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ORIGINAL PAPER

Simplicial slices of the space of minimal $SU(2)$-orbits in spheres

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Abstract Let $(\mathcal{M}_3^k)^{SU(2)}$ denote the DoCarmo–Wallach moduli space of $SU(2)$-equivariant spherical minimal immersions of the three sphere S^3 of degree k. Although the complexity of these moduli increases rapidly with k (for example, $\dim(\mathcal{M}_3^k)^{SU(2)} = \mathcal{O}(k^2)$), we show here that they possess linear slices that are simplices of dimension $\mathcal{O}(k)$. The construction of these simplicial slices depend on the DeTurck–Ziller classification of 3-dimensional spherical space forms imbedded into spheres as minimal $SU(2)$-orbits. The existence of these slices enables us to give asymptotically sharp estimates on a sequence of Grünbaum type measures of symmetry of these moduli.

Keywords Spherical minimal immersion · Eigenmap · Moduli · Simplex.

Mathematics Subject Classification (1991) Primary 53C42

1 Introduction and simplicial slices of the moduli

The curvature of the three-sphere S^3 minimally immersed in the unit sphere S_V of an orthogonal $SU(2)$-module V as an orbit can only take discrete values $\lambda_k/3$, $k \in \mathbf{N}$, where $\lambda_k = k(k+2)$ is the kth eigenvalue of the Laplacian $\triangle$ of S^3 with respect to the standard (curvature 1) metric, see Takahashi (1966) and Wallach (1972). Keeping this standard metric on S^3 (up to the conformality factor $\lambda_k/3$), the components of such an immersion $f : S^3 \to S_V$ become spherical harmonics of order k on S^3, (eigen)functions in the eigenspace $\mathcal{H}_3^k \subset C^\infty(S^3)$ of $\triangle$ corresponding to λ_k. We call f an $SU(2)$-equivariant *spherical minimal immersion of* (algebraic) *degree* k.

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For fixed k, we denote by $(\mathcal{M}_3^k)^{SU(2)}$ the DoCarmo–Wallach moduli space of such immersions. [For the original construction of the moduli, see DoCarmo and Wallach (1971), Wallach (1972). For a detailed up-to-date treatment of the subject, see Chapters 4–5 in Toth (2002) and also Weingart (1999).] This notation is justified as this moduli is the $SU(2)$-fixed point set of the moduli $\mathcal{M}_3^k$ of *all* degree k spherical minimal immersions $f : S^3 \to S_V$ (into various Euclidean vector spaces V) defined by dropping $SU(2)$-equivariance and keeping conformality (with factor $\lambda_k/3$). (The moduli $\mathcal{M}_3^k$ parametrizes full spherical minimal immersions up to congruence on the range. $\mathcal{M}_3^k$ is a compact convex body in an $SO(4)$-submodule of $S^2(\mathcal{H}_3^k)$, the symmetric square of $\mathcal{H}_3^k$. The induced natural action of $SO(4)$ on $S^2(\mathcal{H}_3^k)$ restricted to $\mathcal{M}_3^k$ is given by precomposition of the corresponding immersions. Via restriction $SU(2) \subset SO(4)$, the fixed point set $(\mathcal{M}_3^k)^{SU(2)}$ then parametrizes the $SU(2)$-equivariant spherical minimal immersions. In fact, this moduli is a compact convex body in an $SU(2)'$-submodule of $S^2(\mathcal{H}_3^k)$, where $SO(4) = SU(2) \cdot SU(2)'$ is the natural almost product structure). In their early work DoCarmo and Wallach (1971), showed that $\mathcal{M}_3^k$ is nontrivial iff $k \geq 4$ by giving a lower bound on the dimension (depending on $k \geq 4$), in particular, they showed that $\dim \mathcal{M}_3^4 \geq 18$. They conjectured that their lower bound was sharp. In Muto (1984), for the first nontrivial moduli $\mathcal{M}_3^4$, Muto settled the conjecture by an explicit computation. In Toth (1994) the conjecture has been resolved affirmatively for any $k \geq 4$. Initiated by the work of Mashimo (1984, 1985), individual spherical minimal immersions of S^3 (and also higher odd dimensional spheres) have been constructed by several authors, see DeTurck and Ziller (1967, 1992, 1993), Escher and Weingart (2000), and Toth and Ziller (1999). Using these, in Toth and Ziller (1999) the 18-dimensional moduli $\mathcal{M}_3^4$ has been completely described in geometric terms.

Although there is a natural $SO(4)$-equivariant imbedding $\mathcal{M}_3^k \to \mathcal{M}_3^{k+1}$ defined in Toth (2002) (which restricts to an $SU(2)'$-equivariant imbedding $(\mathcal{M}_3^k)^{SU(2)} \to (\mathcal{M}_3^{k+1})^{SU(2)}$), in the degrees $k \geq 5$, not much is known about the structure of the moduli $\mathcal{M}_3^k$ or the $SU(2)$-equivariant part $(\mathcal{M}_3^k)^{SU(2)}$.

In the study by Toth (1999, 2000) of $SU(2)$-equivariant spherical minimal immersions into spheres of minimum codimension, an interesting phenomenon occurs. In degree $k = 6$, the least codimensional example, the so-called *tetrahedal minimal immersion Tet* : $S^3 \to S^6$ corresponds to a boundary point of $(\mathcal{M}_3^6)^{SU(2)}$ and this point is a vertex of a triangular slice (by a plane through the origin) of the entire moduli. [The tetrahedral minimal immersion gets its name from its full invariance group, the binary tetrahedral group (the lift of the rotation group of a regular tetrahedron via the two-fold cover $SU(2) \to SO(3)$). Factoring, it gives a minimal imbedding of the tetrahedral manifold into the 6-sphere.] In degree $k = 8$, the same phenomenon recurs: $(\mathcal{M}_3^8)^{SU(2)}$ possesses a tetrahedral slice (by a 3-dimensional linear subspace) and one of the vertices of the tetrahedron corresponds to the *octahedral minimal immersion Oct* : $S^3 \to S^8$ (with invariance group the binary octahedral group). Contrary to expectation, as shown by Weingart (1999), for the icosahedral case in degree $k = 12$, the corresponding simplicial slice in $(\mathcal{M}_3^{12})^{SU(2)}$ is only tetrahedral with one of the vertices still corresponding to the *isocahedral minimal immersion Ico* : $S^3 \to S^{12}$ (with binary icosahedral invariance group).

The natural question arises to what extent is this phenomenon general, that is, do the moduli $(\mathcal{M}_3^k)^{SU(2)}$ (and consequently $\mathcal{M}_3^k$) possess simplicial slices of large dimension (across the origin).

