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Critical Points of the Distance Function on the Moduli Space for Spherical Eigenmaps and Minimal Immersions

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Abstract. The boundary of a DoCarmo-Wallach moduli space parametrizing (harmonic) eigenmaps between spheres or spherical minimal immersions carries a natural stratification. In this paper we study the critical points of the distance function on the boundary strata. We show that the critical points provide a natural generalization of eigenmaps with L^2-orthonormal components. We also point out that many classical examples of eigenmaps correspond to critical points.

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1. Introduction and statement of results

Let G be a compact Lie group and $\mathcal{H}$ a finite dimensional orthogonal G-module. Let $S^2(\mathcal{H})$ denote the G-module of symmetric linear endomorphisms of $\mathcal{H}$, and $S^2_0(\mathcal{H})$ the G-submodule of traceless endomorphisms in $S^2(\mathcal{H})$. We define the (reduced) moduli space

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \{C \in S^2_0(\mathcal{H}) \,|\, C + I \geq 0\},$$

where $\geq$ stands for positive semidefinite. $\mathcal{K}_0$ is a G-invariant compact convex body in $S^2_0(\mathcal{H})$ [11]. Given a G-submodule $\mathcal{E} \subset S^2_0(\mathcal{H})$, our main problem is to study the structure of the boundary of the compact convex body $\mathcal{K}_0 \cap \mathcal{E}$ in $\mathcal{E}$.

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A motivation to consider $\mathcal{K}_0 \cap \mathcal{E}$ is the following. Let M be a compact Riemannian manifold, G a compact Lie group of isometries of M, and $\mathcal{H} \subset C^\infty(M)$ a finite dimensional G-submodule. We call a map $f : M \to V$ into a Euclidean vector space V an *$\mathcal{H}$-map* if the components of f belong to $\mathcal{H}$. Then [11] $\mathcal{K}_0$ parametrizes the congruence classes of full $\mathcal{H}$-maps satisfying a normalization condition. In particular, for $K \subset G$ a closed subgroup and $M = G/K$ a compact Riemannian homogeneous space, there is a G-submodule $\mathcal{E}(\mathcal{H}) \subset S^2_0(\mathcal{H})$ such that $\mathcal{K}_0 \cap \mathcal{E}(\mathcal{H})$ parametrizes the congruence classes of full *spherical* $\mathcal{H}$-maps $f : M \to S_V$, where S_V is the unit sphere in V. If, in addition, M is naturally reductive and $\mathcal{H}$ is an eigenspace of the Laplacian acting on $C^\infty(M)$ corresponding to an eigenvalue λ, then an $\mathcal{H}$-map has λ-eigenfunctions as components, and $\mathcal{K}_0 \cap \mathcal{E}(\mathcal{H})$ is a parameter space for *λ-eigenmaps* [14]. Even more specifically, if M is isotropy irreducible then there is a G-submodule $\mathcal{F}(\mathcal{H}) \subset S^2_0(\mathcal{H})$ such that $\mathcal{K}_0 \cap \mathcal{F}(\mathcal{H})$ parametrizes the congruence classes of full *isometric minimal immersions* $f : M \to S_V$ of the $\lambda/\dim M$-multiple of the original metric on M [3,21].

We now return to the general situation and consider $\mathcal{K}_0$. The boundary $\partial\mathcal{K}_0$ consists of those endomorphisms C in $\mathcal{K}_0$ for which $\det(C + I) = 0$. It is therefore natural to consider the function $\Delta : S^2(\mathcal{H}) \to \mathbf{R}$ defined by

$$\Delta(C) = \det(C + I), \quad C \in S^2(\mathcal{H}).$$

We consider the level-sets

$$\mathcal{B}_t = (\Delta|_{\mathcal{K}_0})^{-1}(t) = \{C \in \mathcal{K}_0 \mid \det(C + I) = t\}.$$

Comparison of the geometric and arithmetic means of the eigenvalues of a symmetric endomorphism in $\mathcal{K}_0$ shows that $\mathcal{B}_t$ is nonempty iff $0 \leq t \leq 1$. (For a proof, see Sec.3.) We also have $\mathcal{B}_1 = \{0\}$ and $\mathcal{B}_0 = \partial\mathcal{K}_0$. In particular, $-\log\Delta$ is a nonnegative exhaustion function on the interior of $\mathcal{K}_0$. Clearly, the level-sets of Δ are the same as the level-sets of $-\log\Delta$.

Theorem A. $-\log\Delta$ *is a strictly convex analytic exhaustion function on the interior of* $\mathcal{K}_0$. *Given a G-submodule $\mathcal{E}$ of $S^2_0(\mathcal{H})$, for $0 < t < 1$, the level-set $\mathcal{B}_t \cap \mathcal{E}$ is a strictly convex real-analytic hypersurface of $\mathcal{E}$. Radial projection $C \mapsto C/|C|$, $0 \neq C \in S^2(\mathcal{H})$, restricted to $\mathcal{B}_t \cap \mathcal{E}$ establishes an analytic diffeomorphism between $\mathcal{B}_t \cap \mathcal{E}$ and the unit sphere of $\mathcal{E}$.*

Let $\dim\mathcal{H} = n$. The level-sets of the integer-valued function $\nu : S^2(\mathcal{H}) \to \{0, 1, \ldots, n\}$, $\nu(C) = \operatorname{rank}(C + I)$, $C \in S^2(\mathcal{H})$, restricted to $\mathcal{K}_0$ form a stratification of $\mathcal{K}_0$. We call this the *rank-stratification* of the moduli space. By abuse of notation, for $\nu \in \{1, \ldots, n\}$, we denote the level-set by

$$\mathcal{B}^\nu_0 = \{C \in \mathcal{K}_0 \mid \operatorname{rank}(C + I) = \nu\}. \tag{1}$$

(Since $\mathcal{K}_0 \subset S^2_0(\mathcal{H})$, $\nu = 0$ is not realized.) Clearly, $\mathcal{B}^n_0$ is the interior of $\mathcal{K}_0$ so that

$$\partial\mathcal{K}_0 = \mathcal{B}_0 = \bigcup_{\nu=1}^{n-1} \mathcal{B}^\nu_0.$$

Theorem B. (a) *Let $\mathcal{E} \subset S^2_0(\mathcal{H})$ be a G-submodule. Given $C_0 \in \mathcal{B}_0 \cap \mathcal{E}$, if*

$$S^2(\ker(C_0 + I)) \cap \mathcal{E}^\perp = \{0\} \tag{2}$$

then the rank-stratum passing through C_0 intersected with $\mathcal{E}$ is a real-analytic submanifold near C_0. Here $S^2(\ker(C_0+I))$ is considered as a linear subspace of $S^2(\mathcal{H})$ via the inclusion $\ker(C_0+I) \subset \mathcal{H}$. *In particular, the corank-one stratum $\mathcal{B}_0^{n-1}$ intersected with $\mathcal{E}$ is a real-analytic submanifold.*

(b) *The rank-strata $\mathcal{B}_0^\nu$, $\nu = 1, \ldots, n$, in (1) are real-analytic submanifolds of $S_0^2(\mathcal{H})$. For $C_0 \in \mathcal{B}_0^\nu$, the tangent space to $\mathcal{B}_0^\nu$ at C_0 is*

$$T_{C_0}(\mathcal{B}_0^\nu) = S^2(\ker(C_0+I))^\perp \cap S_0^2(\mathcal{H}). \tag{3}$$

Remark. It is not known whether the rank-stratification of the DoCarmo-Wallach moduli spaces $\mathcal{K}_0 \cap \mathcal{E}(\mathcal{H})$ and $\mathcal{K}_0 \cap \mathcal{F}(\mathcal{H})$ have singular points at all.

Example. Identify $S_0^2(\mathbf{R}^2)$ with $\mathbf{R}^2$ by associating to the matrix $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ the point $(a,b) \in \mathbf{R}^2$. With this, $\mathcal{K}_0(\mathbf{R}^2)$ is the closed unit disk in $\mathbf{R}^2$. The boundary unit circle forms a single rank-stratum corresponding to $\nu = 1$. For $0 < t < 1$, $\mathcal{B}_t$ is the circle with radius $\sqrt{1-t}$ and center at the origin. The eigenvalues of any $C_0 \in \mathcal{B}_t$ are $\pm\sqrt{1-t}$.

Let $C, C' \in \mathcal{K}_0$. We say that C' is *derived* from C, written as $C' \leftharpoonup C$ if $\mathrm{im}(C'+I) \subset \mathrm{im}(C+I)$. Then $\leftharpoonup$ is clearly reflexive and transitive. Given $C \in \mathcal{K}_0$, the set

$$I_C = \{C' \in \mathcal{K}_0 \,|\, C' \leftharpoonup C\}$$

is a compact convex body in its affine span. We call I_C the *cell* of C. The relative interior

$$I_C^\circ = \{C' \in \mathcal{K}_0 \,|\, C' \rightleftharpoons C\}$$

is called the *open cell* of C. Then I_C° consists of those traceless symmetric endomorphisms C' of $\mathcal{H}$ for which $\mathrm{im}(C'+I) = \mathrm{im}(C+I)$. By definition, the tangent space $T_C(I_C^\circ) = S^2(\mathrm{im}(C+I))$. Since the symmetrized relation $\rightleftharpoons$ is an equivalence, the open cells give a partition of $\mathcal{K}_0$. The interior of $\mathcal{K}_0$ is the cell of the origin. Finally, it is clear that each open cell is contained in a single rank-stratum so that the rank-strata $\mathcal{B}_0^\nu$, $\nu = 1, \ldots, n$, are unions of open cells.

