Geom Dedicata (2012) 160:1–14
DOI 10.1007/s10711-011-9667-z

ORIGINAL PAPER

A measure of symmetry for the moduli of spherical minimal immersions

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Received: 1 September 2010 / Accepted: 4 October 2011 / Published online: 15 October 2011

Abstract A Grünbaum type of measure of symmetry is calculated and estimated for the DoCarmo-Wallach moduli spaces for eigenmaps and spherical minimal immersions. The DeTurck-Ziller classification of minimal imbeddings of 3-dimensional space forms is used to obtain exact determination of the measure for the $SU(2)$-equivariant moduli.

Keywords DoCarmo-Wallach moduli · DeTurck-Ziller classification · Measure of symmetry · Convex set · Distortion

Mathematics Subject Classification (2000) 52A05 · 52A38 · 52B11

1 Introduction and statement of results

Given a compact homogeneous Riemannian manifold M and an eigenvalue λ of the Laplace-Beltrami operator acting on $C^\infty(M)$, a (spherical) λ-eigenmap $f : M \to S_V$ into the unit sphere S_V of a Euclidean vector space V is a map whose components $\alpha \circ f, \alpha \in V^*$, are in the eigenspace $\mathcal{H}_\lambda$ corresponding to the eigenvalue λ. (Such maps are harmonic in the sense of Eells-Sampson with constant energy-density $\lambda/2$. For details, see [9] or Appendix 2 in [23].) Assuming that M is isotropy irreducible, a conformal λ-eigenmap $f : M \to S_V$ is called a spherical minimal immersion. The conformality factor is then $\lambda/\dim M$ and f is an isometric minimal immersion of M into S_V with respect to $\lambda/\dim M$-times the original metric on M. (See [8,23,27,28] or the brief summary in Section 1 of [26].)

The DoCarmo-Wallach moduli spaces parametrize spherical eigenmaps and spherical minimal immersions $f : M \to S_V$ for various Euclidean vector spaces V.

For a given eigenvalue λ, let $S^2_0(\mathcal{H}_\lambda)$ denote the space of traceless symmetric endomorphisms of $\mathcal{H}_\lambda$. Within $S^2_0(\mathcal{H}_\lambda)$, the DoCarmo-Wallach moduli spaces are linear slices of the convex body

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$$\mathcal{K}_\lambda = \{C \in S_0^2(\mathcal{H}_\lambda) \mid C + I \geq 0\},$$

where $\geq$ means positive semi-definite. The slices are by linear subspaces $\mathcal{E}_\lambda$ (spherical eigenmaps) and $\mathcal{F}_\lambda$ (spherical minimal immersions), where $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda \subset S_0^2(\mathcal{H}_\lambda)$ are defined by certain orthogonality relations in terms of the Dirac delta $\delta_\lambda : M \to S_{\mathcal{H}^*}$ and its differential [23,28].

Beyond the fact that the moduli $\mathcal{L}_\lambda = \mathcal{K}_\lambda \cap \mathcal{E}_\lambda$ and $\mathcal{M}_\lambda = \mathcal{K}_\lambda \cap \mathcal{F}_\lambda$ are convex bodies in their ambient linear spans $\mathcal{E}_\lambda$ and $\mathcal{F}_\lambda$, very little is known about their structure.

If G is a transitive group of isometries of M, then the eigenspace $\mathcal{H}_\lambda$ carries a natural G-module structure, and $\mathcal{E}_\lambda$ and $\mathcal{F}_\lambda$ are G-submodules with respect to the extended G-module structure on $S_0^2(\mathcal{H}_\lambda)$. On the level of the spherical maps, this G-action is given by precomposition so that the moduli $\mathcal{L}_\lambda$, and $\mathcal{M}_\lambda$ are also naturally G-invariant.

For a compact rank one symmetric space $M = G/K$ [2], the eigenspaces $\mathcal{H}_\lambda$ are irreducible [13–15], and the structure of the quotient $S_0^2(\mathcal{H}_\lambda)/\mathcal{E}_\lambda$, in particular, $\dim \mathcal{E}_\lambda$ is known [22]. In fact, the finite sums of products $\mathcal{H}_\lambda \cdot \mathcal{H}_\lambda$ of functions in $\mathcal{H}_\lambda$ is a G-submodule of $S^2(\mathcal{H}_\lambda)$, and

$$\mathcal{E}_\lambda = S^2(\mathcal{H}_\lambda)/\mathcal{H}_\lambda \cdot \mathcal{H}_\lambda. \tag{1}$$

If $\{\lambda_k\}_{k\geq 1}$ denotes the sequence of eigenvalues in increasing order, then we have

$$\mathcal{H}_{\lambda_k} \cdot \mathcal{H}_{\lambda_k} = \begin{cases} \sum_{i=0}^{k} \mathcal{H}_{\lambda_{2i}} & \text{if } M = S^m \\ \sum_{i=0}^{2k} \mathcal{H}_{\lambda_i} & \text{otherwise} \end{cases} \tag{2}$$

Combining (1)–(2) gives $\dim \mathcal{E}_\lambda = \dim \mathcal{L}_\lambda$.

For the Euclidean sphere $M = S^m$ and $G = SO(m+1)$, we write $\mathcal{H}_m^k = \mathcal{H}_{\lambda_k}$, $\mathcal{E}_m^k = \mathcal{E}_{\lambda_k}$, etc. The decomposition of $\mathcal{E}_m^k$ into irreducible $SO(m+1)$-components (in terms of highest weights) has been calculated in [8,23,27]. This shows that the moduli space $\mathcal{L}_m^k$ parametrizing spherical λ_k-eigenmaps $f : S^m \to S_V$ is nontrival if and only if $m \geq 3$ and $k \geq 2$. (Triviality of the moduli for $m = 2$ is known as Calabi's rigidity [4].) The first nontrivial domain S^3 is special in view of the splitting of the acting group $SO(4) = SU(2) \cdot SU(2)'$. The fixed point sets $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{L}_3^k)^{SU(2)'}$ are linear slices of $\mathcal{L}_3^k$. Moreover, by restriction, they are mutually orthogonal $SU(2)'$- and $SU(2)$-submodules of $\mathcal{L}_3^k$. Since they parametrize $SU(2)$- and $SU(2)'$-equivariant eigenmaps, they are called *equivariant moduli*. Note that $SU(2)'$ is a conjugate of $SU(2)$ within $SO(4)$, and the module structures on the respective equivariant moduli are isomorphic via this conjugation.

