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On the Structure of Convex Sets with Applications to the Moduli of Spherical Minimal Immersions

Gabor Toth

Department of Mathematics, Rutgers University
Camden, New Jersey 08102
e-mail: gtoth@crab.rutgers.edu

Abstract. We study the properties of certain affine invariant measures of symmetry associated to a compact convex body $\mathcal{L}$ in a Euclidean vector space. As functions of the interior of $\mathcal{L}$, these measures of symmetry are proved or disproved to be concave in specific situations, notably for the reduced moduli of spherical minimal immersions.

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1. Introduction and statement of results

Let $\mathcal{E}$ be a Euclidean vector space of dimension n. If $\mathcal{K} \subset \mathcal{E}$ then $[\mathcal{K}]$ and $\langle \mathcal{K} \rangle$ denote the convex hull and the affine span of $\mathcal{K}$, respectively. Since $[\mathcal{K}]$ has a nonempty relative interior in $\langle \mathcal{K} \rangle$, it is a convex body in $\langle \mathcal{K} \rangle$.

Let $\mathcal{L}$ be a compact convex body in $\mathcal{E}$ and $\mathcal{O}$ a point in the interior of $\mathcal{L}$. As in [9, 10], we define an affine invariant $\sigma(\mathcal{L}, \mathcal{O})$ as follows. Given $C \in \partial\mathcal{L}$, the line passing through $\mathcal{O}$ and C intersects $\partial\mathcal{L}$ in another point that we call the *opposite of* C with respect to $\mathcal{O}$ and denote it by C^o. Clearly, $(C^o)^o = C$.

We define the *distortion function* $\Lambda : \partial\mathcal{L} \times \operatorname{int}\mathcal{L} \to \mathbf{R}$ as

$$\Lambda_{\mathcal{L}}(C, \mathcal{O}) = \Lambda(C, \mathcal{O}) = \frac{d(C, \mathcal{O})}{d(C^o, \mathcal{O})}, \quad C \in \partial\mathcal{L}, \ \mathcal{O} \in \operatorname{int}\mathcal{L},$$

where $d(X, X') = |X - X'|$ is the Euclidean distance between the points X and X' in $\mathcal{E}$, and C^o is the opposite of C with respect to $\mathcal{O}$. The distortion Λ is a continuous function [3]. By definition, $\Lambda(C^o, \mathcal{O}) = 1/\Lambda(C, \mathcal{O})$.

The minimum distortion $\lambda(\mathcal{O}) = \inf_{C\in\partial\mathcal{L}} \Lambda(C, \mathcal{O})$, as a function of the interior of $\mathcal{L}$ has an extensive literature. (We refer to the survey article of Grünbaum [5] and the references therein.) In particular, the structure of the level sets and the critical set of Minkowski's measure of symmetry $\sup_{\mathcal{O}\in \operatorname{int}\mathcal{L}} \lambda(\mathcal{O}) \geq 1/n$ have been studied by many authors. The derived measures $\lambda(\mathcal{O}_0)$, where $\mathcal{O}_0$ is a specific center of $\mathcal{L}$ also have an extensive literature. Choices of $\mathcal{O}_0$ include the centroid, the centers of circumscribed and inscribed ellipsoids, the centroid of the surface area of $\partial\mathcal{L}$, and the curvature centroid.

A multi-set $\{C_0, \dots, C_n\}$ is called a *configuration* with respect to $\mathcal{O}$ if $\{C_0, \dots, C_n\} \subset \partial\mathcal{L}$ and $\mathcal{O} \in [C_0, \dots, C_n]$. Let $\mathcal{C}(\mathcal{L}, \mathcal{O})$ denote the set of all configurations of $\mathcal{L}$. We define

$$\sigma(\mathcal{L}, \mathcal{O}) = \inf_{\{C_0,\dots,C_n\}\in\mathcal{C}(\mathcal{L},\mathcal{O})} \sum_{i=0}^{n} \frac{1}{1 + \Lambda(C_i, \mathcal{O})}. \tag{1}$$

A configuration $\{C_0, \dots, C_n\}$ for which the infimum is attained is called *minimal*. Minimal configuration exists since $\mathcal{L}$ is compact.

$\sigma(\mathcal{L}, \mathcal{O})$ is a continuous function on the space of compact convex bodies with specified interior point, and it is also invariant under affine transformations. In addition, by [10], $1 \leq \sigma(\mathcal{L}, \mathcal{O}) \leq (n + 1)/2$. The lower bound is attained by simplices, and the upper bound is realized by symmetric $\mathcal{L}$ (with respect to $\mathcal{O}$). Because of these properties, $\sigma(\mathcal{L}, \mathcal{O})$ is a measure of symmetry in the sense of Grünbaum [5].

$\sigma(\mathcal{L}, \mathcal{O})$ is related to the measures of symmetry

$$\inf_{\{C_0,\dots,C_n\}\in\mathcal{C}(\mathcal{L},\mathcal{O})} \sum_{i=0}^{n} \Lambda(C_i, \mathcal{O}) \quad \text{and} \quad \inf_{\{C_0,\dots,C_n\}\in\mathcal{C}(\mathcal{L},\mathcal{O})} \prod_{i=0}^{n} \Lambda(C_i, \mathcal{O})$$

for which upper and lower bounds have been derived. (See again Grünbaum [5].)

According to [9] (Theorem D), $\sigma(\mathcal{L}, .) : \operatorname{int}\mathcal{L} \to \mathbf{R}$ is continuous and has the property

$$\lim_{d(\mathcal{O},\partial\mathcal{L})\to 0} \sigma(\mathcal{L}, \mathcal{O}) = 1.$$

In particular, it extends continuously to the boundary of $\mathcal{L}$ with value 1.

A point $\mathcal{O} \in \operatorname{int}\mathcal{L}$ is said to be *regular* if, for any minimizing configuration $\{C_0, \dots, C_n\} \in \mathcal{C}(\mathcal{L}, \mathcal{O})$, the convex hull $[C_0, \dots, C_n]$ is a simplex which contains $\mathcal{O}$ in its interior. The set $\mathcal{R}$ of regular points is called the *regular set* of $\mathcal{L}$. This is an open (possibly empty) subset of $\operatorname{int}\mathcal{L}$.

If $\mathcal{O}$ is a regular point then, at each point C in a minimal configuration, $\Lambda(., \mathcal{O})$ assumes a local maximum. Therefore, there exists a pair of parallel supporting hyperplanes passing through C and C^o. (This follows from the description of $\partial\mathcal{L}$ near C in Section 7 of [9].) Thus, the segment $[C, C^o]$ is an affine diameter of $\mathcal{L}$. We obtain that if $\mathcal{O}$ is a regular point then there are $n+1$ (affinely) independent affine diameters of $\mathcal{L}$ passing through $\mathcal{O}$. Note that it is a difficult and unsolved problem to characterize those points in the interior of $\mathcal{L}$ at which $n+1$ affine diameters pass through. (See Grünbaum [5] and Koziński [6, 7].)

In [9] (Proposition 2 and Theorem E), we proved the following:

Theorem A. *The function $\sigma(\mathcal{L}, .)$ is concave on the regular set $\mathcal{R}$. For $n = 2$, it is concave on the entire $\mathcal{L}$.*

One of the main purposes of this paper is to see how far can this result be extended to specific classes of compact convex bodies. The main technical result is to calculate $\sigma(\mathcal{L}, .)$ for any convex cone $\mathcal{L}$ in terms of certain affine invariants on the base of the cone. This will give the following:

Theorem B. *For any 3-dimensional compact convex cone $\mathcal{L}$, the function $\sigma(\mathcal{L}, .)$ is concave on $\operatorname{int}\mathcal{L}$. There exists a 4-dimensional compact convex cone $\mathcal{L}$ such that $\sigma(\mathcal{L}, .)$ is not concave on $\operatorname{int}\mathcal{L}$.*

The key to prove Theorem B is to extend σ to a sequence of invariants $\{\sigma_m\}_{m\geq 1}$, with $\sigma_n = \sigma$, and study the monotonicity properties of this sequence.

Let $m \in \mathbf{N}$. A multi-set $\{C_0, \dots, C_m\}$ is an *m-configuration* with respect to $\mathcal{O}$ if $\{C_0, \dots, C_m\} \subset \partial\mathcal{L}$ and $\mathcal{O} \in [C_0, \dots, C_m]$. For an m-configuration $\{C_0, \dots, C_m\}$ the convex hull $[C_0, \dots, C_m]$ is a convex polytope in its affine span $\langle C_0, \dots, C_m\rangle$. This polytope has maximum dimension m iff it is a simplex. In this case, we call $\{C_0, \dots, C_m\}$ *simplicial.*

We define

$$\sigma_m(\mathcal{L}, \mathcal{O}) = \inf_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L},\mathcal{O})} \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i, \mathcal{O})},$$

where $\mathcal{C}_m(\mathcal{L}, \mathcal{O})$ denotes the set of all m-configurations of $\mathcal{L}$. An m-configuration for which the infimum is attained is *minimal.* Compactness implies that minimal configurations exist. $\sigma_m(\mathcal{L}, .) : \operatorname{int}\mathcal{L} \to \mathbf{R}$ is continuous [9] (Theorem D), and extends continuously to $\partial\mathcal{L}$ to 1 since

$$\lim_{d(\mathcal{O},\partial\mathcal{L})\to 0} \sigma_m(\mathcal{L}, \mathcal{O}) = 1. \tag{2}$$

Since a 1-configuration of $\mathcal{L}$ is an opposite pair of points $\{C, C^o\} \subset \partial\mathcal{L}$, we have $\sigma_1(\mathcal{L}, \mathcal{O}) = 1$.

In what follows, we suppress $\mathcal{O}$ when no confusion arises.

By [9] (Theorem B), for $m \in \mathbf{N}$, we have

$$1 \leq \sigma_m(\mathcal{L}) \leq \frac{m+1}{2}. \tag{3}$$

If $\sigma_m(\mathcal{L}) = 1$ then $m \leq n$ and there exists an affine subspace $\mathcal{F} \subset \mathcal{E}$, $\mathcal{O} \in \mathcal{F}$, of dimension m such that $\mathcal{L}\cap\mathcal{F}$ is an m-simplex. In this case a minimal configuration $\{C_0, \dots, C_m\} \in \mathcal{C}_m(\mathcal{L} \cap \mathcal{F})$ is simplicial. Moreover, we have

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} = 1 \quad \text{and} \quad \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} C_i = \mathcal{O}. \tag{4}$$

Conversely, if $\mathcal{L}$ has a simplicial intersection with an m-dimensional affine subspace $\mathcal{F} \ni \mathcal{O}$ then $\sigma_m(\mathcal{L}) = 1$.
For $m \geq 2$, $\sigma_m(\mathcal{L}) = (m+1)/2$ if and only if $\Lambda = 1$ on $\partial\mathcal{L}$, that is, if and only if $\mathcal{L}$ is symmetric.

As before, $\sigma_m(\mathcal{L}, \mathcal{O})$, $m \geq 1$, are measures of symmetry.

In general, for $m' \geq m$, we obviously have

$$\sigma_{m'}(\mathcal{L}) \leq \sigma_m(\mathcal{L}) + \frac{m' - m}{1 + \max_{\partial\mathcal{L}} \Lambda}. \tag{5}$$

For $m \geq n$, $n = \dim \mathcal{L}$, equality holds in (5) [10]:

$$\sigma_m(\mathcal{L}) = \sigma_n(\mathcal{L}) + \frac{m - n}{1 + \max_{\partial\mathcal{L}} \Lambda}.$$

Equivalently, the sequence $\{\sigma_m(\mathcal{L})\}_{m \geq 1}$ is *arithmetic* (with difference $1/(1 + \max_{\partial\mathcal{L}} \Lambda)$) from the n-th term onwards.