Let $G = O(\mathcal{H})$. The action of G on $\mathcal{H}$ extends to an action of G on $S^2(\mathcal{H})$ by conjugation: $U \cdot C = UCU^{-1}$, $U \in O(\mathcal{H})$, $C \in S^2(\mathcal{H})$. Then $S_0^2(\mathcal{H})$ is an invariant linear subspace, and $\mathcal{K}_0$ is $O(\mathcal{H})$-invariant. The action of $O(\mathcal{H})$ on $\mathcal{K}_0$ respects the cell-decomposition, that is, for $g \in O(\mathcal{H})$ and $C \in \mathcal{K}_0$, we have $g(I_C^\circ) = I_{g \cdot C}^\circ$.

Let $G_\nu(\mathcal{H})$ denote the Grassmann manifold of ν-dimensional linear subspaces of $\mathcal{H}$. For $\nu \in \{1, \ldots, n\}$, we define

$$\Gamma_\nu : \mathcal{B}_0^\nu \to G_\nu(\mathcal{H})$$

by

$$\Gamma_\nu(C) = \mathrm{im}(C+I), \quad C \in \mathcal{B}_0^\nu.$$

The setup in the proof of Theorem B gives the following byproduct:

Corollary. *Let $\nu \in \{1, \ldots, n\}$. The map $\Gamma_\nu : \mathcal{B}_0^\nu \to G_\nu(\mathcal{H})$ is an $O(\mathcal{H})$-equivariant submersion with fibres the open cells in $\mathcal{B}_0^\nu$.*

We consider the *distance square function* $\Lambda^2 : S^2(\mathcal{H}) \to \mathbf{R}$, defined by

$$\Lambda^2(C) = |C|^2 = \operatorname{trace}(C^2), \quad C \in S^2(\mathcal{H}).$$

We have

$$d\Lambda^2(C)(C') = 2\langle C, C' \rangle, \quad C, C' \in S^2(\mathcal{H}). \tag{4}$$

We will be interested in the critical points of Λ^2 on $\mathcal{B}_t$, $0 \le t < 1$.

Theorem C. (Critical Points in the Interior) *Let $\mathcal{E} \subset S_0^2(\mathcal{H})$ be a G-submodule and $0 < t < 1$. Then $C_0 \in \mathcal{B}_t \cap \mathcal{E}$ is a critical point of $\Lambda^2|_{\mathcal{B}_t \cap \mathcal{E}}$ iff*

$$C_0 - \lambda t (C_0 + I)^{-1} \in \mathcal{E}^\perp, \tag{5}$$

for some $\lambda \in \mathbf{R}$. The critical value of Λ^2 on C_0 is

$$\Lambda^2(C_0) = |C_0|^2 = \lambda t \left(2n - \sum_{j=1}^n \left(\lambda_j + \frac{1}{\lambda_j} \right) \right), \quad t = \det(C_0 + I),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $C_0 + I > 0$.
For $\mathcal{E} = S_0^2(\mathcal{H})$, the critical points of $\Lambda^2|_{\mathcal{B}_t}$ are those endomorphisms of $\mathcal{H}$ that have exactly two distinct eigenvalues, one negative and one positive. For each $\nu = 1, \ldots, n-1$, the set of critical endomorphisms in $\mathcal{B}_t$ with multiplicity ν of the positive eigenvalue forms a single critical $O(\mathcal{H})$-orbit. The eigenvalues of an endomorphism on this orbit are

$$\alpha \in \left(0, \frac{n}{\nu} - 1\right) \quad \text{and} \quad -\frac{\nu}{n-\nu}\alpha \in (-1, 0), \tag{6}$$

where α is the unique solution of the equation

$$\left(1 - \frac{\nu}{n-\nu}\alpha\right)^{n-\nu} (1+\alpha)^\nu = t \tag{7}$$

on $(0, n/\nu - 1)$. The critical value of Λ^2 on this orbit is $\nu n/(n-\nu) \cdot \alpha^2$.

Let $\mathcal{E} \subset S_0^2(\mathcal{H})$ be a G-submodule. Let $\nu \in \{1, \ldots, n-1\}$, and consider the rank-stratum $\mathcal{B}_0^\nu \subset \mathcal{B}_0$ corresponding to ν. Let $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}$. We say that C_0 is a *critical point* of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}}$ if $d\Lambda^2(C_0)$ vanishes on $T_{C_0}(\mathcal{B}_0^\nu) \cap \mathcal{E}$. When C_0 is a smooth point on $\mathcal{B}_0^\nu \cap \mathcal{E}$ then this is equivalent to the usual definition of a critical point. Geometrically, C_0 is a critical point iff the sphere $\{C \,|\, |C| = |C_0|\}$ is tangent to $\mathcal{B}_0^\nu \cap \mathcal{E}$ at C_0.

Remark. Let $C_0 \in \mathcal{B}_0 \cap \mathcal{E}$ be a (critical) point where $\Lambda^2|_{\mathcal{B}_0 \cap \mathcal{E}}$ takes a local minimum. Then C_0 is automatically a smooth point. In fact, by convexity, near C_0, the topological sphere $\mathcal{B}_0 \cap \mathcal{E}$ lies between the Euclidean sphere $\{C \in \mathcal{E} \,|\, |C| = |C_0|\}$ and its tangent space at C_0.

Theorem D. (Critical Points on the Boundary) *Let $\mathcal{E} \subset S_0^2(\mathcal{H})$ be a linear subspace. Then $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}$ is a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}}$ iff*

$$C_0 \in S^2(\ker(C_0 + I)) + \mathcal{E}^\perp. \tag{8}$$

In particular, each open cell I_C° in $\mathcal{B}_0^\nu$ intersected with $\mathcal{E}$ contains at most one critical point. Let $\mathcal{E} = S_0^2(\mathcal{H})$. On each rank-stratum $\mathcal{B}_0^\nu$, $\nu \in \{1, \ldots, n-1\}$, the set of critical points of $\Lambda^2|_{\mathcal{B}_0^\nu}$, forms a single $O(\mathcal{H})$-orbit. This orbit consists of endomorphisms that have eigenvalues $n/\nu - 1$ and -1 with corresponding multiplicities ν and $n - \nu$. The critical value of Λ^2 on this orbit is $n(n/\nu - 1)$. In terms of the cell-decomposition of $\mathcal{B}_0^\nu$, the critical $O(\mathcal{H})$-orbit passes through the centroid of each open cell contained in $\mathcal{B}_0^\nu$. The map $\Gamma^\nu : \mathcal{B}_0^\nu \to G_\nu(\mathcal{H})$ restricted to the critical $O(\mathcal{H})$-orbit is an analytic diffeomorphism, and the rest of the $O(\mathcal{H})$-orbits are fibre bundles over $G_\nu(\mathcal{H})$ with nontrivial fibres.

Remark. Notice that $\mathcal{B}_t \to \mathcal{B}_0$ as $t \to 0$, and by (6)–(7):

$$\lim_{t\to 0} \alpha = \frac{n}{\nu} - 1, \quad \lim_{t\to 0} \left(-\frac{\nu}{n-\nu}\alpha \right) = -1, \quad \lim_{t\to 0} \frac{\nu n}{n-\nu}\alpha^2 = n\left(\frac{n}{\nu} - 1\right).$$

Corollary. *For any $C \in \mathcal{B}_0$, we have*

$$\sqrt{\frac{n}{n-1}} \leq |C| \leq \sqrt{n(n-1)}.$$

Let $\mathcal{E} \subset S_0^2(\mathcal{H})$ be a G-submodule, and $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}$. We say that C_0 is a *weak critical point* (of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}}$) if C_0 is orthogonal to $I_{C_0} \cap \mathcal{E}$, where I_{C_0} is the cell containing C_0. By (4), a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}}$ is also a weak critical point. Moreover, again by (4), since $T_{C_0}(I_{C_0}^\circ) = S^2(\operatorname{im}(C_0 + I))$, C_0 is a weak critical point iff

$$C_0 \in (T_{C_0}(I_{C_0}^\circ) \cap \mathcal{E})^\perp = S^2(\ker(C_0 + I)^\perp)^\perp + \mathcal{E}^\perp.$$

Given any $C_0 \in \mathcal{B}_0 \cap \mathcal{E}$, it is clear that there is a weak critical point $C_0' \in \mathcal{B}_0 \cap \mathcal{E}$ derived from C_0. In fact, C_0' can be chosen to be the point where $\Lambda^2|_{I_{C_0} \cap \mathcal{E}}$ attains its minimum.
In Section 2 we will derive a more transparent necessary and sufficient condition for weak criticality for points of the DoCarmo-Wallach moduli space $\mathcal{K}_0 \cap \mathcal{E}(\mathcal{H})$.