Let $d((\mathcal{M}_3^k)^{SU(2)}) \geq 1$ denote the maximum dimension of simplicial slices of $(\mathcal{M}_3^k)^{SU(2)}$ (by linear subspaces). As shown in Toth (2006) (see also Sect. 2 below), we have

$$d((\mathcal{M}_3^k)^{SU(2)}) \leq \max_{\partial(\mathcal{M}_3^k)^{SU(2)}} \Lambda(.,0), \tag{1}$$

where Λ is the *distortion function*. (For the general definition of the distortion, see Sect. 2). For $C \in \partial(\mathcal{M}_3^k)^{SU(2)}$, $\Lambda(C,0)$ is also the largest eigenvalue of C viewed as a symmetric endomorphism in $S^2(\mathcal{H}_3^k)$. This largest eigenvalue has been calculated in Toth (2002) (Example 2.3.12, p. 121) for more general $SU(2)$-equivariant (λ_k-eigen)maps. This computation along with the existence of $SU(2)$-equivarant degree k spherical minimal immersions $f : S^3 \to S^{2k+1}$, for $k \geq 5$ odd, and $f : S^3 \to S^k$, for $k \geq 6$ even, shows that for $k > 4$, we have

$$\max_{\partial(\mathcal{M}_3^k)^{SU(2)}} \Lambda(.,0) = \begin{cases} k & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \text{ is odd} \end{cases} \tag{2}$$

(Although the first moduli $(\mathcal{M}_3^4)^{SU(2)}$ is is completely described in Toth 1999, due to the existence of a 6-dimensional extremal set on its boundary, for the maximum distortion only the lower bound 3/2 is known). Note that, in contrast to (2), we have

$$\dim(\mathcal{M}_3^k)^{SU(2)} = (2[k/2]+5)([k/2]-1) = \mathcal{O}(k^2).$$

(This dimension formula was first derived heuristically in DeTurck and Ziller 1992 and precisely in Toth 1999.) Note that, as an interesting consequence, for $k \geq 4$ even, $(\mathcal{M}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^{k+1})^{SU(2)}$ are equidimensional.

The main result of this paper is the following:

Theorem A *For $k \geq 7$ odd, let*

$$m(k) = \left[\frac{1}{2}\left(k - \sqrt{\frac{k(k+2)}{3}}\right)\right]. \tag{3}$$

Then $m(k) < d((\mathcal{M}_3^k)^{SU(2)})$, that is, $(\mathcal{M}_3^k)^{SU(2)}$ has a simplicial slice of dimension $m(k)+1$ across 0.

For $k \geq 10$ even, let

$$m(k) = \left[\frac{1}{2}\min\left(k - \sqrt{\frac{k(k+2)}{3}}, \frac{k-10}{2}\right)\right]. \tag{4}$$

Then, $2m(k) < d((\mathcal{M}_3^k)^{SU(2)})$, that is, $(\mathcal{M}_3^k)^{SU(2)}$ has a simplicial slice of dimension $2m(k)+1$ across 0.

For k odd, the theorem implies that $(\mathcal{M}_3^7)^{SU(2)}$ has a triangular slice, $(\mathcal{M}_3^{11})^{SU(2)}$ has a tetrahedral slice, $(\mathcal{M}_3^{17})^{SU(2)}$ has a pentatopal slice, etc. The moduli $(\mathcal{M}_3^5)^{SU(2)}$ actually has a triangular slice; see the remark in Sect. 4. Note that, for $k \geq 25$ odd, $m(k)$ in (3) can be replaced by the simpler lower bound $[k/5]$. For k even, the minimum in (4) is attained by $(k-10)/2$ if and only if $0 \leq k \leq 56$. Equivalently, for $k \geq 57$, the formula for $m(k)$ in (3) holds in both cases, regardless the parity of k. As noted above, the existence of the tetrahedral, octahedral, and icosahedral immersions imply that $(\mathcal{M}_3^6)^{SU(2)}$ has a triangular slice, and that $(\mathcal{M}_3^8)^{SU(2)}$ and $(\mathcal{M}_3^{12})^{SU(2)}$ both have tetrahedral slices. The theorem above implies that $(\mathcal{M}_3^{14})^{SU(2)}$ and $(\mathcal{M}_3^{16})^{SU(2)}$ still have tetrahedral slices, and that $(\mathcal{M}_3^{18})^{SU(2)}$ has a 5-dimensional simplical slice, etc.

The concept of a spherical minimal immersion naturally extends to any domain dimension, and one arrives at the moduli $\mathcal{M}_m^k$, a compact convex body in an $SO(m+1)$-submodule of $S^2(\mathcal{H}_m^k)$, where $\mathcal{H}_m^k$ is the space of spherical harmonics of order k on S^m. This moduli parametrizes the spherical minimal immersions $f : S^m \to S_V$ into the unit sphere of a Euclidean vector space V, for various V (up to congruence on the range). By DoCarmo and Wallach (1971), $\mathcal{M}_m^k$ is nontrivial if and only if $m \geq 3$ and $k \geq 4$. The *domain dimension raising operator* defined on such minimal immersions in Toth (2002) gives rise to a linear imbedding $\mathcal{M}_m^k \to \mathcal{M}_{m+1}^k$ *onto a linear slice of* $\mathcal{M}_{m+1}^k$. We thus have the following:

Corollary *Theorem A holds with* $(\mathcal{M}_3^k)^{SU(2)}$ *replaced by* $\mathcal{M}_m^k$, $m \geq 3$.

Dropping the condition of conformality in the definition of a spherical minimal immersion one arrives at the concept of a λ_k-eigenmap $f : S^m \to S_V$, the only assumption being that all components $\alpha \circ f$, $\alpha \in V^*$, belong to $\mathcal{H}_m^k$. The DoCarmo–Wallach moduli $\mathcal{L}_m^k$ parametrizing the λ_k-eigenmaps of S^m into spheres (up to congruence) is once again a compact convex body in an $SO(m+1)$-submodule of $S^2(\mathcal{H}_m^k)$, and $\mathcal{M}_m^k$ is a linear slice of $\mathcal{L}_m^k$. By Toth (1994), $\mathcal{L}_m^k$ is nontrivial if and only if $m \geq 3$ and $k \geq 2$. We will be primarily interested in the moduli $\mathcal{L}_3^k$, and its $SU(2)$-equivariant part $(\mathcal{L}_3^k)^{SU(2)}$. As before, let $d((\mathcal{L}_3^k)^{SU(2)})$ denote the maximum dimension of a simplicial slice of $(\mathcal{L}_3^k)^{SU(2)}$ across 0. Once again, we have by Toth (2006)

$$d((\mathcal{L}_3^k)^{SU(2)}) \leq \max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(.,0) = \max_{\partial(\mathcal{M}_3^k)^{SU(2)}} \Lambda(.,0), \tag{5}$$

where the last equality holds as the eigenvalue computations do not reflect the difference between eigenmaps and spherical minimal immersions.

The analogue of Theorem A for eigenmaps indicates that the moduli $(\mathcal{L}_3^k)^{SU(2)}$ have simpler structure:

Theorem B *The maximum dimension* $d((\mathcal{L}_3^k)^{SU(2)})$ *of a simplicial slice of the moduli* $(\mathcal{L}_3^k)^{SU(2)}$, $k \geq 2$, *is the largest possible, that is, equality holds in* (5). *Thus, for* $k \geq 3$ *odd,* $(\mathcal{L}_3^k)^{SU(2)}$ *has a simplicial slice of dimension* $(k-1)/2$, *and for* $k \geq 2$ *even, it has a simplicial slice of dimension* k.

