle$ for some $O_0 = \operatorname{int}\mathcal{K} \setminus \langle C_1, \ldots, C_m\rangle$. For $\mathcal{P}_m$ we need to show that $\{C_1, \ldots, C_m\}$ is *affinely independent* and $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{E}$ is an *m-simplex.*

First, by the induction hypothesis, $\{C_1, \ldots, C_{m-1}\}$ is affinely independent and $\mathcal{K} \cap \langle C_1, \ldots, C_{m-1}, O_0\rangle$ is an $(m-1)$-simplex with $[C_1, \ldots, C_{m-1}]$ as a side. In particular, we have $\mathcal{K} \cap \langle C_1, \ldots, C_{m-1}\rangle = [C_1, \ldots, C_{m-1}] \subset \partial\mathcal{K}$. If $\{C_1, \ldots, C_m\}$ were affinely dependent then we would have $C_m \in \langle C_1, \ldots, C_{m-1}\rangle$ so that $C_m \in [C_1, \ldots, C_{m-1}] \subset \partial\mathcal{K}$. Since $C_1, \ldots, C_m$ are distinct, this would contradict to the assumption that C_m is an extremal point. We obtain that $\{C_1, \ldots, C_m\}$ is an *affinely independent* set. It follows that $\dim \mathcal{E} = m$, the set $\Delta = [C_1, \ldots, C_m] \subset \mathcal{K}_0$ is an m-simplex, and $\mathcal{H} = \langle \Delta\rangle = \langle C_1, \ldots, C_m\rangle$ a hyperplane in $\mathcal{E}$. (For the most part of the proof below we will work within $\mathcal{E}$ so that all the concepts are understood in this affine subspace.)

We denote by $\mathcal{G} \subset \mathcal{E}$ the closed half-space with $\partial\mathcal{G} = \mathcal{H}$ and $O_0 \in \operatorname{int}\mathcal{G}$. For $1 \le i \le m$, we let $\Delta_i = [C_1, \ldots, \widehat{C_i}, \ldots, C_m]$; the ith face of Δ opposite to the vertex C_i. For $1 \le i \ne j \le m$, we let $\Delta_{ij} = \Delta_i \cap \Delta_j$.

We will repeatedly use the induction hypothesis in the following setting:

For $O \in \operatorname{int}\mathcal{K}_0 \cap \operatorname{int}\mathcal{G}$, we have $\mathcal{K}_0 \cap \langle \Delta_i, O\rangle = [\Delta_i, B_i]$ for some $B_i \in \partial\mathcal{K}_0 \cap \operatorname{int}\mathcal{G}$. Taking the respective boundaries, we have $\Delta_i \subset \partial\mathcal{K}_0$, $1 \le i \le m$. In particular, we have $\mathcal{K}_0 \cap \mathcal{H} = \Delta$ and $[\Delta_{ij}, B_i] \subset \partial\mathcal{K}_0$, $1 \le i \ne j \le m$.

We now turn to the proof of the second statement of $\mathcal{P}_m$ above: $\mathcal{K}_0$ is an m-simplex. Let $\mathcal{H}' \subset \operatorname{int}\mathcal{G}$ be a hyperplane parallel to $\mathcal{H}$ and supporting $\mathcal{K}_0$ at some point $C_0 \in \partial\mathcal{K}_0$. Choose a sequence $\{O_k\}_{k\ge 1} \subset \operatorname{int}\mathcal{K}_0 \cap \operatorname{int}\mathcal{G}$ such that $\lim_{k\to\infty} O_k = C_0$. By the induction hypothesis, for each $1 \le i \le m$, we have

$$\mathcal{K}_0 \cap \langle \Delta_i, O_k\rangle = [\Delta_i, B_{i,k}], \tag{15}$$

for some $B_{i,k} \in \partial\mathcal{K}_0 \cap \operatorname{int}\mathcal{G}$, $k \ge 1$. Since $\mathcal{H}'$ supports $\mathcal{K}_0$ at C_0, for each $1 \le i \le m$, we clearly have $\lim_{k\to\infty} B_{i,k} = C_0$, $1 \le i \le m$. (Otherwise, by compactness, $\{B_{i,k}\}_{k\ge 1}$ would subconverge to a point $C_0' \in \partial\mathcal{K} \cap \mathcal{H}'$, $C_0' \ne C_0$, contradicting to $O_k \in [\Delta_i, B_{i,k}]$ and $\lim_{k\to\infty} O_k = C_0$.) Letting $k \to \infty$ in (15), we obtain

$$\mathcal{K}_0 \cap \langle \Delta_i, C_0\rangle = [\Delta_i, C_0], \quad 1 \le i \le m. \tag{16}$$

Since

$$\partial[\Delta_i, C_0] = \Delta_i \cup \bigcup_{1\le j\ne i\le m} [\Delta_{ij}, C_0], \tag{17}$$

as a byproduct, we have

$$[\Delta_{ij}, C_0] \subset \partial\mathcal{K}_0, \quad 1 \le i \ne j \le m. \tag{18}$$

We now claim that

$$[\Delta_i, C_0] \subset \partial\mathcal{K}_0, \quad 1 \le i \le m. \tag{19}$$

Assume on the contrary that $[\Delta_i, C_0] \not\subset \partial\mathcal{K}_0$ for a specific $1 \le i \le m$. This means that the closed half-space $\mathcal{G}_i \subset \mathcal{E}$ with boundary hyperplane $\mathcal{H}_i = \langle \Delta_i, C_0\rangle \subset \mathcal{E}$ and $C_i \notin \mathcal{G}_i$ intersects the interior of $\mathcal{K}_0$.

