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Spherical Minimal Immersions with Prescribed Codimension

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Abstract. We describe a general construction of manufacturing new spherical minimal immersions between round spheres out of old ones. The new immersions have higher domain dimension and degree and the construction has a precise control on the codimension. Applied to classified and recent examples, the construction gives an abundance of new spherical minimal immersions with prescribed codimensions.

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1. Introduction and Statement of Results

Minimal isometric immersions of round spheres into round spheres are noteworthy from a geometric point of view as they provide a rich source of examples of immersed minimal submanifolds in spheres with high degree of symmetry [1–5, 8, 16, 19, 20]. Scaling the domain and range spheres to unit radius, a minimal immersion between round spheres can be viewed as a homothetic minimal immersion $f: S^m \to S_V$ into the unit sphere S_V of a Euclidean vector space V. We call f a *spherical minimal immersion*. The value of the homothety can run through only discrete values λ_p/m, $p \geqslant 1$, where $\lambda_p = p(p+m-1)$ is the pth eigenvalue of the Laplacian $\triangle^{S^m}$. In this case the components $\alpha \circ f, \alpha \in V^*$, are eigenfunctions of $\triangle^{S^m}$ corresponding to λ_p so that f is a *p-eigenmap*. We call p the *degree of f*. We denote by $\mathcal{H}_m^p$ the eigenspace of $\triangle^{S^m}$ corresponding to λ_p; this is the space of *spherical harmonics* of order p on S^m. If $V_f = \{\alpha \circ f \mid \alpha \in V^*\}$ denotes the *space of components* of f then $V_f \subset \mathcal{H}_m^p$ is the definition relation for a p-eigenmap f. Conversely, a conformal p-eigenmap $f: S^m \to S_V$ is automatically a spherical minimal immersion (with conformality λ_p/m). $m=2$ or $p \leqslant 3$ correspond to rigid ranges [1, 4, 14, 19], that is for these values, a spherical minimal immersion is given by the classical and generalized *Veronese maps*. For $m \geqslant 3$ and $p \geqslant 4$, however, there are infinitely many geometrically distinct spherical minimal immersions; in fact they fill a moduli space $\mathcal{M}_m^p$, a compact convex body in a finite-dimensional $\mathrm{SO}(m+1)$-module. The exact dimension of $\mathcal{M}_m^p$ has been determined in [10]. (For another proof, see

[20].) The dimension increases rapidly with m and p. The lowest (18-) dimensional moduli space $\mathcal{M}_3^4$ has been completely described in [16].

Little is known in higher domain dimensions m and higher degree p. Operators that associate to a given spherical minimal immersion other spherical minimal immersions of lower and higher degrees provided an important technical tool in calculating $\dim \mathcal{M}_m^p$ but the operator loses control on the range dimension. The *domain dimension raising operator* [14] does have a control on the range dimension but it does not change the degree. In view of this it is natural to ask if there are operators that associate to a given spherical minimal immersion new spherical minimal immersions with higher domain and range dimensions and precise control on the range dimension. As a generalization of domain dimension raising, the purpose of this paper is to construct such operators.

THEOREM A. *Let $f_\ell : S^m \to S_{V_\ell}, \ell = 0, \dots, N$, be p-eigenmaps and $\chi_\ell \in \mathcal{H}_{n-1}^q, \ell = 0, \dots, N$ orthogonal spherical harmonics suitably normalized to a common norm (depending on m,n,p,q). Then there exists a $(p+q)$-eigenmap $f^\chi = (f_0, \dots, f_N)^{\chi_0, \dots, \chi_N}$: $S^{m+n} \to S_V$ such that for the space of components we have*

$$\dim V_{f^\chi} = \sum_{\ell=0}^{N} V_{f_\ell} + \dim \mathcal{H}_{m+n}^{p+q} - (N+1) \dim \mathcal{H}_m^p. \tag{1}$$

If $f_\ell, \ell = 0, \dots, N$, are spherical minimal immersions of degree p then f^χ is a spherical minimal immersion of degree $p+q$. Finally, f^χ also inherits a common degree of isotropy of $f_\ell, \ell = 0, \dots, N$ (Section 2.3).

The proof of Theorem A can immediately be generalized to the case when p and q are both varying but $p+q$ stays constant. For notational simplicity, however, we kept p and q constant separately.

Let $f : S^m \to S_V$ be a p-eigenmap, that is, $V_f \subset \mathcal{H}_m^p$. We call the codimension of V_f in $\mathcal{H}_m^p$ the *complementary range dimension* of f, and denote it $c(f)$. With this, (1) can be written as

$$c(f^\chi) = \sum_{\ell=0}^{N} c(f_\ell). \tag{2}$$

The lowest range dimension for f^χ occurs when $N+1 = \dim \mathcal{H}_{n-1}^q$.

Several particular cases of Theorem A are of interest. For explicit examples, see Section 4. For $n = 1, \mathcal{H}_0^q$ is nontrivial iff $q = 0, 1$. In this case $N = 0$, $\dim \mathcal{H}_0^q = 1$ and $\chi_0 \in \mathcal{H}_0^q$ is unique (up to sign) due to the normalizing condition. Given a p-eigenmap $f_0 : S^m \to S_{V_0}$, for $q = 0$, the associated p-eigenmap $f^\chi : S^{m+1} \to S_V$ is given by the domain dimension raising operator applied to f_0 [14], and, for $q = 1$, we obtain a $(p+1)$-eigenmap $f^\chi : S^{m+1} \to S_V$. In both cases the complementary range dimensions are preserved: $c(f^\chi) = c(f)$.

For $n=2$ and $q \geqslant 1$, we have dim $\mathcal{H}_1^q = 2$, and the spherical harmonics in $\mathcal{H}_1^q$ are restrictions of linear combinations of $\Re(z^p)$ and $\Im(z^p)$ of a complex variable $z \in \mathbf{C}$. Given p-eigenmaps $f_0 : S^m \to S_{V_0}$, $f_1 : S^m \to S_{V_1}$ and $\chi_0 = \Re(z^q)$, $\chi_1 = \Im(z^q)$ suitably normalized then the associated $(p+q)$-eigenmap f^χ: $S^{m+2} \to S_V$ satisfies $c(f^\chi) = c(f_0) + c(f_1)$.

As noted above, the structure of $\mathcal{M}_3^4$ has been completely described in [16]. The possible complementary range dimensions of quartic minimal immersions $f : S^3 \to S_V$ are $c(f) = 0-6, 9-10, 15$. (f is SU(2)-or SU(2)$'$-equivariant iff dim V_f is divisible by 5.) Combining this with the discussion above we obtain the following:

COROLLARY. *There exist spherical minimal immersions* f: $S^{n+3} \to S_V$ *of degree* $q+4$ *with complementary range dimensions* $0 \leqslant c(f) \leqslant 15(\dim \mathcal{H}_{n+1}^q - 1) + 6$ *and* $c(f) = 15(\dim \mathcal{H}_{n-1}^q - 1) + k, k = 9, 10, 15,$ *provided that* $\dim \mathcal{H}_{n-1}^q \geqslant 1$, *i.e.* $n \geqslant 2$ *or, for* $n=1$, *we have* $q \leqslant 1$.

It is a difficult and largely unsolved problem to give suitable lower and upper bounds for the (complementary) range dimension of spherical minimal immersions $f : S^m \to S_V$. In 1976, J.D. Moore [8] gave the lower bound $2m+1 \leqslant \dim V_f$ $(\leqslant V)$. The tetrahedral minimal immersion Tet: $S^3 \to S^6$ [2, 3, 14, 16] shows that this lower bound is sharp. Using a technique of moduli spaces, the author gave various lower bounds depending on both the domain dimension and the degree [12, 13]. For eigenmaps many partial results exist [6, 14, 17, 21].