Example. Identify $S_0^2(\mathbf{R}^3)$ with $\mathbf{R}^5$ by associating to the matrix $\begin{bmatrix} \alpha & a & b \\ a & \beta & c \\ b & c & -\alpha-\beta \end{bmatrix}$ the point $(\alpha, \beta, a, b, c) \in \mathbf{R}^5$. The boundary $\partial\mathcal{K}_0(\mathbf{R}^3)$ consists of two rank-strata $\mathcal{B}_0^\nu$, $\nu = 1, 2$. The orthogonal group $O(3)$ acts on $S_0^2(\mathbf{R}^3)$ irreducibly by conjugation. In fact, as an $O(3)$-module, $S_0^2(\mathbf{R}^3)$is isomorphic to $\mathcal{H}^2$, the space of quadratic spherical harmonics on S^2. This action of $O(3)$ leaves $\mathcal{K}_0(\mathbf{R}^3)$ invariant. Since $\mathcal{K}_0(\mathbf{R}^3)$ is compact and convex, radial projection $\partial\mathcal{K}_0(\mathbf{R}^3) \to S^4$ is an $O(3)$-equivariant homeomorphism. Since $S_0^2(\mathbf{R}^3)$ is irreducible as an $O(3)$-module, the $O(3)$-orbits on $S^4 \approx \partial\mathcal{K}_0(\mathbf{R}^3)$ form a homogeneous family of isoparametric hypersurfaces of degree 3 [2]. It is well-known that there are two singular orbits each analytically diffeomorphic to the real projective plane $\mathbf{R}P^2$. The rest of the orbits are principal.

The rank-stratum $\mathcal{B}_0^1$ is a single $O(3)$-orbit of the diagonal matrix diag $(2,-1,-1)$. Since the isotropy subgroup is $O(1)\times O(2)$, this $O(3)$-orbit is one of the singular orbits: $O(3)/(O(1)\times O(2)) = \mathbf{R}P^2$. The rank-stratum $\mathcal{B}_0^2$ is the $O(3)$-orbit of the open cell of the 1-parameter family of matrices diag $(\alpha,\beta,-1)$, $\alpha+\beta=1$, $\alpha,\beta>-1$. The unique Λ^2-extremal $O(3)$-orbit in $\mathcal{B}_0^2$ corresponds to $\alpha=\beta=1/2$. The isotropy subgroup at this point is $O(2)\times O(1)$, so that this is the other singular orbit. Note that the two singular orbits are antipodal (since $\mathrm{Fix}_{SO(2)}(S_0^2(\mathbf{R}^3)) \cong \mathrm{Fix}_{SO(2)}(\mathcal{H}^2)$ is 1-dimensional).

2. Applications to DoCarmo-Wallach moduli spaces

In this section we study critical points of the distance function on the DoCarmo-Wallach moduli spaces. Let $M=G/K$ be a compact Riemannian homogeneous space, and consider $C^\infty(M)$, the space of smooth functions on M. We endow $C^\infty(M)$ with a G-invariant scalar product $\langle\cdot,\cdot\rangle$. Let $\mathcal{H}\subset C^\infty(M)$ be a finite dimensional G-submodule. We restrict the scalar product on $C^\infty(M)$ to a scalar product on $\mathcal{H}$. If $\mathcal{H}$ is irreducible, this scalar product is unique up to a constant multiple. In particular, up to scaling, this scalar product is the L^2-scalar product defined by integration with respect to the G-invariant volume form.
We define the Dirac delta $\delta=\delta_{\mathcal{H}}: M\to\mathcal{H}^*$ by evaluating the elements of $\mathcal{H}$ on points of M. Since M is homogeneous, $\delta_{\mathcal{H}}$ restricts to a spherical $\mathcal{H}^*$-map $\delta: M\to S_{\mathcal{H}^*}$ [11]. (The scalar product on $\mathcal{H}^*$ is induced by the G-invariant scalar product on $\mathcal{H}$ suitably scaled.)
As noted above, there exists a G-submodule $\mathcal{E}(\mathcal{H})\subset S^2(\mathcal{H})$ such that $\mathcal{L}(\mathcal{H})=\mathcal{K}_0\cap\mathcal{E}(\mathcal{H})$ parametrizes the (congruence classes of) full spherical $\mathcal{H}$-maps $f: M\to S_V$ for various V. The parametrization is given by associating to f the symmetric endomorphism $C=\langle f\rangle = A^*A-I\in S^2(\mathcal{H})$, where $A:\mathcal{H}^*\to V$ is the (unique) surjective linear map with $f=A\circ\delta_{\mathcal{H}}$. An $\mathcal{H}$-map $f:M\to V$ maps into the unit sphere S_V iff

$$|f(x)|^2-|\delta_{\mathcal{H}}(x)|^2=\langle (A^*A-I)\delta_{\mathcal{H}}(x),\delta_{\mathcal{H}}(x)\rangle=\langle C,\delta_{\mathcal{H}}(x)\odot\delta_{\mathcal{H}}(x)\rangle=0,$$

for all $x\in M$. Here $\odot$ denotes the symmetric tensor product. This shows that

$$\mathcal{E}(\mathcal{H})=\{\delta_{\mathcal{H}}(x)\odot\delta_{\mathcal{H}}(x)\,|\,x\in M\}^\perp\subset S^2(\mathcal{H}), \tag{9}$$

since an $\mathcal{H}$-map $f:M\to V$ maps into S_V iff $C\in\mathcal{E}(\mathcal{H})$.
Let

$$\Psi^0: S^2(\mathcal{H})\to C^\infty(M)$$

be the G-module homomorphism given by multiplication [11]. The image of Ψ^0 consists of finite sums of products of functions in $\mathcal{H}$. We denote this G-submodule by $\mathcal{H}\cdot\mathcal{H}$. An easy consequence of the parametrization of $\mathcal{L}(\mathcal{H})$ above is that the kernel of Ψ^0 is $\mathcal{E}(\mathcal{H})$. We obtain

$$\mathcal{E}(\mathcal{H})=S^2(\mathcal{H})/(\mathcal{H}\cdot\mathcal{H}).$$

Thus, to determine $\mathcal{E}(\mathcal{H})$, one needs to decompose $\mathcal{H}\cdot\mathcal{H}$ into irreducible components.

Remark. For $M=G/K$ a compact rank 1 symmetric space, every irreducible component of $C^\infty(M)$ is the full eigenspace V_λ corresponding to an eigenvalue λ. If λ_p, $p\geq 0$, denotes the p-th eigenvalue, we also have [11]

$$V_{\lambda_p} \cdot V_{\lambda_p} = \begin{cases} \sum_{j=0}^{p} V_{\lambda_{2j}} & \text{if } M = S^m \\ \sum_{j=0}^{2p} V_{\lambda_j} & \text{otherwise.} \end{cases}$$

We now work out the adjoint $(\Psi^0)^* : \mathcal{H} \cdot \mathcal{H} \to S^2(\mathcal{H})$. Let $\{\chi^j\}_{j=1}^n \subset \mathcal{H}$ be an orthonormal basis. With respect to this basis, we write the elements in $S^2(\mathcal{H})$ as matrices. Given $\xi \in \mathcal{H}\cdot\mathcal{H}$, we have

$$\begin{aligned} \langle C, (\Psi^0)^*(\xi)\rangle &= \sum_{j,l=1}^{n} c_{jl}(\Psi^0)^*(\xi)_{jl} \\ = \langle \Psi^0(C), \xi\rangle &= \left\langle \sum_{j,l=1}^{n} c_{jl}\chi^j\chi^l, \xi \right\rangle = \sum_{j,l=1}^{n} c_{jl}\langle \xi, \chi^j\chi^l\rangle. \end{aligned}$$

We thus have

$$(\Psi^0)^*(\xi) = \sum_{j,l=1}^{n} \langle \xi, \chi^j\chi^l\rangle \chi^j \odot \chi^l.$$

In coordinates:

$$(\Psi^0)^*(\xi)_{jl} = \langle \xi, \chi^j\chi^l\rangle, \quad j, l = 1, \ldots, n. \tag{10}$$

Since Ψ^0 is onto $\mathcal{H}\cdot\mathcal{H}$, the adjoint $(\Psi^0)^*$ is a linear isomorphism between $\mathcal{H}\cdot\mathcal{H}$ and $\mathcal{E}(\mathcal{H})^\perp$. In particular, we have

$$\mathcal{E}(\mathcal{H})^\perp = \operatorname{im}(\Psi^0)^* = \{(\Psi^0)^*(\xi) \mid \xi \in \mathcal{H}\cdot\mathcal{H}\}.$$

Remark. An orthonormal basis $\{\chi^j\}_{j=1}^n \subset \mathcal{H}$ gives the isomorphisms $\mathcal{H} \cong \mathcal{H}^* \cong \mathbf{R}^n$. With respect to these, the Dirac delta $\delta_{\mathcal{H}}$ has components $\chi^1, \ldots, \chi^n$, that is, we have

$$\delta_{\mathcal{H}}(x) = \sum_{j=1}^{n} \chi^j(x)\chi^j. \tag{11}$$

In particular, from the defining equality $\langle C\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle = 0$ of $\mathcal{E}(\mathcal{H})$, we see that $\mathcal{E}(\mathcal{H})$ consists of traceless symmetric endomorphisms: $\mathcal{E}(\mathcal{H}) \subset S_0^2(\mathcal{H})$.

Since $\delta_{\mathcal{H}}$ is spherical, $\sum_{j=1}^n (\chi^j)^2 = 1$. On the other hand, this is an element in $\mathcal{H}\cdot\mathcal{H}$. We obtain that the constants always belong to $\mathcal{H}\cdot\mathcal{H}$.