Let $\mathcal{H}_i' \subset \operatorname{int}\mathcal{G}_i$ be a hyperplane parallel to $\mathcal{H}_i$ and supporting $\mathcal{K}_0$ at some point $V_i \in \partial\mathcal{K}_0 \cap \operatorname{int}\mathcal{G}_i$. Repeating the previous argument (in the use of a sequence $\{O_k\}_{k\ge 1} \subset \operatorname{int}\mathcal{K}_0 \cap \operatorname{int}\mathcal{G}_i$ converging to V_i) we obtain $\mathcal{K}_0 \cap \langle \Delta_j, V_i\rangle = [\Delta_j, V_i]$, $1 \le j \le m$, and $[\Delta_{jk}, V_i] \subset \partial\mathcal{K}_0$, $1 \le j \ne k \le m$.

Now, C_i and V_i are on different sides of $\mathcal{H}_i$, therefore $[C_i, V_i]$ and $\mathcal{H}_i$ intersect in a point $X \in \mathcal{K}_0$. By (16), $\mathcal{K}_0 \cap \mathcal{H}_i = [\Delta_i, C_0]$ so that $X \in [\Delta_i, C_0]$. In addition, since $m \ge 3$, $C_i \in \Delta_{jk}$ for some (actually any) $1 \le j \ne k \le m$ distinct from i, we are in the position to apply (18) to get $[C_i, V_i] \subset \partial\mathcal{K}_0$. In particular, $X \in \partial\mathcal{K}_0$. Combining the last two inclusions for X, we have $X \in \partial[\Delta_i, C_0]$. Thus, by (17), we finally have $X \in [\Delta_{ij}, C_0]$, for some $1 \le j \ne i \le m$.

Summarizing, we obtain that $[\Delta_{ij}, C_0]$ and $[C_i, V_i]$ are both contained in the boundary of $\mathcal{K}_0$ and intersect (transversally) at X. By convexity, the convex hull $[\Delta_j, C_0, V_i]$, $\Delta_j = [\Delta_{ij}, C_i]$, is also contained in the boundary of $\mathcal{K}_0$, and, for reasons of dimension, $\langle \Delta_j, C_0, V_i \rangle$ is a supporting hyperplane of $\mathcal{K}_0$.

Once again, let $\{O_k\}_{k \geq 1} \subset \operatorname{int} \mathcal{K}_0 \cap \operatorname{int} \mathcal{G}$ be a sequence converging to X. By the induction hypothesis, $\mathcal{K}_0 \cap \langle \Delta_j, O_k \rangle$ is an $(m-1)$-simplex with Δ_j as a side. Taking the limit as $k \to \infty$ we obtain that the limiting intersection is an $(m-1)$-simplex with Δ_j as a side and an extra vertex W. On the other hand, the limit of the hyperplanes $\langle \Delta_j, O_k \rangle$ as $k \to \infty$ is the hyperplane $[\Delta_j, C_0, V_i]$ supporting $\mathcal{K}_0$. Thus, the limiting simplex $[\Delta_j, W]$ must contain C_0, and V_i. Due to the extremal choices of the latter two points, we must have $W = C_0$ and $W = V_i$ simultaneously. This is a contradiction, so that we finally arrive at (19).

Since (19) holds for all $1 \leq i \leq m$, we see that $\mathcal{K}_0 \cap \mathcal{G}$ is the m-simplex $[\Delta, C_0]$. Let $\mathcal{G}'$ be the closed half-space complementary to $\operatorname{int} \mathcal{G}$ in $\mathcal{E}$. If $\mathcal{G}'$ is disjoint from the interior of $\mathcal{K}_0$ then $\mathcal{K}_0$ is the m-simplex $[\Delta, C_0]$, and $\mathcal{P}_m$ follows. Otherwise, applying the argument above to $\mathcal{G}'$ instead of $\mathcal{G}$, we obtain that $\mathcal{K}_0 \cap \mathcal{G}'$ is another m-simplex $[\Delta, C_0']$. In this case $\mathcal{K}_0$ is then a double m-simplex with base Δ (that is, two m-simplices with disjoint interiors joined at their common side Δ.) It remains to show that this cannot occur.

First assume that $m < n$. Then $C_{m+1} \in \partial \mathcal{K}$ exists. Let $O_0 \in \operatorname{int} \Delta$ and apply the construction above to $\Delta' = [\Delta_1, C_{m+1}] = [C_2, \ldots, C_m, C_{m+1}]$ and $\mathcal{E}' = \langle \Delta', O_0 \rangle$. We obtain that $\mathcal{K}_0' = \mathcal{K} \cap \mathcal{E}'$ is an m-simplex or a double m-simplex with base Δ'. On the other hand, we have $\mathcal{K} \cap \langle \Delta_1, O_0 \rangle = [\Delta_1, C_1] = \Delta$ with O_0 an interior point of $\mathcal{K}_0'$ away from Δ'. This contradicts to the extremality of C_1.

