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ASYMMETRY OF CONVEX SETS WITH ISOLATED EXTREME POINTS

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ABSTRACT. When measuring asymmetry of convex sets $\mathcal{L} \subset \mathbf{R}^n$ in terms of inscribed simplices, the interior of $\mathcal{L}$ naturally splits into regular and singular sets. Based on examples, it may be conjectured that the singular set is empty iff $\mathcal{L}$ is a simplex. In this paper we prove this conjecture with the additional assumption that $\mathcal{L}$ has at least n isolated extreme points on its boundary.

1. Introduction and statement of results

Throughout, we use standard notation and basic concepts in the theory of convex sets and functions [1, 5]. Let $\mathcal{E}$ be a Euclidean vector space of dimension n. (We usually take $\mathcal{E} = \mathbf{R}^n$.) If $\mathcal{K} \subset \mathcal{E}$, then $\langle \mathcal{K} \rangle$ and $[\mathcal{K}]$ denote the *affine span* and *convex hull* of $\mathcal{K}$, respectively. For $\mathcal{K} = \{B_0, \ldots, B_m\}$ finite, $[\mathcal{K}]$ is a convex *polytope*. This polytope is an m-*simplex* if $B_0, \ldots, B_m$ are *affinely independent*, or equivalently, if $\dim[\mathcal{K}] = \dim\langle \mathcal{K} \rangle = m$. A convex set $\mathcal{L} \subset \mathcal{E}$ is a *convex body* if it has nonempty interior. Every convex set is a convex body in its affine span.

Let $\mathcal{L}$ be a compact convex body in $\mathbf{R}^n$ and O a point in the interior of $\mathcal{L}$. As in [6, 7], we define a sequence of (affine) invariants $\{\sigma_m(\mathcal{L}, O)\}_{m \geq 1}$. Intuitively, $\sigma_m(\mathcal{L}, O)$ measures how lopsided $\mathcal{L}$ is in dimension m viewed from O. Since $\mathcal{L}$ is compact and convex, given $C \in \partial\mathcal{L}$, we have $\langle O, C \rangle \cap \partial\mathcal{L} = \{C, C^o\}$, where C^o is called the *opposite* of C (with respect to O). We define the *distortion function* $\Lambda_{\mathcal{L}} = \Lambda : \partial\mathcal{L} \times \operatorname{int}\mathcal{L} \to \mathbf{R}$ by

$$\Lambda(C, O) = \frac{d(C, O)}{d(C^o, O)}, \quad C \in \partial\mathcal{L},\ O \in \operatorname{int}\mathcal{L},$$

where d is the Euclidean distance in $\mathbf{R}^n$. The distortion Λ is a continuous function [6, 7]. By definition, $\Lambda(C^o, O) = 1/\Lambda(C, O)$.

The minimum distortion $\lambda(O) = \inf_{C \in \partial\mathcal{L}} \Lambda(C, O)$, as a function on the interior of $\mathcal{L}$, has been studied by many authors. (See Grünbaum [2] and the extensive references therein.) In particular, there are many lower estimates on *Minkowski's measure of symmetry* $\sup_{O \in \operatorname{int}\mathcal{L}} \lambda(O)$ and the *derived measures* $\lambda(O_0)$, where O_0 is the centroid, the centers of circumscribed and inscribed ellipsoids, the centroid of the surface area of $\partial\mathcal{L}$, and the curvature centroid.

Let $m \geq 1$. A finite multi-set $\{C_0, \ldots, C_m\}$ is called an m-*configuration* with respect to O if $\{C_0, \ldots, C_m\} \subset \partial\mathcal{L}$ and $O \in [C_0, \ldots, C_m]$. The set of m-configurations

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is denoted by $\mathcal{C}_m(\mathcal{L}, O)$. We define

$$\sigma_m(\mathcal{L}, O) = \inf_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L},O)} \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i, O)}.$$

Since a 1-configuration is an opposite pair of points, we have $\sigma_1(\mathcal{L}, O) = 1$.

An m-configuration $\{C_0, \dots, C_m\}$ for which the infimum is attained is called *minimal*. Compactness implies that minimal configurations exist. $\sigma_m(\mathcal{L}, .) : \operatorname{int}\mathcal{L} \to \mathbf{R}$ is a continuous function ([7], Theorem D). In what follows, we suppress O when no confusion arises. In addition, we also suppress the dimension n; in particular, we write $\sigma(\mathcal{L})$ for $\sigma_n(\mathcal{L})$, etc.