The construction of f^χ can also be used to obtain an insight of the structure of the respective moduli spaces as follows:

THEOREM B. *We have the isometry*

$$\prod_{q=0}^{r} \left(\mathcal{M}_m^{r-q}\right)^{\dim \mathcal{H}_{n-1}^q} \cong \mathcal{M}_{m+n}^r \cap \bigoplus_{q=0}^{r} S_0^2\left(\mathcal{H}_m^{r-q} \cdot \mathcal{H}_{n-1}^q\right), \tag{3}$$

where $\mathcal{H}_m^{r-q} \cdot \mathcal{H}_{n-1}^q$ *is the linear subspace of* $\mathcal{H}_{m+n}^r$ *consisting of finite sums of products of spherical harmonics in* $\mathcal{H}_m^{r-q}$ *and* $\mathcal{H}_{n-1}^q$, *and, for a Euclidean vector space* $\mathcal{H}$, $S_0^2(\mathcal{H})$ *is the space of traceless symmetric endomorphisms of* $\mathcal{H}$.

Once again two particular cases are of interest. For $n=1$, (3) reduces to

$$\mathcal{M}_m^r \times \mathcal{M}_m^{r-1} \cong \mathcal{M}_{m+1}^r \cap S_0^2\left(\mathcal{H}_m^r\right) \oplus S_0^2\left(\mathcal{H}_m^{r-1} \cdot \mathcal{H}_0^1\right),$$

and, for $n=2$, we have

$$\prod_{q=0}^{r} \left(\mathcal{M}_m^{r-q}\right)^2 \cong \mathcal{M}_{m+2}^r \cap \bigoplus_{q=0}^{r} S_0^2\left(\mathcal{H}_m^{r-q} \cdot \mathcal{H}_1^q\right).$$

Remark. The treatment of the case $n = 1$ can be extended to prove that $\prod_{q=4}^{r} \mathcal{M}_m^q$, $r \geqslant 4$, is the intersection of $\mathcal{M}_{m+1}^r$ with a certain linear subspace of $S_0^2(\mathcal{H}_{m+1}^r)$ (see [11]).

2. Preliminaries

2.1. SPHERICAL HARMONICS

Consider the ring of polynomials $\mathbf{R}[x] = \mathbf{R}[x_0, \ldots, x_m]$ with real coefficients in $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$. Precomposing polynomials with linear transformations of $\mathbf{R}^{m+1}$ gives rise to a $\mathrm{GL}(m+1, \mathbf{R})$-module structure on $\mathbf{R}[x]$. In addition, $\mathbf{R}[x]$ is graded by the degree and the grading is preserved by this action. We denote by $\mathbf{R}[x]^p$ the $\mathrm{GL}(m+1, \mathbf{R})$-submodule of $\mathbf{R}[x]$ consisting of homogeneous polynomials of degree p. The Laplacian

$$\triangle_x = \sum_{i=0}^{m} \frac{\partial^2}{\partial x_i^2}$$

gives the decomposition

$$\mathbf{R}[x]^p = \mathcal{H}[x]^p \oplus \mathbf{R}[x]^{p-2} \cdot |x|^2,$$

where the kernel $\mathcal{H}[x]^p$ is the space of harmonic homogeneous polynomials of degree p in $x \in \mathbf{R}^{m+1}$. This decomposition is orthogonal with respect to the L^2-scalar product (defined by integration over $S^m \subset \mathbf{R}^{m+1}$). Comparison of $\triangle_x$ and the spherical Laplacian $\triangle^{S^m}$ shows that the restrictions of the polynomials in $\mathcal{H}[x]^p$ to S^m are precisely the spherical harmonics of order p on S^m, the eigenfunctions of $\triangle^{S^m}$ corresponding to the pth eigenvalue $\lambda_p = p(p+m-1)$. Suppressing the variable x, we denote this eigenspace by $\mathcal{H}_m^p$. We will identify $\mathcal{H}[x]^p$ and $\mathcal{H}_m^p$ (by restriction or extension); for example, a spherical harmonic will also be viewed as a harmonic homogeneous polynomial of degree p on $\mathbf{R}^{m+1}$. Since the Laplacian in invariant under orthogonal transformations, $\mathcal{H}_m^p$ is an $\mathrm{SO}(m+1)$-submodule of $\mathbf{R}[x]^p$.

We now consider the ring of polynomials

$$\mathbf{R}[x, y] = \mathbf{R}[x_0, \ldots, x_m, y_1, \ldots, y_n] \cong \mathbf{R}[x] \otimes \mathbf{R}[y] = \mathbf{R}[x_0, \ldots, x_m] \otimes \mathbf{R}[y_1, \ldots, y_n]$$

with real coefficients in the variables $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$ and $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. The isomorphism is given by multiplication $\xi \otimes \chi \mapsto \xi \cdot \chi$, $\xi \in \mathbf{R}[x]$ and $\chi \in \mathbf{R}[y]$.$\mathbf{R}[x, y]$ is also a $\mathrm{GL}(m+1, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R})$-module in a natural way. In addition, $\mathbf{R}[x, y]$ is bigraded by the bidegree, and the bigrading is preserved by this action. We denote by $\mathbf{R}[x, y]^{p,q}$ the $\mathrm{GL}(m+1, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R})$-submodule of $\mathbf{R}[x, y]$ of polynomials that are homogeneous of degree p in x and homogeneous of degree q in y. Clearly, $\mathbf{R}[x, y]^{p,0} = \mathbf{R}[x]^p$ and $\mathbf{R}[x, y]^{0,q} = \mathbf{R}[y]^q$. We also have

$$\mathbf{R}[x]^p \otimes \mathbf{R}[y]^q \cong \mathbf{R}[x, y]^{p,q} \subset \mathbf{R}[x, y]^{p+q},$$

where the isomorphism is again given by multiplication.

We consider the Laplacians

$$\triangle_x = \sum_{i=0}^{m} \frac{\partial^2}{\partial x_i^2} \quad \text{and} \quad \triangle_y = \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}.$$

The joint kernel

$$\mathcal{H}[x, y]^{p,q} = \ker(\triangle_x | \mathbf{R}[x, y]^{p,q}) \cap \ker(\triangle_y | \mathbf{R}[x, y]^{p,q})$$

is an $\mathrm{SO}(m+1) \times \mathrm{SO}(n)$-module in a natural way. Clearly, $\mathcal{H}[x, y]^{p,0} = \mathcal{H}[x]^p$ and $\mathcal{H}[x, y]^{0,q} = \mathcal{H}[y]^q$. We have

$$\mathcal{H}[x]^p \otimes \mathcal{H}[y]^q \cong \mathcal{H}[x, y]^{p,q} \subset \mathcal{H}[x, y]^{p+q}.$$

As before, restricting to the respective spheres (and suppressing the variables x and y), we have $\mathcal{H}[x]^p = \mathcal{H}_m^p$, $\mathcal{H}[y]^q = \mathcal{H}_{n-1}^q$, $\mathcal{H}[x, y]^{p+q} = \mathcal{H}_{m+n}^{p+q}$, and we obtain the $\mathrm{SO}(m+1) \times \mathrm{SO}(n)$-submodule

$$\mathcal{H}_m^p \otimes \mathcal{H}_{n-1}^q \cong \mathcal{H}_m^p \cdot \mathcal{H}_{n-1}^q \subset \mathcal{H}_{m+n}^{p+q},$$

where the isomorphism is given by multiplication and $\mathcal{H}_m^p \cdot \mathcal{H}_{n-1}^q$ consists of finite sums of products of spherical harmonics in $\mathcal{H}_m^p$ and $\mathcal{H}_{n-1}^q$. In particular, for any $0 \neq \chi \in \mathcal{H}_{n-1}^q$, $\mathcal{H}_m^p \cdot \chi$ is a linear subspace of $\mathcal{H}_{m+n}^{p+q}$.

Finally, varying p and q, we get the direct sum

$$\sum_{q=0}^{r} \mathcal{H}_m^{r-q} \otimes \mathcal{H}_{n-1}^q \cong \sum_{q=0}^{r} \mathcal{H}_m^{r-q} \cdot \mathcal{H}_{n-1}^q \subset \mathcal{H}_{m+n}^r \tag{4}$$

as an $\mathrm{SO}(m+1) \times \mathrm{SO}(n)$-submodule. The second sum is orthogonal (Corollary in Section 2.4).

