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ORIGINAL PAPER

Fine structure of convex sets from asymmetric viewpoint

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Abstract We study a sequence of measures of symmetry $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\geq 1}$ for a convex body $\mathcal{L}$ with a specified interior point $\mathcal{O}$ in an n-dimensional Euclidean vector space $\mathcal{E}$. The mth term $\sigma_m(\mathcal{L}, \mathcal{O})$ measures how far the m-dimensional affine slices of $\mathcal{L}$ (across $\mathcal{O}$) are from an m-simplex (viewed from $\mathcal{O}$). The interior of $\mathcal{L}$ naturally splits into regular and singular sets, where the singular set consists of points $\mathcal{O}$ with largest possible $\sigma_n(\mathcal{L}, \mathcal{O})$. In general, to calculate the singular set is difficult. In this paper we derive a number of results that facilitate this calculation. We show that concavity of $\sigma_n(\mathcal{L}, .)$ viewed as a function of the interior of $\mathcal{L}$ occurs at points $\mathcal{O}$ with highest degree of singularity, or equivalently, at points where the sequence $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\geq 1}$ is arithmetic. As a byproduct, these results also shed light on the structure and connectivity properties of the regular and singular sets.

Keywords Convex set · Distortion · Measure of symmetry

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1 Preliminaries and statement of results

1.1 A sequence of measures of symmetry and its properties

Let $\mathcal{E}$ be a Euclidean vector space of dimension n. For an arbitrary subset $\mathcal{K}$ of $\mathcal{E}$ we define $[\mathcal{K}]$ and $\langle\mathcal{K}\rangle$ the *convex hull* and the *affine span* of $\mathcal{K}$, respectively. For $\mathcal{K}$ finite, say $\mathcal{K} = \{B_0, \dots, B_m\}$, $[\mathcal{K}]$ is a convex *polytope*. This polytope is an *m-simplex*

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if $B_0, \ldots, B_m$ are *affinely independent*, or equivalently, if $\dim[\mathcal{K}] = \dim\langle\mathcal{K}\rangle = m$. A compact convex set $\mathcal{L} \subset \mathcal{E}$ with nonempty interior is called a *convex body*. Every compact convex set is a convex body in its affine span.

Let $\mathcal{L} \subset \mathcal{E}$ be a convex body and $\mathcal{O} \in \operatorname{int} \mathcal{L}$, an interior point of $\mathcal{L}$. Given a boundary point $C \in \partial\mathcal{L}$, we define the *opposite* C^o of C (with respect to $\mathcal{O}$) to be the unique point of $\partial\mathcal{L}$ with $\mathcal{O}$ in the interior of the line segment $[C, C^o]$. The ratio $\Lambda_{\mathcal{L}}(C, \mathcal{O}) = \Lambda(C, \mathcal{O})$ of lengths that $\mathcal{O}$ splits the line segment $[C, C^o]$ is called the *distortion* of C with respect to $\mathcal{O}$. Clearly, $\Lambda(C^o, \mathcal{O}) = 1/\Lambda(C, \mathcal{O})$, and

$$C + \Lambda(C, \mathcal{O})C^o = (1 + \Lambda(C, \mathcal{O}))\mathcal{O}.$$

Let $m \geq 1$. A multi-set $\{C_0, \ldots, C_m\} \subset \partial\mathcal{L}$ of boundary points of $\mathcal{L}$ is called an *m-configuration* if the convex hull $[C_0, \ldots, C_m]$ contains $\mathcal{O}$. The set of m-configurations is denoted by $\mathcal{C}_m(\mathcal{L}, \mathcal{O})$. With this we let

$$\sigma_m(\mathcal{L}, \mathcal{O}) = \inf_{\{C_0,\ldots,C_m\} \in \mathcal{C}_m(\mathcal{L},\mathcal{O})} \sum_{i=0}^{m} \frac{1}{1 + \Lambda(C_i, \mathcal{O})}, \quad m \geq 1. \tag{1}$$

A configuration at which the infimum is attained is called *minimal*. Minimal configurations exist since, by compactness of $\mathcal{L}$, a minimizing sequence of configurations subconverges. Minimal configurations are by no means unique.

For the simplest example, let $\Delta \subset \mathcal{E}$ be an n-simplex with $\mathcal{O} \in \operatorname{int} \Delta$. A simple computation in the use of projectivities (from the vertices of Δ) shows that the set of vertices of Δ forms a minimal n-configuration, and $\sigma_n(\Delta, \mathcal{O}) = 1$.

A 1-configuration is an antipodal pair of points $\{C, C^o\} \subset \partial\mathcal{L}$. Since

$$\frac{1}{1 + \Lambda(C, \mathcal{O})} + \frac{1}{1 + \Lambda(C^o, \mathcal{O})} = 1,$$

any 1-configuration is minimal, and $\sigma_1(\mathcal{L}, \mathcal{O}) = 1$.

A (minimal) configuration in $\mathcal{C}_k(\mathcal{L}, \mathcal{O})$ can always be extended to a configuration in $\mathcal{C}_{k+\ell}(\mathcal{L}, \mathcal{O})$, $\ell \geq 1$, by adding ℓ copies of a boundary point at which $\Lambda(., \mathcal{O})$ attains global maximum. We thus have

$$\sigma_{k+\ell}(\mathcal{L}, \mathcal{O}) \leq \sigma_k(\mathcal{L}, \mathcal{O}) + \frac{\ell}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}, \quad k, \ell \geq 1. \tag{2}$$

Since $\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}) \geq 1$, for $k = 1$ and $\ell = m - 1$, (2) gives

$$\sigma_m(\mathcal{L}, \mathcal{O}) \leq \frac{m+1}{2}, \quad m \geq 1. \tag{3}$$

Clearly, for $m \geq 2$, equality holds if and only if $\mathcal{L}$ is symmetric with respect to $\mathcal{O}$.

Let $k = n$ and consider a minimal configuration in $\mathcal{C}_{n+\ell}(\mathcal{L}, \mathcal{O})$, $\ell \geq 1$. The convex hull of this minimal configuration is a convex polytope in $\mathcal{E}$ containing $\mathcal{O}$. According to a theorem of Carathéodory, it contains an n-configuration. (For a recent proof and generalizations, see Boltyanski and Martini (2001).) This subconfiguration is necessarily minimal, and $\Lambda(., \mathcal{O})$ attains global maximum at the ℓ complementary configuration points. We obtain

$$\sigma_{n+\ell}(\mathcal{L}, \mathcal{O}) = \sigma_n(\mathcal{L}, \mathcal{O}) + \frac{\ell}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}, \quad \ell \geq 1.$$

Equivalently, the sequence $\{\sigma_m\}_{m\geq 1}$ is arithmetic from the nth term onwards [with difference $1/(1 + \max_{\partial\mathcal{L}} \Lambda)$].

For brevity, we suppress the index when it is equal to the dimension of the respective convex body. With this, it is clear from the definition in (1) that, for $1 \leq m < n$, we have

$$\sigma_m(\mathcal{L}, \mathcal{O}) = \inf_{\mathcal{O}\in\mathcal{F}\subset\mathcal{E},\, \dim\mathcal{F}=m} \sigma(\mathcal{L} \cap \mathcal{F}), \tag{4}$$

where the infimum is over affine subspaces $\mathcal{F} \subset \mathcal{E}$. In view of these, the sequence $\{\sigma_m\}_{m\geq 1}$ can be derived from σ.

An m-configuration of $\mathcal{L}$ (with respect to $\mathcal{O}$) is called *simplicial* if its convex hull is an m-simplex with $\mathcal{O}$ in its relative interior. We denote by $\Delta_m(\mathcal{L}, \mathcal{O}) \subset \mathcal{C}_m(\mathcal{L}, \mathcal{O})$ the subset of simplicial configurations. Clearly, in (1), the infimum can be taken over $\Delta_m(\mathcal{L}, \mathcal{O})$, but a minimizing sequence in $\Delta_m(\mathcal{L}, \mathcal{O})$ may not subconverge in $\Delta_m(\mathcal{L}, \mathcal{O})$.

Let $\{C_0, \ldots, C_m\} \in \Delta_m(\mathcal{L}, \mathcal{O})$ with convex hull $\Delta = [C_0, \ldots, C_m]$ and $\mathcal{O}$ in the relative interior of Δ. Comparing distortions, we have

$$1 = \sigma(\Delta, \mathcal{O}) = \sum_{i=0}^{m} \frac{1}{1 + \Lambda_\Delta(C_i, \mathcal{O})} \leq \sum_{i=0}^{m} \frac{1}{1 + \Lambda_\mathcal{L}(C_i, \mathcal{O})}.$$

Taking a minimizing sequence in $\Delta_m(\mathcal{L}, \mathcal{O})$, at the limit, we obtain

$$1 \leq \sigma_m(\mathcal{L}, \mathcal{O}), \quad m \geq 1. \tag{5}$$

By convexity, equality holds in (5) if and only if $\mathcal{L}$ has a simplicial intersection by an m-dimensional affine subspace that passes through the reference point $\mathcal{O}$. (For details, see Toth (2004).) In particular, $\sigma(\mathcal{L}, \mathcal{O}) = 1$ if and only if $\mathcal{L}$ is an n-simplex.