It is also clear that, for $m < n$, we have

$$\sigma_m(\mathcal{L}) = \inf_{\mathcal{O} \in \mathcal{F} \subset \mathcal{E}, \, \dim \mathcal{F} = m} \sigma(\mathcal{L} \cap \mathcal{F}),$$

where the infimum is over affine subspaces $\mathcal{F} \subset \mathcal{E}$.

In [9] (Theorem B) the following superadditivity was proved:

$$\sigma_{m+m'}(\mathcal{L}) - \sigma_{m+1}(\mathcal{L}) \geq \sigma_{m'}(\mathcal{L}) - \sigma_1(\mathcal{L}), \quad m \geq 0, \;\; m' \geq 2.$$

In particular, setting $m' = 2$ and using $\sigma_2(\mathcal{L}) \geq 1$, we see that the sequence $\{\sigma_m(\mathcal{L})\}_{m \geq 1}$, is *nondecreasing.*

In Section 2 we will derive a stronger statement, namely, that, after a possible string of 1's, this sequence is strictly increasing.

The original motivation for the measures of symmetry $\{\sigma_m(\mathcal{L})\}_{m \geq 1}$ is the *bulging phenomenon* observed in moduli spaces of spherical minimal immersions [11, 13]. The general setting for the moduli is as follows [1, 2, 11]. Let $\mathcal{H}$ be a Euclidean vector space and $S_0^2(\mathcal{H})$ the space of tracefree symmetric endomorphisms of $\mathcal{H}$. We define the *reduced moduli space* by

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \{C \in S_0^2(\mathcal{H}) \,|\, C + I \geq 0\},$$

where ≥ 0 means positive semi-definite. Then $\mathcal{K}_0$ is a compact convex body in $S_0^2(\mathcal{H})$. The distortion $\Lambda(C)$ (with respect to the origin) of an endomorphism $C \in \partial\mathcal{K}_0$ is the maximal eigenvalue of C. Finally, the moduli space corresponding to a linear subspace $\mathcal{E} \subset S_0^2(\mathcal{H})$ is the intersection $\mathcal{L} = \mathcal{E} \cap \mathcal{K}_0$.

It is a difficult and important problem to describe the geometry of $\mathcal{L}$. More specifically, let M be a compact (isotropy irreducible) Riemannian homogeneous manifold, λ an eigenvalue of the Laplace-Beltrami operator acting on functions of M, and $\mathcal{H}$ the eigenspace of functions corresponding to λ. Then the DoCarmo-Wallach moduli space of spherical minimal immersions of M is the moduli space of $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H})$ corresponding to a specific choice $\mathcal{E}$ of a linear subspace in $S^2_0(\mathcal{H})$. For details, see [11].

Theorem C. *We have* $\sigma_1(\mathcal{K}_0, .) = \cdots = \sigma_{h-1}(\mathcal{K}_0, .) = 1$, $h = \dim \mathcal{H}$. *For* $m \geq h$, *we have*

$$\sigma_m(\mathcal{K}_0, \mathcal{O}) = \inf_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{K}_0)} \sum_{i=0}^{m} \frac{1 + \langle \mathcal{O}, C_i\rangle/h}{1 + \Lambda(C_i)},$$

where $\mathcal{C}_m(\mathcal{K}_0)$ *is the set of* m*-configurations of* $\mathcal{K}_0$ *with respect to the origin. Consequently,* $\sigma_m(\mathcal{K}_0, .)$ *is concave on* $\operatorname{int} \mathcal{K}_0$ *and attains its unique maximum* $(m+1)/h$ *at the origin.*

The structure of a moduli space $\mathcal{L} = \mathcal{E} \cap \mathcal{K}_0$ is much more subtle. Even the first statement of Theorem C leads to an unsolved problem. Let $r(\mathcal{L})$ be the largest positive integer such that $\sigma_m(\mathcal{L}) = 1$ for $1 \leq m \leq r(\mathcal{L})$. We have $r(\mathcal{L}) \leq \dim \mathcal{K} - n(\mathcal{L})$, where $n(\mathcal{L}) = \min\{\operatorname{rank}(C + I) \mid C \in \partial\mathcal{L}\}$. In the setting of spherical minimal immersions, $n(\mathcal{L}) - 1$ is the minimal range dimension for such immersions. To determine $n(\mathcal{L})$ is an old and difficult problem [1], [2], [8], [11], [12], [13].

Corollary. *Let* M *be a compact isotropy irreducible Riemannian homogeneous space,* λ *an eigenvalue of the Laplace-Beltrami operator on* M, $\mathcal{H}$ *the corresponding eigenspace, and* $\mathcal{L}$ *the moduli space of spherical minimal immersions of* M. *Assume that the sequence* $\{\sigma_m(\mathcal{L})\}_{m\geq 1}$ *starts with a string of 1's of length* r. *Then there exists a spherical minimal immersion of* M *into a sphere of dimension* $< \dim \mathcal{H} - r$.

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2. The measures $\sigma(\mathcal{L}, \mathcal{O})$ and $\sigma_{n-k}(\mathcal{L}, \mathcal{O})$, $0 \leq k < n$

In view of (5) we define the *regular set* $\mathcal{R} \subset \operatorname{int} \mathcal{L}$ as

$$\mathcal{R} = \left\{ \mathcal{O} \in \operatorname{int} \mathcal{L} \mid \sigma(\mathcal{L}, \mathcal{O}) < \sigma_{n-1}(\mathcal{L}, \mathcal{O}) + \frac{1}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})} \right\}.$$

By continuity of the functions in the defining inequality of $\mathcal{R}$, the set $\mathcal{R}$ is open in $\operatorname{int} \mathcal{L}$ and hence in $\mathcal{E}$.

Again by (5), for $1 \leq k < n$, we define

$$\mathcal{S}_k = \left\{ \mathcal{O} \in \operatorname{int} \mathcal{L} \mid \sigma(\mathcal{L}, \mathcal{O}) = \sigma_{n-k}(\mathcal{L}, \mathcal{O}) + \frac{k}{1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O})} \right\}.$$

We call $\mathcal{S} = \mathcal{S}_1$ the *singular set.* $\mathcal{S}$ is the complement of $\mathcal{R}$ in $\operatorname{int} \mathcal{L}$ and therefore it is relatively closed in $\operatorname{int} \mathcal{L}$. Clearly, we have

$$\mathcal{S}_{k+1} \subset \mathcal{S}_k, \quad k = 1, \dots, n-2.$$

Let $\mathcal{O} \in \mathcal{S}$. We define the *degree of singularity of* $\mathcal{O}$ as the largest k such that $\mathcal{O} \in \mathcal{S}_k$. By the above, the degree of singularity of $\mathcal{O}$ is $n - m$ if and only if the sequence $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m \geq 1}$ is arithmetic (with difference $1/(1 + \max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}))$) exactly from the m-th term onwards.

Proposition 1. *$\mathcal{O} \in \mathcal{R}$ if and only if every minimal configuration is simplicial and $\mathcal{O}$ is contained in the interior of the corresponding n-simplex. In this case, $\Lambda(., \mathcal{O})$ attains its local maximum at each configuration point.*

Let $\mathcal{O} \in \mathcal{S}$. Then $\mathcal{O} \in \mathcal{S}_{n-m}$ if and only if there exists a minimal n-configuration which contains an m-configuration. In this case, the m-configuration is also minimal, and at each n-configuration point complementary to the m-configuration $\Lambda(., \mathcal{O})$ attains absolute maximum. If, in addition, the degree of singularity of $\mathcal{O}$ is $n - m$ (equivalently, $\mathcal{O} \notin \mathcal{S}_{n-m+1}$), then the m-configuration is simplicial, $\mathcal{O}$ is contained in the relative interior of the corresponding m-simplex, and $\Lambda(., \mathcal{O})$ restricted to the affine span of the m-configuration attains local maxima at every m-configuration point.

Proof. We first observe that any subconfiguration of a minimal configuration is also minimal, and the distortion attains absolute maximum at the complementary points in the respective affine span.

Now let $1 \leq m \leq n-1$. We claim that $\mathcal{O} \in \mathcal{S}_{n-m}$ if and only if there exists a minimal n-configuration $\{C_0, \ldots, C_n\} \in \mathcal{C}_n(\mathcal{L}, \mathcal{O})$ which contains an m-configuration $\{C_0, \ldots, C_m\} \in \mathcal{C}_m(\mathcal{L}, \mathcal{O})$, where we relabeled the configuration points if necessary. The entire proposition follows from this claim.

Indeed, setting $m = n - 1$, the claim implies that $\mathcal{O} \in \mathcal{R}$ if and only if every minimal n-configuration $\{C_0, \ldots, C_n\} \in \mathcal{C}_n(\mathcal{L}, \mathcal{O})$ contains no subconfiguration, or equivalently, the corresponding convex hull $[C_0, \ldots, C_n]$ is an n-simplex and $\mathcal{O}$ is not on its boundary. Thus the claim implies the first statement of the proposition.

The degree of singularity of $\mathcal{O}$ is $n - m$ if and only if m is the least number in the claim. Indeed, if the subconfiguration $\{C_0, \ldots, C_m\} \in \mathcal{C}_m(\mathcal{L}, \mathcal{O})$ is not simplicial or $\mathcal{O}$ is on the boundary of its convex hull then this contradicts the minimality of m. Thus, the claim implies the remaining part of the proposition.

By the observation above, we need to prove the "only if" part of the claim. Let $\mathcal{O} \in \mathcal{S}_{n-m}$. Choose a minimal m-configuration $\{C_0, \ldots, C_m\} \in \mathcal{C}_m(\mathcal{L}, \mathcal{O})$ and extend it to an n-configuration $\{C_0, \ldots, C_n\} \in \mathcal{C}_n(\mathcal{L}, \mathcal{O})$ by adding $C_{m+1} = \cdots = C_n$ at which $\Lambda(., \mathcal{O})$ attains absolute maximum. With this, we have

$$\begin{aligned} \sum_{i=0}^{n} \frac{1}{1 + \Lambda(C_i)} &= \sum_{i=0}^{m} \frac{1}{1 + \Lambda(C_i)} + \frac{n - m}{1 + \max_{\partial\mathcal{L}} \Lambda} \\ &= \sigma_m(\mathcal{L}, \mathcal{O}) + \frac{n - m}{1 + \max_{\partial\mathcal{L}} \Lambda} \\ &= \sigma(\mathcal{L}, \mathcal{O}). \end{aligned}$$

Thus, $\{C_0, \ldots, C_n\}$ is minimal and the claim follows.