2.2. EIGENMAPS

Recall that a map $f\colon S^m \to S_V$ into the unit sphere S_V of a Euclidean vector space V is said to be a p-eigenmap if the space of components $V_f = \{\alpha \circ f \mid \alpha \in V^*\}$ is contained in $\mathcal{H}_m^p$. Any p-eigenmap can thus be viewed as a harmonic homogeneous polynomial map $f\colon \mathbf{R}^{m+1} \to V$ of degree p. f is said to be *full* if it has no nonzero component, that is $\alpha \circ f \neq 0$ if $\alpha \neq 0$. Restricting f to the linear span of its image, it becomes full. For full f, we have $V \cong V^* \cong V_f$.

Two full p-eigenmaps $f_1\colon S^m \to S_{V_1}$ and $f_2\colon S^m \to S_{V_2}$ are said to be *congruent* if there exists an isometry $U\colon V_1 \to V_2$ such that $f_2 = U \circ f_1$.

We endow $\mathcal{H}_m^p$ with the scaled L^2-scalar product

$$\langle \xi, \xi' \rangle = \frac{\dim \mathcal{H}_m^p}{\mathrm{vol}(S^m)} \int_{S^m} \xi \xi' v_{S^m}, \quad \xi, \xi' \in \mathcal{H}_m^p,$$

where

$$\dim \mathcal{H}_m^p = (2p + m - 1) \frac{(p+m-2)!}{p!(m-1)!},$$

v_{S^m} is the volume form of S^m and

$$\mathrm{vol}(S^m) = \int_{S^m} v_{S^m} = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}$$

is the volume of S^m.

The *standard p-eigenmap* $f_{m,p} : S^m \to S_{(\mathcal{H}_m^p)^*}$ is the Dirac delta defined by evaluating spherical harmonics at points of S^m. With respect to an orthonormal basis $\{f_{m,p}^j\}_{j=0}^{N(p,m)} \subset \mathcal{H}_m^p$, $\dim \mathcal{H}_m^p = N(m, p) + 1$, that identifies $\mathcal{H}_m^p$ and $(\mathcal{H}_m^p)^*$, we have

$$f_{m,p}(x) = \sum_{j=0}^{N(m,p)} f_{m,p}^j(x) f_{m,p}^j.$$

Clearly, $f_{m,p}$ is full since $V_{f_{m,p}} = \mathcal{H}_m^p$.

The complementary range dimension of a p-eigenmap $f\colon S^m \to S_V$ is defined as $c(f) = \dim \mathcal{H}_m^p - \dim V_f$. It is clear that $c(f_{m,p}) = 0$. For f full, we have $c(f) = \dim \mathcal{H}_m^p - \dim V$.

Let $f : S^m \to S_V$ be a p-eigenmap. By construction of $f_{m,p}$, there is a (unique) linear map $A : \mathcal{H}_m^p \to V$ such that $f = A \circ f_{m,p}$. f is full iff A onto.

We associate to f the symmetric linear endomorphism $\langle f \rangle = A^\top A - I \in S^2(\mathcal{H}_m^p)$ of $\mathcal{H}_m^p$. It follows that $\langle f \rangle$ is traceless and it depends only on the congruence class of f. (See Lemma 2.3.2 in [14], p. 113.) For the complementary range dimension of a full p-eigenmap $f : S^m \to S_V$ we have $c(f) = \mathrm{corank}(\langle f \rangle + I)$.

The map $f \mapsto \langle f \rangle$ then gives rise to a parametrization of the space of congruence classes of a full p-eigenmaps $f\colon S^m \to S_V$ (for various V). The range of the parametrization is $S_0^2(\mathcal{H}_m^p)$, the space of traceless symmetric endomorphisms of $\mathcal{H}_m^p$. Since f maps to the unit sphere, $\langle f \rangle$ is contained in the linear subspace

$$\mathcal{E}_m^p = \{f_{m,p}(x) \odot f_{m,p}(x) \mid x \in S^m\}^\perp \subset S^2(\mathcal{H}_m^p),$$

where $\odot$ denotes the symmetric tensor product and the orthogonal complement is taken with respect to the natural scalar product

$$\langle C, C' \rangle = \mathrm{trace}(CC'), \quad C, C' \in S^2(\mathcal{H}_m^p).$$

(See Theorem 2.3.1 in [14], p. 111.) Since $A^\top A$ is always positive semi-definite, the image of the parametrization is contained in the set

$$\mathcal{L}_m^p = \{C \in \mathcal{E}_m^p \mid C + I \geqslant 0\}.$$

It turns out that the image is the entire $\mathcal{L}_m^p$. Clearly $\mathcal{L}_m^p$ is a convex body in $\mathcal{E}_m^p$, and it is also compact as the eigenvalues of the endomorphisms in $\mathcal{L}_m^p$ are bounded. For more details, see [14]. $\mathcal{L}_m^p$ is called the *standard moduli space* for eigenmaps.

2.3. SPHERICAL MINIMAL IMMERSIONS AND ISOTROPY

Recall that a spherical minimal immersion of degree p is a homothetic minimal immersion $f: S^m \to S_V$ with homothety λ_p/m. The condition of homothety is

$$\langle f_*(X), f_*(Y)\rangle = \frac{\lambda_p}{m}\langle X, Y\rangle,$$

for any vector fields X, Y on S^m.

Since S^m is isotropy irreducible, the standard p-eigenmap $f_{m,p}: S^m \to S_{\mathcal{H}_m^p}$ is a spherical minimal immersion.

As noted above a homothetic minimal immersion f: $S^m \to S_V$ is automatically a p-eigenmap. The construction of the moduli space above carries over to spherical minimal immersions. We obtain that the space of congruence classes of spherical minimal immersions. $f: S^m \to S_V$ of degree p can be parametrized by a compact convex body $\mathcal{M}_m^p$ in a linear subspace $\mathcal{F}_m^p \subset \mathcal{E}_m^p$, where

$$\mathcal{F}_m^p = \{(f_{m,p})_*(X) \odot (f_{m,p})_*(Y) \mid X, Y \in T(S^m)\}^\perp$$

and

$$\mathcal{M}_m^p = \mathcal{L}_m^p \cap \mathcal{F}_m^p = \{C \in \mathcal{F}_m^p \mid C + I \geqslant 0\}.$$

$\mathcal{M}_m^p$ is called the *standard moduli space* for spherical minimal immersions.

Let $f: S^m \to S_V$ be a spherical minimal immersion of degree p. We denote by $\beta_k(f)$ and $\mathcal{O}_f^k, k \leqslant p$, the (densely defined) kth *fundamental form* and the kth *osculating bundle* of f. f is said to be *isotropic of order* $k, 2 \leqslant k \leqslant p$, if, for $2 \leqslant l \leqslant k$, we have

$$\begin{aligned}&\langle \beta_l(f)(X_1, \ldots, X_l), \beta_l(f)(X_{l+1}, \ldots, X_{2l})\rangle \\ &\quad = \langle \beta_l(f_{m,p})(X_1, \ldots, X_l), \beta_l(f_{m,p})(X_{l+1}, \ldots, X_{2l})\rangle,\end{aligned}$$

where $X_1, \ldots, X_{2l}$ are vector fields on S^m [14]. This condition implies that the osculating bundles $\mathcal{O}_f^l$ and $\mathcal{O}_{f_{m,p}}^l$ are isomorphic for $2 \leqslant l \leqslant k$.

If $p \leqslant 2k+1$ then an isotropic minimal immersion of order k is standard. For $p \geqslant 2(k+1)$, the space of congruence classes of full isotropic minimal immersions of order k can be parametrized by the intersection

$$\mathcal{M}_m^{p;k} = \mathcal{M}_m^p \cap \mathcal{F}_m^{p;k} = \{C \in \mathcal{F}_m^{p;k} \mid C + I \geqslant 0\},$$

where $\mathcal{F}_m^{p;k} \subset \mathcal{F}_m^p$ is a linear subspace. The dimension $\mathcal{M}_m^{p;k}$ has been calculated in [14]. (See also [20].)