Let $C_0 \in \mathcal{B}_0 \cap \mathcal{E}$. We claim that $(\Psi^0)^*$ establishes a linear isomorphism between the orthogonal complement of $(C_0 + I)(\mathcal{H})\cdot\mathcal{H}$ in $\mathcal{H}\cdot\mathcal{H}$ and $S^2(\ker(C_0+I)) \cap \mathcal{E}^\perp(\mathcal{H})$. Here $(C_0+I)(\mathcal{H})\cdot\mathcal{H}$ denotes the linear subspace of $C^\infty(M)$ consisting of finite sums of products of functions from the image of $C_0 + I$ and from $\mathcal{H}$.

To prove the claim, we first note that $C \in S^2(\ker(C_0+I))$ iff $(C_0+I)C = 0$. According to our computations above, $C \in \mathcal{E}^\perp(\mathcal{H})$ iff $c_{jl} = \langle \xi, \chi^j\chi^l\rangle$ for some $\xi \in \mathcal{H}\cdot\mathcal{H}$, where $\{\chi^j\}_{j=1}^n \subset \mathcal{H}$ is an L^2-orthonormal basis. We thus have

$$\begin{aligned} ((C_0+I)C)_{jl} &= \sum_{k=1}^{n} (C_0+I)_{jk}\langle \xi, \chi^k\chi^l\rangle \\ &= \langle \xi, \sum_{k=1}^{n} (C_0+I)_{jk}\chi^k\chi^l\rangle \\ &= \langle \xi, (C_0+I)(\chi^j)\chi^l\rangle. \end{aligned}$$

This vanishes iff $C \in S^2(\ker(C_0 + I))$ iff ξ is orthogonal to $(C_0 + I)(\mathcal{H}) \cdot \mathcal{H}$. The claim follows.

The claim just proved establishes the equivalence of the condition in (2) with the condition $(C_0 + I)(\mathcal{H}) \cdot \mathcal{H} = \mathcal{H} \cdot \mathcal{H}$. Theorem B gives the following

Corollary 1. *Let $M = G/K$ and $\mathcal{H} \subset C^\infty(M)$ a finite dimensional G-submodule. Given $C_0 \in \mathcal{B}_0 \cap \mathcal{E}(\mathcal{H})$, if $(C_0 + I)(\mathcal{H}) \cdot \mathcal{H} = \mathcal{H} \cdot \mathcal{H}$ then the rank-stratum passing through C_0 intersected with $\mathcal{E}(\mathcal{H})$ is a real-analytic submanifold near C_0.*

Theorems C and D imply the following:

Corollary 2. *Let $M = G/K$ and $\mathcal{H} \subset C^\infty(M)$ a finite dimensional G-submodule. For $0 < t < 1$, the endomorphism $C_0 \in \mathcal{B}_t \cap \mathcal{E}(\mathcal{H})$ is a critical point of $\Lambda^2|_{\mathcal{B}_t \cap \mathcal{E}(\mathcal{H})}$ iff*

$$C_0 - \lambda t (C_0 + I)^{-1} = (\Psi^0)^* \xi, \tag{12}$$

for some $\lambda \in \mathbf{R}$ and $\xi \in \mathcal{H} \cdot \mathcal{H}$. For $t = 0$, $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})$ is a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})}$ iff

$$C_0 - (\Psi^0)^* \xi \in S^2(\ker(C_0 + I)),$$

or equivalently

$$(C_0 + I)(C_0 - (\Psi^0)^* \xi) = 0 \tag{13}$$

for some $\xi \in \mathcal{H} \cdot \mathcal{H}$.

According to the parametrization of $\mathcal{L}(\mathcal{H})$, we have

$$\operatorname{rank}(C + I) = \operatorname{rank}(A^* A) = \operatorname{rank}(A) = \dim V.$$

Thus (1) implies that the intersection $\mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})$ parametrizes the full spherical $\mathcal{H}$-maps $f : M \to S^{\nu-1}$.

It is a difficult and largely unsolved problem to obtain all possible range dimensions $\nu - 1$ for which a full spherical $\mathcal{H}$-map f exists. By the parametrization, this problem is equivalent to finding all $1 \le \nu \le n$, $n = \dim \mathcal{H}$, such that $\mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})$ is nonempty. In this case we say that ν is *feasible*.

The most studied case is $M = G/K = S^m$, the Euclidean m-sphere, and $\mathcal{H} = V_{\lambda_p} = \mathcal{H}_m^p \subset C^\infty(S^m)$, $\lambda_p = p(p + m - 1)$, the space of spherical harmonics of order p on S^m. According to our definitions, a spherical $\mathcal{H}_m^p$-map $f : S^m \to S_V$ is a λ_p-eigenmap [4]. Hence $1 \le \nu \le n(m,p)$, $n(m,p) = \dim \mathcal{H}_m^p$, is feasible iff a full λ_p-eigenmap $f : S^m \to S^{\nu-1}$ exists iff $\mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H}_m^p) \neq \emptyset$. A general lower bound for a feasible range is $\nu > m/2 + 1$ [22] (in fact, this applies to all homogeneous polynomial maps without harmonicity). According to a result in [7], for any $p \ge 2$, if ν is feasible for m, then $\nu + m + 2$ is feasible for $m + 1$. For quadratic eigenmaps ($p = 2$), many feasible ranges $2 \le \nu \le m(m+3)/2$ are known. For example, for $m \ge 5$, the higher ranges $\nu = m(m+3)/2 - r$ are feasible for $r = 0 - 4, 6, 10 - 12, 15$ [7,19]. In the following table we tabulate some of the known results [7,19,23]:

m	ν
3	3,5-9
4	5,8,10-14
5	5,8-20
6	13,15-27
7	5,8-9,11,13-25

On the other hand, for $m = 3, 4$, $\nu = 4$ is not feasible, and, for $m = 6$, $\nu = 6$ is not feasible.

Assume now that $M = G/K$ is a compact isotropy irreducible Riemannian homogeneous space, and $\mathcal{H} \subset C^\infty(M)$ is an irreducible G-submodule. Then $\mathcal{H}$ is contained in the full eigenspace V_λ for some eigenvalue λ [21]. As noted above, there exists a G-submodule $\mathcal{F}(\mathcal{H})$ of $S^2(\mathcal{H})$ such that $\mathcal{M}(\mathcal{H}) = \mathcal{K}_0 \cap \mathcal{F}(\mathcal{H})$ parametrizes the isometric minimal immersions $f : M \to S_V$ of the $\lambda/\dim M$-multiple of the metric on M. According to the parametrization, $\mathcal{B}_0^\nu \cap \mathcal{F}(\mathcal{H})$ parametrizes the full spherical minimal immersions $f : M \to S^{\nu-1}$. Once again, the problem is to list all feasible ν's, that is, those values of ν for which $\mathcal{B}_0^\nu \cap \mathcal{F}(\mathcal{H})$ is nonempty.

A general result of Moore asserts that $\nu \geq 2m + 1$ [8]. For quartic minimal immersions ($p = 4$), this lower bound can be improved: $\nu \geq m(m + 5)/6$ [12,13].

Returning to the general setting, let $f : M \to S_V$ be a full spherical $\mathcal{H}$-map, where $M = G/K$ and let $\mathcal{H} \subset C^\infty(M)$ be a finite dimensional G-module. We call $V_f = \operatorname{im}(\langle f\rangle + I) \subset \mathcal{H}$ the *space of components* of f, where $\langle f\rangle = A^*A - I$ and $f = A \circ \delta_{\mathcal{H}}$ with $A : \mathcal{H}^* \to V$ linear and surjective. Once and for all, we identify $\mathcal{H}$ with $\mathcal{H}^*$ via a fixed orthonormal basis $\{\chi^j\}_{j=1}^n \subset \mathcal{H}$. Since $V_f = \operatorname{im}(A^*A)$, a typical element of V_f is of the form $(A^*A)(\chi)$, where $\chi \in \mathcal{H}$. For $x \in M$, we have

$$\begin{aligned}(A^*A)(\chi)(x) &= \sum_{j=1}^n \langle (A^*A)(\chi), \chi^j\rangle \chi^j(x)\\ &= \langle (A^*A)(\chi^j), \chi\rangle \chi^j(x)\\ &= \langle (A^*A)\delta_{\mathcal{H}}(x), \chi\rangle\\ &= \langle A^*(f(x)), \chi\rangle\\ &= \langle f(x), A(\chi)\rangle = \alpha(f(x)),\end{aligned}$$

where we used (11), and $\alpha \in V^*$ is the linear functional of V defined by the last equality. We obtain that a typical element of V_f can be written as $\alpha \circ f$, where $\alpha \in V^*$.

Since f is full, precomposition with f gives rise to an isomorphism $f^* : V^* \to V_f$.