Proposition 2. *The sequence* $\{\sigma_m(\mathcal{L}, \mathcal{O})\}_{m\geq 1}$*, after a possible initial string of 1's, is strictly increasing.*

First, for $1 \leq m \leq n$, we define $\mathcal{S}_m(\mathcal{L}, \mathcal{O}) \subset \mathcal{C}_m(\mathcal{L}, \mathcal{O})$ as the subset of all simplicial m-configurations (with respect to $\mathcal{O}$). By continuity, $\mathcal{C}_m(\mathcal{L}, \mathcal{O})$ can be replaced by $\mathcal{S}_m(\mathcal{L}, \mathcal{O})$ in the definition of $\sigma_m(\mathcal{L}, \mathcal{O})$. We need the following:

Lemma 1. *Let* $\{C_0, \dots, C_m\} \in \mathcal{S}_m(\mathcal{L}, \mathcal{O})$ *with* $\mathcal{O}$ *in the relative interior of the* m*-simplex* $[C_0, \dots, C_m]$*. Setting*

$$\mathcal{O} = \sum_{i=0}^{m} \mu_i C_i, \quad 0 < \mu_i < 1, \ i = 0, \dots, m,$$

we have

$$\mu_i \leq \frac{1}{1 + \Lambda(C_i, \mathcal{O})}, \quad i = 0, \dots, m.$$

For a specific index i*, equality holds iff* $[C_0, \dots, \widehat{C}_i, \dots, C_m] \subset \partial\mathcal{L}$*.*

Proof. The lemma follows easily by comparing the distortion functions of $\mathcal{L}$ and that of the simplex $[C_0, \dots, C_m]$ and using (4).

Proof of Proposition 2. Assume that $\sigma_{m-1} = \sigma_m(\mathcal{L}, \mathcal{O})$ for some $m \geq 3$. Clearly, $\mathcal{O}$ is a regular point. Let $\{C_0, \dots, C_m\} \in \mathcal{C}_m(\mathcal{L}, \mathcal{O})$ be minimizing. Thus, $[C_0, \dots, C_m]$ is an m-simplex with $\mathcal{O}$ in its relative interior, and Lemma 1 applies. Let $0 \leq i, j \leq m$ be distinct, and $\mathcal{F}_{ij} = \langle C_i, C_j, \mathcal{O}\rangle$. Then $\mathcal{F}_{ij}$ is an affine plane passing through $\mathcal{O}$. As in Section 3 of [9], superadditivity implies that $\mathcal{F}_{ij}$ intersects $\mathcal{L}$ in a triangle with vertices C_i, C_j and another vertex, say, C_{ij}. This latter vertex appears in the decomposition of $\mathcal{O}$ in Lemma 1 by compressing all points C_k, $k \neq i, j$ into a single point

$$\mu_i C_i + \mu_j C_j + \mu_{ij} C_{ij} = \mathcal{O} \quad \text{with} \quad \mu_i + \mu_j + \mu_{ij} = 1,$$

where

$$C_{ij} = \frac{1}{\mu_{ij}} \sum_{k=0;\, k\neq i,j}^{m} \mu_k C_k.$$

Applying Lemma 1 to the triangle $[C_i, C_j, C_{ij}]$, we obtain

$$\mu_i = \frac{1}{1 + \Lambda(C_i, \mathcal{O})} \quad \text{and} \quad \mu_j = \frac{1}{1 + \Lambda(C_j, \mathcal{O})}.$$

Since i, j are arbitrary, the proposition follows.

Finally we will need the following:

Proposition 3. *Let* C *be a smooth point of* $\partial\mathcal{L}$ *and assume that* $\Lambda(., \mathcal{O})$ *has a local maximum at* C*. Then* C^o *is also a smooth point of* $\partial\mathcal{L}$ *and the tangent spaces at* C *and at* C^o *are parallel. If, in addition,* C *is a* k*-flat point (the tangent space at* C *contains a maximal* k*-dimensional affine subspace* $\mathcal{A}_C$ *and* C *is contained in the relative interior of* $\partial\mathcal{L} \cap \mathcal{A}_C$*) then* C^o *is* ℓ*-flat, where* $\ell \geq k$*, and a translate of* $\mathcal{A}_C$ *is contained in* $\mathcal{A}_{C^o}$*.*

For the proof, see Section 7 of [9].

3. The measures $\tau(\mathcal{L},\mathcal{O})$ and $\tau_{n-k}(\mathcal{L},\mathcal{O})$, $1 \le k < n$

Let $\mathcal{L}$ be a compact convex body in a Euclidean vector space $\mathcal{E}$ of dimension n, and $\mathcal{O}$ an interior point of $\mathcal{L}$. We let $\mathcal{S}^0_m(\mathcal{L},\mathcal{O})$ denote the set of all simplicial m-configurations (with respect to $\mathcal{O}$) with a distinguished element. We usually index the elements of a simplicial m-configuration $\{B_0,\dots,B_m\}$ such that the distinguished element is B_0. Then B_0 is a vertex of the m-simplex $[B_0,\dots,B_m]$ with opposite face $[B_1,\dots,B_m]$.

We now introduce another sequence of invariants $\tau_m(\mathcal{L},\mathcal{O})$, $1 < m \le n$, which will be useful in calculating distortions of cones in the next section. We let

$$\tau_m(\mathcal{L},\mathcal{O}) = \inf_{\{B_0,\dots,B_m\}\in\mathcal{S}^0_m(\mathcal{L},\mathcal{O})} \left[\frac{1}{1+\Lambda_{[B_0,\dots,B_m]}(B_0,\mathcal{O})} + \sum_{i=1}^m \frac{1}{1+\Lambda_{\mathcal{L}}(B_i,\mathcal{O})}\right] \tag{6}$$

where $\Lambda_{[B_0,\dots,B_m]}(.,\mathcal{O})$ is the distortion function of the m-simplex $[B_0,\dots,B_m]$ if $\mathcal{O}$ is in the relative interior of $[B_0,\dots,B_m]$, otherwise it is defined by the obvious limit. In particular, if $\mathcal{O}\in[B_1,\dots,B_m]$ then $\Lambda_{[B_0,\dots,B_n]}(B_0,\mathcal{O})=\infty$.

Alternatively, $\Lambda_{[B_0,\dots,B_m]}(B_0,\mathcal{O})$ and $\Lambda_{\mathcal{L}}(B_i,\mathcal{O})$, $i=1,\dots,m$, can be interpreted as the distortions of the intersection $\mathcal{L}\cap\langle B_0,\dots,B_m\rangle$ truncated with $\langle B_1,\dots,B_m\rangle$.

An easy application of Lemma 1 gives $\tau_m(\mathcal{L},\mathcal{O})\ge 1$. It is also clear from the limiting behavior that

$$\tau_m(\mathcal{L},\mathcal{O}) \le \sigma_{m-1}(\mathcal{L},\mathcal{O}). \tag{7}$$

We have $\tau_2(\mathcal{L},\mathcal{O})=1$. For uniformity, we define $\tau_1(\mathcal{L},\mathcal{O})=1$. (It seems to be a difficult problem to decide whether equality holds in (7).)

By (3) and (7), we have

$$1 \le \tau_m(\mathcal{L},\mathcal{O}) \le \frac{m}{2}.$$

$\tau_m(\mathcal{L},\mathcal{O})=1$ if and only if $\sigma_{m-1}(\mathcal{L},\mathcal{O})=1$. (Once again, see Lemma 1.) Also, if $\tau_m(\mathcal{L},\mathcal{O})=m/2$ for some $m\ge 3$, then $\mathcal{L}$ is symmetric with respect to $\mathcal{O}$.

As usual, we suppress the index $m=n$. With this, we have

$$\tau_m(\mathcal{L},\mathcal{O}) = \inf_{\mathcal{O}\in\mathcal{F}\subset\mathcal{E}\,;\,\dim\mathcal{F}=m} \tau(\mathcal{L}\cap\mathcal{F},\mathcal{O}), \tag{8}$$

where the infimum is over affine subspaces $\mathcal{F}\subset\mathcal{E}$.

Clearly, (5) holds for σ replaced by τ:

$$\tau_{m'}(\mathcal{L},\mathcal{O}) \le \tau_m(\mathcal{L},\mathcal{O}) + \frac{m'-m}{1+\max_{\partial\mathcal{L}}\Lambda(.,\mathcal{O})}, \quad 1\le m\le m'\le n. \tag{9}$$

It will be convenient to define the out-of-range invariant $\tau_{n+1}(\mathcal{L},\mathcal{O})=\sigma(\mathcal{L},\mathcal{O})$.

Given $\{B_0,\dots,B_m\}\in\mathcal{S}^0_m(\mathcal{L},\mathcal{O})$, we define the *bulging*

$$\beta_{[B_0,\dots,B_m]}(B_i,\mathcal{O}) = \frac{1}{1+\Lambda_{\mathcal{L}}(B_i,\mathcal{O})} - \frac{1}{1+\Lambda_{[B_0,\dots,B_n]}(B_i,\mathcal{O})}. \tag{10}$$

We then have

$$\tau_m(\mathcal{L},\mathcal{O}) = \inf_{\{B_0,\ldots,B_m\}\in\mathcal{S}^0_m(\mathcal{L},\mathcal{O})} \sum_{i=1}^m \beta_{[B_0,\ldots,B_m]}(B_i,\mathcal{O}) + 1. \tag{11}$$

Proposition 4. *For $1 < m \le n$, we have*

$$m\tau_{m-1}(\mathcal{L},\mathcal{O}) \le (m-1)\tau_m(\mathcal{L},\mathcal{O}) + 1. \tag{12}$$

Proof. Let $\epsilon > 0$ and consider a simplicial m-configuration $\{B_0,\ldots,B_m\}$ with respect to $\mathcal{O}$ such that

$$\frac{1}{1+\Lambda_{[B_0,\ldots,B_m]}(B_0,\mathcal{O})} + \sum_{i=1}^m \frac{1}{1+\Lambda_{\mathcal{L}}(B_i,\mathcal{O})} < \tau_m(\mathcal{L},\mathcal{O}) + \epsilon. \tag{13}$$

We may assume that $\mathcal{O}$ is in the relative interior of the m-simplex $[B_0,\ldots,B_m]$. With this, we have

$$\mathcal{O} = \sum_{i=0}^m \lambda_i B_i, \quad \sum_{i=0}^m \lambda_i = 1, \quad 0 < \lambda_i < 1.$$

We write

$$\mathcal{O} = (\lambda_0+\lambda_1)B_{01} + \sum_{i=2}^m \lambda_i B_i,$$

where

$$B_{01} = \frac{\lambda_0}{\lambda_0+\lambda_1}B_0 + \frac{\lambda_1}{\lambda_0+\lambda_1}B_1.$$

Let $\bar{B}_{01} = \mu B_{01} \in \partial\mathcal{L}$, where $\mu \ge 1$.
Clearly, $[\bar{B}_{01}, B_2,\ldots,B_m]$ is an $(m-1)$-simplex with $\mathcal{O}$ in its interior. Let $\bar{B}^o_{01} \in [B_2,\ldots,B_m]$ be the opposite of $\bar{B}_{01}$ with respect to this simplex.
$[B_0, B_1, \bar{B}^o_{01}]$ is a triangle with $\mathcal{O}$ in its interior. We now add the terms

$$\frac{1}{1+\Lambda_{[\bar{B}_{01},B_2,\ldots,B_m]}(\bar{B}_{01},\mathcal{O})} + \frac{1}{1+\Lambda_{[\bar{B}_{01},B_2,\ldots,B_m]}(\bar{B}^o_{01},\mathcal{O})} - 1$$

(amounting to zero) to the left-hand side of the inequality in (13), and split the terms into two groups. The terms

$$\frac{1}{1+\Lambda_{[\bar{B}_{01},B_2,\ldots,B_m]}(\bar{B}_{01},\mathcal{O})} + \sum_{i=2}^m \frac{1}{1+\Lambda_{\mathcal{L}}(B_i,\mathcal{O})} - 1$$

can be estimated below by $\tau_{m-1}(\mathcal{L},\mathcal{O}) - 1$ since $\bar{B}_{01}$ can be taken as the distinguished element in the $(m-1)$-configuration $\{\bar{B}_{01}, B_2,\ldots,B_m\}$. The remaining terms in the second group are

$$\frac{1}{1+\Lambda_{[B_0,\ldots,B_m]}(B_0,\mathcal{O})} + \frac{1}{1+\Lambda_{\mathcal{L}}(B_1,\mathcal{O})} + \frac{1}{1+\Lambda_{[\bar{B}_{01},B_2,\ldots,B_m]}(\bar{B}^o_{01},\mathcal{O})}. \tag{14}$$