2.4. AN INTEGRAL FORMULA

PROPOSITION. *Let*

$$\xi \in \mathbf{R}[x]^p, \quad x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}, \quad \text{and} \quad \chi \in \mathbf{R}[y]^q, \;\; y = (y_1, \ldots, y_n) \in \mathbf{R}^n.$$

Then we have

$$\int_{S^{m+m}} \xi \chi v_{S^{m+n}} = \frac{1}{2} \beta \left(\frac{p+m+1}{2}, \frac{q+n}{2} \right) \int_{S^m} \xi v_{S^m} \int_{S^{n-1}} \chi v_{S^{n-1}}, \tag{5}$$

where the β-function is given by

$$\beta(a,b) = 2 \int_0^{\pi/2} \sin^{2a-1} \phi \cos^{2b-1} \phi \, \mathrm{d}\phi.$$

Proof. Consider the map γ: $[0, \pi/2] \times \mathbf{R}^{m+1} \times \mathbf{R}^n \to \mathbf{R}^{m+n+1}$ defined by

$$\gamma(\phi, x, y) = \sin \phi \cdot x + \cos \phi \cdot y, \quad x \in \mathbf{R}^{m+1}, \quad y \in \mathbf{R}^n.$$

We denote the restriction γ: $[0, \pi/2] \times S^m \times S^{n-1} \to S^{m+n}$ by the same symbol. Clearly, γ is a diffeomeorphism between $(0, \pi/2) \times S^m \times S^{n-1}$ and S^{m+n} with the great spheres $S^m \times \{0\}$ and $\{0\} \times S^{n-1}$ deleted. Transforming the integral on the left-hand side of (5) by γ, and using homogeneity, we obtain

$$\int_{S^{m+m}} \xi \chi v_{S^{m+n}} = \int_0^{\pi/2} \sin^p \phi \cos^q \phi \int_{S^m} \xi \int_{S^{n-1}} \chi \, |\operatorname{Jac}(\gamma)| v_{S^{n-1}} v_{S^m} \, \mathrm{d}\phi.$$

It remains to calculate the determinant of the Jacobian of γ at a point $(\phi, x, y) \in (0, \pi/2) \times S^m \times S^{n-1}$. To do this, we first calculate the Jacobian of γ as a map $(0, \pi/2) \times \mathbf{R}^{m+1} \times \mathbf{R}^n \to \mathbf{R}^{m+n+1}$ and then restrict it to $\mathbf{R} \times T_x(S^m) \times T_y(S^{n-1})$. Taking partial derivatives, we obtain

$$\operatorname{Jac}(\gamma)(\phi, x, y) = \begin{bmatrix} \cos \phi \cdot x & \sin \phi I_{m+1} & 0 \\ -\sin \phi \cdot y & 0 & \cos \phi I_n \end{bmatrix}.$$

Here $x \in \mathbf{R}^{m+1}$ and $y \in \mathbf{R}^n$ are column vectors and the dimension of the identity matrix is indicated by a subscript. We now evaluate this on $(t, u, v) \in \mathbf{R} \times T_x(S^m) \times$

$T_y(S^{n-1})$, where $u \in T_x(S^m)$ is viewed as a vector in $\mathbf{R}^{m+1}$ with $\langle x, u\rangle = 0$, and $v \in T_y(S^{n-1})$ as a vector in $\mathbf{R}^n$ with $\langle y, v\rangle = 0$. We obtain

$$\mathrm{Jac}(\gamma)(\phi, x, y)(t, u, v) = (\cos\phi \cdot x - \sin\phi \cdot y)t + \sin\phi \cdot u + \cos\phi \cdot v \in T_{\gamma(\phi,x,y)}(S^{m+n}).$$

An orthonormal basis

$$(1, 0, 0), (0, u_1, 0), \ldots, (0, u_m, 0), (0, 0, v_1), \ldots, (0, 0, v_{n-1})$$

is mapped by the Jacobian to the *orthogonal basis*

$$\cos\phi \cdot x - \sin\phi \cdot y, \sin\phi \cdot u_1, \ldots, \sin\phi \cdot u_m, \cos\phi \cdot v_1, \ldots, \cos\phi \cdot v_{n-1}.$$

Thus the determinant is

$$|\mathrm{Jac}(\gamma)(\phi, x, y)| = \pm \sin^m \phi \cos^{n-1} \phi.$$

The integral formula (5) follows (since the sign is positive by inspection). □

COROLLARY. *If $\chi \in \mathcal{H}^q_{n-1}$ and $\chi' \in \mathcal{H}^{q'}_{n-1}$ are orthogonal spherical harmonics then the linear subspaces $\mathcal{H}^{r-q}_m \cdot \chi$ and $\mathcal{H}^{r-q'}_m \cdot \chi'$ are orthogonal in $\mathcal{H}^r_{m+n}$. This holds, in particular, if $q \neq q'$, so that the sum $\sum_{q=0}^r \mathcal{H}^{r-q}_m \cdot \mathcal{H}^{r-q}_{n-1}$ in* (4) *is orthogonal.*

2.5. CONSTRUCTION OF f^χ

A spherical harmonic $\chi \in \mathcal{H}^q_{n-1}$ is said to be *normalized* if

$$|\chi|^2 = \nu(m, n, p, q),$$

where

$$\nu(m, n, p, q) = \frac{\beta\left(\frac{m+1}{2}, \frac{n}{2}\right)}{\beta\left(p + \frac{m+1}{2}, q + \frac{n}{2}\right)} \frac{\dim \mathcal{H}^p_m \dim \mathcal{H}^q_{n-1}}{\dim \mathcal{H}^{p+q}_{m+n}}. \tag{6}$$

For $\ell = 0, \ldots, N$, $N \leqslant N(n-1, q)$, let $f_\ell : S^m \to S_{V_\ell}$ be p-eigenmaps, and $\chi_\ell \in \mathcal{H}^q_{n-1}$ mutually orthogonal normalized spherical harmonics. Without loss of generality we may assume that f_ℓ, $\ell = 0, \ldots, N$, are full. We define the map

$$f^\chi = (f_0, \ldots, f_N)^{\chi_0, \ldots, \chi_N} : \mathbf{R}^{m+n} \to V_\chi$$

as follows

$$f^\chi(x, y) = (f_0(x)\chi_0(y), \ldots, f_N(x)\chi_N(y), \pi_\chi(f_{m+n, p+q}(x, y))), \quad (x, y) \in \mathbf{R}^{m+n} \tag{7}$$

where π_χ is the orthogonal projection in $\mathcal{H}^{p+q}_{m+n}$ to the linear subspace $\left(\sum_{\ell=0}^N \mathcal{H}^p_m \cdot \chi_\ell\right)^\perp$. We thus have

$$V_\chi = \sum_{\ell=0}^N V_\ell \oplus \left(\sum_{\ell=0}^N \mathcal{H}^p_m \cdot \chi_\ell\right)^\perp.$$

It is clear that f^χ is a harmonic polynomial map of degree $p+q$.

Remark. For $n=1$ and $q=0, \mathcal{H}^0_0 = \mathbf{R}$. Setting $N=0$, we recover the domain dimension raising operator in [14]. Note that $v(m,1,p,0)$ is the normalizing constant $c_{m,p,p}$ in (2.8.6) of [14], p. 150. (Note that the right-hand side of (2.8.6) gives $c^2_{m,p,q}$ rather than $c_{m,p,q}$.)

3. Proofs

We begin with the proof of Theorem A. We first show that f^χ is spherical in the sense that it maps S^{m+n} to S_{V_χ}. It will then follow that the restriction is a $(p+q)$-eigenmap $f^\chi\colon\ S^{m+n} \to S_{V_\chi}$.