Let $f' : M \to S_{V'}$ be another full spherical $\mathcal{H}$-map. We say that f' is *derived* from f, written as $f' \leftharpoonup f$, if $V_{f'} \subset V_f$. Equivalently, according to the definition of a cell in Section 1, $f' \leftharpoonup f$ iff $\langle f'\rangle \leftharpoonup \langle f\rangle$. We define the *relative moduli space* $\mathcal{L}_f \subset \mathcal{L}(\mathcal{H})$ as the set of parameter points $\langle f'\rangle$, where $f' : M \to S_{V'}$ is a full spherical $\mathcal{H}$-map derived from f. Comparing this with the definition of a cell in Section 1, we see that

$$\mathcal{L}_f = I_{\langle f\rangle} \cap \mathcal{E}(\mathcal{H}).$$

In particular, the relative moduli space is a compact convex body, cut out from the moduli space $\mathcal{L}(\mathcal{H})$ by an affine subspace of $\mathcal{E}(\mathcal{H})$. The *open relative moduli space* $\mathcal{L}_f^\circ = I_{\langle f\rangle}^\circ \cap \mathcal{E}(\mathcal{H})$

is the relative interior of $\mathcal{L}_f$ in its affine span.
We call f *linearly rigid* [21] if $\mathcal{L}_f$ is trivial. By a result in [17], f is linearly rigid iff $\langle f\rangle = A^*A - I$, $f = A \circ \delta_{\mathcal{H}}$, is an *extremal point* of $\mathcal{L}(\mathcal{H})$ in the sense of convex geometry.
Let $f' \leftharpoondown f$ as above. Then there exists a surjective linear map $A' : V \to V'$ such that $f' = A \circ f$. Combining this with $f = A \circ \delta_{\mathcal{H}}$, we obtain $f' = A' \circ A \circ \delta_{\mathcal{H}}$. Letting $B = A'^*A' - I \in S^2(V)$, the parameter point $\langle f'\rangle$ can be rewritten as

$$\begin{aligned} \langle f'\rangle &= (A'A)^*(A'A) - I = A^*(A'^*A')A - I \\ &= A^*(B+I)A - I = A^*BA + \langle f\rangle, \end{aligned} \tag{14}$$

where we used $\langle f\rangle = A^*A - I$. We see that the tangent space to the relative interior $\mathcal{L}_f^\circ = I^\circ_{\langle f\rangle} \cap \mathcal{E}(\mathcal{H})$ at $\langle f\rangle$ is

$$T_{\langle f\rangle}(\mathcal{L}_f^\circ) = \{A^*BA \in \mathcal{E}(\mathcal{H}) \,|\, B \in S^2(V)\} + \langle f\rangle. \tag{15}$$

The condition $A^*BA \in \mathcal{E}(\mathcal{H})$ can be reformulated in terms of f as follows:

$$\begin{aligned} \langle (A^*BA)\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle &= \langle BA\delta_{\mathcal{H}}(x), A\delta_{\mathcal{H}}(x)\rangle \\ &= \langle Bf(x), f(x)\rangle = 0, \quad x \in M. \end{aligned}$$

Under the injective linear map $S^2(V) \to S^2(\mathcal{H})$ defined by $B \mapsto A^*BA$, $B \in S^2(V)$, the (linearized) tangent space above corresponds to

$$\mathcal{E}_f = \{B \in S^2(V) \,|\, \langle Bf, f\rangle = 0\}. \tag{16}$$

We obtain that f is linearly rigid iff $\mathcal{E}_f$ is trivial.

Theorem E. *Let $M = G/K$ and $\mathcal{H} \subset C^\infty(M)$ a finite dimensional G-submodule. Let $f : M \to S^{\nu-1}$ be a full spherical $\mathcal{H}$-map of boundary type ($\nu < n$). Then $\langle f\rangle$ is a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})}$ iff there exists $\xi \in \mathcal{H}\cdot\mathcal{H}$ such that the following two conditions are satisfied:*

(i) $\xi \perp V_f \cdot V_f^\perp$;

(ii) *For any $B \in S^2(\mathbf{R}^\nu)$, we have*

$$\langle G(f)^2 - G(f), B\rangle = \langle \xi, \langle Bf, f\rangle\rangle,$$

where $G(f) = \left(\langle f^l, f^k\rangle\right)_{l,k=1}^\nu$ is the Gram matrix of the components $(f^1, \ldots, f^\nu)$ of f, and the right-hand side is the scalar product of ξ and $\langle Bf, f\rangle \in \mathcal{H}\cdot\mathcal{H}$.

Remark 1. Conditions (i)–(ii), for $B \in S^2(\mathbf{R}^\nu)$, imply

$$\langle Bf, f\rangle \in V_f \cdot V_f^\perp \Rightarrow G(f)^2 - G(f) \perp B.$$

In particular, setting $\langle Bf, f\rangle = 0$ which is the defining equality for $\mathcal{E}_f$, we obtain that a critical point $\langle f\rangle$ must satisfy

$$G(f)^2 - G(f) \in \mathcal{E}_f^\perp. \tag{17}$$

(Note that this is automatically satisfied if f is linearly rigid.) Conversely, if this condition is satisfied then there is a unique $\xi \in V_f \cdot V_f$ that satisfies (ii). Indeed, let Ξ be a linear functional on $V_f \cdot V_f$ defined by $\Xi(\langle Bf, f\rangle) = \langle G(f)^2 - G(f), B\rangle$, $B \in V_f \cdot V_f$. Since $\langle Bf, f\rangle = 0$ iff $B \in \mathcal{E}_f$, we see that $G(f)^2 - G(f) \in (\mathcal{E}_f)^\perp$ guarantees that Ξ is well-defined. The existence of $\xi \in V_f \cdot V_f$ follows.

There is a simple geometric interpretation of the scalar product $\langle G(f)^2 - G(f), B\rangle$ in (ii) of Theorem E as follows. Let $B \in S^2(\mathbf{R}^\nu)$, and consider A^*BA in the linearized tangent space $T_{\langle f\rangle}(\mathcal{L}_f^\circ) - \langle f\rangle$ (cf. (15)). Using $\langle f\rangle = A^*A - I$ and $G(f) = AA^*$, we have

$$\begin{aligned} \langle\langle f\rangle, A^*BA\rangle &= \langle A^*A - I, A^*BA\rangle \\ &= \langle A(A^*A - I)A^*, B\rangle \\ &= \langle AA^*AA^* - AA^*, B\rangle \\ &= \langle G(f)^2 - G(f), B\rangle. \end{aligned}$$

Thus, $\langle G(f)^2 - G(f), B\rangle$ is the scalar product of $\langle f\rangle$ with the vector $A^*BA \in T_{\langle f\rangle}(\mathcal{L}_f^\circ) - \langle f\rangle$ corresponding to B.

Recall from Section 1 that $\langle f\rangle$ is a weak critical point if $\langle f\rangle$ is orthogonal to the relative moduli space $\mathcal{L}_f = I_{\langle f\rangle} \cap \mathcal{E}(\mathcal{H})$. By the computation above, $\langle f\rangle$ is a weak critical point iff (17) is satisfied.

Remark 2. Let $\mathcal{H} \subset C^\infty(M)$ be irreducible. Given a full spherical $\mathcal{H}$-map $f : M \to S^{\nu-1}$, for $B \in \mathcal{E}_f$, integrating the defining equality $\langle Bf, f\rangle = \sum_{l,k=1}^{\nu} b_{lk} f^l f^k = 0$, we obtain $\sum_{l,k=1}^{\nu} b_{lk} G(f)_{lk} = 0$ so that $\langle B, G(f)\rangle = 0$. (By irreducibility of $\mathcal{H}$, up to a constant multiple, the scalar product on $\mathcal{H}$ is the same as the L^2-scalar product.) We obtain that

$$G(f) \in \mathcal{E}_f^\perp$$

is always satisfied. Combining this with the above, we see that

$$G(f)^2 \in \mathcal{E}_f^\perp$$

holds for a (weak) critical point $\langle f\rangle$.

Endow $C^\infty(M)$ with the L^2-scalar product. We say that a full $\mathcal{H}$-map $f : M \to S_V$ *has orthonormal components* if, with respect to an orthonormal basis in V that identifies V with $\mathbf{R}^\nu$, the components $f^1, \ldots, f^\nu$ of f are L^2-orthogonal with the same L^2-norm. Since $\sum_{j=1}^{\nu} (f^j)^2 = 1$, integrating we find that this common L^2-norm square is $\mathrm{vol}\,(M)/\dim V = \mathrm{vol}\,(M)/\nu$.

If f has orthonormal components then (relative to an L^2-orthonormal basis) its Gram matrix $G(f)$ is a constant multiple of the identity: $G(f) = (\mathrm{vol}\,(M)/\nu)I$. Then $G(f)^2 - G(f)$ is also a constant multiple of the identity. We see that (i)–(ii) are satisfied for ξ constant, in fact, $\xi = \mathrm{vol}\,(M)/\nu - 1$. Theorem E implies that if f is of boundary type then $\langle f\rangle$ is a critical point. Conversely, if $\xi = \lambda_0 \in \mathcal{H} \cdot \mathcal{H}$ is constant then (i) is automatically satisfied while (ii) gives $G(f)(G(f) - (1 + \lambda_0)I) = 0$. Since f is full, $G(f)$ is positive definite, and $G(f) = (1 + \lambda_0)I$ follows. We obtain that f has orthonormal components (and $\lambda_0 = \mathrm{vol}\,(M)/\nu$).

Corollary 3. *Choose the L^2-scalar product on $C^\infty(M)$. In Corollary 2, $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})$ is a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})}$ with $\xi = \lambda_0 \in \mathcal{H} \cdot \mathcal{H}$ constant iff the corresponding full spherical $\mathcal{H}$-map $f : M \to S_V$ has orthonormal components.*

The classification of spherical $\mathcal{H}$-maps with orthonormal components, in particular, the classification of eigenmaps with orthonormal components is a difficult and largely unsolved problem posed by R. T. Smith [18] in the late seventies.