The first term in (14) is equal to

$$\frac{1}{1+\Lambda_{[\bar{B}^o_{01},B_0,B_1]}(B_0,\mathcal{O})}.$$

By the same token, the second term in (14) is equal to

$$\beta_{[B_0,\dots,B_m]}(B_1,\mathcal{O})+\frac{1}{1+\Lambda_{[\bar{B}^o_{01},B_0,B_1]}(B_1,\mathcal{O})}.$$

Finally, the third term in (14) can be estimated as

$$\frac{1}{1+\Lambda_{[\bar{B}_{01},B_2,\dots,B_m]}(\bar{B}^o_{01},\mathcal{O})}\geq\frac{1}{1+\Lambda_{[\bar{B}^o_{01},B_0,B_1]}(\bar{B}^o_{01},\mathcal{O})}$$

since

$$\Lambda_{[\bar{B}_{01},B_2,\dots,B_m]}(\bar{B}^o_{01},\mathcal{O})\leq\Lambda_{[\bar{B}^o_{01},B_0,B_1]}(\bar{B}^o_{01},\mathcal{O}).$$

Putting everything together, we obtain

$$\tau_{m-1}(\mathcal{L},\mathcal{O})+\beta_{[B_0,\dots,B_m]}(B_1,\mathcal{O})\leq\tau_m(\mathcal{L},\mathcal{O})+\epsilon$$

since the terms involving distortions at the vertices of the triangle $[\bar{B}^o_{01},B_0,B_1]$ add up to 1. Replacing B_1 by B_i, summing up with respect to $i=1,\dots,m$, and letting $\epsilon\to 0$, we obtain

$$m\tau_{m-1}(\mathcal{L},\mathcal{O})+\sum_{i=1}^{m}\beta_{[B_0,\dots,B_m]}(B_i,\mathcal{O})\leq m\tau_m(\mathcal{L},\mathcal{O}).$$

Finally, using (11), we arrive at (12).

Since $\tau_m(\mathcal{L},\mathcal{O})\geq 1$, (12) implies that the sequence $\{\tau_m(\mathcal{L},\mathcal{O})\}_{m=1}^n$ is nondecreasing. Moreover, again by (12), the only way $\tau_{m-1}(\mathcal{L},\mathcal{O})=\tau_m(\mathcal{L},\mathcal{O})$ can happen is that it is equal to 1. This means that the sequence $\{\tau_m(\mathcal{L},\mathcal{O})\}_{m=1}^n$, after an initial string of 1's, is strictly increasing.

Iterating (12), for $1\leq m'<m\leq n$, we get

$$m\tau_{m-m'}(\mathcal{L},\mathcal{O})\leq(m-m')\tau_m(\mathcal{L},\mathcal{O})+m'.$$

Replacing m' by $m-m'$ and adding, we obtain

$$\tau_{m-m'}(\mathcal{L},\mathcal{O})+\tau_{m'}(\mathcal{L},\mathcal{O})\leq\tau_m(\mathcal{L},\mathcal{O})+1,$$

a subadditive property.

4. Convex cones

Let $\mathcal{L}_0 \subset \mathcal{E}_0$ be a compact convex body and $\mathcal{O}_0$ an interior point of $\mathcal{L}_0$. Let $\mathcal{E} = \mathcal{E}_0 \times \mathbf{R}$ and $\mathcal{O}_1 \in \mathcal{E}$ not contained in $\mathcal{E}_0$. We consider the *cone* $\mathcal{L} = [\mathcal{L}_0, \mathcal{O}_1]$. We let $0 < \lambda < 1$ and $\mathcal{O}_\lambda = (1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1$. In this section we calculate $\sigma(\mathcal{L}, \mathcal{O}_\lambda)$ in terms of $\mathcal{L}_0$. We assume that $\mathcal{L}_0$ and therefore $\mathcal{L}$ are not simplicial since otherwise there is nothing to calculate. It is technically convenient to assume that $\mathcal{O}_0$ is the origin of $\mathcal{E}_0$. We also set $n = \dim \mathcal{L}$ so that $\dim \mathcal{L}_0 = n-1$.

Theorem D. *Let $\mathcal{L}$ be a cone with base $\mathcal{L}_0$ and vertex $\mathcal{O}_1$. Let $\mathcal{O}_\lambda = (1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1$, $0 < \lambda < 1$, be the base point of $\mathcal{L}$, where $\mathcal{O}_0$ is the base point of $\mathcal{L}_0$. Then we have the following:*
I. *If* $0 < \lambda \leq \frac{1}{2+\max_{\partial\mathcal{L}_0}\Lambda(.,\mathcal{O}_0)}$ *then*

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \min_{1\leq m\leq n} \left((n-m+1)\lambda + (1-\lambda)\tau_m(\mathcal{L}_0, \mathcal{O}_0)\right);$$

II. *If* $\frac{1}{2+\max_{\partial\mathcal{L}_0}\Lambda(.,\mathcal{O}_0)} \leq \lambda < 1$ *then*

$$\begin{aligned}\sigma(\mathcal{L}, \mathcal{O}_\lambda) &= \lambda + (1-\lambda)\min_{1\leq m\leq n}\left(\tau_m(\mathcal{L}_0, \mathcal{O}_0) + \frac{n-m}{1+\max_{\partial\mathcal{L}_0}\Lambda(.,\mathcal{O}_0)}\right)\\ &= \lambda + (1-\lambda)\min\left(\sigma(\mathcal{L}_0, \mathcal{O}_0), \tau(\mathcal{L}_0, \mathcal{O}_0) + \frac{1}{1+\max_{\partial\mathcal{L}_0}\Lambda(.,\mathcal{O}_0)}\right).\end{aligned}$$

In these formulas, $\tau_n(\mathcal{L}_0, \mathcal{O}_0) = \sigma_{n-1}(\mathcal{L}_0, \mathcal{O}_0) = \sigma(\mathcal{L}_0, \mathcal{O}_0)$.
In particular, the function $\lambda \mapsto \sigma(\mathcal{L}, \mathcal{O}_\lambda)$, $\lambda \in [0,1]$, is piecewise linear and concave (with limit equal to 1 at the endpoints).

The proof of Theorem D is technical. In what follows we will make a number of observations and reduce Theorem D to a simpler one. To simplify the notation we let γ denote the function $\lambda \mapsto \sigma(\mathcal{L}, \mathcal{O}_\lambda)$, $\lambda \in [0,1]$. In addition, for $1 \leq m \leq n$, we consider the linear functions

$$\begin{aligned}\alpha_m(\lambda) &= (n-m+1)\lambda + (1-\lambda)\tau_m,\\ \beta_m(\lambda) &= \lambda + (1-\lambda)\left(\tau_m + \frac{n-m}{1+M_0}\right),\end{aligned}$$

where we suppressed $\mathcal{L}_0$, $\mathcal{O}_0$, and set $M_0 = \max_{\partial\mathcal{L}_0}\Lambda(.,\mathcal{O}_0)$. Recall here also that $\tau_1 = \tau_2 = 1$, $\tau_{n-1} = \tau$, and $\tau_n = \sigma_{n-1} = \sigma$. With these, I–II of Theorem D rewrites as
I. If $0 < \lambda \leq 1/(2+M_0)$ then $\gamma(\lambda) = \min_{1\leq m\leq n}\alpha_m(\lambda)$.
II. If $1/(2+M_0) \leq \lambda < 1$ then $\gamma(\lambda) = \min_{1\leq m\leq n}\beta_m(\lambda)$.
Clearly, $\alpha_n = \beta_n$. Moreover, for $1 \leq m < n$, we have

$$\alpha_m(0) = \tau_m < \tau_m + \frac{n-m}{1+M_0} = \beta_m(0) \quad \text{and} \quad \alpha_m(1) = n-m+1 > 1 = \beta_m(1),$$

while α_m and β_m attain the same value at $1/(2+M_0)$. It follows that in both cases I–II, we have

$$\gamma(\lambda) = \min_{1\le m\le n} \left(\alpha_m(\lambda), \beta_m(\lambda)\right), \quad \lambda \in [0,1].$$

Thus, γ is piecewise linear and concave. By (2), the one-sided limits of γ at the endpoints of $[0,1]$ are both equal to 1. The second statement of Theorem D follows.

By (9), for $1 \le m < m' < n$, we have $\beta_m \ge \beta_{m'}$. Thus,

$$\min_{1\le m\le n} \beta_m(\lambda) = \min(\beta_n(\lambda), \beta_{n-1}(\lambda)).$$

This is the second equality in case B of Theorem D.

We now give a more detailed analysis of the minimum in case I:

$$\gamma(\lambda) = \min_{1\le m\le n} \alpha_m(\lambda), \quad \lambda \in [0, 1/(2+M_0)].$$

For $1 \le m < m' \le n$, the linear functions α_m and $\alpha_{m'}$ have the same value at

$$\ell_{m,m'} = 1 - \frac{1}{1+\frac{\tau_{m'}-\tau_m}{m'-m}}. \tag{15}$$

For $1 \le m < m' < n$, using (9), we have

$$\ell_{m,m'} \le 1 - \frac{1}{1+\frac{1}{1+M_0}} = \frac{1}{2+M_0}.$$

We say that α_m *participates* in γ if the zero set of $(\alpha_m - \gamma)|_{[0,1/(2+M_0)]}$ has nonempty interior. Since γ is concave, this zero set is a closed interval. We call this the *interval of participation* of α_m in γ. Since γ is piecewise linear, the interval $[0, 1/(2+M_0)]$ is subdivided into intervals of participation for the various α_m, $1 \le m \le n$. We have

$$1 = \alpha_1(0) = \tau_1 = \alpha_2(0) = \tau_2 \le \cdots \le \alpha_{n-1}(0) = \tau \le \sigma_{n-2} \le \sigma = \tau_n = \alpha_n(0),$$

and

$$\alpha_1'(0) = n-1 > \alpha_2'(0) = n-2 > \cdots > \alpha_{n-1}'(0) = 2-\tau \ge \alpha_n'(0) = 1-\sigma.$$

Let $1 \le m < m' \le n$, and assume that both α_m and $\alpha_{m'}$ participate in γ. Comparing the intercepts and slopes above, we see that concavity of γ implies that the interval of participation of α_m precedes that of $\alpha_{m'}$.

We let $1 < m_1 < \cdots < m_s \le n$ denote those indices m for which α_m participates in γ. By the above, the intervals of participation of α_{m_i}, $i = 1, \ldots, s$, subdivide $[0, 1/(2+M_0)]$ in a successive manner.

Comparing the slopes and intercepts above we see that m_1 is the largest index m such that $\tau_m = 1$. (Recall that the sequence $\{\tau_m\}_{m=1}^n$ starts with a string of 1's.)