Once again, the domain dimension raising operator applied to the moduli $\mathcal{L}_m^k$ of λ_k-eigenmaps of S^m gives rise to a linear imbedding $\mathcal{L}_m^k \to \mathcal{L}_{m+1}^k$ onto a linear slice of $\mathcal{L}^k_{m+1}$ as in Toth (2002), and the second statement of Theorem B holds for $(\mathcal{L}^k_3)^{SU(2)}$ replaced by $\mathcal{L}^k_m$.

2 Measure of symmetry for the moduli

In this section we give an application why is it useful to give sharp lower bounds for $d((\mathcal{L}^k_3)^{SU(2)}$ and $d((\mathcal{M}^k_3)^{SU(2)}$.

In Toth (2006) a sequence of measures of symmetry $\{\sigma_\ell(\mathcal{L}, \mathcal{O})\}_{\ell\geq 1}$ was introduced for a (compact) convex body $\mathcal{L}$ of a Euclidean vector space $\mathcal{E}$ with a specified base point $\mathcal{O} \in \operatorname{int}(\mathcal{L})$. (For a comprehensive study of measures of symmetries, see Grünbaum 1963). The ℓth term is defined as

$$\sigma_\ell(\mathcal{L}, \mathcal{O}) = \inf_{\substack{C_0,\ldots,C_\ell\in\partial\mathcal{L}\\ \mathcal{O}\in[C_0,\ldots,C_\ell]}} \sum_{i=0}^{\ell} \frac{1}{1+\Lambda(C_i, \mathcal{O})},$$

where the square bracket means *convex hull* and $\Lambda : \partial\mathcal{L} \to \mathbf{R}$ is the distortion function. [At $C \in \partial\mathcal{L}$, $\Lambda(C, \mathcal{O})$ is the ratio that the point $\mathcal{O}$ splits the line segment, the portion of the line passing through C and $\mathcal{O}$ in $\mathcal{L}$]. Roughly speaking, $\sigma_\ell(\mathcal{L}, \mathcal{O})$ measures how distorted $\mathcal{L}$ is in dimension ℓ viewed from the base point $\mathcal{O}$. Clearly, $\sigma_1(\mathcal{L}, \mathcal{O}) = 1$.

Although they are affine invariants, the computation of $\sigma_\ell(\mathcal{L}, \mathcal{O})$, $\ell \geq 2$, (even for planar convex bodies) is difficult. In general, by Toth (2004, 2006), we have

$$1 \leq \sigma_\ell(\mathcal{L}, \mathcal{O}) \left(\leq \frac{\ell+1}{2}\right).$$

and the lower bound is attained if and only if $\mathcal{L}$ has an ℓ-dimensional simplicial slice across $\mathcal{O}$. In addition, for $\ell' \leq \ell$, we obviously have

$$\sigma_\ell(\mathcal{L}, \mathcal{O}) \leq \sigma_{\ell'}(\mathcal{L}, \mathcal{O}) + \frac{\ell-\ell'}{1+\max_{\partial\mathcal{L}}\Lambda(.,\mathcal{O})}. \tag{6}$$

Moreover, equality holds for $\ell' = \dim\mathcal{L}$, or equivalently, the sequence $\{\sigma_\ell(\mathcal{L}, \mathcal{O})\}_{\ell\geq 1}$ is arithmetic from the $\dim\mathcal{L}$-term onwards with difference $1/(1 + \max_{\partial\mathcal{L}}\Lambda(.,\mathcal{O}))$. In fact, for $\ell' \geq \dim\mathcal{L}$, as a consequence of the Carathéodory theorem, a minimal configuration contains a (necessarily minimal) simplicial configuration. (For a recent proof and generalizations, see Boltyanski and Martini 2001).

Let $d(\mathcal{L}, \mathcal{O})$ be the maximum dimension of a simplicial slice of $\mathcal{L}$ across $\mathcal{O}$. By the above, the sequence $\{\sigma_\ell(\mathcal{L}, \mathcal{O})\}_{\ell\geq 1}$ then starts with a string of 1's of length $d(\mathcal{L}, \mathcal{O})$, and, by a result in Toth (2008), the sequence is *strictly increasing* from the $d(\mathcal{L}, \mathcal{O})$th term onwards.

Setting $\ell' = d(\mathcal{L}, \mathcal{O})$ in (6), we obtain

$$\frac{\ell+1}{1+\max_{\partial\mathcal{L}}\Lambda(.,\mathcal{O})} \leq \sigma_\ell(\mathcal{L}, \mathcal{O}) \leq 1 + \frac{\ell-d(\mathcal{L}, \mathcal{O})}{1+\max_{\partial\mathcal{L}}\Lambda(.,\mathcal{O})}, \quad \ell \geq d(\mathcal{L}, \mathcal{O}), \tag{7}$$

where the first inequality is a trivial estimate of the defining equality of σ_ℓ. Comparison of the lower and upper bounds in (7) immediately implies that

$$d(\mathcal{L}, \mathcal{O}) \leq \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}). \tag{8}$$

Combining the upper bound in (7) and (8), as a byproduct, we obtain

$$\sigma_\ell(\mathcal{L}, \mathcal{O}) \leq \frac{\ell+1}{1+d(\mathcal{L}, \mathcal{O})}, \quad \ell \geq d(\mathcal{L}, \mathcal{O}). \tag{9}$$

If equality holds in (8) (that is, $\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})$ is an integer and it is the dimension of a maximal simplicial slice of $\mathcal{L}$ across $\mathcal{O}$) then

$$\sigma_\ell(\mathcal{L}, \mathcal{O}) = \max\left(1, \frac{\ell+1}{1+\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}\right), \quad \ell \geq 1, \tag{10}$$

or equivalently, the sequence $\{\sigma_\ell(\mathcal{L}, \mathcal{O})\}_{\ell\geq 1}$ is arithmetic from the $d(\mathcal{L}, \mathcal{O})$-term onwards. [Note that (10) is always true for $\ell = 1$]. We will see shortly that the converse is false, that is, (10) does not imply (8).

A natural (albeit difficult) problem is to calculate the measures of symmetry for all moduli $\mathcal{L}_m^k$, $\mathcal{M}_m^k$ and $(\mathcal{L}_3^k)^{SU(2)}$, $(\mathcal{M}_3^k)^{SU(2)}$ at least with respect to the origin (as base point).