Clearly, $\sigma(\mathcal{L}, \mathcal{O})$ is invariant under affine transformations, and is a continuous function on the space of convex bodies $\mathcal{L}$ with specified interior point $\mathcal{O}$. Because of this, and the bounds in (3) and (5), $\sigma(\mathcal{L}, \mathcal{O})$ can be considered as a *measure of symmetry* on the pairs $(\mathcal{L}, \mathcal{O})$, $\mathcal{O} \in \operatorname{int} \mathcal{L}$. Although closely related, due to the presence of the interior point, this differs from the measures of symmetry in the sense of Grünbaum Grünbaum (1963).

As shown in Toth (2006) (Theorem B), the sequence $\{\sigma_m\}_{m\geq 1}$ has the monotonicity property:

$$\sigma_{k+\ell} - \sigma_{k+1} \geq \sigma_\ell - \sigma_1, \quad k \geq 0, \ell \geq 1.$$

In particular, $\{\sigma_m\}_{m\geq 1}$ is increasing. A stronger statement was proved in Toth (2008): After a possible string of ones, $\{\sigma_m\}_{m\geq 1}$ is strictly increasing. By the above, the length of the initial string of ones is equal to the maximum dimension of the simplicial intersections that pass through the reference point.

Letting $n = k + \ell$ in (2), we have

$$\sigma(\mathcal{L}, \mathcal{O}) \leq \sigma_{n-\ell}(\mathcal{L}, \mathcal{O}) + \frac{\ell}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}, \quad 1 \leq \ell \leq n. \tag{6}$$

1.2 Regular points

We call $\mathcal{O} \in \operatorname{int} \mathcal{L}$ a *regular* point if every minimizing sequence in $\Delta(\mathcal{L}, \mathcal{O})$ subconverges in $\Delta(\mathcal{L}, \mathcal{O})$. By (6), $\mathcal{O}$ is regular if and only if

$$\sigma(\mathcal{L}, \mathcal{O}) < \sigma_{n-1}(\mathcal{L}, \mathcal{O}) + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}. \tag{7}$$

On $\operatorname{int} \mathcal{L}$, the functions $\sigma_m, m \geq 1$, are continuous, and the family of functions $\{\Lambda(C, .)\}_{C\in\partial\mathcal{L}}$ is equicontinuous Toth (2006). Hence, by (7), the regular set $\mathcal{R}$, the set of regular points, is open in $\operatorname{int} \mathcal{L}$ and hence in $\mathcal{L}$.

Example Let $\Delta \subset \mathcal{E}$ be an n-simplex, and $\mathcal{O} \in \operatorname{int} \Delta$. As noted above, $\sigma_n(\Delta, \mathcal{O}) = 1$. By (5) (for $m = n - 1$) and the definition of regularity (7), we see that $\mathcal{O} \in \mathcal{R}$. We obtain that the regular set $\mathcal{R}$ of an n-simplex is the entire interior.

Let $\mathcal{O} \in \mathcal{R}$ be a regular point and $\{C_0, \ldots, C_n\}$ a minimal configuration. By definition, $[C_0, \ldots, C_n]$ is an n-simplex, and by (7), $\mathcal{O}$ is in its interior. It follows that Λ attains a local maximum at every configuration point $C_i, i = 0, \ldots, n$. Moreover Grünbaum (1963); Koziński (1954, 1958), the line segment $[C_i, C_i^o]$ is an *affine diameter* of $\mathcal{L}$ in the sense that there exist parallel supporting hyperplanes at C_i and at C_i^o. (This follows from the local study of extrema of Λ, see Sect. 7 of Toth (2006).) In addition (Toth (2009b), Lemma 2.1), the minimal configuration can be chosen such that every configuration point is an extreme point of the convex body $\mathcal{L}$ in the sense of convex geometry Grünbaum (2003).

Actually, somewhat more can be said about these local maxima. Taking the opposites in a minimal configuration $\{C_0, \ldots, C_n\}$ we obtain another simplicial configuration $\{C_0^o, \ldots, C_n^o\}$. The ith face $[C_0^o, \ldots, \widehat{C_i^o}, \ldots, C_n^o]$ of the corresponding simplex, projected to the boundary $\partial\mathcal{L}$ from $\mathcal{O}$ is a domain $\mathcal{D}_i \subset \partial\mathcal{L}$ which clearly contains C_i in its relative interior. Restricted to $\mathcal{D}_i$, Λ assumes its global maximum at C_i. (This follows easily, since otherwise C_i in the minimal configuration could be replaced by a point in $\mathcal{D}_i$ at which Λ assumes its global maximum over $\mathcal{D}_i$, a contradiction.)

Since $\cup_{i=0}^{n}\mathcal{D}_i = \partial\mathcal{L}$, we obtain that *in a minimal configuration there is at least one configuration point at which* $\Lambda(., \mathcal{O})$ *assumes its global maximum (over* $\partial\mathcal{L}$*).*

In dimension two this gives the following simple criterion: An interior point $\mathcal{O}$ of a planar convex body $\mathcal{L}$ (with non-constant $\Lambda(., \mathcal{O})$) is *not* regular if and only if the two sets

$$\mathcal{M}_{\mathcal{O}} = \{C \in \partial\mathcal{L} \mid \Lambda(C, \mathcal{O}) = \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})\}$$
$$\mathcal{N}_{\mathcal{O}} = \{C \in \partial\mathcal{L} \mid \Lambda(C, \mathcal{O}) = \min_{\partial\mathcal{L}} \Lambda(., \mathcal{O})\}$$

are connected, and along the two complementary boundary arcs Λ decreases from $\mathcal{M}_{\mathcal{O}}$ to $\mathcal{N}_{\mathcal{O}}$. (The "if" part follows by assuming that $\mathcal{O}$ is regular and considering a simplicial minimal configuration with one element in $\mathcal{M}_{\mathcal{O}}$. The "only if" part was proved in Toth (2006), Lemma 2.)

Returning to the general situation of an n-dimensional convex body $\mathcal{L} \subset \mathcal{E}$, in Toth (2009b) we studied the existence of regular points near an isolated extreme point of $\mathcal{L}$. Our first result gives a description of the convex body near a flat point on the boundary $\partial\mathcal{L}$ with nearby regular points. Recall that $\mathcal{O}_0 \in \partial\mathcal{L}$ is a flat point if there is a unique supporting hyperplane $\mathcal{H}_0$ to $\mathcal{L}$ such that $\mathcal{O}_0$ is in the relative interior of $\mathcal{L}_0 = \mathcal{H}_0 \cap \partial\mathcal{L}$.

Theorem A *Let* $\mathcal{L} \subset \mathcal{E}$ *be a convex body, and* $\{\mathcal{O}_k\}_{k\geq 1} \subset \operatorname{int}\mathcal{L}$ *a sequence of regular points converging to a flat point* $\mathcal{O}_0 \in \partial\mathcal{L}$. *If* $\mathcal{H}_0$ *is the unique supporting hyperplane at* $\mathcal{O}_0$ *to* $\mathcal{L}$ *then* $\mathcal{L}_0 = \mathcal{H}_0 \cap \partial\mathcal{L}$ *is an* $(n-1)$*-simplex. If* $C_0, \ldots, C_{n-1}$ *denote the vertices of this simplex then, for each* $i = 0, \ldots, n-1$, *there are parallel supporting hyperplanes* $\mathcal{K}_i$ *and* $\mathcal{K}_i^o$ *to* $\mathcal{L}$ *such that* $C_i \in \mathcal{K}_i$ *and the face* $[C_0, \ldots, \widehat{C_i}, \ldots, C_{n-1}]$ *opposite to* C_i *is contained in* $\mathcal{K}_i^o$.

Example The simplest illustration to Theorem A is a proper trapezoid $\mathcal{L} \subset \mathbf{R}^2$. The diagonals split $\mathcal{L}$ into four triangles. The interior of the triangle corresponding to the longer parallel side is the regular set $\mathcal{R}$ of $\mathcal{L}$. (See the proof of Theorem B in Toth (2009a) for details.) Decreasing the length of the shorter parallel side, we see that $\mathcal{R}$ can be made arbitrarily large within $\mathcal{L}$.

Example (Puffy Simplex) Let $\Delta = [C_0, \ldots, C_n]$ be an n-simplex in $\mathcal{E}$. Let $\mathcal{L} = \Delta \cup \mathcal{L}'$, where $\mathcal{L}'$ is a convex set obtained by 'puffing up' the side $[C_1, \ldots, C_n]$ of Δ as follows. We have $\Delta \cap \mathcal{L}' = \partial\Delta \cap \partial\mathcal{L}' = [C_1, \ldots, C_n]$, and the rest of the boundary of $\mathcal{L}'$ is given by the graph of a nonnegative function $f : [C_1, \ldots, C_n] \to \mathbf{R}$ measured as a distance along rays emanating from C_0 and passing through $[C_1, \ldots, C_n]$. We assume that $\mathcal{L}' \setminus [C_1, \ldots, C_n]$ is contained in the interior of the n-simplex $[C_1, \ldots, C_n, V]$, where $V = (C_1 + \cdots + C_n - C_0)/(n-1)$. (This condition just fails in the example above.) For simplicity, we assume that f is smooth. In Sect. 3 we will show that the regular set of $\mathcal{L}$ is $\mathcal{R} = \operatorname{int}\Delta$. In addition, letting $\mathcal{O} = (1-\lambda)C_0 + \lambda X \in \mathcal{R}$, $X \in \operatorname{int}[C_1, \ldots, C_n]$ and $0 < \lambda < 1$, we will also show that

$$\sigma(\mathcal{L}, \mathcal{O}) = 1 + \lambda \frac{f(X)}{|X| + f(X)}.$$

Summarizing, for f nonzero, this is an example of a non-simplicial convex body with n affinely independent flat cells and nearby regular points.