The first nontrivial moduli $\mathcal{L}_3^2$ is particularly simple, as it is the convex hull of $(\mathcal{L}_3^2)^{SU(2)}$ and $(\mathcal{L}_3^2)^{SU(2)'}$. In addition, $(\mathcal{L}_3^2)^{SU(2)}$ is the convex hull of the $SU(2)'$-orbit of the parameter point $\langle Hopf \rangle$ corresponding to the Hopf map $Hopf : S^3 \to S^2$. This orbit, in turn, is the real projective plane imbedded into a copy of the 4-sphere in $(\mathcal{E}_3^2)^{SU(2)}$ as a Veronese surface. In particular, $\dim \mathcal{L}_3^2 = 2 \dim (\mathcal{L}_3^2)^{SU(2)} = 10$. (For more details, see [23].)

The moduli $\mathcal{M}_\lambda$ has been extensively studied only for the Euclidean m-sphere S^m and $G = SO(m+1)$. (This is partially due to the complexity of the decomposition of $\mathcal{F}_\lambda$ into irreducible components for non-spherical compact rank one symmetric spaces. For example, for the complex projective space, $\mathcal{F}_\lambda$ fails to have multiplicity one decomposition [22].) For $M = S^m$, in [8], DoCarmo and Wallach gave a lower bound for the dimension of $\mathcal{M}_m^k$. They showed that the quotient $\mathcal{E}_m^k/\mathcal{F}_m^k$ is contained in the sum of all class one subrepresentations in $S_0^2(\mathcal{H}_m^k)$ with respect to the pair $(SO(m+1), SO(m))$. They conjectured that the lower estimate was sharp. This has first been proved in [25]. (For a more recent detailed proof, see [23]. For a different approach and proof, see [28].) With this the decomposition of $\mathcal{F}_m^k$

into irreducible $SO(m+1)$-components is determined, in particular, the exact dimension $\dim \mathcal{M}_m^k = \dim \mathcal{F}_m^k$ is known.

The moduli $\mathcal{M}_m^k$ is nontrivial if and only if $m \geq 3$ and $k \geq 4$. The first nontrivial moduli $\mathcal{M}_3^4$, an 18-dimensional convex body, has been described in [26]. (See also [23,28].) Once again, $\mathcal{M}_3^4$ is the convex hull of the orthogonal 9-dimensional slices $(\mathcal{M}_3^4)^{SU(2)}$ and $(\mathcal{M}_3^4)^{\bar{SU}(2)}$, but the structure of these slices is more subtle.

In general, very little is known about the geometry of the moduli $\mathcal{L}_m^k$ and $\mathcal{M}_m^k$, and even the simpler $SU(2)$-equivariant moduli $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^k)^{SU(2)}$. The degree raising operator gives rise to $SO(m+1)$-equivariant linear imbeddings $\mathcal{L}_m^k \to \mathcal{L}_m^{k+1}$ and $\mathcal{M}_m^k \to \mathcal{M}_m^{k+1}$, but the images are only *properly contained* in linear slices of $\mathcal{L}_m^{k+1}$ and $\mathcal{M}_m^{k+1}$ [24]. As a related problem, not much is known how the range dimensions change under these imbeddings. The domain dimension raising operator [23] does give linear imbeddings $\mathcal{L}_m^k \to \mathcal{L}_{m+1}^k$ and $\mathcal{M}_m^k \to \mathcal{M}_{m+1}^k$ *onto* linear slices of the respective moduli but it increases the range dimension by $\dim \mathcal{H}_{m+1}^k / \mathcal{H}_m^k = \mathcal{O}(k^m)$.

A sequence of measures of symmetry $\{\sigma_m\}_{m \geq 1}$ for convex bodies à la Grünbaum [11,12] was introduced and studied in [19–21]. For a convex body $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$, and a point $\mathcal{O}$ in the interior of $\mathcal{L}$, $\sigma_m(\mathcal{L}, \mathcal{O})$ measures how far the m-dimensional affine slices of $\mathcal{L}$ (through $\mathcal{O}$) are from being symmetric (viewed from $\mathcal{O}$). The measure of symmetry $\sigma_m(\mathcal{L}, \mathcal{O})$ is defined as follows.

First, convexity of $\mathcal{L}$ implies that any line passing through $\mathcal{O}$ intersects the boundary of $\mathcal{L}$ at two *antipodal* points. If $C \in \partial\mathcal{L}$ with antipodal $C^o \in \partial\mathcal{L}$ then $\mathcal{O}$ splits the line segment $[C, C^o]$ into the ratio

$$\Lambda(C, \mathcal{O}) = \frac{d(C, \mathcal{O})}{d(C^o, \mathcal{O})},$$

where d is the distance function on $\mathcal{E}$. This defines the *distortion function* $\Lambda : \partial\mathcal{L} \to \mathbf{R}$. Clearly, $\Lambda(C^o, \mathcal{O}) = 1/\Lambda(C, \mathcal{O})$.

Second, a multi-set $\{C_0, \ldots, C_m\} \subset \partial\mathcal{L}$ is called an *m-configuration* if the convex hull $[C_0, \ldots, C_m]$ contains $\mathcal{O}$. The set of all m-configurations is denoted by $\mathcal{C}_m(\mathcal{L}, \mathcal{O})$. We then define

$$\sigma_m(\mathcal{L}, \mathcal{O}) = \inf_{\{C_0,\ldots,C_m\} \in \mathcal{C}_m(\mathcal{L},\mathcal{O})} \sum_{i=0}^{m} \frac{1}{1 + \Lambda(C_i, \mathcal{O})}. \tag{3}$$

For $m = \dim \mathcal{L}$ the subscript is suppressed and we write $\sigma(\mathcal{L}, \mathcal{O})$.