Theorem B and the previous discussion however imply that, for $\mathcal{L} = (\mathcal{L}_3^k)^{SU(2)}$ and $\mathcal{O} = 0$ the origin, we have

$$\sigma_\ell((\mathcal{L}_3^k)^{SU(2)}, 0) = \max\left(1, \frac{\ell+1}{1+\max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(., 0)}\right), \quad \ell \geq 1, \tag{11}$$

where [according to (2) and (5)] $\max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(., 0)$ is k for k even, and $(k-1)/2$ for k odd. Once again a natural question is whether (11) is true with $(\mathcal{L}_3^k)^{SU(2)}$ replaced by $(\mathcal{M}_3^k)^{SU(2)}$. For example, as noted above (see also the remark in Sect. 5), the 9-dimensional moduli $(\mathcal{M}_3^5)^{SU(2)}$ has a *maximal* (two) dimensional slice, so that we have

$$\sigma_\ell((\mathcal{M}_3^5)^{SU(2)}, 0) = \max\left(1, \frac{\ell+1}{3}\right), \quad \ell \geq 1,$$

In general, Theorem A along with (9) give the following:

Corollary *For $k \geq 7$ odd, we have*

$$\sigma_\ell((\mathcal{M}_3^k)^{SU(2)}, 0) \leq \max\left(1, \frac{\ell+1}{2+m(k)}\right), \quad \ell \geq 1,$$

where $m(k)$ is given in (3). For $k \geq 10$ even, we have

$$\sigma_\ell((\mathcal{M}_3^k)^{SU(2)}, 0) \leq \max\left(1, \frac{\ell+1}{2+2m(k)}\right), \quad \ell \geq 1,$$

where $m(k)$ is given in (4).

We now return to the general case and consider the G-invariant setting, in which G is a compact (not necessarily connected) Lie group, $\mathcal{E}$ is an orthogonal G-module and $\mathcal{L} \subset \mathcal{E}$ is a G-invariant (compact) convex body with origin 0 in the interior of $\mathcal{L}$. Under a mild condition, we can calculate the measure of symmetry $\sigma_\ell(\mathcal{L}, 0)$ at least for $\ell \geq \dim \mathcal{L}$.

Proposition *If G acts on $\mathcal{L}$ with no nonzero fixed point (or equivalently, $\mathcal{E}$ has no trivial G-component) then, for $\ell \geq \dim \mathcal{L}$, we have*

$$\sigma_\ell(\mathcal{L}, 0) = \frac{\ell+1}{1+\max_{\partial\mathcal{L}} \Lambda(., 0)}.$$

Proof Let $C \in \partial\mathcal{L}$ such that $\Lambda(., 0)$ attains its maximum at C. The center of mass of the G-orbit $G(C)$ passing through C is G-fixed, and, due to our assumption, it must be the origin. The center of mass is also in the convex hull of this orbit, and we obtain $0 \in [G(C)]$. By Carathéodory's Theorem (Berger 1987), there exist $C_0, \ldots, C_n \in G(C)$, $n = \dim \mathcal{L}$ such that $0 \in [C_0, \ldots, C_n]$. Since $\mathcal{L}$ is G-invariant, $G(C) \subset \partial\mathcal{L}$ so that we have

$$\sigma_n(\mathcal{L}, 0) \leq \sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i, 0)} = \frac{n+1}{1+\max_{\partial\mathcal{L}} \Lambda(., 0)}.$$

Here we used that G acts on $\mathcal{L} \subset \mathcal{E}$ by isometries, and therefore $\Lambda(C_i, 0) = \Lambda(C, 0) = \max_{\partial\mathcal{L}} \Lambda(., 0)$. The opposite inequality obviously holds, so that the statement in the proposition holds for $\ell = n = \dim \mathcal{L}$. As noted above, the seguence $\{\sigma_l\}_{\ell \geq 1}$ is arithmetic for $\ell \geq \dim \mathcal{L}$ with difference $1/(1 + \max_{\partial\mathcal{L}} \Lambda(., 0))$ and the proposition now follows.

Remarks
1. The simplest example to the proposition is a regular polytope $\mathcal{P}$ (centered at the origin) since its symmetry group acts with no nonzero fixed points. The proposition then gives the measure of symmetry $\sigma_\ell(\mathcal{P}, 0)$ for $\ell = \dim \mathcal{P}$ (and base point the origin) in terms of the maximal distortion. The maximal distortion is attained at any vertex. Note that, in dimension two, for a regular odd-sided polygon, sharp inequality holds in (8), whereas, using the proposition for $\ell \geq 2$, (10) clearly holds. [For even sided polygons both (8) and (10) hold].
2. By construction, all moduli $\mathcal{L}_m^k$, $\mathcal{M}_m^k$, and $(\mathcal{L}_3^k)^{SU(2)}$, $(\mathcal{M}_3^k)^{SU(2)}$ satisfy the setting in the proposition, see Toth (2002). The Lie groups $SO(m+1)$ and $SU(2)'$ act in their respective spaces with no nonzero fixed points (Toth 2002, Corollary 2.3.4, p. 144). The proposition gives the measure of symmetry σ_ℓ of these moduli but only in their respective dimension ℓ (and with respect to the origin as base point). Note that to determine the maximal distortion for these moduli is a difficult problem

seemingly related to the long standing problem of finding minimum codimensional spherical minimal immersions (DeTurck and Ziller 1967, 1992, 1993; DoCarmo and Wallach 1971; Mashimo 1984; Toth 1999, 2002, 2000; Wallach 1972; Weingart 1999).

3 Representations of $SU(2)$

The irreducible complex $SU(2)$-modules are parametrized by their dimension, and they can be realized as submodules appearing in the (multiplicity one) decomposition of the $SU(2)$-module of complex homogeneous polynomials $\mathbf{C}[z, w]$ in two variables. (We use here some basic facts in representation theory, see Fulton and Harris 1991; Knapp 1986). For $k \geq 0$, the kth submodule W_k, $\dim_{\mathbf{C}} W_k = k+1$, comprises the homogeneous polynomials of degree k. With respect to the L^2-scalar product (suitably scaled) the standard orthonormal basis for W_k is $\{\xi_j\}_{j=0}^k$, where $\xi_j = z^{k-j}w^j/\sqrt{(k-j)!j!}$, $j = 0, \ldots, k$. For k odd, W_k is irreducible *as a real $SU(2)$-module*. For k even, the fixed point set R_k of the complex anti-linear self map of W_k given on the basis by $\xi_j \mapsto (-1)^j \xi_{k-j}$, $j = 0, \ldots, k$, is an *irreducible real submodule* with $W_k = R_k \otimes_{\mathbf{R}} \mathbf{C}$. Given a polynomial

$$\xi = \sum_{j=0}^{k} c_j \xi_j \in W_k \tag{12}$$

of unit norm, the orbit map $f_\xi : S^3 \to W_k$, $f_\xi(g) = g \cdot \xi = \xi \circ g^{-1}$, $g \in SU(2)$, (through ξ) maps into the unit sphere. In coordinates, we have

$$f_\xi(a, b)(z, w) = \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad a, b \in \mathbf{C}, |a|^2 + |b|^2 = 1, \ z, w \in \mathbf{C}. \tag{13}$$

Here $g = (a, b) \in S^3 \subset \mathbf{C}^2$ is identified with $\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \in SU(2)$, or equivalently, the unit quaternion $a + \jmath b \in S^3 \subset \mathbf{H}$. [On the right-hand side the inverse $g^{-1} = (\bar{a}, -b) = \bar{a} - \jmath b$ acts on $(z, w) = z + \jmath w$ by multiplication]. From the complex form of the Laplacian, it is clear that the components of f_ξ are harmonic, so that $f_\xi : S^3 \to S_{W_k}$ is a λ_k-eigenmap. If k is even and $\xi \in R_k$, then f_ξ maps into the unit sphere S_{R_k}, so that we obtain a λ_k-eigenmap $f_\xi : S^3 \to S_{R_k}$. (As we will see below, in both cases the components of f_ξ are orthonormal, with respect to an orthonormal basis on the range).