In general, we obviously have

$$\sigma_{m+k}(\mathcal{L}) \leq \sigma_m(\mathcal{L}) + \frac{k}{1+\max_{\partial\mathcal{L}} \Lambda}, \quad m, k \geq 1. \tag{1.1}$$

For $m = n$, a configuration in $\mathcal{C}_{n+k}(\mathcal{L}, O)$, $k \geq 1$, always contains a subconfiguration in $\mathcal{C}(\mathcal{L}, O)$ so that equality holds in (1.1). (See [8] for details.) In other words, the sequence $\{\sigma_m(\mathcal{L})\}_{m\geq 1}$ is *arithmetic* (with difference $1/(1+\max_{\partial\mathcal{L}} \Lambda)$) from the n-th term onwards.

By [7] (Theorem B), for $m \geq 1$, we have

$$1 \leq \sigma_m(\mathcal{L}) \leq \frac{m+1}{2}.$$

The lower bound $\sigma_m(\mathcal{L}) = 1$ is realized iff there exists an affine subspace $\mathcal{F} \subset \mathbf{R}^n$, $O \in \mathcal{F}$, of dimension m such that $\mathcal{L} \cap \mathcal{F}$ is an m-simplex. For $m \geq 2$, the upper bound $\sigma_m(\mathcal{L}) = (m+1)/2$ is realized iff $\mathcal{L}$ is symmetric (with respect to O).

Thus, up to scaling, $\sigma_m(\mathcal{L}, O)$, $m \geq 1$, are *measures of symmetry* in the sense of Grünbaum [2] since σ_m is clearly continuous on the space of compact convex bodies with specified interior points and is also invariant under similarity transformations.

For estimates on the related symmetries of measure

$$\inf_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L},O)} \sum_{i=0}^{m} \Lambda(C_i, O) \text{ and } \inf_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L},O)} \prod_{i=0}^{m} \Lambda(C_i, O)$$

(at least for $m = n$) see also Grünbaum [2].

We define the *regular set* $\mathcal{R} \subset \operatorname{int}\mathcal{L}$ as

$$\mathcal{R} = \left\{ O \in \operatorname{int}\mathcal{L} \,|\, \sigma(\mathcal{L}, O) < \sigma_{n-1}(\mathcal{L}, O) + \frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(., O)} \right\}.$$

An element of $\mathcal{R}$ is called a *regular point*. An interior point is called *singular* if it is not regular. By continuity of the functions in the defining inequality of $\mathcal{R}$, the set $\mathcal{R}$ is open in $\operatorname{int}\mathcal{L}$ (and hence in $\mathbf{R}^n$). The structure of a compact convex body viewed from a regular point is technically much easier to deal with than when viewed from a singular point. For example, as shown below (Lemma 2.1), if O is regular, then there exists a minimal n-configuration consisting of *extreme points* only. (Recall that a point on the boundary of $\mathcal{L}$ is called an extreme point if it is not contained in the interior of a boundary line segment.) This, for $\mathcal{L}$ a convex polytope, reduces the determination of $\sigma(\mathcal{L}, O)$ to a *finite* enumeration on the vertices of $\mathcal{L}$. Moreover, according to a result in [7], the distortion function $\Lambda(\mathcal{L}, .)$ is *concave on* $\mathcal{R}$. Irrespective of regularity, concavity of the distortion function holds in 2-dimensions [7]. By contrast, there exists a 4-dimensional cone in which, due to the existence of singular points near the base, the distortion function is not concave [6]. It is therefore important to analyze when and where singularity does occur.

It is easy to show that $O \in \mathcal{R}$ iff the convex hull of every minimal configuration is an n-simplex with O in its interior. (See [8] for details.) In addition, $\Lambda(.,O)$ attains its local maximum at each configuration point.