The purpose of this paper is to calculate and derive various upper estimates for the measures of symmetry $\sigma(\mathcal{L}_\lambda, 0)$ and $\sigma(\mathcal{M}_\lambda, 0)$. Our starting point is the following:

Theorem 1 *Let $M = G/K$ be a Riemannian homogeneous space. Assume that the eigenspace $\mathcal{H}_\lambda \subset C^\infty(M)$ is an irreducible G-submodule. Then, we have*

$$\frac{\dim \mathcal{L}_\lambda + 1}{\dim \mathcal{H}_\lambda} \leq \sigma(\mathcal{L}_\lambda, 0) = \frac{\dim \mathcal{L}_\lambda + 1}{1 + \max_{\partial\mathcal{L}_\lambda} \Lambda(., 0)} \leq \frac{\dim V_{\min}}{\dim \mathcal{H}_\lambda}(\dim \mathcal{L}_\lambda + 1), \tag{4}$$

where $f : M \to S_{V_{\min}}$ is a spherical λ-eigenmap with minimum range dimension. If $M = G/K$ is isotropy irreducible then we have

$$\frac{\dim \mathcal{M}_\lambda + 1}{\dim \mathcal{H}_\lambda} \leq \sigma(\mathcal{M}_\lambda, 0) = \frac{\dim \mathcal{M}_\lambda + 1}{1 + \max_{\partial\mathcal{M}_\lambda} \Lambda(., 0)} \leq \frac{\dim V_{\min}}{\dim \mathcal{H}_\lambda}(\dim \mathcal{M}_\lambda + 1), \tag{5}$$

where $f : M \to S_{V_{\min}}$ is a spherical minimal immersion (inducing $\lambda/\dim M$ times the metric on M) with minimum range dimension. In either case of (4)–(5), if equality holds in *the upper estimate then the respective map $f : M \to S_{V_{\min}}$ has L^2-orthonormal components (up to scaling and with respect to an orthonormal basis in $V_{\min}$).*

Theorem 1 will be proved in Sects. 3 and 4. In particular, the middle equalities in (4)–(5) are consequences of the more general formula (14) of Theorem 3, the lower bounds in (4)–(5) will follow from (18) and the subsequent Remark 3, and Theorem 4 will imply the upper bounds in (4)–(5).

In general, for a convex body $\mathcal{L}$ with interior point $\mathcal{O}$, we always have [21]

$$\sigma(\mathcal{L}, \mathcal{O}) \le \frac{\dim \mathcal{L} + 1}{2}.$$

For $\dim \mathcal{L} \ge 2$, equality holds if and only if $\mathcal{L}$ is symmetric with respect to $\mathcal{O}$. Thus, in the upper estimates (4)–(5), we need to look for minimal ranges for which $\dim V_{\min} < \dim \mathcal{H}_\lambda/2$. As noted above, for compact rank one symmetric spaces $M = G/K$ the eigenspaces $\mathcal{H}_\lambda$ are irreducible so that Theorem 1 applies.

For $M = S^m$, in view of (4)–(5), to calculate $\sigma(\mathcal{L}_m^k, 0)$ ($m \ge 3, k \ge 2$) and $\sigma(\mathcal{M}_m^k, 0)$ ($m \ge 3, k \ge 4$), one needs to know the maximal distortion for eigenmaps and spherical minimal immersions. This is a difficult and largely unsolved problem [10,24]. To obtain upper bounds for these measures of symmetry, one needs to know the minimal range dimensions of such maps. This is the so-called DoCarmo problem. In general, to give bounds on the minimum range dimension is an old and difficult problem [8] (Remark 1.6) and [5–7,18,24,28].

The arithmetic properties of the sequence $\{\sigma_m\}_{m\ge1}$ (to be discussed in Sect. 2) will imply the following:

Corollary *Let $d_\lambda = d(\mathcal{H}_\lambda)$ be the maximum dimension such that $\mathcal{L}_\lambda$ has a d_λ-dimensional simplex as a linear slice (across the origin 0). Then*

$$d(\mathcal{H}_\lambda) \le \max_{\partial \mathcal{L}_\lambda} \Lambda(., 0). \tag{6}$$

Analogous statement holds for $\mathcal{M}_\lambda$ (with $\mathcal{L}_\lambda$ replaced by $\mathcal{M}_\lambda$ in (6)). Equality holds if and only if the sequence $\{\sigma_m\}_{m\ge1}$ is arithmetic from the d_λ-th term onward.

In the lowest non-trivial case of quadratic eigenmaps of the three-sphere, the Hopf map $Hopf : S^3 \to S^2$ corresponds to both maximal distortion 2 and minimal range dimension. Hence, we obtain

$$\sigma(\mathcal{L}_3^2, 0) = \frac{\dim \mathcal{L}_3^2 + 1}{1 + \Lambda(\langle Hopf \rangle, 0)} = 3\frac{2}{3}.$$

The explicit description of $\mathcal{L}_3^2$ shows [23] that $\mathcal{L}_3^2$ (in fact, $(\mathcal{L}_3^2)^{SU(2)}$) has a triangular slice across 0. ($\langle Hopf \rangle$ can be chosen as one of the vertices of the triangle. Its antipodal is the parameter point $\langle Ver^{\mathbf{C}} \rangle$, where $Ver^{\mathbf{C}}$ is the complex Veronese map. The latter is the center of a disk on the boundary of the moduli, and the boundary circle of the disk is on the $SU(2)'$ orbit of $\langle Hopf \rangle$.) Thus, equality holds in (6), and we obtain

$$\sigma_m(\mathcal{L}_3^2, 0) = \frac{m+1}{3}, \quad m \ge 2.$$

In the lowest non-trivial case of moduli $\mathcal{M}_3^4$ for quartic spherical minimal immersions of the three sphere, a role similar to the Hopf map is played by the (minimum range-dimensional) quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ [26,23]. The corresponding point $\langle \mathcal{I} \rangle$ on the moduli has distortion $3/2$ and this gives the *upper bound*

$$\sigma(\mathcal{M}_3^4, 0) \leq \frac{\dim \mathcal{M}_3^4 + 1}{1 + \Lambda(\langle \mathcal{I} \rangle, 0)} = 7\frac{3}{5}. \tag{7}$$

Remark 1 For some $SU(2)$-equivariant moduli, low dimensional simplicial slices can be constructed explicitly. For example, $(\mathcal{M}_3^6)^{SU(2)}$ has a triangular slice, and $(\mathcal{M}_3^8)^{SU(2)}$ and $(\mathcal{M}_3^{12})^{SU(2)}$ both have tetrahedral slices (across 0). These are constructed using the tetrahedral, octahedral and icosahedral spherical minimal immersions [5–7,24,28].

The *minimal orbit method* [5–7] for $SU(2)$ (or *equivariant construction* originally introduced by Mashimo [17]) has been used by DeTurck and Ziller to obtain a large number of low range-dimensional $SU(2)$-equivariant eigenmaps and spherical minimal immersions of the three sphere. They constructed these with specific invariance properties to prove that every homogeneous spherical space form (of S^3 and also of higher dimensional odd dimensional spheres) admits a minimal isometric imbedding into a Euclidean sphere (of sufficiently high dimension). For our purposes here these immersions, in turn, enable us to calculate the measures of symmetry for the equivariant moduli $(\mathcal{L}_3^k)^{SU(2)}$, $k \geq 2$, and $(\mathcal{M}_3^k)^{SU(2)} k \geq 4$.