As examples, up to congruence on the range, the Hopf map $Hopf : S^3 \to S_{R_2}$ is the $SU(2)$-orbit of the polynomial $\xi = izw \in R_2$, and the quartic minimal immersion $\mathcal{I} : S^3 \to S_{W_4}$ is the $SU(2)$-orbit map of the polynomial $\xi = (\sqrt{6}/24)(z^4 - w^4) + (\sqrt{2}/4)z^2w^2 \in W_4$. (There is no minimal $SU(2)$-orbit in R_4. Note that, even though W_4 is reducible as a real $SU(2)$-module, $\mathcal{I}$ still has L^2-orthonormal components).

Substituting (12) into (13) we obtain

$$f_\xi(a, b)(z, w) = \sum_{j=0}^{k} \frac{c_j}{\sqrt{(k-j)!j!}} (\bar{a}z + \bar{b}w)^{k-j}(-bz + aw)^j. \tag{14}$$

To get a more specific expression, we introduce the coefficients $\{\chi_l^j\}_{j,l=0}^k$ through the generating function

$$(\bar{a}z + \bar{b}w)^{k-j}(-bz + aw)^j = \sum_{l=0}^{k} \chi_l^j(a,b) z^{k-l} w^l \tag{15}$$

As we will see below, up to scaling, $\{\chi_l^j\}_{j,l=0}^k$ are the *matrix coefficients* of the complex $SU(2)$-module W_k. [As a consequence of the Peter–Weyl theorem, see Fulton and Harris 1991; Knapp 1986, these matrix coefficients form an orthonormal Hilbert basis of $L^2(SU(2))$]. Using binomial expansions, χ_l^j can be (and usually is) expanded into a more specific form but we do not need this. Also, using the standard parametrization of $SU(2)$ one can also write χ_l^j in terms of Jacobi polynomials. (For details, see Vilenkin 1968; Vilenkin and Klimyk 1991).

By (15), χ_l^j is a harmonic homogeneous polynomial in $a, b, \bar{a}, \bar{b}$ with integer coefficients. Moreover, in these variables, it has bidegree $(j, k-j)$, so that $\chi_l^j \in \mathcal{H}^{j,k-j}$. Since $\chi_l^j(1,0) = \delta_{jl}$, we immediately see that, for fixed $j = 0, \ldots, k$, $\{\chi_l^j\}_{l=0}^k$ is a basis in $\mathcal{H}^{j,k-j}$.

The matrix elements of the $SU(2)$-module W_k are

$$\begin{aligned}
\langle (a,b)\cdot \xi_j, \xi_l\rangle &= \frac{1}{\sqrt{(k-j)!j!(k-l)!l!}} \langle (a,b)(z^{k-j}w^j), z^{k-l}w^l)\rangle \\
&= \frac{1}{\sqrt{(k-j)!j!(k-l)!l!}} \langle (\bar{a}z + \bar{b}w)^{k-j}(-bz + aw)^j, z^{k-l}w^l)\rangle \\
&= \frac{1}{\sqrt{(k-j)!j!(k-l)!l!}} \left\langle \sum_{m=0}^{k} \chi_m^j(a,b) z^{k-m}w^m, z^{k-l}w^l \right\rangle \\
&= \sqrt{\frac{(k-l)!l!}{(k-j)!j!}}\, \chi_l^j(a,b).
\end{aligned}$$

Since W_k is irreducible, Schur's orthogonality relations give that

$$\sqrt{\frac{(k-l)!l!}{(k-j)!j!}}\, \chi_l^j(a,b), \quad j, l = 0\ldots, k,$$

are L^2*-orthogonal with the same norm*. In particular, $\{\sqrt{(k-l)!l!}\chi_l^j\}_{l=0}^k$ (up to scaling) is an orthonormal basis for $\mathcal{H}^{j,k-j}$.

The polynomials χ_l^j have many symmetry properties. We only need the following:

$$\overline{\chi_l^j(a,b)} = (-1)^{j+l}\chi_{k-l}^{k-j}(a,b). \tag{16}$$

This follows easily by substituting $(a,b) \mapsto (\bar{a}, \bar{b})$, $(z,w) \mapsto (-w, z)$ in (15) and comparing coefficients.

L^2-orthonormality of χ_l^j along with (16), implies that, for k odd, the set of real polynomials

$$\sqrt{\frac{(k-l)!l!}{(k-j)!j!}}\left\{\Re(\chi_l^j),\Im(\chi_l^j)\right\}_{0\le j\le (k-1)/2,\, 0\le l\le k} \tag{17}$$

form, up to scaling, an *orthonormal* basis for the real spherical harmonics $\mathcal{H}_3^k$. Similarly, for k even, an orthonormal basis is given by

$$\sqrt{\frac{(k-l)!l!}{(k-j)!j!}}\left\{\Re(\chi_l^j),\Im(\chi_l^j)\right\}_{0\le j< k/2,\, 0\le l\le k}$$
$$\bigcup \frac{\sqrt{(k-l)!l!}}{(k/2)!}\left\{\Re(\chi_l^{k/2}),\Im(\chi_l^{k/2})\right\}_{0\le l< k/2}\bigcup\left\{\chi_{k/2}^{k/2}\right\}. \tag{18}$$

4 $SU(2)$-equivariant eigenmaps

For $k\ge 3$ odd, the $SU(2)$-module structure of the space of *real* spherical harmonics $\mathcal{H}_3^k$ is given as follows. First, as complex $U(2)$-modules, we have

$$\mathcal{H}_3^k|_{U(2)}=\sum_{j=0}^{k}\mathcal{H}^{j,k-j}=\sum_{j=0}^{(k-1)/2}(\mathcal{H}^{j,k-j}\oplus\overline{\mathcal{H}^{j,k-j}}),$$

where we used $\mathcal{H}^{k-j,j}=\overline{\mathcal{H}^{j,k-j}}$. Restricting further to $SU(2)$, we have $\mathcal{H}^{j,k-j}|_{SU(2)}=W_k$. Thus, the space of real spherical harmonics decomposes as

$$\mathcal{H}_3^k|_{SU(2)}=\frac{k+1}{2}W_k,$$

where W_k is viewed as a *real* (irreducible) $SU(2)$-module. We will use $\{\xi_j, i\xi_j\}_{j=0}^k$ as a basis for the real $SU(2)$-module W_k.