Because of this, we will need the behavior of the boundary of $\mathcal{L}$ at a local maximum C of $\Lambda(.,O)$ as described in [7] (Section 7). For the moment we only use the fact that if C is a smooth point of $\partial\mathcal{L}$ at which $\Lambda(.,O)$ assumes a local maximum, then C^o is also smooth and the tangent spaces at C and C^o to $\partial\mathcal{L}$ are parallel. If C is not a smooth point, it is still true that there exist parallel supporting hyperplanes at C and C^o. In particular, $[C, C^o]$ is an *affine diameter* in the sense of Grünbaum [2]. Thus, if O is a regular point O belongs to at least $n+1$ (affinely independent) affine diameters. To determine points with this property is an unsolved problem; in particular, it is not known whether or not the centroid has this property. For further results, see Grünbaum [2] and Kosiński [3, 4].

We denote by $\mathcal{L}_0$ the set of extreme points of $\mathcal{L}$. By a theorem of Minkowski, we have $\mathcal{L} = [\mathcal{L}_0]$. (See Theorem D in [5], p. 84.) We call an extreme point $C \in \mathcal{L}_0$ *isolated* if C is not a limit point of $\mathcal{L}_0$. The following simple example is the motivation for our study:

Example. Let $\mathcal{L}$ have an isolated extreme point C, and assume that, away from C, $\partial\mathcal{L}$ is smooth. We claim that there are singular points in the interior of $\mathcal{L}$.

First, since C is an isolated extreme point, the set of supporting hyperplanes $\mathcal{H}$ at C such that $\mathcal{L} \cap \mathcal{H} = \{C\}$ is a nonempty open set (in the respective Grassmann manifold). This follows from the conical structure of $\mathcal{L}$ near C. (In fact, $\mathcal{L}$ is the convex hull of $[\mathcal{L}_0 \setminus \{C\}]$ and the single point C; see Lemma 2.2 below.) For each $\mathcal{H}$ in this set, we consider the set of points $B \in \partial\mathcal{L}$, $B \neq C$, such that the tangent space of $\partial\mathcal{L}$ at B is parallel to $\mathcal{H}$. Since $\mathcal{L}$ is convex and, away from C, its boundary is smooth, the union $\mathcal{B}$ of these sets is open in $\partial\mathcal{L}$. (In fact, $\partial\mathcal{L} \setminus \mathcal{B}$ is closed, as follows again from the conical structure of $\mathcal{L}$ near C.) The convex hull $[\mathcal{B}]$ intersects the interior of $\mathcal{L}$. Any point O in this interior must be singular. Indeed, if O were regular, then at least one point in a minimal configuration $\{C_0, \dots, C_n\} \in \mathcal{C}(\mathcal{L}, O)$ would be contained in $\mathcal{B}$. This is a contradiction, since the tangent space at that point has no parallel translate tangent to $\partial\mathcal{L}$ at another smooth point.

The situation in the nonsmooth case is much more complicated. Our first result asserts that regularity of the interior of $\mathcal{L}$ along with the existence of isolated extreme points impose severe restrictions to the structure of $\mathcal{L}$.

Theorem 1.1. *Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body with all interior points regular. Assume that $\mathcal{L}$ has (at least) two isolated extreme points C_0 and C_1. Then, for any plane τ that contains C_0 and C_1, the intersection $\mathcal{L} \cap \tau$ is either $[C_0, C_1]$ or a triangle with $[C_0, C_1]$ as a side.*

An illustrative example to Theorem 1.1 to be discussed below is the following:

Example. Let $S \subset \mathbf{R}^3$ be the unit circle of the coordinate plane spanned by the first and second coordinate axes, and let $C_{\pm} = (1, 0, \pm 1)$. Let $\mathcal{L}$ be the convex hull $[S, C_+, C_-]$. Then $\mathcal{L}_0 = (S \setminus \{(1, 0, 0)\}) \cup \{C_{\pm}\}$. Clearly, $\mathcal{L}_0$ is not closed. Due to triangular intersections, we have $\sigma_2(\mathcal{L}, .) = 1$. Hence [7], $\sigma(\mathcal{L}, .)$ is concave on the whole interior of $\mathcal{L}$.

Theorem 1.2. *Let $\mathcal{L} \subset \mathbf{R}^n$ be as in Theorem 1.1. Assume that $\mathcal{L}$ has at least n isolated extreme points. Then $\mathcal{L}$ is a simplex.*

For $\mathcal{L}$ a convex polytope, the extreme points are the vertices and they are all isolated. Theorem 1.2 gives the following:

Theorem 1.3. *Let $\mathcal{L} \subset \mathbf{R}^n$ be a convex polytope which is not a simplex. Then there are singular points in the interior of $\mathcal{L}$.*

The proof will actually show that, if nonempty, the set of singular points has nonempty interior and its closure contains part of the boundary of $\mathcal{L}$.