Corollary 1 *Let $\mathcal{L}$ be a convex body, and $\mathcal{L}'$ obtained from $\mathcal{L}$ by truncating $\mathcal{L}$ with a hyperplane $\mathcal{H}$. If $\mathcal{H} \cap \mathcal{L}$ is not a simplex then the relative interior of $\mathcal{H} \cap \mathcal{L}$ has an open neighborhood in $\mathcal{L}'$ consisting of singular points only.*

Remark In Theorem A, applying a small rotation to $\mathcal{K}_i$ away from $\mathcal{L}$ about $\mathcal{H}_0 \cap \mathcal{K}_i$, the rotated hyperplane $\mathcal{K}_i'$ is still supporting $\mathcal{L}$ and satisfies $\mathcal{K}_i' \cap \mathcal{L} = \{C_i\}$. In particular, C_i is an exposed point of $\mathcal{L}$.

As noted above, the regular set of an n-simplex is the entire interior of the simplex. In Toth (2009a) we conjectured that the converse was also true: *A convex body all of whose interior points are regular is a simplex.* In Toth (2009b) this was proved with the additional assumption that $\mathcal{L}$ has at least n isolated extreme points on its boundary.

This combined with the remark above immediately gives the following:

Corollary 2 *Let $\mathcal{L}$ be a convex body with all interior points regular. Then $\mathcal{L}$ is a simplex if and only if the following two conditions are satisfied:*

1. *There is at least one flat point on the boundary.*
2. *All exposed points are extreme points.*

1.3 Singular points and degree of singularity

The complement $\mathcal{S}$ of the regular set $\mathcal{R}$ in int $\mathcal{L}$ is called the *singular set*; its elements are the singular points of $\mathcal{L}$. By (7), $\mathcal{O} \in \mathcal{S}$ if and only if

$$\sigma(\mathcal{L}, \mathcal{O}) = \sigma_{n-1}(\mathcal{L}, \mathcal{O}) + \frac{1}{1 + \max_{\partial \mathcal{L}} \Lambda(., \mathcal{O})}. \tag{8}$$

By the corollary to Theorem A, any non-simplicial truncation of a convex body creates (infinitely many) singular points near the truncating hyperplane. As a related example, let $\mathcal{L}$ be a cone with base $\mathcal{L}_0$ which is not a simplex. Then, arbitrarily near to any point in the relative interior of $\mathcal{L}_0$ in int $\mathcal{L}$ there is a singular point.

Returning to the general case of a convex body $\mathcal{L} \subset \mathcal{E}$, we have the descending sequence

$$\mathcal{S} = \mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots \supset \mathcal{S}_{n-1}$$

defined as follows:

For $1 \leq \ell < n$, $\mathcal{O} \in \mathcal{S}_\ell$ if any of the following equivalent statements hold:

(i) In (6) equality holds:

$$\sigma(\mathcal{L}, \mathcal{O}) = \sigma_{n-\ell}(\mathcal{L}, \mathcal{O}) + \frac{\ell}{1 + \max_{\partial \mathcal{L}} \Lambda(., \mathcal{O})}.$$

(ii) There exists a minimal n-configuration in $\mathcal{C}(\mathcal{L}, \mathcal{O})$ which contains a (necessarily minimal) configuration in $\mathcal{C}_{n-\ell}(\mathcal{L}, \mathcal{O})$ (and at the complementary configuration points $\Lambda(., \mathcal{O})$ assumes global maximum).

(iii) The sequence $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\geq 1}$ is arithmetic [with difference $1/(1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}))$] from the $(n-\ell)$th term onwards.

The degree (of singularity) of a singular point $\mathcal{O} \in \mathcal{S}$ is the largest ℓ such that $\mathcal{O} \in \mathcal{S}_\ell$, that is, $\mathcal{O} \in \mathcal{S}_\ell \setminus \mathcal{S}_{\ell+1}$. In this case, the minimal $(n-\ell)$-subconfiguration asserted in (ii) is simplicial, and the sequence $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\geq 1}$ is arithmetic exactly from the $(n-\ell)$th term onwards. In other words, how soon the sequence becomes arithmetic measures the degree of degeneracy of the convergence to the infimum in (1) (for $m = n$).

The following result and its consequences are useful in calculations:

Theorem B *Let $k < n$ and assume that $\mathcal{O} \notin \mathcal{S}_{n-k}$. If $\{C_0, \ldots, C_n\}$ is a minimal configuration then there are at most k configuration points C_i at which*

$$\Lambda(C_i, \mathcal{O}) \leq \frac{k+1}{\sigma_k(\mathcal{L}, \mathcal{O})} - 1.$$

For $k = 1$ Theorem B asserts that, if $\mathcal{O} \notin \mathcal{S}_{n-1}$ then, with the possible exception of one configuration point, we have $\Lambda(C_i, \mathcal{O}) > 1$.

The exceptional point cannot be avoided as can be seen taking $\mathcal{L}$ a simplex and $\mathcal{O}$ close to one of the vertices. The condition $\mathcal{O} \notin \mathcal{S}_{n-1}$ is also essential: If $\mathcal{L}$ is symmetric with center of symmetry $\mathcal{O}$ then $\mathcal{O} \in \mathcal{S}_{n-1}$ and Λ is identically one on $\partial\mathcal{L}$.

Theorem C *Let $\mathcal{L} \subset \mathcal{E}$ be symmetric. Then $\mathcal{S} = \operatorname{int} \mathcal{L}$.*

Although this was proved in Toth (2009a), we give here a much shorter and more elegant proof based on a comparison principle.

In Toth (2009a) it was also conjectured that the converse of Theorem C is true. In Example 2, Sect. 3, we will give a counterexample to this. In fact, we will show that every interior point of the double of a regular tetrahedron is singular.

Other counterexamples can be found among convex cones as follows. Let $\mathcal{L}_0$ be a convex body in an $(n-1)$-dimensional Euclidean vector space $\mathcal{E}_0$. Let $\mathcal{E} = \mathcal{E}_0 \times \mathbf{R}$, $V \in \mathcal{E} \setminus \mathcal{E}_0$, and $\mathcal{L} = [\mathcal{E}, V]$ the cone with base $\mathcal{L}_0$ and vertex V. In Toth (2008) (Proposition 6) we proved that if $\mathcal{O} \in \operatorname{int} \mathcal{L}$ is regular in $\mathcal{L}$ then its projection $\mathcal{O}_0 \in \mathcal{L}_0$ from V to the base $\mathcal{L}_0$ is also regular in $\mathcal{L}_0$. Now, if $\mathcal{L}_0$ is symmetric then by Theorem C, all its interior points are singular. Hence, the interior of any cone with base $\mathcal{L}_0$ consists of singular points only. Using this, in Example 3, Sect. 3, we calculate σ for the square pyramid.

Regularity of the projected point $\mathcal{O}_0$ does not imply regularity of the point $\mathcal{O}$. An example will be given in Sect. 3 (Example 4); a 3-dimensional cone with base a regular odd-sided polygon.

1.4 Concavity

Beyond the behavior of points with regard to the infimum in (1), it is also interesting to study $\sigma(\mathcal{L}, .)$ as a function of the interior point $\mathcal{O}$. In Toth (2006) we proved that

$$\lim_{d(\mathcal{O},\partial\mathcal{L})\to 0} \sigma(\mathcal{L}, \mathcal{O}) = 1, \tag{9}$$

where d is the Euclidean distance. This implies that $\sigma(\mathcal{L}, .)$ extends continuously to $\partial\mathcal{L}$.

In Toth (2006) we also proved that $\sigma(\mathcal{L}, .)$ is concave on the regular set $\mathcal{R}$. In addition, we also showed that, for $n = 2$, $\sigma(\mathcal{L}, .)$ is a concave on the entire interior of $\mathcal{L}$. In contrast, in Toth (2008) we constructed a 4-dimensional cone, and, by a long computation, we showed that near the base $\sigma(\mathcal{L}, .)$ was not concave.

Theorem D *Let $\mathcal{L} \subset \mathcal{E}$ be a convex body. Assume that $\mathcal{O} \in \mathrm{int}\,\mathcal{L}$ is a singular point of degree $n-1$. Let $C \in \partial\mathcal{L}$ be a point at which $\Lambda(., \mathcal{O})$ assumes its global maximum. Then $\sigma(\mathcal{L}, .)$ is concave on $[\mathcal{O}, C^o]$ if and only if every point of $[\mathcal{O}, C^o)$ is singular of degree $n-1$. In this case, $\sigma(\mathcal{L}, .)$ is linear on $[\mathcal{O}, C^o)$.*

Thus, for a symmetric $\mathcal{L}$, $\dim \mathcal{L} = n \geq 3$, (radial) non-concavity or concavity of $\sigma(\mathcal{L}, .)$ depends on whether the inclusion $\mathcal{S}_{n-1} \subset \mathcal{S}$ is proper or not, or equivalently, whether or not the interior of $\mathcal{L}$ has singular points of different degrees. In Example 6, Sect. 3, we will show that the for the 3-dimensional cube $\mathcal{L}$, $\mathcal{S}_2$ is the union $\mathcal{A}$ of the 3 cross-axes connecting the midpoints of the 3 opposite pairs of faces. It thus follows that σ for the cube is not concave.