We first consider the case when $f_\ell = f_{m,p}\colon\ S^m \to S_{(\mathcal{H}^p_m)^*}$, is the standard p-eigenmap for each $\ell = 0,\ldots,N$. We fix and orthonormal basis $\{f^j_{m,p}\}_{j=0}^{N(m,p)} \subset \mathcal{H}^p_m$ whose elements constitute the components of $f_{m,p}$. In the integral formula (5) we set $\xi = (f^i_{m,p})^2$ and $\chi = \chi^2_\ell$. Using (6), we calculate

$$\begin{aligned}
|f^j_{m,p}\chi\ell|^2 &= \frac{1}{2}\beta\left(p+\frac{m+1}{2}, q+\frac{n}{2}\right)\frac{\dim \mathcal{H}^{p+q}_{m+n}}{\mathrm{vol}(S^{m+n})}\int_{S^m}\left(f^j_{m,p}\right)^2 v_{S^m}\int_{S^{n-1}}\chi^2_\ell v_{S^{n-1}} \\
&= \frac{1}{2}\beta\left(p+\frac{m+1}{2}, q+\frac{n}{2}\right)\frac{\dim \mathcal{H}^{p+q}_{m+n}}{\dim \mathcal{H}^p_m \dim \mathcal{H}^q_{n-1}}\frac{\mathrm{vol}(S^m)\,\mathrm{vol}(S^{n-1})}{\mathrm{vol}(S^{m+n})}|\chi_\ell|^2 \\
&= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n}{2}\right)\frac{\mathrm{vol}(S^m)\,\mathrm{vol}(S^{n-1})}{\mathrm{vol}(S^{m+n})} = 1,
\end{aligned}$$

where the last equality follows from the volume formula for the sphere along with the identity

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

(For the last step we can also use the integral formula (5) for $p=q=0$ and $\xi = 1, \chi = 1$.)

The calculation above and the integral formula (5) shows that the polynomials $f^j_{m,p}\chi_\ell \subset \mathcal{H}^{p+q}_{m+n}$, $j=0,\ldots,N(m,p)$, $\ell=0,\ldots,N$, form an orthonormal basis in the linear subspace $\sum_{\ell=0}^N \mathcal{H}^p_m \cdot \chi_\ell$. We can extend this to an orthonormal basis to the entire $\mathcal{H}^{p+q}_{m+n}$ by adjoining the elements of an orthonormal basis in the orthogonal complement $(\sum_{\ell=0}^N \mathcal{H}^p_m \cdot \chi_\ell)^\perp$. The elements of the extended basis can be used as components of the standard $(p+q)$-eigenmap $f_{m+n,p+q}: S^{m+n} \to S_{\mathcal{H}^{p+q}_{m+n}}$. Therefore, we have

$$\sum_{\ell=0}^{N} |f_{m,p}(x)\chi_\ell(y)|^2 + |\pi_\chi(f_{m+n,p+q}(x,y))|^2$$
$$= |f_{m+n,p+q}(x,y)|^2 = (|x|^2+|y|^2)^{p+q},$$

where π_χ is the orthogonal projection in $\mathcal{H}^{p+q}_{m+n}$ with kernel $\sum_{\ell=0}^{N} \mathcal{H}^p_m \cdot \chi_\ell$, and the last equality is because $f_{m+n,p+q}$ is spherical. Since $f_{m,p}$ is also spherical, the first term can be written as

$$\sum_{\ell=0}^{N} |f_{m,p}(x)\chi_\ell(y)|^2 = |x|^{2p} \sum_{\ell=0}^{N} \chi_\ell(y)^2.$$

We now replace the $N+1$ copies of $f_{m,p}$ by $f_0, \dots, f_N$. Since f_ℓ is spherical, we have $|f_\ell(x)|^2 = |x|^{2p}$ as a homogeneous polynomials of degree $2p$. Hence, we have

$$\sum_{\ell=0}^{N} |f_\ell(x)\chi_\ell(y)|^2 = |x|^{2p} \sum_{\ell=0}^{N} \chi_\ell(y)^2.$$

The computation just carried out gives

$$\sum_{\ell=0}^{N} |f_\ell(x)\chi_\ell(y)|^2 + |\pi_\chi(f_{m+n,p+q}(x,y))|^2 = (|x|^2+|y|^2)^{p+q}$$

as homogeneous polynomials of degree $p+q$. The left-hand side is $|f^\chi(x,y)|^2$ and sphericality of f^χ follows. It is also clear that f^χ is full.

To derive (1), we work with the complementary range dimension and show (2). We have

$$\begin{aligned} c(f^\chi) &= \dim \mathcal{H}^{p+q}_{m+n} - \left(\sum_{\ell=0}^{N} \dim V_\ell + \dim \left(\sum_{\ell=0}^{N} \mathcal{H}^p_m \cdot \chi_\ell \right)^{\perp} \right) \\ &= \dim \mathcal{H}^{p+q}_{m+n} - \left(\sum_{\ell=0}^{N} \dim V_\ell + \dim \mathcal{H}^{p+q}_{m+n} - (N+1) \dim \mathcal{H}^p_m \right) \\ &= (N+1) \dim \mathcal{H}^p_m - \sum_{\ell=0}^{N} \dim V_\ell = \sum_{\ell=0}^{N} c(f_\ell). \end{aligned}$$

Assume now that $f_\ell: S^m \to S_{V_\ell}$, $\ell = 0, \dots, N$, are spherical minimal immersions of degree p.

We now need to introduce an important tool of checking whether a p-eigenmap $f: S^m \to S_V$ is homothetic as follows [10, 14]. For $a \in \mathbf{R}^{m+1}$, we denote by X^a the *conformal vector field on* S^m defined by a. X^a is the uniform extension of a along the inclusion $S^m \subset \mathbf{R}^{m+1}$ followed by projection to the tangent bundle of $S^m : X^a$ naturally extends to a vector field on $\mathbf{R}^{m+1}$ by setting

$$X_x^a = a - \frac{\langle a, x\rangle}{|x|^2} x, \quad x \in \mathbf{R}^{m+1}.$$

For $a, b \in \mathbf{R}^{m+1}$, we now introduce the function

$$\begin{aligned}\Psi(f)(a,b) &= \Psi(f)(X^a, X^b)\\ &= \langle f_*(X^a), f_*(X^b)\rangle - \frac{\lambda_p}{m}\langle X^a, X^b\rangle |x|^{2(p-1)}\\ &= \langle f_*(X^a), f_*(X^b)\rangle - \langle (f_{m,p})_*(X^a), (f_{m,p})_*(X^b)\rangle.\end{aligned}$$

Since the conformal fields span each tangent space of S^m, f is homothetic iff $\Psi(f)$ vanishes for all $a, b \in \mathbf{R}^{m+1}$. As computation shows, $\Psi(f)(a,b)$ is a homogeneous polynomial of degree $2(p-1)$. We will use the following formula for $\Psi(f)$:

$$\Psi(f)(a,b) = \langle \partial_a f, \partial_b f\rangle + \left(\frac{\lambda_p}{m} - p^2\right) a^* b^* |x|^{2(p-2)} - \frac{\lambda_p}{m}\langle a, b\rangle |x|^{2(p-1)}.$$

Here ∂_a is the directional derivative with respect to a and a^* is the linear functional corresponding to a. Since the last two terms on the right-hand side do not depend on f and $\Psi(f_{m,p}) = 0$, we also have

$$\Psi(f)(a,b) = \langle \partial_a f, \partial_b f\rangle - \langle \partial_a f_{m,p}, \partial_b f_{m,p}\rangle. \tag{8}$$