For $1 \leq i < s$, the interval of participation of α_{m_i} in γ is $[\ell_{m_{i-1},m_i}, \ell_{m_i,m_{i+1}}]$. (Here we set $m_0 = 0$ so that $\ell_{m_0,m_1} = 0$.)
The last interval of participation is therefore $[\ell_{m_{s-1},m_s}, 1/(2+M_0)]$. By continuity of γ, at $\lambda = 1/(2+M_0)$, the matching condition

$$\alpha_{m_s}(\lambda) = \min(\beta_n(\lambda), \beta_{n-1}(\lambda))$$

must be satisfied. This works out to be

$$\min\left(\sigma, \tau + \frac{1}{1+M_0}\right) = \tau_{m_s} + \frac{n-m_s}{1+M_0}.$$

If $\sigma < \tau + 1/(1+M_0)$ then $m_s = n$.
If $\sigma \geq \tau + 1/(1+M_0)$ then, for $m_s = n$, we have $\sigma = \tau + 1/(1+M_0)$, and, for $m_s < n$, we have $\tau = \tau_{m_s} + (n-1-m_s)/(1+M_0)$.
There are several consequences of this discussion.

Theorem E. *Assume that* $\tau_3(\mathcal{L}_0,.), \ldots, \tau_{n-1}(\mathcal{L}_0,.) = \tau(\mathcal{L}_0,.), \sigma(\mathcal{L}_0,.)$ *are all concave on* $\operatorname{int}\mathcal{L}_0$. *Then* $\sigma(\mathcal{L},.)$ *is also concave on* $\operatorname{int}\mathcal{L}$.

Proof. Since the minimum of concave functions is concave, we need to show that α_m and β_m, $m = 1, \ldots, n$, are concave functions on $\operatorname{int}\mathcal{L}$. We do this in a slightly more general setting. Assume that ϕ is a concave function on $\operatorname{int}\mathcal{L}_0$, and $c > 0$ is a fixed constant. Define the function $\psi : \operatorname{int}\mathcal{L} \to \mathbf{R}$ by

$$\psi((1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1) = c\lambda + (1-\lambda)\phi(\mathcal{O}_0),\ \mathcal{O}_0 \in \operatorname{int}\mathcal{L}_0,\ 0 < \lambda < 1.$$

Since α_m and β_m are special cases of this (see also the corollary to Proposition 1 in [9]), it remains to show that ψ is concave.
We let $\mathcal{O}_0^0, \mathcal{O}_0^1 \in \operatorname{int}\mathcal{L}_0$, $0 < \mu, \nu < 1$, and

$$\mathcal{O}_\mu^0 = (1-\mu)\mathcal{O}_0^0 + \mu\mathcal{O}_1 \quad \text{and} \quad \mathcal{O}_\nu^1 = (1-\nu)\mathcal{O}_0^1 + \nu\mathcal{O}_1.$$

We need to show that

$$\psi((1-\kappa)\mathcal{O}_\mu^0 + \kappa\mathcal{O}_\nu^1) \geq (1-\kappa)\psi(\mathcal{O}_\mu^0) + \kappa\psi(\mathcal{O}_\nu^1),\ 0 < \kappa < 1.$$

We write

$$(1-\kappa)\mathcal{O}_\mu^0 + \kappa\mathcal{O}_\nu^1 = (1-\lambda)\left(\frac{(1-\kappa)(1-\mu)}{1-\lambda}\mathcal{O}_0^0 + \frac{\kappa(1-\nu)}{1-\lambda}\mathcal{O}_0^1\right) + \lambda\mathcal{O}_1,$$

where $\lambda = (1-\kappa)\mu + \kappa\nu$. With these, we have

$$\begin{aligned}
\psi((1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1) &= c\lambda + (1-\lambda)\phi\left(\frac{(1-\kappa)(1-\mu)}{1-\lambda}\mathcal{O}_0^0 + \frac{\kappa(1-\nu)}{1-\lambda}\mathcal{O}_0^1\right) \\
&\geq c\lambda + (1-\kappa)(1-\mu)\phi(\mathcal{O}_0^0) + \kappa(1-\nu)\phi(\mathcal{O}_0^1) \\
&= (1-\kappa)(c\mu + (1-\mu)\phi(\mathcal{O}_0^0)) + \kappa(c\nu + (1-\nu)\phi(\mathcal{O}_0^1)) \\
&= (1-\kappa)\psi(\mathcal{O}_\mu^0) + \kappa\psi(\mathcal{O}_\nu^1).
\end{aligned}$$

Concavity and Theorem E follow.

Note that Theorem E (along with Theorem A) implies the first statement of Theorem B. For the second statement we have the following:

Example. Let $\Delta \subset \mathbf{R}^2$ be an equilateral triangle inscribed in the unit circle of $\mathbf{R}^2$. Let $\mathcal{L}_0$ be the intersection of the vertical cylinder in $\mathbf{R}^3$ with base Δ and the unit ball. Then $\sigma_2(\mathcal{L}_0, .)$ is not concave. Indeed, for $\mathcal{O} \in \Delta$, we have $\sigma_2(\mathcal{L}_0, \mathcal{O}) = 1$, and, for any other point $\mathcal{O}$ in the interior of $\mathcal{L}_0$, we have $\sigma_2(\mathcal{L}_0, \mathcal{O}) > 1$ as there is no triangular intersection of $\mathcal{L}_0$ away from Δ.

Now consider $\mathcal{L}_0$ as the base of a 4-dimensional cone $\mathcal{L}$ (with vertex $\mathcal{O}_1$). We claim that $\sigma(\mathcal{L}, .)$ is not concave.

Let $\mathcal{O}_t = (0, 0, t)$, $|t| < 1$. We calculate $\sigma(\mathcal{L}, (1-\lambda)\mathcal{O}_t + \lambda\mathcal{O}_1)$ for small $\lambda > 0$. For $t = 0$, $\mathcal{O}_0 \in \Delta$. Since Δ is a triangular intersection of $\mathcal{L}_0$, we have $\sigma_2(\mathcal{L}_0, \mathcal{O}_0) = 1$. Thus, we have

$$1 = \tau_1(\mathcal{L}_0, \mathcal{O}_0) = \tau_2(\mathcal{L}_0, \mathcal{O}_0) = \tau(\mathcal{L}_0, \mathcal{O}_0) < \tau_4(\mathcal{L}_0, \mathcal{O}_0) = \sigma(\mathcal{L}_0, \mathcal{O}_0).$$

Hence, the last 1 in this sequence is at the index $m_1 = 3$. Consequently, for small $\lambda > 0$, we have

$$\sigma(\mathcal{L}, (1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1) = \alpha_3(\lambda) = 2\lambda + (1-\lambda)\tau(\mathcal{L}_0, \mathcal{O}_0) = 1 + \lambda.$$

Now let $t \neq 0$. Since $\mathcal{O}_t \notin \Delta$, $\mathcal{L}_0$ has no triangular intersection passing through $\mathcal{O}_t$, we have $\sigma_2(\mathcal{L}_0, \mathcal{O}_t) > 1$. Thus

$$1 = \tau_1(\mathcal{L}_0, \mathcal{O}_t) = \tau_2(\mathcal{L}_0, \mathcal{O}_t) < \tau(\mathcal{L}_0, \mathcal{O}_t) \leq \tau_4(\mathcal{L}_0, \mathcal{O}_t) = \sigma(\mathcal{L}_0, \mathcal{O}_t).$$

The last 1 in this sequence has index $m_1 = 2$. Consequently, for small $\lambda > 0$, we have

$$\sigma(\mathcal{L}, (1-\lambda)\mathcal{O}_t + \lambda\mathcal{O}_1) = \alpha_2(\lambda) = 3\lambda + (1-\lambda)\tau_2(\mathcal{L}_0, \mathcal{O}_0) = 1 + 2\lambda.$$

(Note that the length of the first interval of participation tends to zero as $t \to 0$.)

Let $0 < t < 1$ be fixed and $\lambda > 0$ small enough so that the formulas above hold for $\pm t$. Then, at the endpoints of the line segment $[\mathcal{O}_{-t}, \mathcal{O}_t]$, the function $\sigma(\mathcal{L}, .)$ is $1 + 2\lambda$ and at the midpoint $\mathcal{O}_0$ it is $1 + \lambda$. Thus, $\sigma(\mathcal{L}, .)$ is not concave.

It remains to prove Theorem D. To do this, in view of the discussion above, we need to show the following:

Theorem F. *Let $\mathcal{L}$, $\mathcal{L}_0$ and $\mathcal{O}_0$ be as in Theorem D. We have the following:*

(i) *If $\mathcal{O}_\lambda \in \mathcal{R}$ is a regular point then*

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \lambda + (1-\lambda)\sigma(\mathcal{L}_0, \mathcal{O}_0). \tag{16}$$

(ii) *If* $\mathcal{O}_\lambda \in \mathcal{S}$ *is singular with degree of singularity* $n-m$ *then*

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \lambda + (1-\lambda)\tau_m(\mathcal{L}_0, \mathcal{O}_0) + \frac{n-m}{1+\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)}, \tag{17}$$

where

$$\frac{1}{1+\max_{\partial\mathcal{L}} \Lambda(., \mathcal{O}_\lambda)} = \begin{cases} \lambda, & \textit{if } 0 < \lambda \leq \frac{1}{2+\max_{\partial\mathcal{L}_0} \Lambda(., \mathcal{O}_0)} \\ \frac{1-\lambda}{1+\max_{\partial\mathcal{L}_0} \Lambda(., \mathcal{O}_0)}, & \textit{if } \frac{1}{2+\max_{\partial\mathcal{L}_0} \Lambda(., \mathcal{O}_0)} \leq \lambda < 1 \end{cases}. \tag{18}$$

In addition, the infimum in $\tau_m(\mathcal{L}_0, \mathcal{O}_0)$ *is attained by a simplicial configuration.*

The proof of Theorem F is given in the rest of this section. We begin with the following:

Lemma 2. *Let* $A_0, A_1 \in \mathbf{R}^2$ *be distinct points and* $A_\lambda = (1-\lambda)A_0 + \lambda A_1$ *with* $0 < \lambda < 1$. *Let* B_0, B_λ, B_1 *be collinear points perspectively related to* A_0, A_λ, A_1 *by a perspectivity with center at* $\mathcal{O}$. *Let*

$$\Lambda_0 = \frac{d(A_0, \mathcal{O})}{d(B_0, \mathcal{O})}, \quad \Lambda_1 = \frac{d(A_1, \mathcal{O})}{d(B_1, \mathcal{O})}, \quad \text{and} \quad \Lambda_\lambda = \frac{d(A_\lambda, \mathcal{O})}{d(B_\lambda, \mathcal{O})}.$$

Then, we have

$$\Lambda_\lambda = (1-\lambda)\Lambda_0 + \lambda\Lambda_1.$$

Proof. We may assume that $\mathcal{O}$ is the origin. Then, we have

$$B_0 = \pm\frac{1}{\Lambda_0}A_0, \quad B_1 = \pm\frac{1}{\Lambda_1}A_1, \quad \text{and} \quad B_\lambda = \pm\frac{1}{\Lambda_\lambda}A_\lambda.$$

We also have $B_\lambda = (1-\mu)B_0 + \mu B_1$ for some $0 < \mu < 1$. Playing this equation back to the definition of A_λ and comparing, Lemma 2 follows.