2. Proofs

Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body. We first recall the notion of *k-flat points* on $\partial\mathcal{L}$ [7]. Let $C \in \partial\mathcal{L}$. We call an affine subspace $\mathcal{A} \subset \mathbf{R}^n$ a *supporting flat at* C if $C \in \mathcal{A}$ and $\mathcal{A}$ is contained in a supporting hyperplane of $\mathcal{L}$ at C. Consider the set of supporting flats $\mathcal{A}$ at C such that $\partial\mathcal{L} \cap \mathcal{A}$ is a compact convex body in $\mathcal{A}$ and C is contained in its relative interior. Since $\mathcal{L}$ is convex, this set has a unique maximal element denoted by $\mathcal{A}_C$. We call C a k-(dimensional) flat point if $\dim \mathcal{A}_C = k$. Clearly, C is an extreme point iff $k = 0$.

Lemma 2.1. *Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body. If O is a regular point of $\mathcal{L}$, then there exists a minimal configuration $\{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, O)$ consisting of extreme points.*

Proof. Let $\{C_0, \ldots, C_n\} \in \mathcal{C}_n(\mathcal{L}, O)$ be minimal. Since O is a regular point, $[C_0, \ldots, C_n]$ is an n-simplex containing O in its interior and $\Lambda(., O)$ attains a relative maximum at each C_i, $i = 0, \ldots, n$. Suppressing the index for simplicity, assume that a configuration point C is not extremal. Then C is a k-flat point for some $k > 0$, $k = \dim \mathcal{A}_C$. Since $\Lambda(., O)$ attains a relative maximum at C, according to a result of [7] (the proposition in Section 7), the antipodal point C^o is l-flat, $l \geq k$, and $\mathcal{A}_C$ is parallel to $\mathcal{A}_{C^o}$ in the sense that a translate of $\mathcal{A}_C$ is contained in $\mathcal{A}_{C^o}$.

Choose a point C' on the boundary of the compact convex body $\partial\mathcal{L} \cap \mathcal{A}_C$ in $\mathcal{A}$. Clearly, C' is a lower dimensional flat point than C. Since $\mathcal{A}_C$ is parallel to $\mathcal{A}_{C^o}$, $\Lambda(., O)$ is constant on $[C, C']$. Moving C toward C' and replacing C with the moved point, the configuration condition $O \in [C_0, \ldots, C_n]$ stays intact since O is a regular point. Thus, replacing C by C' in the configuration, we arrive at a minimal configuration with C' being a lower dimensional flat point than C. Proceeding inductively, we can replace each nonextremal point of the configuration with and extremal point without altering minimality. Lemma 2.1 follows. $\square$

Corollary. *Let $\mathcal{L} \subset \mathbf{R}^n$ be a convex polytope and denote by $\mathcal{V}$ the set of vertices. Assume that O is a regular point of $\mathcal{L}$. Then, we have*

$$\sigma(\mathcal{L}, O) = \min_{\{V_0, \ldots, V_n\} \in \mathcal{V}} \sum_{i=0}^{n} \frac{1}{1 + \Lambda(V_i, O)}.$$

Returning to the general setting, as in Section 1, we let $\mathcal{L}_0 \subset \partial\mathcal{L}$ denote the set of extreme points. We have $\mathcal{L} = [\mathcal{L}_0]$. Recall that an extreme point C is isolated if C has an open neighborhood disjoint from $\mathcal{L}_0 \setminus \{C\}$. Our first task is to describe $\mathcal{L}$ near an isolated extreme point.

Lemma 2.2. *Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body and $\mathcal{L}_0$ the set of extreme points. Let $C \in \mathcal{L}_0$ be an isolated extremal point. Then*

$$U_C = \mathcal{L} \setminus \overline{[\mathcal{L}_0 \setminus \{C\}]} \tag{2.1}$$

is a relatively open set in $\mathcal{L}$ that contains C. For any $C' \in U_C \cap \partial\mathcal{L}$, $C' \neq C$, the line segment $[C, C']$ is on the boundary of $\mathcal{L}$, and it extends to a boundary line segment $[C, C'']$ with $C'' \in [\mathcal{L}_0 \setminus \{C\}]$.