Varying degrees of singularity is also indicated by how soon does the sequence $\{\sigma_m\}_{m\geq 1}$ begin to be arithmetic at different points of the interior. For example, if $\mathcal{L}$ has a simplicial intersection of dimension $k \geq 2$, then at the points of the simplicial intersection the sequence $\{\sigma_m\}_{m\geq 1}$ starts with a string of at least k ones. In particular, the sequence cannot be arithmetic from the $k-1 \geq 1$ term onwards. We obtain the following:

Corollary 1 *Assume that $\mathcal{L} \subset \mathcal{E}$ is symmetric. If $\mathcal{L}$ has a simplicial intersection of dimension ≥ 2 then $\sigma(\mathcal{L}, .)$ is not concave on* $\mathrm{int}\,\mathcal{L}$.

Once again, Example 6 in Sect. 3, the n-dimensional cube $\mathcal{L}$, $n \geq 3$, is a good illustration to this. If the interior point $\mathcal{O}$ is away from the inscribed cross-polytope then $\mathcal{L}$ has a simplicial intersection (parallel to a vertex figure) in dimension $(n-1)$. Thus $\sigma(\mathcal{L}, .)$ is not concave for $n \geq 3$.

As noted in the introduction $\sigma(\mathcal{L}, .)$ is always concave in two dimensions. Theorem D immediately gives the following description of the topologies of $\mathcal{S}$ and $\mathcal{R}$:

Corollary 2 *Let $\mathcal{L}$ be a planar convex body and $\mathcal{O}$ an interior point. Let $\Lambda(., \mathcal{O})$ attain its global maximum at $C \in \partial\mathcal{L}$. Then either the line segment $[\mathcal{O}, C^o)$ is contained in the regular set $\mathcal{R}$ or there exists $\mathcal{O}' \in [\mathcal{O}, C^o)$ such that $\mathcal{S} \cap [\mathcal{O}, C^o) = [\mathcal{O}', C^o)$. In particular, $\mathcal{S} \cup \partial\mathcal{L}$ is path-connected and $\mathcal{R}$ is simply connected.*

As in Example 4, Sect. 3, the regular ℓ-sided polygon $\mathcal{P}_\ell \subset \mathbf{R}^2$ with $\ell = 2m+1$ odd, has regular set the interior of the inscribed star-polygon with Schläfli symbol $\{\frac{2m+1}{m}\}$. The singular set consists of $(2m+1)$ triangles each with one common side with $\mathcal{P}_{2m+1}$. In particular $\partial\mathcal{P}_\ell$ is a deformation retract of $\mathcal{S}$.

2 Proofs

Proof of Theorem A. Let $\{\mathcal{O}_k\}_{k\geq 1} \subset \mathcal{R}$ be a sequence of regular points converging to the flat point $\mathcal{O}_0$. As in the statement of Theorem A, let $\mathcal{L}_0 = \mathcal{H}_0 \cap \partial\mathcal{L}$. $\mathcal{L}_0$ is a convex body in its affine span $\mathcal{H}_0$. Since $\mathcal{O}_0$ is a flat point, it is contained in the relative interior of $\mathcal{L}_0$.

For each $k \geq 1$, let $\{C_{0,k}, \dots, C_{n,k}\}$ be a minimal simplicial configuration for $\sigma(\mathcal{L}, \mathcal{O}_k)$. As noted above, we may assume that $C_{i,k}$, $i = 0, \dots, n$, $k \geq 1$, are all extreme points. Selecting a subsequence if necessary, we may also assume that, for each $i = 0, \dots, n$, the sequence $\{C_{i,k}\}_{k\geq 1}$ converges to a point $C_i \in \partial\mathcal{L}$. Since all configuration points are extreme points, C_i is not in the relative interior of $\mathcal{L}_0$, $i = 0, \dots, n$. We now consider the possible locations of each C_i.

Case I. $C_i \notin \mathcal{L}_0$. Since $\mathcal{O}_k \to \mathcal{O}_0$, as $k \to \infty$, for large k, the opposite $C_{i,k}^o$ of $C_{i,k}$ *with respect to* $\mathcal{O}_k$ is in the relative interior of $\mathcal{L}_0$. In particular, $\mathcal{H}_0$ is the unique supporting hyperplane at $C_{i,k}^o$. Since $\mathcal{O}_k$ is a regular point, $[C_{i,k}, C_{i,k}^o]$ is an affine diameter. Let $\mathcal{H}$ be the supporting hyperplane at $C_{i,k}$ parallel to $\mathcal{H}_0$. Clearly, $\mathcal{H}$ depends only on $\mathcal{H}_0$ and $\mathcal{L}$.

Assume, for a moment, that there are two points that belong to Case I: $C_i,\ C_j \notin \mathcal{L}_0$. By the above, for large k, the points $C_{i,k}$ and $C_{j,k}$ both are in $\mathcal{H}$. If this is the case, we can replace $C_{j,k}$ with $C_{i,k}$ in the configuration without changing minimality by moving $C_{j,k}$ continuously along the line segment to $C_{i,k}$. (Note that during the move the configuration condition will stay intact since $\mathcal{O}_k$ is a regular point.) We then obtain a minimal configuration with repeated configuration points. This contradicts to regularity of $\mathcal{O}_k$. We obtain that there may be at most one point belonging to Case I. Finally, if C_i is such a point then $\Lambda(C_{i,k}, \mathcal{O}_k) \to \infty$ so that $\frac{1}{1+\Lambda(C_{i,k},\mathcal{O}_k)} \to 0$.

Case II. $C_i \in \partial\mathcal{L}_0$. As noted above, with the possible exception of one, all points belong to this case. We may assume that the possible missing point is C_n and let $n' = n-1$ or $n' = n$ according as C_n is missing or not.

For $i = 0, \dots, n'$, by continuity of the distortion, we have $\Lambda_{\mathcal{L}}(C_{i,k}, \mathcal{O}_k) \to \Lambda_{\mathcal{L}_0}(C_i, \mathcal{O}_0)$. Using the results of Case I and summing up, we obtain

$$\sigma(\mathcal{L}, \mathcal{O}_k) = \sum_{i=0}^{n} \frac{1}{1+\lambda(C_{i,k}, \mathcal{O}_k)} \to \sum_{i=0}^{n'} \frac{1}{1+\Lambda_{\mathcal{L}_0}(C_i, \mathcal{O}_0)} = 1, \quad \text{as} \quad k \to \infty, \tag{10}$$

where the last equality is because of (9). From the study of the possible missing point in Case I it is clear that $\{C_0, \dots, C_{n'}\}$ is an n'-configuration for $\mathcal{O}_0$ in $\mathcal{L}_0$. Since $\mathcal{L}_0$

is $(n-1)$-dimensional, by (5), the only way the last equality in (10) can hold is that $n' = n-1$ and $\mathcal{L}_0$ is a simplex. The first statement of Theorem A follows.

Let $\mathcal{V}$ consist of those boundary points $C \in \partial\mathcal{L} \setminus \mathcal{L}_0$ at which there is a supporting hyperplane parallel to $\mathcal{H}_0$. By the analysis in Case I, it is clear that $C_n \in \mathcal{V}$ and the supporting hyperplane in question is $\mathcal{H}$. We therefore have

$$\mathcal{V} = \mathcal{L} \cup \mathcal{H} = \partial\mathcal{L} \cup \mathcal{H}.$$

We need the following:

Lemma *Let $\mathcal{O} \in \operatorname{int}\mathcal{L}$, and assume that at $C \in \partial\mathcal{L}$ the distortion $\Lambda(., \mathcal{O})$ assumes a local maximum. Then*

$$C^o \in \mathcal{L}_0 \Rightarrow C \in \mathcal{V}.$$

Proof of Lemma. Assume $C^o \in \mathcal{L}_0$. Clearly C itself cannot be in $\mathcal{L}_0$.

If C^o is in the relative interior of $\mathcal{L}_0$ then the only hyperplane supporting $\mathcal{L}$ at C^o is $\mathcal{H}_0$. Since $[C, C^o]$ is an affine diameter, there exists a hyperplane supporting $\mathcal{L}$ at C and parallel to $\mathcal{H}_0$. This hyperplane must be $\mathcal{H}$ so that $C \in \mathcal{V}$.

It remains to consider the case when C^o is on the relative boundary of $\mathcal{L}_0$. Let $\mathcal{H}'$ be the hyperplane passing through C and parallel to $\mathcal{H}_0$. We need to show that $\mathcal{H}'$ supports $\mathcal{L}$. To do this, let τ be an affine (2-dimensional) oriented plane containing $[C, C^o]$. Let $\alpha_\tau, \alpha'_\tau$ be the two asymptotic angles between $[C, C^o]$ and the two tangent lines of the boundary $\partial\mathcal{L} \cap \tau$ at C. (For details, see Toth (2006), Sect. 7.) Similarly, define the asymptotic angles $\alpha^o_\tau, \alpha'^o_\tau$ at C^o. Choose the orientation of τ such that α'^o_τ is the asymptotic angle between $[C, C^o]$ and $\mathcal{L}_0 \cap \tau$ at C^o. Since $\Lambda(., \mathcal{O})$ assumes local maximum at C, we have $\alpha_\tau \le \alpha^o_\tau$ and $\alpha'_\tau \le \alpha'^o_\tau$. The second inequality implies that $\mathcal{H}' \cap \tau$ supports $\mathcal{L} \cap \tau$ at C. Varying τ, we see that $\mathcal{H}'$ supports $\mathcal{L}$ at C. Thus $\mathcal{H}' = \mathcal{H}$ and $C \in \mathcal{V}$. The lemma follows.