Theorem 2 *For $k \geq 2$, we have*

$$\max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(., 0) = \begin{cases} k & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \text{ is odd.} \end{cases} \tag{8}$$

The dimension $d^k = d((\mathcal{L}_3^k)^{SU(2)})$ of the largest simplicial slice of $(\mathcal{L}_3^k)^{SU(2)}$ (across 0) is equal to this maximal distortion, and we have

$$\sigma_m((\mathcal{L}_3^k)^{SU(2)}, 0) = \begin{cases} 1 & \text{if } m \leq d^k \\ \frac{m+1}{1+d^k} & \text{if } m > d^k. \end{cases} \tag{9}$$

In particular, we have

$$\sigma((\mathcal{L}_3^k)^{SU(2)}, 0) = \begin{cases} \frac{k+2}{2} & \text{if } k \text{ is even} \\ k & \text{if } k \text{ is odd.} \end{cases} \tag{10}$$

For $k \geq 5$ (8) holds with $\mathcal{L}_3^k$ replaced by $\mathcal{M}_3^k$, and we have

$$\sigma((\mathcal{M}_3^k)^{SU(2)}, 0) = \begin{cases} \frac{k+2}{2} - \frac{5}{k+1} & \text{if } k \text{ is even} \\ k - \frac{10}{k+1} & \text{if } k \text{ is odd.} \end{cases} \tag{11}$$

Remark 2 We have [23]

$$\dim(\mathcal{L}_3^k)^{SU(2)} = [k/2](2[k/2] + 3), \tag{12}$$

$$\dim(\mathcal{M}_3^k)^{SU(2)} = (2[k/2] + 5)([k/2] - 1). \tag{13}$$

Since both these dimensions are $\mathcal{O}(k^2)$, (10)–(11) indicate that $\mathcal{L}_3^k$ and $\mathcal{M}_3^k$ are far from symmetric. Note also the interesting byproduct

$$\sigma((\mathcal{L}_3^k)^{SU(2)}, 0) > \sigma((\mathcal{M}_3^k)^{SU(2)}, 0), \quad k \geq 5$$

which is to be expected as $(\mathcal{M}_3^k)^{SU(2)}$ is a linear slice of $(\mathcal{L}_3^k)^{SU(2)}$.

Remark 3 For $k = 4$, the lowest range-dimensional $SU(2)$-equivariant quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ gives

$$\sigma((\mathcal{M}_3^4)^{SU(2)}, 0) \le 4.$$

Compare this with (7). Ironically, this is only an upper estimate because the $SU(2)$-module structure on the (linear) range of $\mathcal{I}$ is reducible, in fact, the double of an irreducible $SU(2)$-module. (See the lemma in Sect. 5.) In addition, on the boundary of the moduli $(\mathcal{M}_3^4)^{SU(2)}$ there is a 6-dimensional set (corresponding to the so-called type $\mathbf{II}_0$ spherical minimal immersions [23]). Their ranges are also reducible, the triple of an irreducible $SU(2)$-module. The corresponding parameter points are all *extremal* (in the sense of convex geometry) and their algebraic description is cumbersome.

Remark 4 Forgetting $SU(2)$-equivariance, the range dimensions of these $SU(2)$-equivariant eigenmaps and spherical minimal immersions can also be used in (4)–(5) for $V_{\min}$ in the upper estimate of the measures of symmetry $\sigma(\mathcal{L}_3^k, 0)$ and $\sigma(\mathcal{M}_3^k, 0)$. Only upper estimates can be expected since a least range-dimensional $SU(2)$-equivariant minimal immersion *among $SU(2)$-equivariant minimal immersions* usually do not have minimal range dimension *among all spherical minimal immersions*. This has been pointed out by Escher and Weingart [10] who, among others, found a spherical minimal immersion $f : S^3 \to S_V$ with $k = 36$ but $\dim V \le 36$. (For $k = 36$, the minimum range dimension for $SU(2)$-equivariant minimal immersions is 37.)

2 The measures of symmetry $\{\sigma_k\}_{k\ge1}$

The sequence $\{\sigma_k(\mathcal{L}, \mathcal{O})\}_{k\ge1}$ has interesting properties.

1. Measure of symmetry [21]. We have

$$1 \le \sigma_m(\mathcal{L}, \mathcal{O}) \le \frac{m+1}{2}.$$

   The lower bound is attained if and only if $m \le \dim \mathcal{L}$ and $\mathcal{L}$ has a *simplicial* intersection with an m-dimensional affine subspace passing through $\mathcal{O}$. For $m \ge 2$, the upper bound is attained if and only if $\mathcal{L}$ is *symmetric* with respect to $\mathcal{O}$.
2. Monotonicity [19,20]. $\sigma_1(\mathcal{L}, \mathcal{O}) = 1$ and after a possible initial string of ones, the sequence $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\ge1}$ is *strictly* increasing. The length of the string of ones is the maximum-dimensional simplicial intersection of $\mathcal{L}$ by an affine subspace passing though $\mathcal{O}$. Weak monotonicity follows from subadditivity of the differences:

$$\sigma_{m+k}(\mathcal{L}, \mathcal{O}) - \sigma_{m+1}(\mathcal{L}, \mathcal{O}) \ge \sigma_k(\mathcal{L}, \mathcal{O}) - \sigma_1(\mathcal{L}, \mathcal{O}), \quad k \ge 1.$$

3. Sub-arithmeticity [21]. For $m, k \ge 1$, we have

$$\sigma_{m+k}(\mathcal{L}, \mathcal{O}) \le \sigma_m(\mathcal{L}, \mathcal{O}) + \frac{k}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}.$$

   The sequence $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\ge1}$ becomes arithmetic after $m = \dim \mathcal{L}$ with difference $1/(1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}))$. (More precisely, the index m after which the sequence becomes arithmetic is the degree of degeneracy in attaining the infimum in (3) by a minimizing *simplicial* sequence of configurations.)

3 The general setting

Let G be a compact Lie group. Recall that an *orthogonal G-module* is a finite dimensional Euclidean vector space $\mathcal{E}$ on which G acts with linear isometries.