Now, for each $j=0,\ldots,(k-1)/2$, we consider the λ_k-eigenmap $f_{\xi_j}: S^3\to S_{W_k}$. An easy computation shows that

$$\begin{aligned} f_{\xi_j}&=\sum_{l=0}^{k}\sqrt{(k-l)!l!}\,\chi_l^j\,\xi_l\\ &=\sum_{l=0}^{k}\sqrt{(k-l)!l!}\,\Re(\chi_l^j)\,\xi_l+\sum_{l=0}^{k}\sqrt{(k-l)!l!}\,\Im(\chi_l^j)\,(i\xi_l). \end{aligned}$$

We see that (up to scaling) the components of f_{ξ_j} form the standard orthonormal basis of $\mathcal{H}^{j,k-j}$ considered as a real $SU(2)$-module. (It is also clear that f_{ξ_j} and $f_{\xi_{k-j}}$ are congruent.)

We now let $k \geq 2$ be even. An orthonormal basis $\{\eta_j\}_{j=1}^{k+1}$ in the real $SU(2)$-module R_k is given by

$$\begin{aligned}
\eta_{2l+1} &= \frac{1}{\sqrt{2}}\left(\xi_l + (-1)^l \xi_{k-l}\right), \quad l = 0, \dots, k/2 - 1, \\
\eta_{2l+2} &= \frac{i}{\sqrt{2}}\left(\xi_l - (-1)^l \xi_{k-l}\right), \quad l = 0, \dots, k/2 - 1, \\
\eta_{k+1} &= i^{k/2} \xi_{k/2}.
\end{aligned}$$

For each $j = 1, \dots, k+1$, we consider the λ_k-eigenmap $f_{\eta_j} : S^3 \to S_{R_k}$. A somewhat tedious computation gives

$$\begin{aligned}
f_{\eta_{2l+1}} &= \sum_{m=0}^{k/2-1} \sqrt{(k-m)!m!}\left(\Re(\chi_m^l) + (-1)^m \Re(\chi_{k-m}^l)\right)\eta_{2m+1} \\
&\quad + \sum_{m=0}^{k/2-1} \sqrt{(k-m)!m!}\left(\Im(\chi_m^l) - (-1)^m \Im(\chi_{k-m}^l)\right)\eta_{2m+2} \\
&\quad + \sqrt{2}(k/2)!\Re((-i)^{k/2}\chi_{k/2}^l)\eta_{k+1}, \\
f_{\eta_{2l+2}} &= -\sum_{m=0}^{k/2-1} \sqrt{(k-m)!m!}\left(\Im(\chi_m^l) + (-1)^m \Im(\chi_{k-m}^l)\right)\eta_{2m+1} \\
&\quad + \sum_{m=0}^{k/2-1} \sqrt{(k-m)!m!}\left(\Re(\chi_m^l) - (-1)^m \Re(\chi_{k-m}^l)\right)\eta_{2m+2} \\
&\quad + -\sqrt{2}(k/2)!\Im((-i)^{k/2}\chi_{k/2}^l)\eta_{k+1}, \\
f_{\eta_{k+1}} &= \sqrt{2}\sum_{m=0}^{k/2-1} \Re(i^{k/2}\chi_m^{k/2})\eta_{2m+1} \\
&\quad + \sqrt{2}\sum_{m=0}^{k/2-1} \Im(i^{k/2}\chi_m^{k/2})\eta_{2m+2} \\
&\quad + (k/2)!\chi_{k/2}^{k/2}\eta_{k+1}.
\end{aligned}$$

Once again, we see that these λ_k-eigenmaps have orthonormal components, and their spaces of components are mutually orthogonal.

We now need a general result which facilitates the location of simplicial slices in the moduli $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^k)^{SU(2)}$. We begin with two lemmas. Given a λ_k-eigenmap $f : S^m \to S_V$, the *space of components* of f is defined by $V_f = \{\alpha \circ f \mid \alpha \in V^*\}$. (Without loss of generality, we will always assume that f is full, that is, the image of f spans V. Equivalently, precomposition by f is a linear isomorphism $V^* \to V_f$. An eigenmap can always be made full by restiction to the linear span of the image.) Under the DoCarmo–Wallach parametrization, f corresponds to a parameter point

$\langle f \rangle$ in $\mathcal{L}_m^k$. This point is on the boundary of $\mathcal{L}_m^k$ if and only if $V_f \subset \mathcal{H}_m^k$ is a proper subspace. In this case we call f an *eigenmap of boundary type*.

Given a λ_k-eigenmap of boundary type $f : S^m \to S_V$, the line $\mathbf{R}\cdot\langle f \rangle$ intersects $\partial\mathcal{L}_m^k$ at another point called the *antipodal* of $\langle f \rangle$. As in Sect. 2, the ratio that the origin splits the line segment between $\langle f \rangle$ and $\langle f^o \rangle$ is the distortion $\Lambda(\langle f \rangle, 0)$. A representative λ_k-eigenmap $f^o : S^m \to S_{V^o}$ of this antipodal point is called an *antipodal eigenmap* of f. f^o is unique up to congruence.

Lemma 1 *Let $f : S^m \to S_V$ be a λ_k-eigenmap of boundary type. If, relative to an orthonormal basis in the range V, f has orthonormal components then the same holds for an antipodal $f^o : S^m \to S_{V^o}$, and $V_f + V_{f^o} = \mathcal{H}_m^k$ is an orthogonal direct sum.*

Proof See Toth (1999) or Theorem 2.3.14 in Toth (2002).

Lemma 2 *Let $f_j : S^m \to S_{V_j}$, $j = 1, \ldots, l$, be λ_k-eigenmaps. Let C be in the convex hull $[\langle f_1 \rangle, \ldots, \langle f_l \rangle]$ with $C = \sum_{j=1}^l \alpha_j \langle f_j \rangle$, $\sum_{j=1}^l \alpha_j = 1$, $0 \leq \alpha_j \leq 1$, $j = 1, \ldots, l$. Then C is represented by the λ_k-eigenmap $f = (\sqrt{\alpha_1} f_1, \cdots, \sqrt{\alpha_l} f_l) : S^m \to S_V$, $V = V_1 \times \cdots \times V_l$ (made full). In particular, we have $V_f \subset V_{f_1} + \ldots + V_{f_l}$.*

Proof The statement follows easily from the definition of the moduli space. (Note that, for $l = 2$, this is the Connecting Lemma in Toth and Ziller 1999).