We now return to the notations in Cases I–II. Since $C^o_{n,k}$ is in the relative interior of $\mathcal{L}_0$, for large k, we have $C_{n,k} \in \mathcal{V}$. Since $\mathcal{O}_k$ is regular, $C_{n,k}$ can be any point in $\mathcal{V}$, in particular, we can choose $C_{n,k} = C_n$ constant.

From now on we may assume that, for $i = 0, \ldots, n-1$, $C_{i,k} \to C_i \in \mathcal{L}_0$, as $k \to \infty$, and the boundary $(n-1)$-simplex is $\mathcal{L}_0 = [C_0, \ldots, C_{n-1}]$. For k large, $C_{i,k} \notin \mathcal{V}$. Hence, by the lemma above, $C^o_{i,k} \notin \mathcal{L}_0$. Let $\delta_{i,k}$ be the dihedral angle of the angular sector containing $\mathcal{L}$ and bounded by $\mathcal{H}_0$ and $\mathcal{K}_{i,k}$. Define similarly $\delta^o_{i,k}$. Since $\mathcal{K}_{i,k}$ and $\mathcal{K}^o_{i,k}$ are parallel, we have $\delta_{i,k} + \delta^o_{i,k} = \pi$, where $0 < \delta_{i,k}, \delta^o_{i,k} < \pi$. Selecting subsequences, we may assume that $\delta_{i,k} \to \delta_i$ and $\delta^o_{i,k} \to \delta^o_i$ as $k \to \infty$. Clearly, $\delta_i + \delta^o_i = \pi$, and, by convexity, $0 < \delta_i, \delta^o_i < \pi$.

Let $\mathcal{K}^o_i$ be the hyperplane that contains $[C_0, \ldots, \widehat{C_i}, \ldots, C_{n-1}]$ and has dihedral angle δ^o_i with $\mathcal{H}_0$. Being the limit of the supporting hyperplanes $\mathcal{K}^o_{i,k}$, the hyperplane $\mathcal{K}^o_i$ also supports $\mathcal{L}$. Let $\mathcal{K}_i$ be the hyperplane passing through C_i and parallel to $\mathcal{K}^o_i$. Clearly, $\mathcal{K}_i$ supports $\mathcal{L}$ at C_i. The second statement of Theorem A follows.

Note that, a simple comparison of distortions (using that $\mathcal{K}_i$ and $\mathcal{K}^o_i$ are parallel) shows that, for k large, $\Lambda(C_i, \mathcal{O}_k) \ge \Lambda(C_{i,k}, \mathcal{O}_k)$. Hence, for large k, $\{C_0, \ldots, C_n\}$ (with $C_n \in \mathcal{V}$ arbitrary) is a minimizing configuration for $\sigma(\mathcal{L}, \mathcal{O}_k)$.

Proof of Theorem B. The proof is by contradiction. Renumbering if necessary, we may assume that, for $i = 0, \ldots, k$, we have $\Lambda(C_i, \mathcal{O}) \le (k+1)/\sigma_k(\mathcal{L}, \mathcal{O}) - 1$. Then $\sum_{i=0}^{k} \frac{1}{1+\Lambda(C_i,\mathcal{O})} \ge \sum_{i=0}^{k} \frac{\sigma_k(\mathcal{L},\mathcal{O})}{1+k} = \sigma_k(\mathcal{L}, \mathcal{O})$. Let $\{B_0, \ldots, B_k\}$ be a minimal configuration such that $\sum_{i=0}^{k} \frac{1}{1+\Lambda(B_i,\mathcal{O})} = \sigma_k(\mathcal{L}, \mathcal{O})$. We extend this k-configuration to the n-configuration as $\{B_0, \ldots, B_k, C_{k+1}, \ldots, C_n\}$. Then we have

$$\sum_{i=0}^{k} \frac{1}{1+\Lambda(B_i, \mathcal{O})} + \sum_{j=k+1}^{n} \frac{1}{1+\Lambda(C_j, \mathcal{O})} = \sigma_k(\mathcal{L}, \mathcal{O}) + \sum_{j=k+1}^{n} \frac{1}{1+\Lambda(C_j, \mathcal{O})}$$
$$\le \sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i, \mathcal{O})} = \sigma(\mathcal{L}, \mathcal{O}).$$

Since the extension is a configuration, equality holds here. Thus, $\mathcal{O} \in \mathcal{S}_{n-k}$. This is a contradiction.

Before the proof of Theorem C we develop a useful comparison formula. Let $\mathcal{L} \subset \mathcal{E}$ be a convex body and $\mathcal{O} \in \operatorname{int} \mathcal{L}$. For $C, C' \in \partial\mathcal{L}$, $C' \ne C, C^o$, we let $\mathcal{N}_{C,C'} = \langle \mathcal{O}, C, C' \rangle$, an affine plane in $\mathcal{E}$. Within $\mathcal{N}_{C,C'}$ we let $\mathcal{M}_{C,C'}$ be the closed half-plane which contains C' and has boundary line $\langle \mathcal{O}, C \rangle$. Taking opposites, we also have the closed half-plane $\mathcal{M}_{C'^o,C^o}$ which contains C^o and has boundary line $\langle \mathcal{O}, C' \rangle$. Finally, we define

$$\mathcal{L}_{C,C'} = \mathcal{M}_{C,C'} \cap \mathcal{M}_{C'^o,C^o} \cap \operatorname{int} \mathcal{L}.$$

Clearly, $\mathcal{L}_{C,C'}$ is a convex body in $\mathcal{N}_{C,C'}$ with boundary consisting of the line segments $[\mathcal{O}, C']$, $[\mathcal{O}, C^o]$ and a boundary arc of $\mathcal{L} \cap \mathcal{N}_{C,C'}$. It is also clear that $\mathcal{L}_{C,C'} = \mathcal{L}_{C'^o,C^o}$.

For fixed $C \in \partial\mathcal{L}$, we have

$$\bigcap_{C' \ne C,\, C^o\; C' \in \partial\mathcal{L}} \mathcal{L}_{C,C'} = [\mathcal{O}, C^o).$$

Comparison Lemma *Let $\mathcal{L} \subset \mathcal{E}$ be a convex body and $\mathcal{O} \in \operatorname{int} \mathcal{L}$. Let $C, C' \in \partial\mathcal{L}$, $C' \ne C, C^o$, and assume that*

$$\Lambda(C, \mathcal{O}) \ge \Lambda(C', \mathcal{O}). \tag{11}$$

Then, for $\mathcal{O}' \in \mathcal{L}_{C,C'}$, we have

$$\Lambda(C, \mathcal{O}') \ge \Lambda(C', \mathcal{O}'). \tag{12}$$

Moreover, sharp inequality in (11) *implies sharp inequality in* (12).

Proof Consider the half-line ℓ emanating from C and passing through C', and the half-line ℓ^o emanating from C'^o and passing through C^o.

First assume sharp inequality in (11). This assumption means that ℓ and ℓ' intersect at a point, say, P. By convexity of $\mathcal{L}$, the quadrangle $[C', \mathcal{O}, C^o, P]$ contains $\mathcal{L}_{C,C'}$.

Let $\mathcal{O}' \in \mathcal{L}_{C,C'}$, and let B and B' be the opposites of C and C' with respect to $\mathcal{O}'$. Consider the half-line ℓ' emanating from B' and passing through B, and the half-line ℓ'' emanating from B and passing through B'. Along the boundary $\partial\mathcal{L} \cap \mathcal{N}_{C,C'}$ B follows C^o in the same direction as B' follows C'^o. Therefore, by convexity of $\mathcal{L}$, ℓ'' intersects ℓ^o. Since $\ell' \cup \ell'' = \langle B, B' \rangle$, ℓ' must intersect ℓ. The sharp inequality in (12) follows.

If equality holds in (11) then ℓ and ℓ' are parallel, and the proof is analogous. □

Corollary 1 *Let $\mathcal{L}$ and $\mathcal{O} \in \operatorname{int} \mathcal{L}$ as above. Assume that $\Lambda(., \mathcal{O})$ attains its local (global) maximum at $C \in \partial\mathcal{L}$. Then, for any $\mathcal{O}' \in [\mathcal{O}, C^o)$, $\Lambda(., \mathcal{O}')$ also attains its local (global) maximum at C.*

Corollary 2 *Let $\mathcal{L} \subset \mathcal{E}$ be symmetric with center of symmetry $\mathcal{O}$. Let $\mathcal{O}' \in \operatorname{int} \mathcal{L}$, $\mathcal{O}' \neq \mathcal{O}$. Let $\langle \mathcal{O}, \mathcal{O}' \rangle \cap \mathcal{L} = [A, A^o]$ with $\mathcal{O} \in [\mathcal{O}', A]$. Then $\Lambda(., \mathcal{O}')$ attains its global maximum at A (and its global minimum at A^o). Moreover, for any affine plane $\mathcal{M}$ that contains $\langle \mathcal{O}, \mathcal{O}' \rangle$, the distortion is increasing along the boundary arcs of $\mathcal{M} \cup \partial\mathcal{L}$ from A^o to A.*

Proof of Corollary 2. For any $C, C' \in \partial\mathcal{L}$, we have $\Lambda(C, \mathcal{O}) = \Lambda(C', \mathcal{O})$, so that the Comparison Lemma applies.