Proof of Theorem E. We write everything in terms of a fixed orthonormal basis $\{\chi^j\}_{j=1}^n \subset \mathcal{H}$. The equation $f = A \circ \delta_{\mathcal{H}}$ can be written as $f^l = \sum_{j=1}^n a_{lj}\chi^j$, $l = 1 \ldots, \nu$. With this, we see that $G(f) = AA^*$. Substituting $C_0 = \langle f \rangle = A^*A - I \in \mathcal{B}_0^\nu \cap \mathcal{E}(\mathcal{H})$ into (13), we obtain

$$\begin{aligned} (\langle f \rangle + I)(\langle f \rangle - (\Psi^0)^*\xi) &= A^*A(A^*A - I - (\Psi^0)^*(\xi)) \\ &= A^*(G(f)A - A - A(\Psi^0)^*(\xi)) = 0. \end{aligned}$$

Since f is full, A is surjective and A^* is injective. We obtain the equation

$$G(f)A = A(I + (\Psi^0)^*(\xi)).$$

In coordinates:

$$\sum_{k=1}^{\nu} G(f)_{lk} a_{kj} = a_{lj} + \langle \xi, f^l \chi^j \rangle,$$

where we used (10) and $f = A \circ \delta_{\mathcal{H}}$. Multiplying by χ^j and summing up with respect to j, we obtain

$$\sum_{k=1}^{\nu} G(f)_{lk} f^k = f^l + \sum_{j=1}^{n} \langle \xi, f^l \chi^j \rangle \chi^j. \tag{18}$$

In particular the sum on the right-hand side must belong to V_f. Taking the scalar product of both sides of (18) with $\chi \in V_f^\perp$, we obtain

$$\langle \xi, f^l \chi \rangle = 0.$$

Since the f^l's span V_f, this is condition (i).
Taking the scalar product of both sides of (18) with another component of f, we arrive at

$$(G(f)^2 - G(f))_{lk} = \langle \xi, f^l f^k \rangle.$$

Given $B \in S^2(\mathbf{R}^\nu)$, multiplying both sides by the lk-entry b_{lk} and summing up with respect to the indices, we obtain (ii). Theorem E follows.

To close this section, we consider the spherical case $M = S^m$ with $\mathcal{H} = \mathcal{H}_m^p$, the space of spherical harmonics of order p on S^m.
We have $C^\infty(S^m) = \sum_{p=0}^\infty \mathcal{H}_m^p$. In addition, $\mathcal{H}_m^p$ can be identified with the space of harmonic homogeneous polynomials of degree p on $\mathbf{R}^{m+1}$. We let $\mathcal{P}_{m+1}^p$ denote the space of homogeneous polynomials of degree p on $\mathbf{R}^{m+1}$. We have $\mathcal{P}_{m+1}^p = \sum_{j=0}^{[p/2]} \mathcal{H}_m^{p-2j}$. With this and a result in [11] noted in Section 1, $\mathcal{H}_m^p \cdot \mathcal{H}_m^p = \mathcal{P}_{m+1}^{2p} \subset C^\infty(S^m)$. For any p, we endow $\mathcal{P}_{m+1}^p$

with the $SO(m+1)$-invariant scalar product $\langle\cdot,\cdot\rangle$, defined by $\langle\xi,\eta\rangle = \partial_\xi \eta$, $\xi,\eta \in \mathcal{P}^p_{m+1}$. Here ∂_ξ is the polynomial differential operator defined by ξ. On $\mathcal{H}^p_m$, up to a constant multiple, this restricts to the L^2-scalar product since $\mathcal{H}^p_m$ is irreducible.
The moduli space $\mathcal{L}(\mathcal{H}^p_m)$ is nontrivial iff $m \geq 3$ and $p \geq 2$. (In fact, the decomposition of $\mathcal{E}(\mathcal{H}^p_m)$ into $SO(m+1)$-irreducible components and therefore $\dim \mathcal{L}(\mathcal{H}^p_m) = \dim \mathcal{E}(\mathcal{H}^p_m)$ are known [14].) The first nontrivial moduli space $\mathcal{L}(\mathcal{H}^2_3)$ is 10-dimensional, and its boundary is completely described in [10,16]. In particular, $\mathcal{B}^\nu_0 \cap \mathcal{E}(\mathcal{H}^2_3) \neq \emptyset$ iff $\nu = 3, 5-8$. Each rank-stratum $\mathcal{B}^\nu_0 \cap \mathcal{E}(\mathcal{H}^2_3)$ is the $SO(4)$-orbit of a single open relative moduli space. Hence, a weak critical point in $\mathcal{B}^\nu_0 \cap \mathcal{E}(\mathcal{H}^2_3)$ is automatically a critical point of $\Lambda^2|_{\mathcal{B}^\nu_0 \cap \mathcal{E}(\mathcal{H}^2_3)}$.

Theorem F. *Each open relative moduli space on the boundary of $\mathcal{L}(\mathcal{H}^2_3)$ contains a unique critical point of $\Lambda^2|_{\mathcal{B}^\nu_o \cap \mathcal{E}(\mathcal{H}^2_3)}$.*

Proof. Once again, from the classification of quadratic eigenmaps $f : S^3 \to S^{\nu-1}$ of boundary type [10,16], it follows easily that if f has orthonormal components then $\nu = 3, 6$. For $\nu = 3$, f is congruent to the Hopf map, and, for $\nu = 6$, f is congruent to the complex Veronese map. By Corollary 3, these are critical points of $\Lambda^2|_{\mathcal{B}^\nu_o \cap \mathcal{E}(\mathcal{H}^2_3)}$. In fact, at the extremal parameter point corresponding to the Hopf map, Λ^2 actually attains a maximum. The relative moduli space of the Veronese map is a flat 2-dimensional disk with center corresponding to the Veronese map itself. By the unicity of critical points in a cell, it remains to exhibit examples of critical points for $\nu = 5, 7, 8$. By what was said above, we can confine ourselves to weak critical points.
We first consider the case of quadratic eigenmaps $f : S^3 \to S^4$ ($\nu = 5$). A well-known example is obtained by applying the Hopf-Whitehead construction to the real tensor product $\otimes : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$ [4,18]. In coordinates, we have

$$f(x,y,u,v) = (x^2 + y^2 - u^2 - v^2, 2xu, 2xv, 2yu, 2yv), \ (x,y,u,v) \in \mathbf{R}^4.$$

Note that the relative moduli space containing the parameter point $\langle f \rangle$ is a line segment with $\langle f \rangle$ at the midpoint. The Gram-matrix $G(f)$ is diagonal:

$$G(f) = \operatorname{diag}(8,4,4,4,4).$$

To show that $\langle f \rangle$ is a weak critical point we need to verify that (17) holds. Using the definition of $\mathcal{E}_f$ in (16), a simple computation shows that $B \in S^2(\mathbf{R}^5)$ belongs to $\mathcal{E}_f$ iff B is of the form

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & -a & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a \in \mathbf{R}$. Taking the scalar product of B with $G(f)^2 - G(f)$, we see that (17) holds. Thus $\langle f \rangle$ is a (weak) critical point.

Remark. To show directly that $\langle f\rangle$ is a critical point via theorem E is more technical. A somewhat tedious computation[1] yields that $\xi \in \mathcal{P}_4^4 = \mathcal{H}_3^2 \cdot \mathcal{H}_3^2$ satisfying (i)–(ii) has the form

$$\begin{aligned}\xi &= a(x^4+y^4)+b(u^4+v^4)+c(x^3y-xy^3)+d(u^3v-uv^3)\\ &\frac{3}{4}\left(x^2u^2+x^2v^2+y^2u^2+y^2v^2\right)+px^2y^2+qu^2v^2,\end{aligned}$$

where

$$6a+6b+p+q=10.$$

Next we consider quadratic eigenmaps $f: S^3 \to S^6$. A typical relative moduli space corresponding to these eigenmaps is the interior of a straight cone [16]. We need to find the critical point in the cone. Since a one-parameter group in $SO(4)$ rotates each cone about its axis, by unicity of critical points in a cell (Theorem D), the critical point must be on the axis. We define the one-parameter family $f_\alpha : S^3 \to S^6$, $0<\alpha<1$, of quadratic eigenmaps by

$$\begin{aligned}f_\alpha(x,y,u,v) &= \Big(\alpha(x^2+y^2-u^2-v^2), \sqrt{1-\alpha^2}(x^2-y^2), \sqrt{1-\alpha^2}(u^2-v^2),\\ &\quad 2\sqrt{1-\alpha^2}xy, 2\sqrt{1-\alpha^2}uv, \sqrt{2+2\alpha^2}(xu-yv), \sqrt{2+2\alpha^2}(xv+yu)\Big).\end{aligned}$$