We now return to the proof. By assumption, $\Psi(f_\ell) = 0$, $\ell = 0, \ldots, N$. We need to calculate $\Psi(f^\chi)$. Using (7) in (8), for $a, b \in \mathbf{R}^{m+n+1}$, we obtain

$$\begin{aligned}\Psi(f^\chi)(a,b) &= \sum_{\ell=0}^{N} \langle \partial_{a'} f_\ell, \partial_{b'} f_\ell\rangle \partial_{a''}\chi_\ell \partial_{b''}\chi_\ell - (N+1)\langle \partial_{a'} f_{m,p}, \partial_{b'} f_{m,p}\rangle \partial_{a''}\chi_\ell \partial_{b''}\chi_\ell\\ &= \sum_{\ell=0}^{N} \Psi(f_\ell)(a', b') \partial_{a''}\chi_\ell \partial_{b''}\chi_\ell = 0,\end{aligned}$$

where $a = a' + a''$ and $b = b' + b''$ with $a', b' \in \mathbf{R}^{m+1}$ and $a'', b'' \in \mathbf{R}^n$. (Notice that the projection component in (7) cancels. Notice also that in case a' or b' vanish, the formula still holds.) Thus, f^χ is a spherical minimal immersion of degree $p+q$. It remains to do the same computation for isotropy. For this we use the fact that a spherical minimal immersion $f : S^m \to S_V$ is isotropic of order k iff it is isotropic of order $k-1$ and

$$\begin{aligned}\Psi^k(f)(a_1, \ldots, a_{2k}) = \langle \partial_{a_1} \ldots \partial_{a_k} f, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f\rangle -\\ - \langle \partial_{a_1} \ldots \partial_{a_k} f_{m,p}, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f_{m,p}\rangle = 0.\end{aligned}$$

Assume now that $f_\ell : S^m \to S_{V_\ell}$ are isotropic of order k for all $\ell = 0, \ldots, N$. We prove by induction with respect to k that f^χ is also isotropic of order k.

(The first step is clear by noting that isotropy of order 1 is actually homothety.) Since $f_\ell, \ell = 0, \ldots, N$, are isotropic of order $k-1$, the induction hypothesis implies that f^χ is also isotropic of order $k-1$. For $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+n+1}$, we now calculate

$$\begin{aligned}
&\Psi^k(f^\chi)(a_1, \ldots, a_{2k}) \\
&= \langle \partial_{a_1} \ldots \partial_{a_k} f^\chi, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f^\chi \rangle - \langle \partial_{a_1} \ldots \partial_{a_k} f^\chi_{m,p}, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f^\chi_{m,p} \rangle \\
&= \sum_{\ell=0}^{N} \langle \partial_{a'_1} \ldots \partial_{a'_k} f_\ell, \partial_{a'_{k+1}} \ldots \partial_{a'_{2k}} f_\ell \rangle \partial_{a''_1} \ldots \partial_{a''_k} \chi_\ell \cdot \partial_{a''_{k+1}} \ldots \partial_{a''_{2k}} \chi_\ell - \\
&\quad -(N+1) \langle \partial_{a'_1} \ldots \partial_{a'_k} f_{m,p}, \partial_{a'_{k+1}} \ldots \partial_{a'_{2k}} f_{m,p} \rangle \partial_{a''_1} \ldots \partial_{a''_k} \chi_\ell \cdot \partial_{a''_{k+1}} \ldots \partial_{a''_{2k}} \chi_\ell \\
&= \Psi^k(f)(a'_1, \ldots, a'_{2k}) \partial_{a''_1} \ldots \partial_{a''_k} \chi_\ell \partial_{a''_{k+1}} \ldots \partial_{a''_{2k}} \chi_\ell,
\end{aligned}$$

where $a_l = a'_l + a''_l$, $a'_l \in \mathbf{R}^{m+1}$, $a''_l \in \mathbf{R}^n$, $l = 1, \ldots, 2k$. This vanishes since f_ℓ are isotropic of order k for $\ell = 0, \ldots, N$. Theorem A follows.

We now turn to the proof of Theorem B. Since $\mathcal{M}_m^p$ is the intersection of $\mathcal{L}_m^p$ with a linear subspace of $S_0^2(\mathcal{H}_m^p)$, it is enough to prove Theorem B for eigenmaps. Note that the result also holds for $\mathcal{M}_m^{p;k}$, that is, for isotropic minimal immersions.

Recall that the parameter point $\langle f \rangle$ that corresponds to f in the moduli space $\mathcal{L}_m^p$ is given by $\langle f \rangle = A^\top A - 1$, where $A : \mathcal{H}_m^p \to V$ is the linear surjection satisfying $f = A \circ f_{m,p}$.

Assume that $f_\ell : S^m \to S_{V_\ell}$, $\ell = 0, \ldots, N$, are full p-eigenmaps. We have $\langle f_\ell \rangle = A_\ell^\top A_\ell - I \in \mathcal{H}_m^p$ where $A_\ell : \mathcal{H}_m^p \to V_\ell$ is linear and onto with $f_\ell = A_\ell \circ f_{m,p}$. We need to work out $\langle f^\chi \rangle \in \mathcal{L}_{m+n}^{p+q}$. Comparing (7) for f_ℓ and $f_{m,p}$, we obtain

$$f^\chi = (A_0 \oplus \cdots \oplus A_N \oplus 0) \circ f^\chi_{m,p},$$

where 0 is the zero endomorphism of $(\sum_{\ell=0}^N \mathcal{H}_m^p \cdot \chi_\ell)^\perp$. On the other hand, by construction, we have $f^\chi_{m,p} = f_{m+n,p+q}$. We obtain

$$\langle f^\chi \rangle = \langle f_0 \rangle \oplus \cdots \oplus \langle f_N \rangle \oplus 0.$$

In terms of the moduli spaces this means that

$$(\mathcal{L}_m^p)^{\dim \mathcal{H}_{n-1}^q} \cong \mathcal{L}_{m+n}^{p+q} \cap S_0^2(\mathcal{H}_m^p \cdot \mathcal{H}_{n-1}^q).$$

The general case now follows from the Corollary in Section 2.4.

4. Examples

In this section we give a variety of explicit examples of eigenmaps and spherical minimal immersions and apply Theorem A to derive the Corollary in Section 1.

We first introduce the *equivariant construction* that produces a large number of examples of eigenmaps and spherical minimal immersions of S^3. (For details, see Section 1.4 in [14].)

The identification $\mathbf{R}^4 = \mathbf{C}^2$, $(\mathbf{R}^4 \ni (x, y, u, v) \mapsto (x+iy, u+iv) = (z, w) \in \mathbf{C}^2)$, gives rise to the local product decomposition $\mathrm{SO}(4) = \mathrm{SU}(2) \cdot \mathrm{SU}(2)'$, where SU(2) is the special unitary group and $\mathrm{SU}(2)'$ is its conjugate (within SO(4)) by $z \mapsto z$, $w \mapsto \bar{w}$. We further identify $\mathbf{C}^2$ with $\mathbf{H}$, the skew-field of quaternions, via $(z, w) \ni \mathbf{C}^2 \mapsto z + jw \in \mathbf{H}$, where $\{1, i, j, k\} \subset \mathbf{H}$ is the canonical basis. This identification gives rise to the isomorphism of SU(2) with the group of unit quaternions $S^3 \subset \mathbf{H}$. With this, $\mathrm{SU}(2) = S^3$ acts on S^3 by left-translations.

Let W_p be the complex (irreducible) SU(2)-module of complex homogeneous polynomials of degree p in z, w. Then $\dim_{\mathbf{C}} W_p = p+1$, and a typical element $\xi \in W_p$ can be expanded as

$$\xi = \sum_{q=0}^{p} c_q z^{p-q} w^q, \tag{9}$$

where the coefficients c_q, $q = 0, \ldots, p$, are complex constants. The equivariant construction simply assigns to $\xi \neq 0$ the orbit map $f_\xi : S^3 \to W_p$ through ξ:

$$f_\xi(g) = g \cdot \xi = \xi \circ L_{g^{-1}}, \quad g \in \mathrm{SU}(2) = S^3.$$

In coordinates, for $g = a + jb \in S^3$, $a, b \in \mathbf{C}$, we have

$$f_\xi(a+jb)(z, w) = \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad z, w \in \mathbf{C}.$$

By construction, f_ξ is SU(2)-equivariant. We endow W_p with the SU(2)-invariant scalar product with respect to which $\{((p-q)!q!)^{-1/2} z^{p-q} w^q\}_{q=0,\ldots,p}$ is an orthonormal basis. With a suitable normalization of ξ, f_ξ maps into the unit sphere of $W_p = \mathbf{C}^{p+1} = \mathbf{R}^{2p+2}$, and we obtain a (not necessarily full) p-eigenmap $f_\xi : S^3 \to S^{2p+1}$. For p even, suitable choices of the coefficients c_q, $q = 0, \ldots, p$, ensure that the image of f_ξ lies in the real SU(2)-submodule $R_p \subset W_p$, where $\dim R_p = p+1$. The resulting p-eigenmap maps into the p-sphere $S_{R_p} = S^p$.