Theorem 3 *Let $\mathcal{E}$ be an orthogonal G-module and assume that G acts on $\mathcal{E}$ with no nonzero fixed points. If $\mathcal{L}$ is a G-invariant convex body with* $0 \in \operatorname{int} \mathcal{L}$ *then*

$$\sigma(\mathcal{L}, 0) = \frac{\dim \mathcal{L} + 1}{1 + \max_{\partial \mathcal{L}} \Lambda(., 0)}. \tag{14}$$

Proof Assume that the distortion function $\Lambda(., 0) : \partial \mathcal{L} \to \mathbf{R}$ attains its maximum on $\partial \mathcal{L}$ at C. Consider the convex hull $[G(C)] \subset \mathcal{L}$ of the orbit $G(C) \subset \partial \mathcal{L}$ passing through C. This is a G-invariant compact convex set. It contains its center of mass which must be G-fixed. Since $\mathcal{E}$ has no nonzero G-fixed points, this center of mass must be the origin 0. Hence there exist $B_0, \dots, B_m \in G(C)$ such that $\sum_{i=0}^m \lambda_i B_i = 0$, $\sum_{i=0}^m \lambda_i = 1$, $0 \leq \lambda_i \leq 1$, $i = 0, \dots, m$. We may assume that $m \geq \dim \mathcal{L}$ (by adding more points, if necessary). What we just concluded means that $\{B_0, \dots, B_m\}$ is an m-configuration (with respect to 0). Therefore, we have

$$\sigma_m(\mathcal{L}, 0) = \inf_{\{C_0, \dots, C_m\} \in \mathcal{C}_m(\mathcal{L}, 0)} \sum_{i=0}^m \frac{1}{1 + \Lambda(C_i, 0)} \leq \sum_{i=0}^m \frac{1}{1 + \Lambda(B_i, 0)}.$$

On the other hand, the points B_i, $i = 0, \dots, m$ are on an orbit of maximal distortion on $\partial \mathcal{L}$, so that $\Lambda(B_i, 0) = \max_{\partial \mathcal{L}} \Lambda(., 0)$, $i = 1, \dots, m$. We thus have

$$\sum_{i=0}^m \frac{1}{1 + \Lambda(B_i, 0)} = \frac{m + 1}{1 + \max_{\partial \mathcal{L}} \Lambda(., 0)} \leq \sigma_m(\mathcal{L}, 0),$$

where the last inequality is a trivial lower estimate for σ_m as follows from the definition (3). Combining the two inequalities above, we find

$$\sigma_m(\mathcal{L}, 0) = \frac{m + 1}{1 + \max_{\partial \mathcal{L}} \Lambda(., 0)}.$$

As noted in Sect. 2, the sequence $\{\sigma_m\}_{m \geq 1}$ is arithmetic with difference $1/(1 + \max_{\partial \mathcal{L}} \Lambda(., 0))$ from the $\dim \mathcal{L}$-th term onward. Counting backwards, we obtain (14). The theorem follows.

Remark 1 In view of applications to moduli spaces, it would be important to know the minimum number of points $B_0, \dots, B_m$ (on an orbit of maximal distortion) whose convex hull contains the origin.

In our present setting, the corollary to Theorem 1 can be generalized as follows:

Corollary *Let $\mathcal{L}$ be as in Theorem 3. Let d be the maximum dimension such that $\mathcal{L}$ has a d-dimensional simplicial intersection across* 0. *Then*

$$d \leq \max_{\partial \mathcal{L}} \Lambda(., 0).$$

Equality holds if and only if $\{\sigma_m(\mathcal{L}, 0)\}_{m \geq 1}$ is arithmetic from the d-th term onwards.

Proof We first claim that the dimension d of the largest simplicial slice cannot exceed the maximum distortion $\max_{\partial \mathcal{L}} \Lambda(.,0)$. We use the properties of the sequence $\{\sigma_m\}_{m\geq 1}$ as follows. First, the existence of a d-dimensional simplicial slice is equivalent to $\sigma_d(\mathcal{L},0)=1$. Second, sub-arithmeticity implies

$$\sigma(\mathcal{L},0) \leq 1 + \frac{\dim \mathcal{L} - d}{1+\max_{\partial \mathcal{L}} \Lambda(.,0)}. \tag{15}$$

Using (14) and rearranging, the claim now follows.

For the second statement, note that sub-arithmeticity implies

$$\sigma_{m+1}(\mathcal{L},0) - \sigma_m(\mathcal{L},0) \leq \frac{1}{1+\max_{\partial \mathcal{L}} \Lambda(.,0)}. \tag{16}$$

Adding up for $m = d, \dots, \dim \mathcal{L} - 1$ and using $\sigma_d(\mathcal{L},0)=1$, we see that equality holds in (15) if and only if equalities hold in (16) for each $m \geq d$. The corollary follows.

Let $\mathcal{H}$ be an orthogonal G-module. The G action on $\mathcal{H}$ naturally extends to the symmetric square $S^2(\mathcal{H})$ and its traceless part $S_0^2(\mathcal{H})$. From now on we will consider $\mathcal{E}$ as a G-submodule of $S_0^2(\mathcal{H})$ and assume that $\mathcal{E}$ has no trivial components.

Remark 2 If $\mathcal{H}$ is irreducible then on $S_0^2(\mathcal{H})$ (and therefore on each submodule $\mathcal{E}$) G acts with no nonzero fixed points. Indeed, if $C \in S_0^2(\mathcal{H})$ is G-fixed then C, as a symmetric endomorphism of $\mathcal{H}$, commutes with the action of G on $\mathcal{H}$. Since G acts irreducibly on $\mathcal{H}$, C must be a (real) constant multiple of the identity. Since the trace of C is zero, it must itself be zero.

We define

$$\mathcal{K}(\mathcal{H}) = \{C \in S_0^2(\mathcal{H}) \mid C + I \geq 0\},$$

where $\geq$ means positive semi-definite. $\mathcal{K}(\mathcal{H})$ is obviously a convex subset of $S_0^2(\mathcal{H})$ with nonempty interior. Given a traceless symmetric endomorphism C of $\mathcal{H}$ satisfying the defining inequality $C + I \geq 0$, the eigenvalues of C are contained in the interval $[-1, \dim \mathcal{H} - 1]$. In particular, $\mathcal{K}(\mathcal{H})$ is compact, therefore a convex body in $S_0^2(\mathcal{H})$.

The moduli that we will study will be of the form

$$\mathcal{L} = \mathcal{K}(\mathcal{H}) \cap \mathcal{E}.$$

We first study $\mathcal{K}(\mathcal{H})$. We first prove the important observation that, for $C \in \partial \mathcal{K}(\mathcal{H})$, the *distortion* $\Lambda(C,0)$ *is the maximal eigenvalue of* C.