Proof of Theorem C. Let the center of symmetry be $\mathcal{O}$. We have $\Lambda(., \mathcal{O}) = 1$, so that $\mathcal{O} \in \mathcal{S}_{n-1}$, a singular point of degree $n-1$. Now let $\mathcal{O}' \in \operatorname{int} \mathcal{L}$, $\mathcal{O}' \neq \mathcal{O}$, and A, A^o as in Corollary 2. Assume that $\mathcal{O}'$ is a regular point. Let $\{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, \mathcal{O}')$ be minimal. Recall that $[C_0, \ldots, C_n]$ is an n-simplex with $\mathcal{O}'$ in its interior. Let C_i, $i = 0, \ldots, n$ be any of the configuration points, different from A and A^o. Apply the Comparison Lemma above to $\mathcal{M} = \langle \mathcal{O}, \mathcal{O}', C_i \rangle$. Then C_i can be moved along one of the boundary arcs of $\mathcal{M} \cap \operatorname{int} \mathcal{L}$ from A to A^o toward A with increasing $\Lambda(., \mathcal{O}')$. By minimality, $\Lambda(., \mathcal{O}')$ must stay constant. Since $\mathcal{O}'$ is regular, the condition $\mathcal{O}' \in [C_0, \ldots, C_n]$ stays intact when replacing C_i with a moved point. Hence C can be moved to A. This is a contradiction.

Proof of Theorem D. Parametrize the line segment $[\mathcal{O}, C^o]$ as

$$\lambda \mapsto \mathcal{O}_\lambda = (1-\lambda)\mathcal{O} + \lambda C^o, \quad 0 \leq \lambda \leq 1.$$

Consider the function $\lambda \mapsto \sigma(\mathcal{L}, \mathcal{O}_\lambda)$, $\lambda \in [0, 1)$. For $\ell = n-1$, (6) gives

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) \leq 1 + \frac{n-1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)}, \quad 0 \leq \lambda < 1. \tag{13}$$

By Corollary 1 to the Comparison Lemma, if at $C \in \partial\mathcal{L}$ the distortion $\Lambda(., \mathcal{O})$ attains its global maximum then, for each $0 \leq \lambda < 1$, $\Lambda(., \mathcal{O}_\lambda)$ also attains its global maximum at C. Hence (13) rewrites as

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) \leq 1 + \frac{n-1}{1 + \Lambda(C, \mathcal{O}_\lambda)} = 1 + (1-\lambda)\frac{n-1}{1 + \Lambda(C, \mathcal{O})}, \quad 0 \leq \lambda < 1. \tag{14}$$

$\mathcal{O} = \mathcal{O}_0$ is a singular point of degree $n-1$ so that, at $\lambda = 0$, equality holds in (14). As noted above, the left-hand side of (14) is continuous in $\lambda \in [0, 1)$, and by (9), it extends continuously to $\lambda = 1$ with value equal to 1. Hence, equality also holds in (14) for $\lambda = 1$. Thus, if $\lambda \mapsto \sigma(., \mathcal{O}_\lambda)$ is concave for $\lambda \in [0, 1]$ then it must be linear, and equality must hold in (14) everywhere. This means that $[\mathcal{O}, C^o) \subset \mathcal{S}_{n-1}$.

The converse follows from the fact Toth (2006) that the function $1/(1 + \max_{\partial \mathcal{L}} \Lambda)$ is always concave on the interior of $\mathcal{L}$. Theorem D follows.

3 Examples

Example 1 We consider the puffy simplex $\mathcal{L} = \Delta \cup \mathcal{L}'$ in details introduced in the second example of Sect. 1.2. Recall that $\Delta = [C_0, \ldots, C_n]$ in an n-simplex, and the 'puffing' $\mathcal{L}'$ is given by the graph of a function f over the side $[C_1, \ldots, C_n]$. To avoid the trivial case of a simplex, we assume that f is nonzero. Our basic condition that controls the size of $\mathcal{L}'$ is that the relative interior of the graph of f is contained in the interior of the simplex $[C_1, \ldots, C_n, V]$, where $V = (C_1 + \cdots + C_n)/(n-1)$. Geometrically, this means that, apart from the side $[C_1, \ldots, C_n]$, each side of the simplex $[C_1, \ldots, C_n, V]$ has the property that it contains a vector parallel to C_i for some $i = 1, \ldots, n$. (Actually, the vector from V to the midpoint of the side $[C_1, \ldots, \widehat{C_i}, \ldots, C_n, V]$ is parallel to C_i.) The condition on the extent of the graph of f then implies that no tangent plane of the (relative interior of the) graph of f is parallel to any of the sides of $[C_0, \ldots, C_n]$ other than $[C_1, \ldots, C_n]$. In particular, no affine diameter emanates from the (relative interior of the) graph of f except those that end in C_0. For $\mathcal{O} \in \mathcal{L}'$, the distortion $\Lambda(., \mathcal{O})$ has no local maxima on the graph of f. (If $\mathcal{O}$ is on an exceptional affine diameter, the local maximum is at C_0.) We obtain that the interior of $\mathcal{L}'$ consists of singular points. Since the singular set is closed, we have $\mathcal{R} \subset \text{int}\, \Delta$. It remains to show that equality holds.

Given $\mathcal{O} \in \text{int}\, \Delta$, we write $\mathcal{O} = \lambda X + (1-\lambda) C_0$ with $0 < \lambda < 1$ and $X \in \text{int}\, [C_1, \ldots, C_n]$. For simplicity, we place C_0 at the origin. We also write $X = \sum_{i=1}^n \lambda_i C_i$, where $\sum_{i=1}^n \lambda_i = 1$, $0 < \lambda_i < 1$, $i = 1, \ldots, n$. An easy computation then shows

$$\frac{1}{1 + \Lambda(C_i, \mathcal{O})} = \lambda \lambda_i, \quad i = 1, \ldots, n.$$

Since Δ is a simplex, we have $\sigma(\Delta, \mathcal{O}) = 1$, so that, as a byproduct, it immediately follows that $\Lambda_\Delta(C_0, \mathcal{O}) = \lambda/(1 - \lambda)$.

The extension of the line segment $[C_0, X]$ beyond X intersects the simplex $[C_1, \ldots, C_n, V]$ at (say) the side opposite to C_i. This intersection point is $X/(1 - \lambda_i)$. The condition to the extent of the graph of f can then be written as

$$\frac{f(X)}{|X|} + 1 < \frac{1}{1 - \lambda_i}.$$

Eliminating λ_i, we obtain

$$\Lambda(C_i, \mathcal{O}) < \frac{1}{\lambda} \frac{|X|}{f(X)} - 1, \quad i = 1, \ldots, n. \tag{15}$$

Since $\mathcal{L}$ has a codimension one simplicial intersection across $\mathcal{O}$, to prove that $\mathcal{O}$ is a regular point, it is enough to show

$$\sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i,\mathcal{O})} < 1 + \frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(.,\mathcal{O})}. \tag{16}$$

To calculate the sum on the left-hand side, we compare it with $\sigma(\Delta,\mathcal{O}) = 1$ as

$$\begin{aligned}\sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i,\mathcal{O})} &= \sum_{i=0}^{n}\left(\frac{1}{1+\Lambda(C_i,\mathcal{O})} - \frac{1}{1+\Lambda_\Delta(C_i,\mathcal{O})}\right) + 1\\ &= \frac{1}{1+\Lambda(C_0,\mathcal{O})} - \frac{1}{1+\Lambda_\Delta(C_0,\mathcal{O})} + 1\\ &= 1 + \lambda\frac{f(X)}{|X|+f(X)},\end{aligned}$$

where we used $\Lambda(C_0,\mathcal{O}) = \lambda|X|/((1-\lambda)|X| + f(X))$. Thus, (16) reduces to

$$\lambda\frac{f(X)}{|X|+f(X)} < \frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(.,\mathcal{O})}.$$

The maximum of the distortion $\Lambda(.,\mathcal{O})$ occurs either at one of the vertices C_i, $i = 0,\ldots,n$, or at a point on the graph of the function f. An easy comparison using (15) and the formula for $\Lambda(C_0,\mathcal{O})$ above, shows that sharp inequality holds if the maximum distortion occurs at C_i, $i = 0,\ldots,n$. It remains to show that maximum distortion cannot occur on the relative interior of the graph of the function f. To do this, we will replace this graph with the boundary of the covering simplex $[C_1,\ldots,C_n,V]$ (except the base $[C_1,\ldots,C_n]$). Since maximum distortion occurs at a vertex, it is enough to show that the distortion at V cannot be greater than the maximum. Once again, an easy computation shows that

$$\frac{1}{1+\Lambda(V,\mathcal{O})} = (n-1)\lambda \min_{1\le i\le n} \lambda_i \ge \lambda\lambda_j = \frac{1}{1+\Lambda(C_j,\mathcal{O})},$$

where $\max_{1\le i\le n}\lambda_i = \lambda_j$. (16) follows.