Nonequivariant examples can be obtained using the Connecting Lemma ([16], p. 90) as follows. Given any two incongruent p-eigenmaps $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ and $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, the definition $f = (\sqrt{\lambda_1} f_1, \sqrt{\lambda_2} f_2)$ gives a p-eigenmap $f : S^m \to S_{V_1 \times V_2}$. (Clearly, f is not necessarily full even if f_1 and f_2 are. The name comes from the fact that the point $\langle f \rangle \in \mathcal{L}_m^p$ is on the segment connecting $\langle f_1 \rangle$ and $\langle f_2 \rangle$.)

We now let $p = 2$ and discuss quadratic eigenmaps of S^3. (This is the first nontrivial case for eigenmaps.) First let $c_0 = c_1 = 0$ and $c_2 = 1/\sqrt{2}$. Then, up to an isometry on the range, the equivariant construction gives the *complex Veronese map* $\mathrm{Ver}^{\mathbf{C}} : S^3 \to S^5$. In coordinates, we have

$$\mathrm{Ver}^{\mathbf{C}}(z, w) = (z^2, \sqrt{2} zw, w^2), \quad (z, w) \in S^3 \subset \mathbf{C}^2.$$

Second, for $c_0 = c_2 = 0$ and $c_1 = i$, f_ξ maps into $R_2 \subset W_2$, and, up to an isometry on the range, we obtain the Hopf map $\mathrm{Hopf} : S^3 \to S^2$, given by

$$\mathrm{Hopf}(z, w) = (|z^2| - |w|^2, 2z\bar{w}), \quad (z, w) \in S^3.$$

Note that, up to isometries on the domain and the range, the Hopf map is the unique lowest range-dimensional eigenmap. (See Corollary 2.7.2 in [14], p. 143.) The Connecting Lemma applied to $Ver^{\mathbf{C}} \circ g$ and Hopf $\circ g$, for various $g \in \mathrm{SO}(4)$, now produces a large number of full quadratic eigenmaps $f : S^3 \to S^n$, where the possible range dimensions are $n = 2, 4-8$. As a sample, using real coordinates, we have

$$f(x,y,u,v) = \begin{cases} (x^2+y^2-u^2-v^2, 2(xu+yv), 2(xv-yu)), \quad n=2 \\ (x^2+y^2-u^2-v^2, 2xu, 2xv, 2yu, 2yv), \quad n=4 \\ (x^2-y^2, u^2-v^2, 2xy, \sqrt{2}(xu-yv), \sqrt{2}(xv+yu), 2uv), \quad n=5 \\ (1/\sqrt{5}(x^2+y^2-u^2-v^2), 2/\sqrt{5}(x^2-y^2), 2/\sqrt{5}(u^2-v^2), \\ 4/\sqrt{5}xy, 4/\sqrt{5}uv, 2\sqrt{3}/\sqrt{5}(xu-yv), 2\sqrt{3}/\sqrt{5}(xv+yu)), \quad n=6 \\ (x^2-y^2, u^2-v^2, 2xy, \sqrt{2}xu, \sqrt{2}xv, \sqrt{2}yu, \sqrt{2}yv, 2uv), \quad n=7 \\ f_{3,2}(x,y,u,v), \quad n=8 \end{cases}$$

Here $n=2$ and $n=5$ just give the Hopf and complex Veronese maps in real coordinates while $n=8$ corresponds to the standard quadratic eigenmap. (This sample is not as arbitrary as it seems. The corresponding points on the moduli $\mathcal{L}_3^2$ are critical points of the distortion function on the boundary; for details, see Theorem F in [15].) Note that the complementary range dimensions are $c(f) = 0–4, 6$.

To apply Theorem A, we let $f_0, \ldots, f_N$ be quadratic eigenmaps of S^3 chosen from the list above. Then formula (7) defines a $(q+2)$-eigenmap $f^\chi : S^{n+3} \to S_V$ with complementary range dimension given by (2). We claim that the possible values of $c(f)$ are given by the following constraints

$$0 \leqslant c(f) \leqslant 6(\dim \mathcal{H}_{n-1}^q - 1) + 4 \quad \text{or} \quad c(f) = 6 \dim \mathcal{H}_{n-1}^q, \tag{10}$$

provided that $\mathcal{H}_{n-1}^q$ is nontrivial, i.e., for $n=1$, we have $q \leqslant 1$.

To show this, we first let $n=1$. If $\mathcal{H}_0^q$ is nontrivial then $q = 0, 1$ with $\dim \mathcal{H}_0^0 = \dim \mathcal{H}_0^1 = 1$. Thus, in both cases, $N = 0$.

If $q=0$ then χ_0 is a constant. We have the branching

$$\mathcal{H}_4^2 = \mathcal{H}_3^2 \oplus \mathcal{H}_3^1 \cdot y \oplus \mathcal{H}_3^0 \cdot H(y^2),$$

where H is the harmonic projection operator (cf. formula (2.1.15) in [14]), so that $H(y^2) = y^2 - (|x|^2 + y^2)/5$. We see that the image of the orthogonal projection $\pi_\chi : \mathcal{H}_4^2 \to (\mathcal{H}_3^2)^\perp$ is $\mathcal{H}_3^1 \cdot y \oplus \mathcal{H}_3^0 \cdot H(y^2)$, and we can write (7) as

$$f^\chi(x, y) = (a_0 f_0(x), a_1 yx, a_2 H(y^2)),$$

where $x = (x_0, x_1, x_2, x_3) \in \mathbf{R}^4$, $y \in \mathbf{R}$, and $|x|^2 + y^2 = 1$. This is because $\mathcal{H}_3^1$ consists of linear functions. Since f^χ maps into the unit sphere, we have $|f^\chi(x, y)|^2 = (|x|^2 + y^2)^2$. Thus, using $|f_0(x)|^2 = |x|^4$, a simple computation gives the coefficients:

$$a_0 = \frac{\sqrt{15}}{4}, \qquad a_1 = \sqrt{\frac{5}{2}}, \qquad a_2 = \frac{5}{4}.$$

Since $c(f^\chi)=c(f_0)$, (10) clearly holds in this case.

Let $q=1$. As noted above, a typical function in $\mathcal{H}_3^1$ is linear. We have the branching:

$$\mathcal{H}_4^3=\mathcal{H}_3^3\oplus\mathcal{H}_3^2\cdot y\oplus\mathcal{H}_3^1\cdot H(y^2)\oplus\mathcal{H}_3^0\cdot H(y^3).$$

The image of the orthogonal projection $\pi_\chi:\mathcal{H}_4^3\to(\mathcal{H}_3^2\cdot y)^\perp$ is the direct sum $\mathcal{H}_3^3\oplus\mathcal{H}_3^1\cdot H(y^2)\oplus\mathcal{H}_3^0\cdot H(y^3)$. An orthonormal basis in $\mathcal{H}_3^3$ is given by the components of the cubic standard eigenmap $f_{3,3}:S^3\to S_{\mathcal{H}_3^3}$, which, in turn, are orthonormal ultraspherical (Gegenbauer) polynomials. (For an explicit basis, see Vilenkin [18].) With this, the defining formula (7) for the cubic eigenmap $f^\chi:S^4\to S_V$ can be written as

$$f^\chi(x,y)=(a_0yf_0(x),a_1f_{3,3}(x),a_2H(y^2)x,a_3H(y^3)),$$

where $x\in\mathbf{R}^4$, $y\in\mathbf{R}$, and $|x|^2+y^2=1$. Since $H(y^3)=y^3-(3/7)(|x|^2+y^2)y$, $|f_0(x)|^2=|x|^4$ and $|f_{3,3}(x)|^2=|x|^6$, a simple computation gives

$$a_0=\frac{5\sqrt{3}}{4},\qquad a_1=\frac{\sqrt{23}}{4\sqrt{2}},\qquad a_2=\frac{15}{4\sqrt{2}},\qquad a_3=\frac{7}{4}.$$

Once again (10) clearly holds in this case.