Indeed, since C is traceless (and nonzero), the maximal eigenvalue of C must be positive. Therefore, there must be a maximal $t_0 > 0$ such that $-tC + I > 0$ for $0 \leq t < t_0$. For these values of t, $-tC$ is in the interior of $\mathcal{K}(\mathcal{H})$. Since the determinant of $-t_0 C + I$ vanishes, $-t_0 C$ is on the boundary of $\mathcal{K}(\mathcal{H})$. We obtain that $C^o = -t_0 C$ and $-t_0 \lambda_{\max}(C) + 1 = 0$, where $\lambda_{\max}$ denotes the maximal eigenvalue. Combining these, we get

$$C^o = -\frac{1}{\lambda_{\max}(C)} C. \tag{17}$$

Taking norms, we arrive at

$$\lambda_{\max}(C) = \frac{|C|}{|C^o|} = \Lambda(C,0),$$

and the observation follows.

Since $\dim \mathcal{H} - 1$ is an upper bound for all eigenvalues, as a byproduct, we also obtain

$$\frac{1}{\dim \mathcal{H} - 1} \leq \Lambda(., 0) \leq \dim \mathcal{H} - 1. \tag{18}$$

Remark 3 The upper estimate in (18) immediately gives the lower estimates in (4)–(5).

Remark 4 Helly's theorem [1] implies that, for a compact convex body $\mathcal{L} \subset \mathcal{E}$ there is an interior point $\mathcal{O}$ such that

$$\frac{1}{\dim \mathcal{L}} \leq \Lambda(., \mathcal{O}) \leq \dim \mathcal{L}. \tag{19}$$

The bounds are the best possible for $\mathcal{L}$ a simplex.

The maximum distortion in (18) for $\mathcal{K}(\mathcal{H})$ is attained. Indeed, setting $h = \dim \mathcal{H}$, the largest possible maximal eigenvalue $h - 1$ occurs for endomorphisms of the form $C_\chi = \chi \odot \chi - I \in \partial\mathcal{K}(\mathcal{H})$, $\chi \in \mathcal{H}$, with $|\chi| = h$.

In addition, given an orthonormal basis $\{\chi_i\}_{i=1}^h \subset \mathcal{H}$, $\{C_{\sqrt{h}\chi_i}\}_{i=1}^h$ is a minimal simplicial $(h-1)$-configuration. Its convex hull, consisting of traceless endomorphisms of $\mathcal{H}$ that are diagonal with respect to this basis, is a simplicial slice of $\mathcal{K}(\mathcal{H})$ of dimension $\dim \mathcal{H} - 1$. Thus, by Corollary to Theorem 3, we obtain

$$\sigma_m(\mathcal{K}(\mathcal{H}), 0) = \begin{cases} 1 & \text{if } m < \dim \mathcal{H} \\ \frac{m+1}{\dim \mathcal{H}} & \text{if } m \geq \dim \mathcal{H} \end{cases}$$

4 $\mathcal{H}$-maps and their moduli

Let M be a compact Riemannian manifold and $C^\infty(M)$ the space of smooth functions on M. As usual, we endow $C^\infty(M)$ with the L^2-scalar product. Given a map $f : M \to V$ into a Euclidean vector space V, the *space of components* of f is $V_f = \{\alpha \circ f \mid \alpha \in V^*\}$. f is *full* if $\dim V_f = \dim V$ (that is, composing f with linear functionals on V is an isomorphism between V^* and V_f). Given a finite dimensional linear subspace $\mathcal{H} \subset C^\infty(M)$, we call f an *$\mathcal{H}$-map* if $V_f \subset \mathcal{H}$. An $\mathcal{H}$-map is called *spherical* if it maps to the unit sphere of the range. In this case we write $f : M \to S_V$, where S_V is the unit sphere of V.

Let $M = G/K$ be Riemannian homogeneous with acting transitive Lie group of isometries G, and assume that $\mathcal{H}$ is G-invariant. With respect to the L^2-scalar product $\mathcal{H}$ is an *orthogonal G-module*. The *Dirac delta* $\delta_\mathcal{H} : M \to \mathcal{H}^*$ (defined by evaluating the functions in $\mathcal{H} \subset C^\infty(M)$ on points of M) is an $\mathcal{H}$-map, and it is maximal in the sense that every function in $\mathcal{H}$ is a component of $\delta_\mathcal{H}$. With respect to the induced scalar product on $\mathcal{H}^*$ suitably scaled, the Dirac delta is spherical, and we can write $\delta_\mathcal{H} : M \to S_{\mathcal{H}^*}$.

Given a full $\mathcal{H}$-map $f : M \to V$ there exists a (unique) linear map $A : \mathcal{H}^* \to V$ such that $f = A \circ \delta_\mathcal{H}$. Since f is full, A is onto. We associate to f the symmetric linear endomorphism $\langle f \rangle = A^*A - I \in S^2(\mathcal{H})$. This association is one to one on the congruence classes of $\mathcal{H}$-maps, where two $\mathcal{H}$-maps belong to the same congruence class if they differ by an isometry between their ranges. An $\mathcal{H}$-map $f : M \to V$ is spherical if and only if

$$|f(x)|^2 - |\delta_\mathcal{H}(x)|^2 = \langle (A^*A - I)\delta_\mathcal{H}(x), \delta_\mathcal{H}(x)\rangle = \langle \langle f \rangle, \delta_\mathcal{H}(x) \odot \delta_\mathcal{H}(x)\rangle = 0 \tag{20}$$

for all $x \in M$. We conclude that an $\mathcal{H}$-map $f : M \to V$ is spherical if and only if the associated $\langle f \rangle$ belongs to the linear subspace

$$\mathcal{E}(\mathcal{H}) = \{\delta_\mathcal{H}(x) \odot \delta_\mathcal{H}(x) \mid x \in M\}^\perp \subset S^2(\mathcal{H}).$$

Integrating the defining equality $\langle C\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle = 0$ of $C \in \mathcal{E}(\mathcal{H})$ over M, we see that the trace of $C \in S^2(\mathcal{H})$ vanishes, so that $\mathcal{E}(\mathcal{H})$ is contained in the traceless part $S^2_0(\mathcal{H})$.

Since $\langle f\rangle + I = A^*A$ is automatically positive semidefinite, we obtain that

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap \mathcal{E}(\mathcal{H}) = \{C \in \mathcal{E}(\mathcal{H}) \mid C + I \geq 0\}$$

parametrizes the congruence classes of full spherical $\mathcal{H}$-maps $f : M \to S_V$.

Since $\mathcal{K}(\mathcal{H})$ is compact, so is $\mathcal{L}(\mathcal{H})$. The origin $0 = \langle \delta_{\mathcal{H}}\rangle$ is in the interior of $\mathcal{L}(\mathcal{H})$.