Proof. Let C be an isolated extreme point. For the first statement we need to show that

$$C \notin \overline{[\mathcal{L}_0 \setminus \{C\}]}. \tag{2.2}$$

Assuming the contrary, we can select a sequence $\{C_k\}_{k\geq 1} \subset [\mathcal{L}_0 \setminus \{C\}]$ converging to C. For each $k \geq 1$, we can write C_k as a convex linear combination $\sum_{i=0}^n \lambda_{ik} C_{ik}$, where $C_{ik} \in \mathcal{L}_0$, $C_{ik} \neq C$. By compactness, we may assume that, for each $0 \leq i \leq n$, $C_{ik} \to C_i$ and $\lambda_{ik} \to \lambda_i$ as $k \to \infty$. Taking the limit, we obtain $C = \sum_{i=0}^n \lambda_i C_i$. Since C is an extreme point, the only way this is possible is that this sum reduces to a single term. We obtain that $C_i = C$ for a specific $0 \leq i \leq n$, and so $C_{ik} \to C_i = C$ as $k \to \infty$. Hence C is not isolated. (2.2) follows.

For the second statement, let $C' \notin \overline{[\mathcal{L}_0 \setminus \{C\}]}$ be a boundary point of $\mathcal{L}$. Since $[\mathcal{L}_0] = \mathcal{L}$, we can certainly write C' as a convex linear combination of C and (finitely many) points in $\mathcal{L}_0 \setminus \{C\}$. The point C must participate in this linear combination with positive coefficient. Hence C' is in the interior of a segment $[C, C'']$, where $C'' \in [\mathcal{L}_0 \setminus \{C\}]$. Finally, since C and C' are both boundary points of $\mathcal{L}$, the entire line segment $[C, C'']$ is on the boundary of $\mathcal{L}$. Lemma 2.2 follows. □

Remark. Consider the second example above. Removing C_- from $\mathcal{L}_0$, we see that $[\mathcal{L}_0 \setminus \{C_-\}]$ is the positive cone $[S, C_+]$ with the half-open segment $[(1,0,0), C_+)$ deleted. Its closure is $\overline{[\mathcal{L}_0 \setminus \{C_-\}]} = [S, C_+]$, and hence $U_{C_-} = [S, C_-] \setminus [S]$. Notice that, for any C' in the interior of $[C_-, (1,0,0)]$, the line segment $[C_-, C']$ extends beyond $\overline{U_{C_-}}$ to $[C_-, C_+]$.

Lemma 2.3. *Let C be an isolated extreme point of $\mathcal{L}$ with associated open set U_C. Then, for every $O \in U_C$, there is a minimal configuration which contains C. In particular, if O is regular, then $\Lambda(., O)$ takes a local maximum at C.*

Proof. Let $O \in U_C$ and, as in Lemma 2.1, choose a minimal configuration consisting of extreme points. If C does not participate in the configuration, then O must be contained in $[\mathcal{L}_0 \setminus \{C\}]$. This contradicts the assumption. Thus, C is a point in the configuration. The last statement is clear. □

For the next step we introduce some notation and recall some results in [7] (Section 7). Let C be an isolated extreme point of $\mathcal{L}$. Let $\tau \subset \mathbf{R}^n$ be a plane passing through C and an interior point $O \in U_C$ of $\mathcal{L}$. We consider the planar convex body $\mathcal{L} \cap \tau$ with isolated extreme point C. As Lemma 2.3 asserts, $\mathcal{L} \cap \tau$ contains an angular domain with vertex at C. We let $[C, P], [C, Q] \subset \partial\mathcal{L} \cap \tau$ denote the maximal side segments of this domain. We orient τ from O such that the positive orientation corresponds to the sequence P, C, Q. As in Section 7 of [7], $\alpha = \alpha_\tau(C)$ is the angle $\angle O\,C\,Q$. In a similar vein, we let $\alpha^o = \alpha_\tau(C^o)$, where α^o is the angle with vertex at C^o between the line segment $[C^o, O]$ and the right tangent at C^o to the boundary of $\mathcal{L} \cap \tau$.