It remains to calculate $\sigma(\mathcal{L},\mathcal{O})$. Let $\{B_0,\ldots,B_n\}$ be a minimizing configuration for $\sigma(\mathcal{L},\mathcal{O})$, $\mathcal{O} \in \mathcal{R} = \operatorname{int}\Delta$. We may assume that the configuration points are extreme points. Since the extreme points are $C_0,\ldots,C_n$ and possibly points on the graph of f, the condition $\mathcal{O} \in [B_0,\ldots,B_n]$ implies that C_0 must be one of the configuration points. No point on the graph of f can be another configuration point since it would then be one end of an affine diameter, with the other end being C_0, a contradiction to the regularity of $\mathcal{O}$. Thus $\{B_0,\ldots,B_m\} = \{C_0,\ldots,C_n\}$ and the stated formula in the second example of Sect. 1.2 for $\sigma(\mathcal{L},\mathcal{O})$ follows.

Example 2 Let $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$ be the double of two regular tetrahedra $\mathcal{L}_\pm = [C_0, C_1, C_2, C_\pm]$. We claim that $\mathcal{S} = \operatorname{int}\mathcal{L}$. By symmetry, we may assume that $\mathcal{O} \in \operatorname{int}\mathcal{L}_+$, and write $\mathcal{O} = \mathcal{O}_\lambda = (1-\lambda)\mathcal{O}_0 + \lambda C_+$, where $0 \leq \lambda < 1$ and $\mathcal{O}_0 \in \operatorname{int}[C_0, C_1, C_2]$. By symmetry again, we may also assume that $\mathcal{O}_\lambda$ is in the fundamental tetrahedron $\mathcal{T} = [C_0, (C_0+C_1)/2, (C_0+C_1+C_2)/3]$. By a suitable translation $(C_0+C_1+C_2)/3$ can be moved to the origin. Then the restriction $\mathcal{O} \in \mathcal{T}$ amounts to $\mathcal{O}_0 = \mathcal{O}_0^{\alpha,\beta} = \alpha C_0 + \beta(C_0 + C_1)/2$, where $\alpha, \beta \geq 0$ and $0 \leq \alpha + \beta < 1$. To show this latter dependency, we also write $\mathcal{O} = \mathcal{O}_\lambda = \mathcal{O}_\lambda^{\alpha,\beta}$. Since $\mathcal{R}$ is open, to prove that $\mathcal{S} = \operatorname{int} \mathcal{L}$, it is enough to show that $\mathcal{O}_\lambda^{\alpha,\beta} \in \mathcal{S}$ for $\lambda > 0$ and $\alpha, \beta > 0$.

Assume, on the contrary, that $\mathcal{O}_\lambda^{\alpha,\beta}$ is a regular point. Consider a minimal configuration. We may assume that the configuration points are extreme points, in our case, vertices of $\mathcal{L}$. Due to the position of $\mathcal{O}_\lambda^{\alpha,\beta}$, for the minimal configuration we have only two cases: $\{C_0, C_1, C_2, C_+\}$ or $\{C_0, C_1, C_-, C_+\}$. We first claim that the second case cannot occur. Indeed, parametrize the line segment $[C_2, C_-]$ by $C(t) = (1-t)C_2 + tC_-$, $0 \leq t \leq 1$. Then a brief computation gives

$$\Lambda(C(t), \mathcal{O}_\lambda^{\alpha,\beta}) = \frac{2 - t + (1-\lambda)(\alpha+\beta) + \lambda}{(1-\lambda)(1-\alpha-\beta)}. \tag{17}$$

This shows that the distortion decreases from C_2 to C_-. The claim follows.

Since $\mathcal{O} = \mathcal{O}_\lambda^{\alpha,\beta}$ is a regular point, we have

$$\sigma(\mathcal{L}, \mathcal{O}) = \sum_{i=0}^{2} \frac{1}{1 + \Lambda(C_i, \mathcal{O})} + \frac{1}{1 + \Lambda(C_+, \mathcal{O})} < 1 + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}. \tag{18}$$

We calculate these terms as follows. First, $\Lambda(C_i, \mathcal{O}) = \Lambda_{\mathcal{L}_+}(C_i, \mathcal{O})$, and since $\mathcal{L}_+$ is a simplex, we have

$$\sigma(\mathcal{L}_+, \mathcal{O}) = \sum_{i=0}^{2} \frac{1}{1 + \Lambda(C_i, \mathcal{O})} + \frac{1}{1 + \Lambda_{\mathcal{L}_+}(C_+, \mathcal{O})} = 1.$$

On the other hand, we clearly have $\Lambda_{\mathcal{L}_+}(C_+, \mathcal{O}) = \frac{1-\lambda}{\lambda}$. Putting these together, we find that the first sum on the left-hand side of (18) equals $1 - \lambda$.

To calculate $\Lambda(C_+, \mathcal{O})$, we write the opposite of C_+ as $C_+^o = (1-s)C_+ + s\mathcal{O}_0$, with $s > 1$. Since this opposite is on the face $[C_0, C_1, C_-]$, a simple computation gives $s = 2/(\alpha + \beta + 1)$. We thus have

$$\Lambda(C_+, \mathcal{O}) = \frac{1-\lambda}{s - (1-\lambda)} = \frac{1-\lambda}{\frac{2}{\alpha+\beta+1} - (1-\lambda)}. \tag{19}$$

With this, the left-hand side of the inequality in (18) is calculated. For the right-hand side, maximum distortion occurs at a vertex, so that a simple comparison gives

$$\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}) = \Lambda(C_2, \mathcal{O}).$$

On the other hand, by (17), the distortion at the configuration point $C_2 = C(0)$ is

$$\Lambda(C_2, \mathcal{O}) = \frac{2 + (1-\lambda)(\alpha+\beta) + \lambda}{(1-\lambda)(1-\alpha-\beta)}.$$

Substituting all the ingredients back to (18), a simple computation reduces this inequality to $1 < \alpha + \beta$. This is a contradiction. $\mathrm{int}\,\mathcal{L} = \mathcal{S}$ follows.

Since every interior point is singular, as a byproduct, we obtain

$$\sigma(\mathcal{L}, \mathcal{O}) = 1 + \frac{(1-\lambda)(1-\alpha-\beta)}{3},$$

where $\mathcal{O} = (1-\lambda)(\alpha C_i + \beta(C_i + C_j)/2) + \lambda C_\pm$, $i, j = 0, 1, 2$, $i \neq j$, with the centroid of $\mathcal{L}$ at the origin.

Example 3 Let $\mathcal{L}_0 \subset \mathbf{R}^2$ be a square with vertices $C_0 = (2, 0)$, $C_1 = (0, 2)$, $C_2 = (-2, 0)$ and $C_3 = (0, -2)$. Let $V = (0, 0, h)$, $h > 0$, and consider the square pyramid $\mathcal{L} = [C_0, C_1, C_2, C_3, V] \subset \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$ with base $\mathcal{L}_0$. Given an interior point $\mathcal{O} \in \mathrm{int}\,\mathcal{L}$, we calculate $\sigma(\mathcal{L}, \mathcal{O})$. First, by Theorem C and the following discussion, $\mathcal{S} = \mathrm{int}\,\mathcal{L}$. Since $\sigma_2(\mathcal{L}, .) = 1$ (as $\mathcal{L}$ has triangular intersections), we have

$$\sigma(\mathcal{L}, \mathcal{O}) = 1 + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})}.$$

It remains to calculate the maximum distortion. Restrict $\mathcal{O}$ to the fundamental tetrahedron $\mathcal{T} = [0, C_0, (C_0 + C_1)/2, V]$ and write $\mathcal{O} = \mathcal{O}_\lambda^{\alpha,\beta} = (1-\lambda)\mathcal{O}_0^{\alpha,\beta} + \lambda V$, where $\mathcal{O}_0^{\alpha,\beta} = \alpha C_0 + \beta(C_0 + C_1)/2$, $0 < \lambda < 1$, $0 \leq \alpha, \beta, \alpha + \beta < 1$. Then a simple computation shows

$$\begin{aligned}
\Lambda(V, \mathcal{O}_\lambda^{\alpha,\beta}) &= \frac{1-\lambda}{\lambda} \\
\Lambda(C_0, \mathcal{O}_\lambda^{\alpha,\beta}) &= \frac{1+\lambda-(1-\lambda)\alpha}{(1-\lambda)(1-\alpha)} \\
\Lambda(C_1, \mathcal{O}_\lambda^{\alpha,\beta}) &= \frac{1+\lambda+(1-\lambda)\alpha}{(1-\lambda)(1-\alpha)} \\
\Lambda(C_2, \mathcal{O}_\lambda^{\alpha,\beta}) &= \Lambda(C_3, \mathcal{O}_\lambda^{\alpha,\beta}) \\
&= \frac{1+\lambda+(1-\lambda)(\alpha+\beta)}{(1-\lambda)(1-\alpha-\beta)}.
\end{aligned}$$

With these, we obtain

$$\sigma(\mathcal{L}, \mathcal{O}) = 1 + \min\left(\lambda, \frac{(1-\lambda)(1-\alpha-\beta)}{2}\right),$$

where $\mathcal{O} = (1-\lambda)(\alpha C_i + \beta(C_i + C_{i+1})/2) + \lambda V$, $i = 0, 1, 2, 3$, and the indices are counted mod 3.

Example 4 Let $\mathcal{P}_\ell \subset \mathbf{R}^2$ be a regular ℓ-sided polygon. Assume that $\ell = 2m+1$ is odd, $m \geq 2$. A careful study involving the possible affine diameters shows Toth (2009a) that the regular set is the interior of the inscribed star-polygon with Schläfli symbol $\{\frac{2m+1}{m}\}$.