Then the points $\langle f_\alpha\rangle$, $0<\alpha<1$, fill the axis of symmetry of a cone. To locate a possible candidate for the weak critical point, notice that $B_\alpha \in S^2(\mathbf{R}^7)$ defined by

$$B_\alpha = \operatorname{diag}\left(\frac{1}{\alpha^2}, -\frac{1}{1-\alpha^2}, -\frac{1}{1-\alpha^2}, -\frac{1}{1-\alpha^2}, -\frac{1}{1-\alpha^2}, \frac{1}{1+\alpha^2}, \frac{1}{1+\alpha^2}\right),$$

belongs to $\mathcal{E}_{f_\alpha}$. By (17), this means that $\langle G(f_\alpha)^2 - G(f_\alpha), B_\alpha\rangle = 0$ must be satisfied. Working out the scalar products of the components of f_α, we obtain

$$G(f_\alpha) = \operatorname{diag}(8\alpha^2, 4(1-\alpha^2), 4(1-\alpha^2), 4(1-\alpha^2), 4(1-\alpha^2), 4(1+\alpha^2), 4(1+\alpha^2)).$$

With this, we have

$$\langle G(f_\alpha)^2 - G(f_\alpha), B\rangle = 160\alpha^2 - 32.$$

This vanishes iff $\alpha = 1/\sqrt{5}$. Therefore only possible weak critical point is at $\langle f\rangle$, where $f = f_{1/\sqrt{5}} : S^3 \to S^6$ is given by

$$\begin{aligned}f(x,y,u,v) &= \Big(\frac{1}{\sqrt{5}}(x^2+y^2-u^2-v^2),\\ &\quad \frac{2}{\sqrt{5}}(x^2-y^2), \frac{2}{\sqrt{5}}(u^2-v^2), \frac{4}{\sqrt{5}}xy, \frac{4}{\sqrt{5}}uv,\\ &\quad \frac{2\sqrt{3}}{\sqrt{5}}(xu-yv), \frac{2\sqrt{3}}{\sqrt{5}}(xv+yu)\Big).\end{aligned}$$

[1] The use of a computer algebra system such as Maple or Mathematica is recommended.

A matrix $B \in S^2(\mathbf{R}^7)$ belongs to $\mathcal{E}_f$ iff it has the form

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a/4 & 0 & b & c & 0 & 0 \\ 0 & 0 & -a/4 & c & -b & 0 & 0 \\ 0 & b & c & -a/4 & 0 & 0 & 0 \\ 0 & c & -b & 0 & -a/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (a-4b)/6 & -2c/3 \\ 0 & 0 & 0 & 0 & 0 & -2c/3 & (a+4b)/6 \end{bmatrix}$$

where $a, b, c \in \mathbf{R}$. The scalar product of this with $G(f)^2 - G(f)$ vanishes. We obtain that $\langle f \rangle$ is a (weak) critical point. Notice that $\langle f \rangle$ is *not* the center of mass of its cell.
Once again, a tedious computation yields that there is a unique $\xi \in \mathcal{P}_4^4$ satisfying (i)–(ii) of Theorem E. The explicit form of ξ is

$$\xi = \frac{11}{40}\left(x^4 + y^4 + u^4 + v^4\right) + \frac{11}{20}\left(x^2y^2 + u^2v^2\right) + \frac{19}{20}\left(x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2\right).$$

It remains to consider quadratic eigenmaps $f : S^3 \to S^7$. Once again, an example to this quadratic eigenmap is given in coordinates as

$$f(x, y, u, v) = (x^2 - y^2, u^2 - v^2, 2xy, 2uv, \sqrt{2}xu, \sqrt{2}xv, \sqrt{2}yu, \sqrt{2}yv).$$

(Note that all the examples to quadratic eigenmaps above were studied in [16] without having recognized their critical behavior; the only exception is $f : S^3 \to S^6$ which needed to be modified here slightly.) A typical element $B \in \mathcal{E}_f \subset S^2(\mathbf{R}^8)$ takes the form

$$\begin{bmatrix} 0 & -a & 0 & b & 0 & 0 & 0 & 0 \\ -a & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & e & 0 & 0 & 0 & 0 \\ b & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & -b & -c & d \\ 0 & 0 & 0 & 0 & -b & -a & -d-2e & c \\ 0 & 0 & 0 & 0 & -c & -d-2e & -a & b \\ 0 & 0 & 0 & 0 & d & c & b & a \end{bmatrix}$$

where $a, b, c, d, e \in \mathbf{R}$. Once again (17) holds, so that $\langle f \rangle$ is a (weak) critical point.
Note also that $\xi \in \mathcal{P}_4^4$ satisfying (i)–(ii) is given by

$$\begin{aligned} \xi \;&=\; \frac{3}{8}\left(x^4 + y^4 + u^4 + v^4\right) + c(x^3y - xy^3) + d(u^3v - uv^3) \\ &\; \frac{1}{4}(x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2) + \frac{3}{4}(x^2y^2 + u^2v^2). \end{aligned}$$

Finally, the minimum value of Λ^2 is attained at the point corresponding to this eigenmap. Theorem F follows.

The structure of the moduli space $\mathcal{L}_3^3$ parametrizing cubic eigenmaps $f : S^3 \to S^{\nu-1}$ is much more subtle, and very little is known even at the level of examples. Here we confine ourselves to construct a few cubic eigenmaps.
A full cubic eigenmap $f : S^3 \to S^5$ is given in complex coordinates $z = x + iy$, $w = u + iv$ by

$$f(z,w) = \left(z^3, \sqrt{3}z^2 w, \sqrt{3}zw^2, w^3\right).$$

Note that f cannot be $SU(2)$-equivariant [12,13]. By a result of Moore [8] it cannot be a minimal immersion. A simple computation shows that f is linearly rigid, that is $\langle f \rangle$ is an extremal point of $\mathcal{L}_3^3$. In particular, $\langle f \rangle$ is a weak critical point.
Another fairly obvious example of a full cubic eigenmap $f : S^3 \to S^7$ is given by

$$f(z,w) = \left(z^3 - \bar{w}^3, , \sqrt{2}(z^2 w + \bar{z}\bar{w}^2), z^2 w - \bar{z}\bar{w}^2\right).$$

The relative moduli $\mathcal{L}_f$ is 6-dimensional. In fact a typical element $B \in \mathcal{E}_f \subset S^2(\mathbf{R}^8)$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & b & -a & d & -c \\ 0 & 0 & -2a/\sqrt{3} & -2b/\sqrt{3} & -c/3 & -d/3 & p & q \\ 0 & 0 & -2b/\sqrt{3} & 2a/\sqrt{3} & -d/3 & c/3 & q & -p \\ a & b & -c/3 & -d/3 & -2p/\sqrt{3} & -2q/\sqrt{3} & 0 & 0 \\ b & -a & -d/3 & c/3 & -2q/\sqrt{3} & -2p/\sqrt{3} & 0 & 0 \\ c & d & p & q & 0 & 0 & 0 & 0 \\ d & -c & q & -p & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a, b, c, d, p, q \in \mathbf{R}$. The scalar product of this with $G(f)^2 - G(f)$ vanishes, and we obtain that $\langle f \rangle$ is a weak critical point. It is an interesting fact that there is yet another full cubic eigenmap $f : S^3 \to S^7$ (with the same range dimension) which is linearly rigid. This eigenmap is given by

$$f(z,w) = \left(z(|z|^2 - 2|w|^2), w(|w|^2 - 2|z|^2), \sqrt{3}z^2\bar{w}, \sqrt{3}w^2\bar{z}\right).$$

Since f is linearly rigid, $\langle f \rangle$ is an extremal point and a weak critical point.
Degree raising [10,14] defines an $SO(4)$-equivariant imbedding $\mathcal{L}_3^2 \to \mathcal{L}_3^3$. Degree raising applied to the quadratic eigenmaps above then gives several cubic eigenmaps. For example, degree raising applied to the Hopf map gives a full cubic eigenmap $f : S^3 \to S^7$ which is also linearly rigid.

The problem of critical points for quartic minimal immersions : $S^3 \to S^{\nu-1}$ of boundary type is subtle, although a full geometric description of the 18-dimensional moduli space $\mathcal{M}(\mathcal{H}_3^4)$ is given in [10]. Note that it is no longer true that a rank stratum is the $SO(4)$-orbit of a single open relative moduli space. (If $SO(4) = SU(2) \cdot SU(2)'$ is the canonical splitting, $\mathcal{M}(\mathcal{H}_3^4)$ is the convex hull of the equivariant moduli spaces $\mathcal{M}(\mathcal{H}_3^4)^{SU(2)}$ and $\mathcal{M}(\mathcal{H}_3^4)^{SU(2)'}$. In $\mathcal{M}(\mathcal{H}_3^4)^{SU(2)}$, the rank stratum $(\mathcal{B}_0^{20})^{SU(2)}$ has a one-parameter family of relative moduli spaces on mutually different $SU(2)'$-orbits.) It follows from this geometric description that if $f : S^3 \to S^{\nu-1}$ has orthonormal components then $\nu = 10$ or $\nu = 15$. In either case, f is congruent to an explicitly known $SU(2)$-equivariant quartic minimal immersion. The two parameter points corresponding to these minimal immersions are antipodal and their orbits are octahedral manifolds minimally imbedded in their respective spheres [10]. By Corollary 3, $\Lambda^2|_{\mathcal{B}_o^\nu \cap \mathcal{F}(\mathcal{H}_3^4)}$ attains critical values at the corresponding points.