From now on we assume that U_C consists of regular points only. By Lemma 2.3, $\Lambda(., O)$ attains a local maximum at C, and so, by Corollary 1 of Section 7 in [7],

$$\alpha \leq \alpha^o. \tag{2.3}$$

For a boundary point B of $\mathcal{L} \cap \tau$, let $0 \leq \phi(C) \leq \pi$ denote the angle between the left and right tangent lines at B to $\partial\mathcal{L} \cap \tau$. Then we have $\phi(C) = \angle PCQ$. For O close to a fixed interior point of $[C, P]$, the right tangent to $\mathcal{L} \cap \tau$ at C^o intersects the extension of the line segment $[C, P]$ beyond P. We let R denote this intersection point. From the triangle $\triangle CC^oR$, we obtain

$$\phi(C) - \alpha + \alpha^o + \beta = \pi,$$

where $\beta = \angle C^oRC$. Combining this with (2.3), we get

$$\phi(C) + \beta \leq \pi.$$

We now let O approach a fixed interior point of $[C, P]$. We claim that β approaches $\phi(P)$. In fact, as O approaches a fixed interior point of $[C, P]$, the antipodal C^o approaches P along the boundary of $\mathcal{L} \cap \tau$, and the *right* tangent line at C^o approaches the *left* tangent line at P. (See formula (6) in [5], p. 7.) We obtain the following:

Lemma 2.4. *Let $\mathcal{L} \subset \mathbf{R}^n$ be a compact convex body, C an isolated extreme point, and assume that U_C consists of regular points. Then, for any plane passing through C and an interior point of $\mathcal{L}$, we have*

$$\phi(C) + \phi(P) \leq \pi, \tag{2.4}$$

where $\phi(C)$ and $\phi(P)$ are the tangential angles of $\mathcal{L} \cap \tau$ at C and P, and $[C, P]$ is a maximal line segment on the boundary of $\mathcal{L} \cap \tau$.

In the lemma above, we call P an *adjacent point* to the isolated extreme point C. P is adjacent to C if $[C, P]$ is a maximal line segment on the boundary of $\mathcal{L}$.

Proof of Theorem 1.1. We may assume that $\mathcal{L} \cap \tau$ is more than $[C_0, C_1]$, in which case $\mathcal{L} \cap \tau$ is a compact convex body with isolated extreme points C_0 and C_1. Let $P_0, Q_0 \in \partial\mathcal{L} \cap \tau$ and $P_1, Q_1 \in \partial\mathcal{L} \cap \tau$ be adjacent to C_0 and C_1, respectively. Orient τ and choose the labels such that (with respect to an(y) interior point of $\mathcal{L} \cap \tau$) P_0, C_0, Q_0 and P_1, C_1, Q_1 are positively oriented. Assume first that the adjacent points are all distinct, the right tangent at Q_0 and the left tangent at P_1 intersect at a point X, and the left tangent at P_0 and the right tangent at Q_1 intersect at a point Y. For the angle sum of the (convex) octagon $[P_0, C_0, Q_0, X, P_1, C_1, Q_1, Y]$ we have

$$\phi(P_0) + \phi(C_0) + \phi(Q_0) + \beta + \phi(P_1) + \phi(C_1) + \phi(Q_1) + \gamma = 6\pi, \tag{2.5}$$

where β and γ are the angles at X and Y, respectively. On the other hand, by (2.4), we have

$$\phi(C_0) + \phi(P_0),\ \phi(C_0) + \phi(Q_0),\ \phi(C_1) + \phi(P_1),\ \phi(C_1) + \phi(Q_1) \leq \pi.$$

Adding these, we obtain

$$2\phi(C_0) + 2\phi(C_1) + \phi(P_0) + \phi(P_1) + \phi(Q_0) + \phi(Q_1) \leq 4\pi.$$

Comparing this with (2.5), we get

$$\phi(C_0) + \phi(C_1) + 2\pi - \beta - \gamma \leq 0.$$

This is a contradiction. Notice that we get a contradiction even when $\beta = \pi$ or $\gamma = \pi$ (the cases when the corresponding tangents coincide), and even when $P_0 = Q_1$ but $P_1 \neq Q_0$, or when $P_1 = Q_0$ but $P_0 \neq Q_1$.

If X or Y do not exist, we can add additional supporting lines to boundary points of $\mathcal{L} \cap \tau$ and get a contradiction again.