Proposition *Let $f_j : S^m \to S_{V_j}$, $j = 1, \ldots, l$, be λ_k-eigenmaps with orthonormal components (relative to orthonormal bases in the ranges). Assume that the spaces of components V_{f_j}, $j = 1, \ldots, l$, are mutually orthogonal and $V_{f_1} + \cdots + V_{f_l} \neq \mathcal{H}_m^k$. Then there exists a λ_k-eigenmap $f_0 : S^m \to S_{V_0}$ such that the convex hull $[\langle f_0 \rangle, \ldots, \langle f_l \rangle]$ is an l-dimensional simplicial slice of $\mathcal{L}_m^k$ across the origin 0.*

Proof By Lemma 2, the assumption on the spaces of components implies that the convex hull $[\langle f_1 \rangle, \ldots, \langle f_l \rangle]$ is an $(l-1)$-simplex. In addition, any λ_k-eigenmap $f : S^m \to S_V$ corresponding to a parameter point in the relative interior of this simplex has space of components $V_f = V_{f_1} + \cdots + V_{f_m}$. Since this sum is a proper subspace of $\mathcal{H}_m^k$, the entire simplex is on the boundary of the moduli space $\mathcal{L}_m^k$. Moreover, passing from the relative interior to the boundary of the simplex $[\langle f_1 \rangle, \ldots, \langle f_l \rangle]$, the space of components of the corresponding eigenmaps decreases, so that the intersection of the affine span of $[\langle f_1 \rangle, \ldots, \langle f_l \rangle]$ with $\mathcal{L}_m^k$ is again $[\langle f_1 \rangle, \ldots, \langle f_l \rangle]$.

Again by Lemma 2, a λ_k-eigenmap representing a point $C = \sum_{j=1}^l \alpha_j \langle f_j \rangle$ in the convex hull $[\langle f_1 \rangle, \ldots, \langle f_l \rangle]$ is of the form $(\sqrt{\alpha_1} f_1, \ldots, \sqrt{\alpha_l} f_l) : S^m \to S_V$, $V = V_1 \times \cdots \times V_l$. We now impose the condition for this map to have orthonormal components. This holds if and only if $\alpha_j^2 / \dim V_j$ does not depend on $j = 1, \ldots, l$. The condition $\sum_{j=1}^l \alpha_j = 1$ then gives $\alpha_j = \sqrt{\dim V_j} / \sum_{s=1}^l \sqrt{\dim V_s}$. The λ_k-eigenmap obtained this way is denoted by $f : S^m \to S_V$.

Let $f_0 : S^m \to S_{V_0}$ be a full λ_k-eigenmap such that $\langle f \rangle$ a$\langle f_0 \rangle$ are antipodal. Using Lemma 1 we see that since f has orthonormal components so does f_0, and V_{f_0} is the orthogonal complement of V_f. We obtain the orthogonal direct sum

$$V_{f_0} + V_{f_1} + \cdots + V_{f_l} = \mathcal{H}_m^k.$$

Finally, for $j = 1, \ldots, l$, the jth face $[\langle f_0\rangle, \ldots, \widehat{\langle f_j\rangle}, \ldots, \langle f_l\rangle]$ opposite to $\langle f_j\rangle$ is on the boundary of $\mathcal{L}_m^k$ since each point in the relative interior of this face represents a λ_k-eigenmap with space of components $V_{f_0} + \cdots + \widehat{V_{f_j}} + \cdots + V_{f_l} \neq \mathcal{H}_m^k$. The proposition follows.

As an immediate application, note that, for $k \geq 3$ odd, the $(k-1)/2$ λ_k-eigenmaps $f_{\xi_j} : S^3 \to S_{W_k}$, $j = 1, \ldots, (k-1)/2$, and, for $k \geq 2$ even, the k λ_k-eigenmaps $f_{\eta_{2l+1}}, f_{\eta_{2l+2}} : S^3 \to S_{R_k}, l = 0, \ldots, k/2-1$, satisfy the conditions of the proposition. Thus Theorem B follows. (Note that, for k odd, the λ_k-eigenmap whose existence is guaranteed by the proposition is (congruent to) f_{ξ_0}, and, for k even, f_0 is (congruent to) the λ_k-eigenmap is $f_{\eta_{k+1}}$.)

5 Proof of Theorem A

First, let $k \geq 5$ be odd. Let $\xi \in W_k$ as in (12). $f_\xi : S^3 \to S_{W_k}$ is a spherical minimal immersion if and only if the coefficients in (12) satisfy the following

$$
\begin{gathered}
\sum_{j=0}^{k} |c_j|^2 = 1 \\
\sum_{j=0}^{k} (2j-k)^2 |c_j|^2 = \frac{k(k+2)}{3}, \\
\sum_{j=0}^{k-2} \sqrt{(j+1)(j+2)(k-j-1)(k-j)}\, c_j \bar{c}_{j+2} = 0, \\
\sum_{j=0}^{k-1} (k-2j-1) \sqrt{(j+1)(k-j)}\, c_j \bar{c}_{j+1} = 0.
\end{gathered}
\tag{19}
$$

(See Mashimo 1984; DeTurck and Ziller 1992, 1993; Toth 2002.) The first equation just means that ξ has unit norm, or equivalently, f_ξ is a spherical λ_k-eigenmap. (Note that it is clear now that f_ξ has L^2-orthonormal components.) The last three equations are conformality conditions of f_ξ on the tangent space of S^3 at 1.

Remarks In a few cases with $k \geq 4$ the system in (19) can be solved explicitly. As an example, let $k = 5$ and in (19) set $c_3 = \pm\bar{c}_2$, $c_4 = \pm\bar{c}_1$, and $c_5 = \pm\bar{c}_0$. The first two equations of (19) give

$$
|c_0|^2 = |c_5|^2 = \frac{3}{16}, \quad |c_1|^2 = |c_4|^2 = \frac{5}{48}, \quad |c_2|^2 = |c_3|^2 = \frac{11}{48}.
$$

The fourth equation is automatically satisfied while the third

$$
\sqrt{5} c_0 \bar{c}_2 \pm 3 c_1 c_2 = 0
$$

can be resolved easily. We obtain two quintic spherical minimal immersions $f_\pm : S^3 \to S_{W_5} = S^{11}$. One easily shows that V_{f_+} and V_{f_-} are L_2-orthogonal. Using the proposition above, we see that $\langle f_\pm \rangle$ are two vertices of a (maximal) triangular slice of $(\mathcal{M}_4^5)^{SU(2)}$.