Corollary. *Let* $[B_0, B_1, C] \subset \mathbf{R}^2$ *be a triangle and choose points* $\mathcal{O}_0$ *and* $\mathcal{O}_1$ *in the interior of the sides* $[B_0, C]$ *and* $[B_1, C]$. *Let* $0 < \lambda < 1$ *and* $\mathcal{O}_\lambda = (1-\lambda)\mathcal{O}_0 + \lambda\mathcal{O}_1$. *Consider the distortions*

$$\Lambda_0 = \frac{d(C, \mathcal{O}_0)}{d(B_0, \mathcal{O}_0)}, \quad \Lambda_1 = \frac{d(C, \mathcal{O}_1)}{d(B_1, \mathcal{O}_1)}, \quad \Lambda_\lambda = \frac{d(C, \mathcal{O}_\lambda)}{d(B_\lambda, \mathcal{O}_\lambda)},$$

where B_λ *is the projection of* $\mathcal{O}_\lambda$ *to the side* $[B_0, B_1]$ *from* C. *Then we have*

$$\frac{1}{1+\Lambda_\lambda} = \frac{1-\lambda}{1+\Lambda_0} + \frac{\lambda}{1+\Lambda_1}.$$

Proof. $\mathcal{O}_0, \mathcal{O}_\lambda, \mathcal{O}_1$ and B_0, B_λ, B_1 are perspectively related with perspectivity centered at C. The corollary now follows from Lemma 2.

Proposition 5. *Let $C_0 \in \partial\mathcal{L}_0$, $0 \le t \le 1$, and define $C = (1-t)C_0 + t\mathcal{O}_1$. Then we have*

$$\frac{1}{1+\Lambda(C,\mathcal{O}_\lambda)} = \begin{cases} \frac{1-\lambda}{1-t}\frac{1}{1+\Lambda(C_0,\mathcal{O}_0)}, & \text{if } 0 \le t \le t^* \\ \frac{\lambda}{t}, & \text{if } t^* \le t \le 1 \end{cases}$$

where

$$t^* = 1 - \frac{1-\lambda}{1+\lambda\Lambda(C_0,\mathcal{O}_0)} \tag{19}$$

is defined by the condition that, for $t = t^*$, the opposite of C with respect to $\mathcal{O}_\lambda$ in on $\partial\mathcal{L}_0$.

Let C be as in the proposition. We call C a *type* I *point* if $0 < t < t^*$, a *type* II *point* if $t^* < t < 1$ and a *type* III *point* if $t = t^*$. In addition, we call a point $C \in \partial\mathcal{L}_0$ *type* 0. This corresponds to the parameter value $t = 0$.

Proof of Proposition 5. Let C^o be the opposite of C with respect to $\mathcal{O}_\lambda$ in $\mathcal{L}$, and C_0^o the opposite of C_0 with respect to $\mathcal{O}_0$ in $\mathcal{L}_0$. The critical value t^* is defined by $C^o = C_0^o$.

We first let $0 \le t \le t^*$. Since $C^o \in [C_0^o, \mathcal{O}_1]$, we have $C^o = (1-s)C_0^o + s\mathcal{O}_1$, for some $0 \le s \le 1$. Since $C, C^o, \mathcal{O}_\lambda$ are collinear, we have

$$\mathcal{O}_\lambda = (1-\mu)C + \mu C^o,$$

where $0 < \mu < 1$. Expanding, using $C_0^o = -C_0/\Lambda(C_0,\mathcal{O}_0)$, and comparing coefficients, we have

$$\begin{aligned} (1-\mu)t + \mu s &= \lambda \\ (1-\mu)(1-t) - \mu\frac{1-s}{\Lambda(C_0,\mathcal{O}_0)} &= 0. \end{aligned}$$

Eliminating s we obtain

$$\mu = \frac{(1-t)\Lambda(C_0,\mathcal{O}_0) + \lambda - t}{(1-t)(1+\Lambda(C_0,\mathcal{O}_0))}.$$

With this, we have

$$\Lambda(C,\mathcal{O}_\lambda) = \frac{\mu}{1-\mu} = \frac{1-t}{1-\lambda}\left(\Lambda(C_0,\mathcal{O}_0) + 1\right) - 1.$$

The proposition follows in this case. The formula for the critical value t^* also follows since this corresponds to $s = 0$.

Now let $t^* \le t \le 1$ and $C^* = (1-t^*)C_0 + t^*\mathcal{O}_1$ the corresponding point. The pencil of points $C^*, C, \mathcal{O}_1$ and $C_0^o, C^o, \mathcal{O}_0$ are perspectively related by the perspectivity with center at $\mathcal{O}_\lambda$. By Lemma 2, we have

$$\Lambda(C,\mathcal{O}_\lambda) = (1-\nu)\Lambda(C^*,\mathcal{O}_\lambda) + \nu\Lambda(\mathcal{O}_1,\mathcal{O}_\lambda),$$

where

$$C = (1-\nu)C^* + \nu\mathcal{O}_1.$$

Comparing this last equation with $C = (1-t)C_0 + t\mathcal{O}_1$ and $C^* = (1-t^*)C_0 + t^*\mathcal{O}_1$, and using the value of t^*, we obtain

$$\nu = 1 - \frac{1-t}{1-\lambda}(1 + \Lambda(C_0, \mathcal{O}_0)).$$

Substituting this into the formula of $\Lambda(C, \mathcal{O}_\lambda)$ above, the proposition follows.

We now return to the cone $\mathcal{L}$ with base $\mathcal{L}_0$. Let $C \in \partial\mathcal{L}_0$. Let C_0 and C_λ be the opposites of C with respect to $\mathcal{O}_0$ and $\mathcal{O}_\lambda$ in $\mathcal{L}_0$ and $\mathcal{L}$. In Corollary to Lemma 2 we set $B_0 = C_0$, $B_\lambda = C_\lambda$ and $B_1 = \mathcal{O}_1$ so that $\Lambda_1 = \infty$. We thus obtain

$$\frac{1}{1 + \Lambda_{\mathcal{L}}(C, \mathcal{O}_\lambda)} = \frac{1-\lambda}{1 + \Lambda_{\mathcal{L}_0}(C, \mathcal{O}_0)}, \quad C \in \partial\mathcal{L}_0. \tag{20}$$

Note that this proves (18). Indeed, by Proposition 3, the local maxima of $\Lambda(., \mathcal{O}_\lambda)$ are located at $\mathcal{O}_1$ or along $\partial\mathcal{L}_0$. Thus, (18) follows from (20) and from the obvious fact that $\lambda = 1/(1 + \Lambda(\mathcal{O}_1, \mathcal{O}_\lambda))$.

We now split the proof proof according to whether $\mathcal{O}_\lambda$ is regular or singular.

Proposition 6. *Let $\mathcal{L}$ be a cone with base $\mathcal{L}_0$ as above. Assume that $\mathcal{O}_\lambda \in \mathcal{R}$. Then $\mathcal{O}_0 \in \mathcal{R}_0$, where $\mathcal{R}_0$ is the regular set of $\mathcal{L}_0$, and*

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \lambda + (1-\lambda)\sigma(\mathcal{L}_0, \mathcal{O}_0). \tag{21}$$

Proof. Let $\{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, \mathcal{O}_\lambda)$ be minimal. By Proposition 1, $[C_0, \ldots, C_n]$ is an n-simplex with $\mathcal{O}_\lambda$ in its interior and, at each C_i, $\Lambda(., \mathcal{O}_\lambda)$ attains its local maximum. By Proposition 3, the possible local maxima are located at the vertex $\mathcal{O}_1$ of the cone $\mathcal{L}_0$ or along the boundary of the base $\mathcal{L}_0$. Thus, one of the points in the configuration must be $\mathcal{O}_1$.Without loss of generality, we may assume that $C_0 = \mathcal{O}_1$. It also follows that $C_1, \ldots, C_n \in \partial\mathcal{L}_0$. Since $\mathcal{O}_\lambda$ is in the interior of the n-simplex $[C_0, \ldots, C_n]$, the point $\mathcal{O}_0$ is in the interior of the $(n-1)$-simplex $[C_1, \ldots, C_n]$. We claim that the configuration $\{C_1, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}_0, \mathcal{O}_0)$ is minimal. Otherwise, let $\{C'_1, \ldots, C'_n\} \in \mathcal{C}(\mathcal{L}_0, \mathcal{O}_0)$ be a minimal configuration so that

$$\sum_{i=1}^n \frac{1}{1 + \Lambda(C'_i, \mathcal{O}_0)} < \sum_{i=1}^n \frac{1}{1 + \Lambda(C_i, \mathcal{O}_0)}.$$

By (20), we also have

$$\sum_{i=1}^n \frac{1}{1 + \Lambda(C'_i, \mathcal{O}_\lambda)} < \sum_{i=1}^n \frac{1}{1 + \Lambda(C_i, \mathcal{O}_\lambda)}.$$

Adding $\frac{1}{1+\Lambda(C_0,\mathcal{O}_\lambda)}$ to both sides, we obtain a contradiction to the minimality of $\{C_0, \ldots, C_n\}$. The claim follows.

Using (20), we have

$$\begin{aligned}\sigma(\mathcal{L},\mathcal{O}_\lambda) &= \sum_{i=0}^{n}\frac{1}{1+\Lambda(C_i,\mathcal{O}_\lambda)}\\ &= \frac{1}{1+\Lambda(C_0,\mathcal{O}_\lambda)}+\sum_{i=1}^{n}\frac{1}{1+\Lambda(C_i,\mathcal{O}_\lambda)}\\ &= \lambda+(1-\lambda)\sigma(\mathcal{L}_0,\mathcal{O}_0).\end{aligned}$$

Finally, working backwards, it is clear that every minimal configuration $\{C_1,\ldots,C_n\}\in\mathcal{C}(\mathcal{L}_0,\mathcal{O}_0)$ is an $(n-1)$-simplex with $\mathcal{O}_0$ in its interior as its extension by adjoining $C_0=\mathcal{O}_1$ is minimal. Thus, we have $\mathcal{O}_0\in\mathcal{R}_0$. The proposition follows.

In the remainder of this section we study the case when $\mathcal{O}_\lambda\in\mathcal{S}$ is a singular point in $\mathcal{L}$. Assume that the degree of singularity of $\mathcal{O}_\lambda$ is $n-m$, $m<n$. By Proposition 1, there is a minimal n-configuration $\{C_0,\ldots,C_n\}\in\mathcal{C}(\mathcal{L},\mathcal{O}_\lambda)$ which contains a simplicial minimal m-configuration $\{C_0,\ldots,C_m\}\in\mathcal{C}_m(\mathcal{L},\mathcal{O}_\lambda)$, m is the least number with this property, the point $\mathcal{O}_\lambda$ is in the relative interior of the m-simplex $[C_0,\ldots,C_m]$, and $\Lambda(.,\mathcal{O}_\lambda)$ attains its absolute maximum at $C_{m+1},\ldots,C_n$ in $\partial\mathcal{L}$. We can exclude the trivial case $m=1$ since then the configuration contains n points at which $\Lambda(.,\mathcal{O}_\lambda)$ attains its maximum and an additional antipodal point. Thus, from now on we assume that $m\geq 2$.