Let $\mathcal{L}$ be a cone with base $\mathcal{P}_\ell$ and vertex V. Since the base is not triangular, given an open set $U_0 \subset \mathcal{R}$ in the star polygon, by Theorem A, the open set $U = [U_0, V] \setminus \{V\}$ must contain a singular point arbitrarily close to U_0.

More specifically, let $\mathcal{O}_0$ be the center of $\mathcal{P}_\ell$ and $\mathcal{O}_\lambda = (1-\lambda)\mathcal{O}_0 + \lambda V$. Then, for any vertex C of $\mathcal{P}_\ell$, we have $\Lambda(C, \mathcal{O}_\lambda) = \frac{\sec(\pi/\ell)+\lambda}{1-\lambda}$. Now, if

$$\lambda \leq \frac{2 - \sec(\pi/\ell)}{3}$$

then $\mathcal{O}_\lambda$ must be singular. (Indeed, if $\mathcal{O}_\lambda$ were regular then there would exist a minimizing configuration with at least three vertices of $\mathcal{P}_\ell$. This would contradict to Theorem B as $\sigma_2(\mathcal{L}, \mathcal{O}_\lambda) = 1$.)

Example 5 Let $\mathcal{L} \subset \mathcal{E}$ be an n-dimensional ball with center $\mathcal{O}$. By Theorem C, $\mathcal{S} = \operatorname{int} \mathcal{L}$. We claim that $\sigma(\mathcal{L}, .)$ is linear along all radial line segments connecting $\mathcal{O}$ with boundary points. Let $C \in \partial\mathcal{L}$, and parametrize the line segment $[\mathcal{O}, C^o)$ as $\mathcal{O}_\lambda = (1-\lambda)\mathcal{O} + \lambda C^o$, $0 \leq \lambda < 1$. The claim amounts to the following

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = 1 + (1-\lambda)\frac{n-1}{2}.$$

We prove this by induction. For $n = 2$ $\sigma(\mathcal{L}, .)$ is always concave Toth (2006), and the claim follows from Theorem D. Assume that the claim holds in dimensions less than n, and let $\mathcal{L}$ and $\mathcal{O}$ be as above. Since $\mathcal{O}_\lambda$ is singular, we have

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \sigma_{n-1}(\mathcal{L}, \mathcal{O}_\lambda) + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)}. \tag{20}$$

By (4), we also have

$$\sigma_{n-1}(\mathcal{L}, \mathcal{O}_\lambda) = \inf_{\mathcal{O}_\lambda \in \mathcal{F} \subset \mathcal{E},\, \dim \mathcal{F} = n-1} \sigma(\mathcal{L} \cap \mathcal{F})$$

where the infimum is over affine subspaces $\mathcal{F} \subset \mathcal{E}$. Now, $\mathcal{L} \cap \mathcal{F}$ is an $(n-1)$-dimensional ball and the induction hypothesis applies. The infimum is attained for those $\mathcal{F}$ for which $\mathcal{O}_\lambda$ is furthest away from the center of $\mathcal{L} \cap \mathcal{F}$, that is, for those $\mathcal{F}$ that contain $\mathcal{O}$. We obtain

$$\sigma_{n-1}(\mathcal{L}, \mathcal{O}_\lambda) = 1 + (1-\lambda)\frac{n-2}{2}.$$

On the other hand, the maximum of $\Lambda(., \mathcal{O}_\lambda)$ is attained at C. Hence we have

$$\frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)} = \frac{1}{1 + \Lambda(C, \mathcal{O}_\lambda)} = \frac{1-\lambda}{2}.$$

Putting these together, the induction is complete, and the claim follows.

Example 6 Let $\mathcal{L}$ be an n-dimensional cube, $n \geq 3$. The center $\mathcal{O}_0$ is a singular point of degree $n-1$. Let $\mathcal{L}_0 \subset \mathcal{L}$ denote the dual of $\mathcal{L}$ viewed as the inscribed cross-polytope with vertices being the centroids of the cells of $\mathcal{L}$. Clearly, through any point in the complement $\operatorname{int} \mathcal{L} \setminus \operatorname{int} \mathcal{L}_0$ there is a codimension one simplicial intersection (parallel to one of the vertex figures). Thus, at these points the sequence $\{\sigma_m\}_{m\geq 1}$ starts with $(n-1)$ ones. By symmetry, $\operatorname{int} \mathcal{L} = \mathcal{S}$, and we obtain that every point in $\operatorname{int} \mathcal{L} \setminus \operatorname{int} \mathcal{L}_0$ is a singular point of degree one.

The calculation of the degree of singularity for points in the interior of $\mathcal{L}_0$ is more difficult. We will do this only for $n = 3$.
Assume that the cube $\mathcal{L}$ is in the standard position in $\mathbf{R}^3$ having vertices $(\pm 1, \pm 1, \pm 1)$. By symmetry, we may restrict $\mathcal{O}$ to the fundamental tetrahedron $\mathcal{T} = [0, V, E, F]$, where $V = (1, 1, 1)$ is a vertex, $E = (0, 1, 1)$ is the midpoint of an edge, and $F = (0, 1, 0)$ is the midpoint of a face. We then let $\mathcal{O} = \mathcal{O}_{\alpha,\beta,\gamma} = \alpha V + \beta E + \gamma F$, where $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma < 1$. An easy calculation gives

$$\max_{\partial \mathcal{L}} \Lambda(., \mathcal{O}_{\alpha,\beta,\gamma}) = \frac{1+\alpha+\beta+\gamma}{1-\alpha-\beta-\gamma}. \tag{21}$$

By the above, we need to consider points only in the intersection $\mathcal{T}_0 = \mathcal{T} \cup \mathcal{L}_0$. This is another tetrahedron cut out from $\mathcal{T}$ by the plane extension of the vertex figure that has normal vector (1,1,1) and passes through the point (1/3, 1/3, 1/3). Since this plane can also be written as $\langle V/3, E/2, F \rangle$, the restriction $\mathcal{O}_{\alpha,\beta,\gamma} \in \mathcal{T}_0$ amounts to $0 \leq 3\alpha + 2\beta + \gamma \leq 1$.

We now consider a specific plane intersection $\mathcal{L}_{\alpha,\beta,\gamma}$ of $\mathcal{L}$ by a plane that passes through $\mathcal{O}_{\alpha,\beta,\gamma}$ and the vertices $A = (1, 1, -1)$ and $B = (1, -1, 1)$.

For $3\alpha + 2\beta + \gamma = 1$, $\mathcal{L}_{\alpha,\beta,\gamma}$ is the vertex figure considered above. For $3\alpha + 2\beta + \gamma < 1$, $\mathcal{L}_{\alpha,\beta,\gamma}$ is a symmetric trapezoid (which, for $\alpha = \beta = \gamma = 0$, becomes a rectangle). As computation shows, we have $\mathcal{L}_{\alpha,\beta,\gamma} = [A, B, P_{\alpha,\beta,\gamma}, Q_{\alpha,\beta,\gamma}]$, where

$$P_{\alpha,\beta,\gamma} = \left(-1, 1, \frac{5\alpha+4\beta+2\gamma-1}{1-\alpha}\right) \quad \text{and} \quad Q_{\alpha,\beta,\gamma} = \left(-1, \frac{5\alpha+4\beta+2\gamma-1}{1-\alpha}, 1\right).$$

Now a somewhat tedious computation shows that

$$\sigma(\mathcal{L}_{\alpha,\beta,\gamma}, \mathcal{O}_{\alpha,\beta,\gamma}) = \frac{3-3\alpha-2\beta-\gamma}{2}. \tag{22}$$

(In fact, $\mathcal{O}_{\alpha,\beta,\gamma}$ is a regular point of $\mathcal{L}_{\alpha,\beta,\gamma}$ with a minimizing configuration $\{A, B, P_{\alpha,\beta,\gamma}\}$.)

Using (2), we now compute

$$\begin{aligned}\sigma(\mathcal{L}, \mathcal{O}_{\alpha,\beta,\gamma}) &\leq \sigma_2(\mathcal{L}, \mathcal{O}_{\alpha,\beta,\gamma}) + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_{\alpha,\beta,\gamma})} \\ &\leq \sigma(\mathcal{L}_{\alpha,\beta,\gamma}, \mathcal{O}_{\alpha,\beta,\gamma}) + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_{\alpha,\beta,\gamma})} \\ &= 2 - \frac{4\alpha + 3\beta + 2\gamma}{2},\end{aligned}$$

where we used (4) and (21)–(22).

On the other hand, assume that $\mathcal{O}_{\alpha,\beta,\gamma} \in \mathcal{S}_2$. (We already know from Theorem C that $\mathcal{O}_{\alpha,\beta,\gamma}$ is a singular point.) By definition, this means that

$$\begin{aligned}\sigma(\mathcal{L}, \mathcal{O}_{\alpha,\beta,\gamma}) &= 1 + \frac{2}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_{\alpha,\beta,\gamma})} \\ &= 2 - \alpha - \beta - \gamma.\end{aligned}$$

Comparing these two computations, we get $\alpha = \beta = 0$, the defining equality for the cross-axes $\mathcal{A}$. We obtain that

$$0 \in \mathcal{S}_2 \subset \mathcal{A}.$$

Finally, due to the symmetric position of $\mathcal{A}$ relative to the parallel pairs of faces, it is easy to see that every point in $\mathcal{A}$ is singular of degree 2. (In fact, the points of any configuration can be continuously moved to antipodal position with increasing distortion.) Thus $\mathcal{S}_2 = \mathcal{A}$.

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