Finally, let $n=2$ and $q\geqslant 1$. Then, $\mathcal{H}_1^q$ is two-dimensional and is spanned by $\Re(z^q)$ and $\Im(z^q)$, where $z\in\mathbf{C}$ is a complex variable. We let $N=1$ and choose quadratic eigenmaps $f_0:S^3\to S_{V_0}$ and $f_1:S^3\to S_{V_1}$. Formula (7) gives the $(q+2)$-eigenmap $f^\chi:S^5\to S_V$ by

$$f^\chi(x,z)=(a_0\Re(z^q)f_0(x),a_1\Im(z^q)f_1(x),\pi_\chi(f_{5,q+2}(x,z))),$$

where $x\in\mathbf{R}^5$, $z\in\mathbf{C}$, and $|x|^2+|z|^2=1$. The image of the orthogonal projection

$$\pi_\chi:\mathcal{H}_5^{q+2}\to(\mathcal{H}_3^2\cdot\Re(z^q)\oplus\mathcal{H}_3^2\cdot\Im(z^q))^\perp$$

can be obtained from $\mathcal{H}_5^{q+2}$ by branching twice, and once again, choosing concrete bases, an explicit formula for $\pi_\chi(f_{5,q+2})$ can be obtained. Now, $c(f^\chi)=c(f_0)+c(f_1)$, and, varying f_0 and f_1, we see that all possible sums of the corresponding numbers $c(f_0)$ and $c(f_1)$ with ranges 0–4, 6 give $c(f_0)+c(f_1)=0-10,12$. This is (10) for $n=2$. The general case, $n\geqslant 2$, follows by a similar argument setting $N+1=\dim\mathcal{H}_{n-1}^p$ and examining the possible ranges in (2), where $c(f_\ell)=0-4,6$, for each $\ell=0,\dots,N$.

The condition of minimality imposed on an orbit map f_ξ of the equivariant construction gives a set of quadratic equations for the coefficients c_q, $q=0,\dots,p$ in (9). (See (1.4.13) in [14], p. 59.)

We now let $p=4$ and discuss quartic spherical minimal immersions of S^3. (This is the first nontrivial case for spherical minimal immersions.) First, we let

$$c_0=\frac{\sqrt{6}}{24},\qquad c_1=0,\qquad c_2=\frac{\sqrt{2}}{4},\qquad c_3=0,\qquad c_4=-\frac{\sqrt{6}}{24}.$$

With the identifications we have made, we obtain the full quartic minimal immersion $\mathcal{I}: S^3 \to S^9$ which, in complex coordinates, is given by

$$\mathcal{I}(z, w) = (1/\sqrt{2}(z^4 - \bar{w}^4), \sqrt{6}z^2\bar{w}^2, \sqrt{2}(z^3 w + \bar{z}\bar{w}^3), \sqrt{6}(z\bar{z}^2 w - \bar{z}w^2\bar{w}), \sqrt{3/2}(z^2w^2 - \bar{z}^2\bar{w}^2), 1/\sqrt{2}(|z|^4 - 4|z|^2|w|^2 + |w|^4)), \quad (z, w) \in S^3.$$

Note that, up to isometries on the domain and the range, $\mathcal{I}$ is the unique lowest dimensional quartic minimal immersion. (This result is due to DeTurck and Ziller [2, 3].) The next lowest range dimensional example $\mathcal{J}$: $S^3 \to S^{14}$ is obtained from $\mathcal{I}$ by the Connecting Lemma. (For details, see [14], p. 224.) In complex coordinates, we have

$$\mathcal{J}(z, w) = (1/\sqrt{2})(z^4, w^4, 2\sqrt{3}z^2\bar{w}^2, 2z^3w, 2zw^3, \sqrt{3}(z\bar{z}^2 w - \bar{z}w^2\bar{w}), \sqrt{6}z^2w^2, |z|^4 - 4|z|^2|w|^2 + |w|^4), \quad (z, w) \in S^3.$$

As before, the Connecting Lemma gives a variety of examples of nonequivariant full quartic spherical minimal immersions f: $S^3 \to S^n$, where the possible range dimensions are $n = 9; 14-15, 18-24$. The complementary range dimensions are $c(f) = 0-6, 9-10, 15$. The entire boundary of the moduli $\mathcal{M}_3^4$ can be mapped out by using these examples; for details, see [14, 16].

The construction of f^χ with source maps $f_\ell, \ell = 0, \ldots, N$, is the same for eigenmaps and spherical minimal immersions. To prove the Corollary in Section 1 we assume that $f_\ell: S^3 \to S_{V_\ell}$, $\ell = 0, \ldots, N$, are quartic minimal immersions chosen from the examples above with complementary range dimensions $c(f_\ell) = 0-6, 9-10, 15$. Then f^χ: $S^{n+3} \to S_V$ is a spherical minimal immersion of degree $q+4$, where $\chi_0, \ldots, \chi_N \in \mathcal{H}_{n-1}^q$ are orthogonal and suitably normalized. Let $n = 1$. If $q = 0$ then the branching

$$\mathcal{H}_4^4 = \mathcal{H}_3^4 \oplus \mathcal{H}_3^3 \cdot y \oplus \mathcal{H}_3^2 \cdot H(y^2) \oplus \mathcal{H}_3^1 \cdot H(y^3) \oplus \mathcal{H}_3^0 \cdot H(y^4).$$

Formula (7) specialized to the following

$$f^\chi(x, y) = (a_0 f_0(x), a_1 y f_{3,3}(x), a_2 H(y^2) f_{3,2}(x), a_3 H(y^3)x, a_4 H(y^4)).$$

The coefficients a_0, a_1, a_2, a_3, a_4 can be determined from the condition that $|f^\chi|^2 = (|x|^2 + y^2)^4$. Once again, the standard minimal immersions $f_{3,3}$ and $f_{3,2}$ have cubic and quadratic ultraspherical polynomial components, orthonormal in $\mathcal{H}_3^3$ and $\mathcal{H}_3^2$. Since $c(f^\chi) = c(f_0)$, the Corollary follows in this case.

For $q = 1$, the appropriate branching gives

$$f^\chi(x, y) = (a_0 y f_0(x), a_1 f_{3,5}(x), a_2 H(y^2) f_{3,3}(x), a_3 H(y^3) f_{3,2}(x), a_4 H(y^4)x, a_5 H(y^5)).$$

The Corollary of Section 1 follows again in this case.

For $n=2$ and $q \geqslant 1$ we choose $N=1$. With a choice of quartic minimal immersions $f_0: S^3 \to S_{V_0}$ and $f_1: S^3 \to S_{V_1}$ formula (7) can be written as

$$f^{\chi}(x,z) = (a_0 \Re(z^q) f_0(x), a_1 \Im(z^q) f_1(x), \pi_{\chi}(f_{5,q+4}(x,z))).$$

The complementary range dimensions $c(f_0), c(f_1) = 0-6, 9-10, 15$ give $c(f^{\chi}) = c(f_0) + c(f_1) = 0-21, 24-25, 30$.

The general case, $n \geqslant 2$, follows by a similar argument setting $N+1 = \dim \mathcal{H}^p_{n-1}$ and examining the possible ranges in (2), where $c(f_{\ell}) = 0-6, 9-10, 15$, for each $\ell = 0, \ldots, N$.

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