We have

$$\sigma(\mathcal{L},\mathcal{O}_\lambda)=\sigma_m(\mathcal{L},\mathcal{O}_\lambda)+\frac{n-m}{1+\max_{\partial\mathcal{L}}\Lambda(.,\mathcal{O}_\lambda)},\tag{22}$$

where

$$\sigma_m(\mathcal{L},\mathcal{O}_\lambda)=\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i,\mathcal{O}_\lambda)}.$$

Let $\mathcal{F}=\langle C_0,\ldots,C_m\rangle$; it is an affine subspace of dimension m. $\mathcal{F}$ is contained in the affine subspace $\langle\mathcal{F},\mathcal{O}_1\rangle$, properly (with codimension 1) iff $\mathcal{O}_1\notin\mathcal{F}$. The intersection $\mathcal{L}\cap\langle\mathcal{F},\mathcal{O}_1\rangle$ is a cone with base $\mathcal{L}_0\cap\langle\mathcal{F},\mathcal{O}_1\rangle\ni\mathcal{O}_0$ and vertex $\mathcal{O}_1$. It is clear that $\{C_0,\ldots,C_m\}$ is not only minimal in $\mathcal{L}$ but also in $\mathcal{L}\cap\langle\mathcal{F},\mathcal{O}_1\rangle$, in particular

$$\sigma_m(\mathcal{L}\cap\langle\mathcal{F},\mathcal{O}_1\rangle,\mathcal{O}_\lambda)=\sigma_m(\mathcal{L},\mathcal{O}_\lambda).$$

Case I. $\mathcal{O}_1\in\mathcal{F}$. We claim that $\mathcal{O}_\lambda$ is a regular point in $\mathcal{L}\cap\langle\mathcal{F},\mathcal{O}_1\rangle$. Assume, on the contrary, that there is a minimal m-configuration $\{C_0',\ldots,C_m'\}\in\mathcal{C}_m(\mathcal{L}\cap\langle\mathcal{F},\mathcal{O}_1\rangle,\mathcal{O}_\lambda)$ which contains a k-configuration with $k<m$. Since

$$\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i',\mathcal{O}_\lambda)}=\sigma_m(\mathcal{L}\cap\langle\mathcal{F},\mathcal{O}_1\rangle,\mathcal{O}_\lambda)=\sigma_m(\mathcal{L},\mathcal{O}_\lambda)$$

the extended configuration $\{C_0',\ldots,C_m',C_{m+1},\ldots,C_n\}\in\mathcal{C}_n(\mathcal{L},\mathcal{O}_\lambda)$ is minimal. By assumption, it contains a k-configuration. Thus, $\mathcal{O}_\lambda\in\mathcal{S}_{n-k}$ so that $k\geq m$. This is a contradiction.

We now apply Proposition 6 to the regular point $\mathcal{O}_\lambda$ of $\mathcal{L} \cap \langle \mathcal{F}, \mathcal{O}_1 \rangle$. We obtain

$$\sigma_m(\mathcal{L}, \mathcal{O}_\lambda) = \lambda + (1-\lambda)\sigma_{m-1}(\mathcal{L}_0, \mathcal{O}_0), \tag{23}$$

where $\sigma(\mathcal{L}_0 \cap \langle \mathcal{F}, \mathcal{O}_1 \rangle, \mathcal{O}_0) = \sigma_{m-1}(\mathcal{L}_0, \mathcal{O}_0)$. Combining (22) and (23), we obtain

$$\sigma(\mathcal{L}, \mathcal{O}_\lambda) = \lambda + (1-\lambda)\sigma_{m-1}(\mathcal{L}_0, \mathcal{O}_0) + \frac{n-m}{1 + \max_{\partial \mathcal{L}} \Lambda(., \mathcal{O}_\lambda)}.$$

By (7) this does not compete in the infimum in (17) unless equality holds in (7).
Case II. $\mathcal{O}_1 \notin \mathcal{F}$. We now study what type of points are possible in the simplicial minimal configuration $\{C_0, \dots, C_m\}$.
Let C_i and C_j be two distinct configuration points and assume that

$$C_i = (1-t_i)C_{i,0} + t_i \mathcal{O}_1 \quad \text{and} \quad C_j = (1-t_j)C_{j,0} + t_j \mathcal{O}_1,$$

where $C_{i,0}, C_{j,0} \in \partial \mathcal{L}_0$ and $0 < t_i, t_j < 1$. By definition, C_i, C_j can be type I, type II, or type III points.
We define a variation in which C_i and C_j move simultaneously along the line segments $[C_{i,0}, \mathcal{O}_1]$ and $[C_{j,0}, \mathcal{O}_1]$ while keeping the condition $\mathcal{O}_\lambda \in [C_0, \dots, C_m]$ intact. We first write

$$\sum_{k=0}^{m} \mu_k C_k = \mathcal{O}_\lambda,$$

where $\sum_{k=0}^m \mu_k = 1$ with $0 < \mu_k < 1$, $k = 0, \dots, m$. For s small, we define

$$\begin{aligned} C_i(s) &= \frac{1}{1+s} C_i + \frac{s}{1+s} \mathcal{O}_1 \\ C_j(s) &= \frac{\mu_j}{\mu_j - \mu_i s} C_j - \frac{\mu_i s}{\mu_j - \mu_i s} \mathcal{O}_1. \end{aligned}$$

Setting

$$\mu_i(s) = \mu_i(1+s) \quad \text{and} \quad \mu_j(s) = \mu_j - \mu_i s,$$

we have

$$\mu_i(s) C_i(s) + \mu_j(s) C_j(s) + \sum_{k=0,\, k \neq i,j}^{m} \mu_k C_k = \mathcal{O}_\lambda.$$

Thus, substituting $C_i(s)$ and $C_j(s)$ for C_i and C_j, the condition $\mathcal{O}_\lambda \in [C_0, \dots, C_m]$ remains in effect. The parameter values change as

$$t_i(s) = \frac{t_i + s}{1+s} \quad \text{and} \quad t_j(s) = \frac{\mu_j t_j - \mu_i s}{\mu_j - \mu_i s}.$$

Finally, we need to see how

$$\frac{1}{1 + \Lambda(C_i, \mathcal{O}_\lambda)} + \frac{1}{1 + \Lambda(C_j, \mathcal{O}_\lambda)}$$

changes under this substitution.

First, let C_i be type I and C_j type II or type III. We set $s < 0$ so that C_i stays type I and C_j becomes or stays type III. We have

$$\begin{aligned}
&\frac{1}{1+\Lambda(C_i(s),\mathcal{O}_\lambda)} + \frac{1}{1+\Lambda(C_j(s),\mathcal{O}_\lambda)} \\
&\frac{1-\lambda}{1-t_i(s)}\frac{1}{1+\Lambda(C_i(s),\mathcal{O}_\lambda)} + \frac{\lambda}{t_j(s)} \\
&\quad = \frac{1}{1+\Lambda(C_i,\mathcal{O}_\lambda)} + \frac{1}{1+\Lambda(C_j,\mathcal{O}_\lambda)} \\
&\quad + s\frac{1-\lambda}{1-t_i}\frac{1}{1+\Lambda(C_{i,0},\mathcal{O}_0)} \\
&\quad + \frac{\lambda}{1-\mu_j\frac{1-t_j}{\mu_j-\mu_i s}} - \frac{\lambda}{t_j}.
\end{aligned}$$

For $s < 0$ the last three terms are negative. This contradicts minimality. We conclude that if there is a type II or III point then there cannot be a type I point. Assume now that there are no type II and type III points. It is clear that there is at least one type I point since otherwise the condition $\mathcal{O}_\lambda \in [C_0, \ldots, C_m]$ could not be satisfied. We claim that actually there are at least two type I points. Suppose that there is only one type I point C_i. Then the line $\langle C_i, \mathcal{O}_\lambda \rangle$ intersects the i-th face of the simplex $[C_0, \ldots, C_m]$ opposite to C_i, and this face is entirely contained in $\mathcal{L}_0$. Based on the geometric meaning of type II and type III points in Proposition 5, C_i cannot be type I.

Now let C_i and C_j be both type I. Performing the variation as above, a brief computation shows that

$$\begin{aligned}
&\frac{1}{1+\Lambda(C_i(s),\mathcal{O}_\lambda)} + \frac{1}{1+\Lambda(C_j(s),\mathcal{O}_\lambda)} \\
&\quad = \frac{1}{1+\Lambda(C_i,\mathcal{O}_\lambda)} + \frac{1}{1+\Lambda(C_j,\mathcal{O}_\lambda)} \\
&\quad + s\left(\frac{1-\lambda}{1-t_i}\frac{1}{1+\Lambda(C_{i,0},\mathcal{O}_0)} - \frac{\mu_i}{\mu_j}\frac{1-\lambda}{1-t_j}\frac{1}{1+\Lambda(C_{j,0},\mathcal{O}_0)}\right).
\end{aligned}$$

The condition of minimality implies that the last term in the parentheses must be zero. This means that the sum

$$\frac{1}{1+\Lambda(C_i(s),\mathcal{O}_\lambda)} + \frac{1}{1+\Lambda(C_j(s),\mathcal{O}_\lambda)}$$

stays constant for all admissible s. Assume, for definiteness, that $s > 0$ increases. This means that $C_i(s)$ moves away from $C_{i,0}$ while $C_j(s)$ approaches $C_{j,0}$. The point $C_i(s)$ cannot hit the type III territory before $C_{j,0}$ hits $\partial\mathcal{L}_0$ since otherwise we would apply the previous variation and get a contradiction. Thus, at a critical value of s, $C_j(s) \in \partial\mathcal{L}_0$. Applying this to every type I point we will be left with only one type I point which again gives a contradiction.

Summarizing, we obtained that there cannot be any type I points among $\{C_0, \dots, C_m\}$. It also follows that there must be at least one type II or type III point.

We now claim that there are no configuration points in the relative interior of $\mathcal{L}_0 \cap \mathcal{F}$. Suppose that C_i is in the relative interior of $\mathcal{L}_0 \cap \mathcal{F}$. Then, by definition, C_i is an $(m-1)$-flat point in $\mathcal{L} \cap \langle \mathcal{F}, \mathcal{O}_1 \rangle$. By Proposition 3, the antipodal must be (at least an) m-flat (since it is on the boundary of a cone away from the base and the vertex), and the $(m-1)$-flat translates into the m-flat. We can thus move C_i within this flat to any point without changing the distortion. Note also that during the move the condition $\mathcal{O}_\lambda \in [C_0, \dots, C_m]$ stays intact due to the minimality of m. On the other hand, the antipodal of a type II or type III point is in this flat. Thus, we can move C_i to this antipodal and obtain an antipodal pair of points in the configuration. This means that $m = 1$, and this trivial case was excluded.

We now claim that there is no type II point and that there is a unique type III point.

Let $C = C_i$ be any type II or type III point with $C = (1-t)C_0 + t\mathcal{O}_1$, $C_0 \in \partial\mathcal{L}_0$ and $0 < t < 1$. (Here we suppress the index i and briefly revert to the notation of Proposition 5; in particular, C_0 is the point corresponding to C in $\partial\mathcal{L}_0$ not the first point in the configuration.) We assume that $\Lambda(., \mathcal{O}_\lambda)$ restricted to $\mathcal{F}$ attains a local maximum at C.

In what follows, we will work within the subspace $\langle \mathcal{F}, \mathcal{O}_1 \rangle$ in which $\mathcal{F}$ is a codimension one affine subspace. Let N be the unit normal vector to the hyperplane $\mathcal{F}$ pointing to the half-space that contains $\mathcal{O}_1$.

We first claim that the function on $\partial\mathcal{L}_0$ defined by $C' \mapsto \langle C', N \rangle$ attains a local minimum at C_0.

Central projection from $\mathcal{O}_1$ gives a one-to-one correspondence between a neighborhood of C in $\partial\mathcal{L} \cap \mathcal{F}$ and a neighborhood of C_0 in $\partial\mathcal{L}_0 \cap \langle \mathcal{F}, \mathcal{O}_1 \rangle$. Let $C' \in \partial\mathcal{L} \cap \mathcal{F}$ near C with corresponding projection C'_0. Setting $C' = (1-t')C'_0 + t'\mathcal{O}_1$ the condition $C' \in \mathcal{F}$ reduces to $\langle C' - \mathcal{O}_\lambda, N \rangle = 0$. Working this out in terms of t' we obtain

$$t' = 1 + \frac{1-\lambda}{\frac{\langle C'_0, N \rangle}{\langle \mathcal{O}_1, N \rangle} - 1}. \tag{24}$$

(Note that $\langle C'_0, N \rangle \neq 0$ as $\mathcal{O}_1 \notin \mathcal{F}$.) This formula holds for any type I, II, or III point C'.