Summarizing, we obtain $P_0 = Q_1$ and $P_1 = Q_0$. As a byproduct, we also obtain that $\mathcal{L} \cap \tau = [P_0, C_0, P_1, C_1]$. If P_0, C_0, P_1, C_1 are all distinct, then, by (2.4), $[P_0, C_0, P_1, C_1]$ is a parallelogram with $[C_0, C_1]$ as a diagonal. Finally, if these points are not distinct, then $\mathcal{L} \cap \tau$ is a triangle with $[C_0, C_1]$ as a side (and P_0 or P_1 is the other vertex).

It remains to show that the parallelogram intersection is impossible. As in Lemmas 2.1-2.3, we let $O \in U_{C_0}$ and consider a minimal configuration $\{C_0, C_1, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, O)$ consisting of extreme points only. By the last statement of Lemma 2.2, O is contained in the interior of the triangle $[P_0, C_0, P_1]$. Thus, the opposite P_0^o is contained in $[C_0, P_1]$. Any point in the segment $[C_0, P_0^o]$ has the same distortion as C_0 since $\mathcal{L} \cap \tau$ is a parallelogram. Since O and $\overline{[\mathcal{L}_0 \setminus C_0]}$ are disjoint, there must be a point $C_0' \in [C_0, P_1]$ for which O is on the boundary of $\overline{[(\mathcal{L}_0 \setminus \{C_0\}) \cup \{C_0'\}]}$. Thus, O is on the boundary of $[C_0', C_1, \ldots, C_n]$. Hence $\{C_0', \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, O)$. It must be minimal with $C_0' \in [C_0, P_0^o]$ since the distortion along $[P_0^o, P_1]$ increases. This, however, contradicts the regularity of O. Theorem 1.1 follows. □

Proof of Theorem 1.2. Let $\mathcal{L} \subset \mathbf{R}^n$ be as in Theorem 1.1. For $1 \leq m < n$, let $\mathcal{P}_m$ be the following statement: If $C_0, \ldots, C_m \in \mathcal{L}$ are (distinct) isolated extreme points, then they are affinely independent, and, for any $(m+1)$-dimensional affine subspace $\tau \subset \mathbf{R}^n$ that contains $C_0, \ldots, C_m$, the intersection $\mathcal{L} \cap \tau$ is either $[C_0, \ldots, C_m]$ or an $(m+1)$-simplex with $[C_0, \ldots, C_m]$ as a side.

Notice that $\mathcal{P}_1$ is Theorem 1.1, and the second statement of $\mathcal{P}_{n-1}$ is Theorem 1.2. Therefore, Theorem 1.2 will follow by proving $\mathcal{P}_m$ by induction with respect to $m = 1, \ldots, n-1$. Before the general induction step, it is convenient to have an intermediate step:

Lemma 2.5. *Let $\mathcal{L}$ be as in Theorem 1.1. Assume that, for a fixed $2 \leq m < n$, $\mathcal{P}_i$, $1 \leq i < m$, hold. Let $C_0, \ldots, C_m$ be isolated extreme points of $\mathcal{L}$. Then, $C_0, \ldots, C_m$ are affinely independent and*

$$\mathcal{L} \cap \langle C_0, \ldots, C_m \rangle = [C_0, \ldots, C_m]. \tag{2.6}$$

Proof. We first show affine independence. Since $\mathcal{P}_{m-1}$ holds, $C_0, \ldots, C_{m-1}$ are certainly affinely independent. Thus, the affine span $\tau = \langle C_0, \ldots, C_{m-1} \rangle \subset \mathbf{R}^n$ is $(m-1)$ dimensional. Applying $\mathcal{P}_{m-2}$ to τ, we obtain that $\mathcal{L} \cap \tau$ is an $(m-1)$-simplex with $[C_0, \ldots, C_{m-2}]$ as a side. Since $C_{m-1} \notin \langle C_0, \ldots, C_{m-2} \rangle$ is an extreme point of $\mathcal{L}$, it is also an extreme point of $\mathcal{L} \cap \tau$. Thus, we have

$$\mathcal{L} \cap \tau = [C_0, \ldots, C_{m-1}].$$

Now, C_m cannot be in this set since it is an extreme point of $\mathcal{L}$ and thereby also an extreme point of $\mathcal{L} \cap \tau$. Thus, $C_0, \ldots, C_m$ are affinely independent. We now add C_m to τ and set $\tau = \langle C_0, \ldots, C_m \rangle \subset \mathbf{R}^n$, an m dimensional affine subspace. Applying $\mathcal{P}_{m-1}$, once again, $\mathcal{L} \cap \tau$ must be an m-simplex with $[C_0, \ldots, C_{m-1}]$ as a side. C_m is an extreme point in $\mathcal{L}$ and also in $\mathcal{L} \cap \tau$. Equation (2.6) follows. □