Turning to the proof of Theorem A, we now let

$$\zeta_j = c_j \xi_j + c_{(k+1)/2+j} \xi_{(k+1)/2+j}, \quad 0 \le j < (k-1)/4.$$

We have

$$\begin{aligned} f_{\zeta_j} &= \sum_{l=0}^k \sqrt{(k-l)!l!}\, \Re \left(c_j \chi_l^j + c_{(k+1)/2+j} \chi_l^{(k+1)/2+j} \right) \xi_l \\ &\quad + \sum_{l=0}^k \sqrt{(k-l)!l!}\, \Im \left(c_j \chi_l^j + c_{(k+1)/2+j} \chi_l^{(k+1)/2+j} \right) (i \xi_l). \end{aligned}$$

Using (16), we obtain

$$\begin{aligned} f_{\zeta_j} &= \sum_{l=0}^k \sqrt{(k-l)!l!} \left(\Re(c_j \chi_l^j) + (-1)^{(k+1)/2+j+l} \Re(\bar{c}_{(k+1)/2+j} \chi_{k-l}^{(k-1)/2-j}) \right) \xi_l \\ &\quad + \sum_{l=0}^k \sqrt{(k-l)!l!} \left(\Im(c_j \chi_l^j) - (-1)^{(k+1)/2+j+l} \Im(\bar{c}_{(k+1)/2+j} \chi_{k-l}^{(k-1)/2-j}) \right) (i \xi_l). \end{aligned}$$

The gap between the indices of the coefficients of c_j and $c_{(k+1)/2+j}$ is ≥ 3 so that the third and fourth equations in (19) are automatically satisfied. Consequently, f_{ζ_j} is a spherical minimal immersion if and only if the first two equations hold:

$$\begin{aligned} |c_j|^2 + |c_{(k+1)/2+j}|^2 &= 1, \\ (k-2j)^2 |c_j|^2 + (2j+1)^2 |c_{(k+1)/2+j}|^2 &= \frac{k(k+2)}{3}. \end{aligned} \tag{20}$$

The constraint $0 \le j < (k-1)/4$ guarantees that the components of f_{ζ_j} are orthonormal, and that the space of components $V_{f_{\zeta_j}}$ are mutually orthogonal. To apply the proposition, it remains to see under what conditions does the system (20) have solution $x = |c_j|^2$ and $y = |c_{(k+1)/2+j}|^2$.

The determinant of this system is $(k+1)(4j+1-k) < 0$ so that x and y are unique. The additional condition $0 \le x \le 1$ gives

$$(2j+1)^2 \le \frac{k(k+2)}{3} \le (k-2j)^2, \quad 0 \le j < \frac{k-1}{4}.$$

This system can easily be resolved and we obtain

$$0 \le 2j \le k - \sqrt{\frac{k(k+2)}{3}}.$$

Theorem A follows in this case.

For $k \geq 4$ even, $\xi \in R_k$ in (12) if and only if $c_{k-j} = (-1)^j \bar{c}_j, \quad j = 0, \ldots, k/2-1$, and $c_{k/2} = i^{k/2} a_{k/2}$, $a_{k/2} \in \mathbf{R}$. With these, we have

$$\xi = \sum_{j=0}^{k/2-1} \left(c_j \xi_j + (-1)^j \bar{c}_j \xi_{k-j} \right) + i^{k/2} a_{k/2} \xi_{k/2},$$

or equivalently

$$\xi = \sqrt{2} \sum_{l=0}^{k/2-1} (\Re(c_l)\eta_{2l+1} + \Im(c_l)\eta_{2l+2}) + a_{k/2}\eta_{k/2}. \tag{21}$$

Incorporating the new conditions on the coefficients, we see that $f_\xi : S^3 \to S_{R_k}$ is a spherical minimal immersion if and only if we have

$$\begin{aligned}
2 \sum_{l=0}^{k/2-1} |c_l|^2 + a_{k/2}^2 &= 1 \\
2 \sum_{l=0}^{k/2-1} (2l-k)^2 |c_l|^2 &= \frac{k(k+1)}{3} \\
\sum_{l=0}^{k/2-3} \sqrt{(l+1)(l+2)(k-l-1)(k-l)}\, c_l \bar{c}_{l+2} + (-1)^{k/2+1}(k/4)(k/2+1) c_{k/2-1}^2 & \\
+ (-i)^{k/2} \sqrt{(k/2-1)(k/2)(k/2+1)(k/2+2)}\, c_{k/2-2} a_{k/2} &= 0 \\
\sum_{l=0}^{k/2-2} (k-2l-1)\sqrt{(l+1)(k-l)}\, c_l \bar{c}_{l+1} + (-i)^{k/2} \sqrt{(k/2)(k/2+1)}\, c_{k/2-1} a_{k/2} &= 0
\end{aligned} \tag{22}$$

Turning to the proof of the second case, for $l = 0, \ldots, k/2-2$, we define

$$\begin{aligned}
\zeta_{2l+1} &= \sqrt{2}(a_l \eta_{2l+1} + a_{k/2-l-2} \eta_{k-2l-3}) \\
\zeta_{2l+2} &= \sqrt{2}(b_l \eta_{2l+2} + b_{k/2-l-2} \eta_{k-2l-2}),
\end{aligned}$$

where a_l, $b_l \in \mathbf{R}$. The reason for omitting $l = k/2-1$ is the presence of the corresponding coefficient $c_{k/2-1}$ in the third equation of (22). ζ_{2l+1} and ζ_{2l+2} are special cases of (12) with $c_l = a_l$, $c_{k/2-l-2} = a_{k/2-l-2} \in \mathbf{R}$ real and $c_l = ib_l$, $c_{k/2-l-2} = ib_{k/2-l-2} \in i\mathbf{R}$ purely imaginary. Thus, we have

$$\begin{aligned}
f_{\zeta_{2l+1}} &= \sqrt{2}(a_l f_{\eta_{2l+1}} + a_{k/2-l-2} f_{\eta_{k-2l-3}}) \\
f_{\zeta_{2l+2}} &= \sqrt{2}(b_l f_{\eta_{2l+2}} + b_{k/2-l-2} f_{\eta_{k-2l-2}}).
\end{aligned}$$

Just as in the previous case, to simplify the system in (22), we impose a gap ≥ 3 between the indices of the coefficients. This condition amounts to the restriction

$0 \le l \le [(k-10)/4]$. As before, we then see that $f_{\zeta_{2l+1}}$ and $f_{\zeta_{2l+2}}$ are spherical minimal immersions if and only if the first two equations in (22) are satisfied, that is, if and only if $(|a_l|^2, |a_{k/2-l-2}|^2)$ and $(|b_l|^2, |b_{k/2-l-2}|^2)$ are both solutions (x, y) of the system

$$\begin{aligned} x + y &= 1/2 \\ (k-2l)^2 x + (2l+4)^2 y &= k(k+2)/6. \end{aligned} \tag{23}$$

The constraint $0 \le l \le [(k-10)/4]$ guarantees that the components of $f_{\zeta_{2l+1}}$ and $f_{\zeta_{2l+2}}$ are orthonormal, and that the space of components $V_{f_{\zeta_{2l+1}}}, V_{f_{\zeta_{2l+2}}}, 0 \le l \le [(k-10)/4]$ are all mutually orthogonal. To apply the proposition, it remains to study solvability of the system in (23) with nonnegative solutions.

Once again, the determinant of this system is $(k+4)(4l+4-k) < 0$ so that x and y are unique. The condition $0 \le x \le 1/2$ gives

$$(2l+4)^2 \le \frac{k(k+2)}{3} \le (k-2l)^2.$$

This can be easily resolved and, along with the previous constraint, we finally arrive at

$$0 \le 2l \le \min\left(k - \sqrt{\frac{k(k+2)}{3}}, \frac{k-10}{2}\right).$$

Theorem A follows.

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