First assume that C is of type II. Since C' is close to C, it is also a type II point. Thus

$$\frac{1}{1 + \Lambda(C', \mathcal{O}_\lambda)} = \frac{\lambda}{t'}.$$

Comparing these formulas and recalling the assumption on $\Lambda(., \mathcal{O}_\lambda)$ restricted to $\mathcal{F}$ we see that, keeping N constant, $C'_0 \mapsto \langle C'_0, N \rangle$ attains a local maximum at C_0. The claim follows in this case.

Now let C be a type III point. Since $t^* = t$, comparing (19) and (24) (for $t' = t$), we find

$$\Lambda(C_0, \mathcal{O}_0) = -\frac{1}{\lambda}\frac{\langle C_0, N\rangle}{\langle \mathcal{O}_1, N\rangle}.$$

Let $C' \in \partial\mathcal{L} \cap \mathcal{F}$ be near C. By assumption

$$\frac{1}{1+\Lambda(C, \mathcal{O}_\lambda)} \leq \frac{1}{1+\Lambda(C', \mathcal{O}_\lambda)}. \tag{25}$$

Since we have no information about the type of C' we split the study into two cases.

First assume that C' is a type I point. We rewrite (25) using Proposition 5 and (24) (twice) as

$$\left(1 - \frac{\langle C_0, N\rangle}{\langle \mathcal{O}_1, N\rangle}\right)\frac{1}{1+\Lambda(C_0, \mathcal{O}_0)} \leq \left(1 - \frac{\langle C_0', N\rangle}{\langle \mathcal{O}_1, N\rangle}\right)\frac{1}{1+\Lambda(C_0', \mathcal{O}_0)}. \tag{26}$$

In addition, since C' is a type I point, with obvious notation, we also have $t'^* > t'$. By (19) and (24), this gives

$$\frac{1}{1+\Lambda(C_0', \mathcal{O}_0)} < \frac{1}{1 - \frac{1}{\lambda}\frac{\langle C_0', N\rangle}{\langle \mathcal{O}_1, N\rangle}}.$$

Putting everything together in (26), we obtain

$$\left(1 - \frac{\langle C_0, N\rangle}{\langle \mathcal{O}_1, N\rangle}\right)\frac{1}{1 - \frac{1}{\lambda}\frac{\langle C_0, N\rangle}{\langle \mathcal{O}_1, N\rangle}} < \left(1 - \frac{\langle C_0', N\rangle}{\langle \mathcal{O}_1, N\rangle}\right)\frac{1}{1 - \frac{1}{\lambda}\frac{\langle C_0', N\rangle}{\langle \mathcal{O}_1, N\rangle}}.$$

This gives

$$\langle C_0, N\rangle < \langle C_0', N\rangle \tag{27}$$

as stated.

Assume now that C' is a type II point. By Proposition 5, we have $t \geq t'$ so that using (24), we again obtain (27). The claim follows. Note finally that since $\mathcal{L}_0 \cap \langle \mathcal{F}, \mathcal{O}_0\rangle$ is convex, the local minimum at C is actually global.

The above applies to all type II and type III configuration points. Let C_i be a type II or type III point. Let $\mathcal{H}_0 \subset \mathcal{E}_0 \cap \langle \mathcal{F}, \mathcal{O}_1\rangle$ be the hyperplane that contains $C_{i,0}$ and orthogonal to N. (The projection of N to $\mathcal{E}_0 \cap \langle \mathcal{F}, \mathcal{O}_1\rangle$ is nonzero since otherwise $\mathcal{F}$ were parallel to $\mathcal{E}_0 \cap \langle \mathcal{F}, \mathcal{O}_1\rangle$ and it would contain only type I configuration points.) Since $C' \mapsto \langle C', N\rangle$ has a minimum at $C_{i,0}$, $\mathcal{H}_0$ is a supporting hyperplane to $\mathcal{L}_0 \cap \langle \mathcal{F}, \mathcal{O}_1\rangle$. It follows that the projections (from $\mathcal{O}_1$ to $\mathcal{L}_0$) of all the type II and type III configuration points are on $\mathcal{H}_0$, and so is their convex hull. Thus the convex hull $\mathcal{C}$ of the type II and type III configuration points themselves is contained in $\partial\mathcal{L}$. Taking opposites, we see that the opposite convex hull is contained in $\mathcal{L}_0 \cap \langle \mathcal{F}, \mathcal{O}_1\rangle$. The distortion function Λ then must be linear on $\mathcal{C}$. Since Λ has a local maximum at every vertex, it must be constant on $\mathcal{C}$. We obtain that all type II and type III points can be compressed into a single point listed with multiplicity. This contradicts the minimality of m. Thus there is only one type II or type III configuration point. Finally, this cannot be a type II point since then its opposite must comprise the rest of the configuration points, and this is possible only for $m = 1$.

Summarizing, we see that in the configuration $\{C_0, \dots, C_m\}$ there is a unique type III point and the rest of the points are contained in $\partial\mathcal{L}_0$. We let C_0 denote the type III point so that $\{C_1, \dots, C_m\} \subset \partial\mathcal{L}_0$.
We now set

$$C_0 = (1 - t_0)C_{0,0} + t_0\mathcal{O}_1, \quad 0 < t_0 < 1,\ C_{0,0} \in \partial\mathcal{L}_0, \tag{28}$$

and consider the m-configuration $\{B_0, \dots, B_m\} \in \mathcal{S}^0_m(\mathcal{L}_0, \mathcal{O}_0)$, where $B_0 = C_{0,0}$ is the distinguished element and $B_i = C_i$, $i = 1, \dots, m$. We claim that

$$\sum_{i=0}^m \frac{1}{1 + \Lambda(C_i, \mathcal{O}_\lambda)} = \lambda + (1 - \lambda)\left[\frac{1}{1 + \Lambda_{[B_0,\dots,B_m]}(B_0, \mathcal{O}_0)} + \sum_{i=1}^m \frac{1}{1 + \Lambda_{\mathcal{L}_0}(B_i, \mathcal{O}_0)}\right]. \tag{29}$$

With this, it will follow that

$$\sigma_m(\mathcal{L}, \mathcal{O}_\lambda) \geq \lambda + (1 - \lambda)\tau_m(\mathcal{L}_0, \mathcal{O}_0), \tag{30}$$

and the infimum in $\tau_m(\mathcal{L}_0, \mathcal{O}_0)$ is attained by a simplicial configuration. We first note that, by (20), for $i = 1, \dots, m$, we have

$$\frac{1}{1 + \Lambda(C_i, \mathcal{O}_\lambda)} = \frac{1 - \lambda}{1 + \Lambda_{\mathcal{L}_0}(B_i, \mathcal{O}_\lambda)}. \tag{31}$$

Thus, to prove (29), using Proposition 5 and reverting to the C's, we need to show that

$$\frac{1}{1 + \Lambda_{\mathcal{L}}(C_0, \mathcal{O}_\lambda)} = \lambda + \frac{1 - \lambda}{1 + \Lambda_{[C_{0,0},\dots,C_m]}(C_{0,0}, \mathcal{O}_0)}. \tag{32}$$

We need to calculate the distortion at $C_{0,0}$ with respect to the simplex $[C_{0,0}, \dots, C_m]$. To do this, we first determine the intersection of the line through $C_{0,0}$ and $\mathcal{O}_0$ and the $(m-1)$-simplex $[C_1, \dots, C_m]$.
We have

$$\sum_{i=0}^m \mu_i C_i = \mathcal{O}_\lambda \quad \text{and} \quad \sum_{i=0}^m \mu_i = 1,\ 0 < \mu_i < 1.$$

Expanding, we obtain

$$\mu_0(1 - t_0)C_{0,0} + \sum_{i=1}^m \mu_i C_i = 0 \quad \text{and} \quad \mu_0 t_0 = \lambda,$$

where we used that $\mathcal{O}_0$ is the origin. Since $\sum_{i=1}^m \mu_i = 1 - \mu_0 = 1 - \lambda/t_0$, we have

$$-\frac{\lambda(1 - t_0)}{t_0 - \lambda} C_{0,0} = \sum_{i=1}^m \frac{\mu_i}{1 - \lambda/t_0} C_i \in [C_1, \dots, C_m].$$

Thus, we have

$$\Lambda_{[C_{0,0},\ldots,C_m]}(C_{0,0},\mathcal{O}_0) = \frac{t_0-\lambda}{\lambda(1-t_0)}.$$

With this, we obtain

$$\lambda + \frac{1-\lambda}{1+\Lambda_{[C_{0,0},\ldots,C_m]}(C_{0,0},\mathcal{O}_0)} = \frac{\lambda}{t_0}.$$

Now Proposition 6 gives (32).

We now show that equality holds in (30). Let $\{B_0,\ldots,B_m\} \in \mathcal{S}^0_m(\mathcal{L}_0,\mathcal{O}_0)$ be simplicial. We 'lift' this configuration to an m-configuration $\{C_0,\ldots,C_m\} \in \mathcal{C}_m(\mathcal{L},\mathcal{O}_\lambda)$ such that (29) holds with $C_{0,0} = B_0$ and $C_i = B_i$, $i = 1,\ldots,m$. To do this, we write $\sum_{i=0}^m \nu_i B_i = \mathcal{O}_0$ with $\sum_{i=0}^m \nu_i = 1$ and $0 < \nu_i < 1$. We define $\lambda < t_0 < 1$ such that $1/t_0 - 1 = \nu_0(1/\lambda - 1)$, and then $\mu_0 = \lambda/t_0$ and $\mu_i = (1-\lambda)\nu_i$, $i = 1,\ldots,m$. We finally let $C_0 = (1-t_0)B_0 + t_0\mathcal{O}_1$ and $C_i = B_i$, $i = 1,\ldots,m$. With these we have

$$\begin{aligned}\sum_{i=0}^m \mu_i C_i &= \frac{\lambda}{t_0}C_0 + (1-\lambda)\sum_{i=1}^m \nu_i C_i \\ &= \lambda\left(\frac{1}{t_0}-1\right) + \lambda\mathcal{O}_1 + (1-\lambda)(\mathcal{O}_0 - \nu_0 B_0) = \mathcal{O}_\lambda.\end{aligned}$$

In addition, we also have

$$\begin{aligned}\sum_{i=0}^m \mu_i &= \frac{\lambda}{t_0} + (1-\lambda)\sum_{i=1}^m \nu_i \\ &= \frac{\lambda}{t_0} + (1-\lambda)(1-\nu_0) = 1.\end{aligned}$$

These show that $\{C_0,\ldots,C_m\} \in \mathcal{C}_m(\mathcal{L},\mathcal{O}_\lambda)$. On the other hand, reversing the steps in the computation for (29) above, we obtain (29) again. With this, we have

$$\lambda + (1-\lambda)\left[\frac{1}{1+\Lambda_{[B_0,\ldots,B_m]}(B_0,\mathcal{O}_0)} + \sum_{i=1}^m \frac{1}{1+\Lambda_{\mathcal{L}_0}(B_i,\mathcal{O}_0)} \geq \sigma_m(\mathcal{L}_0,\mathcal{O}_0)\right].$$

Taking the infimum, we arrive at

$$\lambda + (1-\lambda)\tau_m(\mathcal{L}_0,\mathcal{O}_0) \geq \sigma_m(\mathcal{L}_0,\mathcal{O}_0).$$

Thus, equality holds in (30). Theorem F follows.

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