We say that a full spherical $\mathcal{H}$-map $f : M \to S_V$ is of *boundary type* if $\dim V < \dim \mathcal{H}$ or equivalently $\langle f\rangle \in \partial\mathcal{L}(\mathcal{H})$.

Theorem 4 *Let $f : M \to S_V$ be a full spherical $\mathcal{H}$-map of boundary type. Then the distortion $\Lambda(\langle f\rangle, 0)$ (with respect to the origin 0) is the maximal eigenvalue of $\langle f\rangle$. We have*

$$\frac{\dim \mathcal{H}}{\dim V} \leq \Lambda(\langle f\rangle, 0) + 1 \leq \frac{\dim \mathcal{H}}{\nu(\langle f\rangle)}, \tag{21}$$

where $\nu(\langle f\rangle)$ is the multiplicity of the maximal eigenvalue. Equality holds if and only if $f : M \to S_V$ has L^2-orthonormal components with respect to an orthonormal basis in V.

Proof The first statement was proved in the previous section in a more general setting. To prove (21) we let $C = A^*A - I = \langle f\rangle, f = A \circ \delta_{\mathcal{H}}, \nu = \nu(C), \dim \mathcal{H} = h$ and $\dim V = n$. As noted above, the eigenvalues of C are contained in $[-1, h-1]$. Since f is full, $A : \mathcal{H}^* \to V$ is onto. Thus, rank $(C + I) =$ rank $(A^*A) =$ rank$A = \dim V = n$ so that the multiplicity of the minimal eigenvalue -1 of C is $h - n$. Thus, for the multiplicity ν of the maximal eigenvalue $\lambda_{\max}$, we must have $\nu \leq n$. Let $\lambda_1, \ldots, \lambda_{n-\nu}$ denote the non-minimal and non-maximal eigenvalues. The condition that C is traceless can be written as

$$\nu\lambda_{\max} + \sum_{i=1}^{n-\nu} \lambda_i = h - n.$$

Finally, since $-1 < \lambda_i < \lambda_{\max}$, we obtain (21).

By Theorem 4, in either case of (4)–(5), for the minimum range-dimensional map $f : M \to S_{V_{\min}}$, we have

$$\frac{\dim \mathcal{H}_\lambda}{\dim V_{\min}} \leq \Lambda(\langle f\rangle, 0) + 1 \leq \max \Lambda(., 0) + 1,$$

and Theorem 1 follows.

5 $SU(2)$-equivariant eigenmaps and their moduli

The irreducible complex $SU(2)$-modules are parametrized by their dimension, and they can be realized as submodules appearing in the (multiplicity one) decomposition of the $SU(2)$-module of complex homogeneous polynomials $\mathbf{C}[z, w]$ in two variables [3,16]. For $k \geq 0$, the k-th submodule W_k, $\dim_{\mathbf{C}} W_k = k + 1$, comprises the homogeneous polynomials of degree k. With respect to the L^2-scalar product (suitably scaled) the standard orthonormal basis for W_k is $\{z^{k-j}w^j/\sqrt{(k-j)!j!}\}_{j=0}^k$. For k odd, W_k is irreducible *as a real $SU(2)$-module.* For k even, the fixed point set R_k of the complex anti-linear self map $z^j w^{k-j} \mapsto (-1)^j z^{k-j} w^j$, $j = 0, \ldots, k$, of W_k is an irreducible real submodule with $W_k = R_k \otimes_{\mathbf{R}} \mathbf{C}$.

Given a (nonzero) polynomial

$$\xi = \sum_{j=0}^{k} c_j z^{k-j} w^j \in W_k \tag{22}$$

the orbit map $f_\xi : S^3 \to W_k$, $f_\xi(g) = g \cdot \xi = \xi \circ g^{-1}$, $g \in SU(2)$, (through ξ) is (up to scaling) a spherical λ_k-eigenmap [23]. In coordinates, we have

$$f_\xi(a,b)(z,w) = \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad a, b \in \mathbf{C},\ |a|^2 + |b|^2 = 1,\ z, w \in \mathbf{C}. \tag{23}$$

Here $g = (a, b) \in S^3 \subset \mathbf{C}^2$ is identified with $\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \in SU(2)$, or equivalently, the unit quaternion $a + \jmath b \in S^3 \subset \mathbf{H}$. (On the right-hand side the inverse $g^{-1} = (\bar{a}, -b) = \bar{a} - \jmath b$ acts on $(z, w) = z + \jmath w$ by multiplication.) f_ξ is a spherical minimal immersion if and only if the coefficients in (22) satisfy the following

$$\begin{aligned}
&\sum_{j=0}^{k} (k-j)!j!|c_j|^2 = 1\\
&\sum_{j=0}^{k} (2j-k)^2 (k-j)!j!|c_j|^2 = \frac{k(k+2)}{3},\\
&\sum_{j=0}^{k-2} (j+2)!(k-j)! c_j \bar{c}_{j+2} = 0,\\
&\sum_{j=0}^{k-1} (k-2j-1)(j+1)!(k-j)! c_j \bar{c}_{j+1} = 0.
\end{aligned}$$

(The first equation means that f_ξ maps into the unit sphere of W_k. The last three equations are conformality conditions on the tangent space of S^3 at 1 [5–7].)

In the examples of Sect. 1, up to congruence, the Hopf map $Hopf : S^3 \to S_{R_2}$ is the $SU(2)$-orbit of the polynomial $\xi = izw \in R_2$, and the quartic minimal immersion $\mathcal{I} : S^3 \to S_{W_4}$ is the $SU(2)$-orbit map of the polynomial $\xi = (\sqrt{6}/24)(z^4 - w^4) + (\sqrt{2}/4)z^2 w^2 \in W_4$. (As noted in Sect. 1, there is no minimal $SU(2)$-orbit in R_4. Note that, even though W_4 is reducible as a real $SU(2)$-module, $\mathcal{I}$ still has L^2-orthonormal components.)

To prove Theorem 2, we first note that Theorem 3 (with $G = SU(2)'$) applies to $SU(2)$-equivariant moduli and gives

$$\sigma((\mathcal{L}_3^k)^{SU(2)}, 0) = \frac{\dim(\mathcal{L}_3^k)^{SU(2)} + 1}{1 + \max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(.,0)} \le \frac{V_{\min}}{(k+1)^2}([k/2](2[k/2]+3)+1), \tag{24}$$

and [23]

$$\sigma((\mathcal{M}_3^k)^{SU(2)}, 0) = \frac{\dim(\mathcal{M}_3^k)^{SU(2)} + 1}{1 + \max_{\partial(\mathcal{M}_3^k)^{SU(2)}} \Lambda(.,0)} \le \frac{V_{\min}}{(k+1)^2}((2[k/2]+5)([k/2]-1)+1), \tag{25}$$

where we used $\dim \mathcal{H}_3^k = (k+1)^2$ and (12)–(13).