Finally, let $m = n$. In this case $\mathcal{K}_0 = \mathcal{K} = [\Delta, C_0, C_0']$ is a double n-cone in $\mathbb{E}^n$. This clearly cannot happen as double cones cannot have all their interior points regular. (If $[C_0, C_0']$ intersects the interior of Δ then this intersection point must be singular as it does not have a simplicial minimal configuration consisting of extremal points only. If $[C_0, C_0']$ meets the boundary of Δ then all interior points of Δ are singular for the same reason.) Part II of Theorem C follows.

4 Examples

We first return to Example 1 of Sect. 1; the unit half-disk $\mathcal{K} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ y \geq 0\}$. As noted there, we claim that the regular set $\mathcal{R}$ is the interior of the triangle $\Delta = [C_-, C_0, C_+]$.

Indeed, given $(a, b) \in \operatorname{int} \mathcal{K}$, there are exactly three affine diameters passing through (a, b); those that also pass through C_-, C_0 or C_+. It immediately follows that any point away from the interior of Δ must be singular. On the other hand, if (a, b) is in the interior of Δ, a simple computation shows that $\{C_-, C_0, C_+\}$ is a minimizing configuration. This gives $\sigma(\mathcal{K}, (a, b)) < 1 + \frac{1}{1+\ell(a,b)}$, that is, $(a, b) \in \mathcal{R}$. (By symmetry, it is enough to consider the case $(a, b) \in \Delta_+ = [C_0, 0, C_+]$. The curve $\{(a, b) \mid \Lambda(C_-, (a, b)) = \Lambda(C_0, (a, b))\} \cap \Delta_+$ is a portion of an ellipse connecting O^* and C_+. It splits Δ_+ into two domains, and one needs to consider the respective two cases.) Having identified the regular and singular sets, we obtain

$$\sigma(\mathcal{K}, (a, b)) = \begin{cases} \frac{1-a^2-b^2}{1-a^2} + b, & \text{if} (a, b) \in \mathcal{R} = \operatorname{int} \Delta; \\ 1 + \frac{1-a^2-b^2}{1+|a|}, & \text{otherwise.} \end{cases}$$

As a second example, let $\mathcal{K} = [\mathcal{K}_0, V]$ be a cone with base $\mathcal{K}_0 = \mathcal{P}_{2m+1} \subset \mathbb{R}^2$, $m \geq 2$, the regular $(2m+1)$-sided polygon (in Example 2 of Sect. 1), and $V \in \mathbb{R}^3 \setminus \mathbb{R}^2$ its vertex. We write $\ell_0^* = \ell_{\mathcal{K}_0}^*$, $\ell^* = \ell_{\mathcal{K}}^*$, $O_0^* = O_{\mathcal{K}_0}^*$ etc., where we indicated the respective convex body by subscript. In [10] (Sect. 4) Klee calculated the critical ratio and the critical set of a convex cone in terms of those of the base. According to this, we have $\ell_{\mathcal{K}}^* = 1 + \ell_{\mathcal{K}_0}^* = 1 + \sec\left(\frac{\pi}{2m+1}\right) > 2$ ((4.5) in [10]). In addition, parametrizing the line segment $[O_0^*, V]$ by $\lambda \mapsto O_\lambda = (1-\lambda)O_0^* + \lambda V$, $0 \leq \lambda \leq 1$, the critical set $\mathcal{K}^*$ also consists of a single point, $O^* = O_{\lambda^*}$, where $\lambda^* = \left(2 + \sec\left(\frac{\pi}{2m+1}\right)\right)^{-1}$. The specific parameter value λ^* is given by the condition that the distortion values of V and any of the vertices of $\mathcal{K}_0$ *with respect to* O_{λ^*} are the same.

According to Theorem B, O^* is a regular point. In [24] (Proposition 6), we proved in general that the projection of any regular point of $\mathcal{K}$ to $\mathcal{K}_0$ along the ray from V is also regular: $\mathcal{R} \subset \bigcup_{O_0 \in \mathcal{R}_0} (O_0, V)$. In our present case the inclusion is proper. In fact, for $0 < \lambda \leq (2 - \sec(\frac{\pi}{2m+1}))/3$, $O_\lambda \in (O^*, V)$ is a *singular point* [27, Example 4]. Note that $(2 - \sec(\frac{\pi}{2m+1}))/3 < \lambda^* = (2 + \sec(\frac{\pi}{2m+1}))^{-1}$ with both sides converging to $1/3$ as $m \to \infty$. This is in striking contrast to the 2-dimensional case; the distance of the regular critical point $O^* = O_{\lambda^*}$ from the singular set decreases to 0 as $m \to \infty$. (In the limit as $m \to \infty$, $\mathcal{K}$ converges to a circular cone all of whose interior points are singular.)

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