We now return to the proof of the general induction step. Assume that, for a fixed $2 \le m < n$, $\mathcal{P}_i$, $1 \le i < m$, hold. The first statement in $\mathcal{P}_m$ is contained in Lemma 2.5. To prove the second statement, let $\tau \subset \mathbf{R}^n$ be an $(m+1)$ dimensional affine subspace that contains $C_0, \ldots, C_m$. We may assume that $\mathcal{L} \cap \tau \neq [C_0, \ldots, C_m]$, since otherwise we are done. Equation (2.6) and $[\mathcal{L}_0] = \mathcal{L}$ show that $\mathcal{L}$ contains an extreme point C away from $\langle C_0, \ldots, C_m \rangle$. In other words, $C_0, \ldots, C_m, C$ are affinely independent, and the $(m+1)$-simplex $[C_0, \ldots, C_m, C]$ is contained in $\mathcal{L} \cap \tau$. It remains to show that

$$\mathcal{L} \cap \tau = [C_0, \ldots, C_m, C]. \tag{2.7}$$

To do this, we will show that

$$[C_0, \ldots, \widehat{C_i}, \ldots, C_m, C] \subset \partial \mathcal{L}, \quad 0 \le i \le m. \tag{2.8}$$

First note that (2.8) implies (2.7). Indeed, (2.8) says that all the faces of $[C_0, \ldots, C_m, C]$ opposite to $C_0, \ldots, C_m$ are on the boundary of $\mathcal{L}$. If the face $[C_0, \ldots, C_m]$ were not on the boundary of $\mathcal{L}$, then there would be another extreme point of $\mathcal{L}$, say $C' \in \partial \mathcal{L} \cap \tau$, on the side of $\langle C_0, \ldots, C_m \rangle \subset \tau$ opposite to C. By (2.8) with C replaced by C', we would obtain that $\mathcal{L} \cap \tau = [C_0, \ldots, C_m, C, C']$ is a double simplex with common base $[C_0, \ldots, C_m]$. This clearly contradicts $\mathcal{P}_1$.

It remains to show (2.8). To do this, for $0 \le i \le m$, we let $\tau_i = \langle C_0, \ldots, \widehat{C_i}, \ldots, C_m, C \rangle \subset \tau$ and apply $\mathcal{P}_{m-1}$. Then (2.6) in Lemma 2.5 gives

$$\mathcal{L} \cap \tau_i = [C_0, \ldots, \widehat{C_i}, \ldots, C_m, C], \quad 0 \le i \le m.$$

In particular, for $0 \le i < j \le m$, the $(m-1)$-simplex

$$[C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$$

is on the boundary of $\mathcal{L}$. Let $C' \in \partial \mathcal{L}$ be a point in the interior of this $(m-1)$-simplex and consider the plane $\sigma = \langle C_i, C_j, C' \rangle$. By $\mathcal{P}_1$, $\mathcal{L} \cap \sigma = [C_i, C_j, C'']$ for some $C'' \in \partial \mathcal{L} \cap \sigma$ with $C' \in [C_i, C_j, C'']$. We claim that $C'' = C'$. This will clearly imply (2.8).

First, C' cannot be in the interior of $[C_i, C_j, C'']$ since otherwise C', C'' and the unique intersection point $C''' = \langle C', C'' \rangle \cap [C_i, C_j]$ would be three collinear points on $\partial \mathcal{L}$, so that, by convexity, $[C'', C''']$ would be on the boundary of $\mathcal{L}$.

Thus, C' is on the boundary of $[C_i, C_j, C'']$, say $C' \in [C_i, C'']$. On the other hand, $C' \in [C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$ and $C'' \in [C_0, \ldots, \widehat{C_j}, \ldots, C_m, C]$ since C_i, C', C'' are collinear. As $[C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$ is the side of the m-simplex $[C_0, \ldots, \widehat{C_j}, \ldots, C_m, C]$ opposite to C_i, $C' = C''$ follows. The second statement of $\mathcal{P}_m$ and hence Theorem 1.2 follow. □

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