According to the next lemma, equality holds if and only if $V_{\min}$ is the minimum range dimension among the respective $SU(2)$-equivariant eigenmaps or spherical minimal immersions:

Lemma *Let $f_0 : S^3 \to S_{V_0}$ be an $SU(2)$-equivariant λ_k-eigenmap. V_0 with its natural $SU(2)$-module structure (given by the equivariance of f) is irreducible if and only if, within the moduli $(\mathcal{L}_3^k)^{SU(2)}$, the range dimension* $\dim V_0$ *is minimal. In this case, the distortion*

$$\Lambda(\langle f_0\rangle, 0) = \frac{(k+1)^2}{\dim V_0} - 1$$

is maximal. The same statement holds for spherical minimal immersions.

Proof Let $f : S^3 \to S_V$ be any full $SU(2)$-equivariant λ_k-eigenmap. Depending on the parity of k, as an $SU(2)$-module, V is the sum of finitely many copies of W_k (k odd) or R_k (k even) (while V_0 is a single copy). In fact, $V = \oplus_{i=1}^N V_i$, and each irreducible component V_i is isomorphic with W_l (l odd) or R_l (l even). Since f is a k-homogeneous polynomial map, so are its components $f_i : S^3 \to V_i$, $i = 1, \ldots, N$. Then, $SU(2)$-equivariance gives $f_i = f_{\xi_i}$, $\xi_i = f_i(1)$, so that $l = k$. The first statement follows.

Due to equivariance, the multiplicity of each eigenvalue of $\langle f\rangle$, including the maximal one, is a multiple of $2(k+1)$ (k odd) or $k+1$ (k even). In particular, using the notations in Theorem 4, we have $\nu(\langle f\rangle) \geq \nu(\langle f_0\rangle) = \dim V_0$, and we obtain

$$\Lambda(\langle f\rangle) + 1 \leq \frac{(k+1)^2}{\nu(\langle f\rangle)} \leq \frac{(k+1)^2}{\dim V_0} \leq \Lambda(\langle f_0\rangle) + 1.$$

The lemma follows.

Any (nontrivial) $SU(2)$-orbit map for W_k (k odd) and R_k (k even) gives a full λ_k-eigenmap of minimum range dimension among the $SU(2)$-equivariant eigenmaps:

$$\dim V_{\min} = \begin{cases} k+1 & \text{if } k \text{ is even} \\ 2(k+1) & \text{if } k \text{ is odd.} \end{cases}$$

Since these eigenmaps have $SU(2)$-irreducible ranges, the lemma above applies, and we obtain (10).

To prove (9), we need to decompose the space of real spherical harmonics $\mathcal{H}_3^k|_{SU(2)}$ into *real* $SU(2)$-irreducible components. We have $\mathcal{H}_3^k|_{SU(2)} = (k+1) R_k$ (k even) and $\mathcal{H}_3^k|_{SU(2)} = (k+1)/2\, W_k$ (k odd). Mimicking the construction of the Dirac delta $\delta : S^3 \to S_{(\mathcal{H}_3^k)^*}$, in each irreducible component $V_j \subset \mathcal{H}_3^k$, $j = 0, \ldots, d^k$, $d^k = d((\mathcal{L}_3^k)^{SU(2)})$, we select an (appropriately scaled) L^2-orthonormal basis, and define an $SU(2)$-equivariant λ_k-eigenmap $f_j : S^3 \to S_{V_j}$ by declaring these basis elements as its components (with respect to the orthonormal basis). On the moduli, the Dirac delta corresponds to the origin, and we obtain

$$0 \in [\langle f_0\rangle, \ldots, \langle f_{d^k}\rangle].$$

This convex hull is a d^k-simplex whose faces are contained in the boundary of $(\mathcal{L}_3^k)^{SU(2)}$. This is because any interior point of the j-th face (opposite to $\langle f_j\rangle$) corresponds to a λ_k-eigenmap whose space of components does not contain the components of f_j. Therefore this λ_k-eigenmap must be of boundary type. We obtain that $[\langle f_0\rangle, \ldots, \langle f_{d^k}\rangle]$ is a d^k-dimensional simplicial intersection of $(\mathcal{L}_3^k)^{SU(2)}$. Now, (9) follows from Corollary to Theorem 3.

The tetrahedral, octahedral, and icosahedral immersions $Tet : S^3 \to S_{R_6}$, $Oct : S^3 \to S_{R_8}$, and $Ico : S^3 \to S_{R_{12}}$ [5–7,23] have minimal range dimensions and maximal distortions (again by the lemma above):

$$\Lambda(\langle Tet\rangle, 0) = 6, \quad \Lambda(\langle Oct\rangle, 0) = 8, \quad \Lambda(\langle Ico\rangle, 0) = 12.$$

These cover the cases $k = 6, 8, 12$ in (11).

In general, the minimal $SU(2)$-orbits of polynomials invariant under the cyclic or binary dihedral groups give $SU(2)$-equivariant minimal immersions with $SU(2)$-irreducible range (and minimal imbeddings of the corresponding lens spaces and dihedral manifolds into sphere). A quick check of Table 1 in [7] (or [24], page 65) shows that, for k even, all values $k \geq 8$ are covered. It remains to treat the case when $k \geq 5$ is odd, that is, we need to construct an $SU(2)$-equivariant minimal immersion $f : S^3 \to S_{W_k}$. The table cited above covers a particular case $k \equiv 3 \pmod 6$ (giving an imbedding of the lens space $L(k/3, 1)$ into S^{2k+1}). In general, the polynomial $\xi = c_0 z^k + c_3 w^{k-3} w^3 \in \mathbf{C}[z, w]$ satisfies the conformality conditions imposed on the orbit map $f_\xi : S^3 \to S_{W_k}$ (with unique real c_0 and c_3). (Because of the gap between the coefficients, the last two conformality conditions are automatically satisfied.) Then f_ξ defines an $SU(2)$-equivariant minimal immersion. Theorem 2 follows.

Acknowledgments The author wishes to thank the referee for the careful reading of the manuscript and for pointing out several improvements.

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