3. Proofs of Theorems A–D

Proof of Theorem A. Let $C \in \mathcal{K}_0$, and let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $C+I$, $\dim \mathcal{H} = n$. Since $C + I \geq 0$, we have $\lambda_1, \ldots, \lambda_n \geq 0$. Since $\operatorname{trace}(C + I) = \operatorname{trace} I = n$, comparison of geometric and arithmetic means gives

$$(0 \leq) \sqrt[n]{\det(C + I)} = \sqrt[n]{\lambda_1 \cdots \lambda_n} \leq \frac{\lambda_1 + \ldots + \lambda_n}{n} = \frac{\operatorname{trace}(C + I)}{n} = 1,$$

with equality iff $\lambda_1 = \ldots = \lambda_n (= 1)$. We obtain that $\mathcal{B}_t$, $t = \det(C + I)$, is nonempty iff $0 \leq t \leq 1$ as stated in Section 1.

For the proof of Theorem A we need the differentiation formula

$$\frac{d^n}{dt^n} \log(\det(A + tB))|_{t=0} = (-1)^{n-1}(n-1)! \operatorname{trace}\left((A^{-1}B)^n\right). \tag{19}$$

where $A, B \in S^2(\mathcal{H})$ with $A > 0$. (This is obtained by expanding log into a power series.) We will make use of this formula in two specific instances. First, let $C_0 \neq 0$ be in the interior of $\mathcal{K}_0$. Setting $A = C_0 + I$, we have $A > 0$ and (19) applies. For $n = 1$, we obtain

$$\begin{aligned} \frac{d}{dt} \log \Delta(C_0 + tB)|_{t=0} &= \frac{d}{dt} \log\det(C_0 + I + tB)|_{t=0} \\ &= \operatorname{trace}\left((C_0 + I)^{-1}B\right) = \langle (C_0 + I)^{-1}, B\rangle. \end{aligned}$$

Thus, we have

$$(\operatorname{grad} \Delta)(C_0) = t(C_0 + I)^{-1} = \operatorname{Adj}(C_0 + I), \quad \det(C_0 + I) = t, \tag{20}$$

where Adj stands for the adjoint matrix. ($\operatorname{Adj}(C_0 + I)_{jl}$ is the determinant obtained from the determinant of $C_0 + I$ by deleting the j-th row and l-th column.)

Let $\mathcal{E} \subset S_0^2(\mathcal{H})$ be a G-submodule. The gradient $\operatorname{grad} \Delta|_{\mathcal{E}}$ of the restriction $\Delta|_{\mathcal{E}}$ at $C_0 \in \mathcal{K}_0 \cap \mathcal{E}$ is the orthogonal projection of $(\operatorname{grad} \Delta)(C_0)$ to $\mathcal{E}$. We now show that this projection is nonzero by working out its scalar product with $C_0 \in \mathcal{E}$. This scalar product is equal to

$$\langle C_0, t(C_0 + I)^{-1}\rangle = t \operatorname{trace}(C_0(C_0 + I)^{-1}) = tn - t \operatorname{trace}(C_0 + I)^{-1} \tag{21}$$

since $\operatorname{trace} I = n$.

As before, let λ_j, $j = 1, \ldots, n$, be the eigenvalues of $C_0 + I$. Since C_0 is in the interior of $\mathcal{K}_0$, $C_0 + I > 0$, and these eigenvalues are all positive. Since C_0 is traceless, we have

$$\operatorname{trace}(C + I) = \sum_{j=1}^n \lambda_j = \operatorname{trace} I = n.$$

On the other hand, we have

$$\operatorname{trace}(C_0+I)^{-1} = \sum_{j=1}^{n} \frac{1}{\lambda_j}.$$

Omitting t, the scalar product in (21) therefore rewrites as

$$\langle C_0, (C_0+I)^{-1}\rangle = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n} \frac{1}{\lambda_j} = 2n - \sum_{j=1}^{n}\left(\lambda_j + \frac{1}{\lambda_j}\right). \tag{22}$$

Now, $x + 1/x \geq 2$, $x > 0$, with equality iff $x = 1$. Since $C_0 \neq 0$, $\lambda_j \neq 1$ for at least one j, and the last sum is $> 2n$. Summarizing, we obtain

$$\langle C_0, (\operatorname{grad}\Delta)(C_0)\rangle = \langle C_0, t(C_0+I)^{-1}\rangle < 0. \tag{23}$$

In particular, $\operatorname{grad}\Delta|_{\mathcal{E}}$ is nonzero along $\mathcal{B}_t \cap \mathcal{E}$. By the implicit function theorem, $\mathcal{B}_t \cap \mathcal{E}$ is a real-analytic manifold.
Second, we set $n = 2$ in (19) and work out the Hessian of $-\log\Delta$. We obtain

$$\operatorname{Hess}(-\log\Delta)(C_0)(B,B) = \frac{d^2}{dt^2}\log\det(C_0+I+tB)|_{t=0} = \operatorname{trace}\left(((C_0+I)^{-1}B)^2\right).$$

Evaluating the trace on an orthonormal basis of eigenvectors of $(C_0+I)^{-1} > 0$, we see that the Hessian is positive definite. We obtain that $-\log\Delta$ is strictly convex on $\operatorname{int}\mathcal{K}_0$. Theorem A follows. (For the last statement note also that (23) expresses the fact that $\operatorname{grad}\Delta|_{\mathcal{E}}$ at C_0 points inside the sphere $\{C \in \mathcal{E} \,|\, |C| = |C_0|\}$. Hence, the tangent space of $\mathcal{B}_t \cap \mathcal{E}$ at C_0 is transversal to the radial line $\mathbf{R}\cdot C_0$.)

Proof of Theorem B. Given $C_0 \in \mathcal{B}_0^\nu$, choose an orthonormal basis in $\operatorname{im}(C_0+I) \subset \mathcal{H}$, and extend it to an orthonormal basis to the whole $\mathcal{H}$. With this $\mathcal{H} = \mathbf{R}^n$, and we can view the endomorphisms of $\mathcal{H}$ as $(n\times n)$-matrices. By the choice of the orthonormal basis, we have

$$C_0 + I = D_0 \oplus 0_{n-\nu}, \tag{24}$$

where $0 < D_0 \in S^2(\mathbf{R}^\nu)$ and we indicated the size of the zero matrix by a subscript. We choose $\epsilon > 0$ such that $|D - D_0| < \epsilon$, $D \in S^2(\mathbf{R}^\nu)$, implies $D > 0$, in particular, D^{-1} exists and is also positive definite. Finally, let

$$\mathcal{N} = \left\{ C \in S^2(\mathbf{R}^n) \;\middle|\; C + I = \begin{bmatrix} D & E \\ E^\top & \tilde{D} \end{bmatrix}, |D - D_0| < \epsilon \right\}.$$

Clearly, $\mathcal{N}$ is an open neighborhood of C_0 in $S^2(\mathbf{R}^n)$.
We have

$$C + I = \begin{bmatrix} D & E \\ E^\top & \tilde{D} \end{bmatrix} = \begin{bmatrix} D & 0 \\ E^\top & \tilde{D} - E^\top D^{-1}E \end{bmatrix}\begin{bmatrix} I_\nu & D^{-1}E \\ 0 & I_{n-\nu} \end{bmatrix}.$$

This shows that $\operatorname{rank}(C+I) = \nu$ iff $\tilde{D} = E^\top D^{-1}E$. Hence

$$\mathcal{B}_0^\nu \cap \mathcal{N} = \left\{ C \in S_0^2(\mathcal{H}) \;\middle|\; C + I = \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1}E \end{bmatrix}, |D - D_0| < \epsilon \right\} \tag{25}$$

is an open neighborhood of C_0 in $\mathcal{B}_0^\nu$. This is because $C+I = \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1}E \end{bmatrix} \geq 0$. Indeed, for $x \in \mathbf{R}^\nu$ and $y \in \mathbf{R}^{n-\nu}$, we have

$$\left\langle \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1}E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \left\langle D^{-1}(Dx+Ey), Dx+Ey \right\rangle \geq 0.$$

Assume now that $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}$, where $\mathcal{E} \subset S_0^2(\mathcal{H})$ is a G-submodule. Then $\mathcal{B}_0^\nu \cap \mathcal{E} \cap \mathcal{N}$ is an open neighborhood of C_0 in $\mathcal{B}_0^\nu \cap \mathcal{E}$, and we need to show that it contains an analytic neighborhood of C_0. The relations that define $\mathcal{B}_0^\nu \cap \mathcal{E} \cap \mathcal{N}$ are

$$\left\langle \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1}E \end{bmatrix} - I, Q \right\rangle = 0,\ Q \in \mathcal{E}^\perp, \text{ and } |D - D_0| < \epsilon.$$

As before, we use the implicit function theorem. In view of (25), analyticity will follow once we show that

$$\mathrm{grad}_{[D,E]} \left\langle \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1}E \end{bmatrix} - I, Q \right\rangle (D_0 \oplus 0_{n-\nu}) \neq 0, \tag{26}$$

for any $Q \in \mathcal{E}^\perp$. Here the gradient is to be taken with respect to the variables that are the entries of the rectangular matrix $[D,E]$ indicated by the subscript. To work out the gradient, we set

$$Q = \begin{bmatrix} Q_0 & F \\ F^\top & Q_1 \end{bmatrix} \in \mathcal{E}^\perp,\ Q_0 \in S^2(\mathbf{R}^\nu),\ Q_1 \in S^2(\mathbf{R}^{n-\nu}).$$

Substituting this into (26), and writing the scalar product as a trace, the gradient becomes

$$\begin{aligned} &\mathrm{grad}_{[D,E]} \Big(\mathrm{trace}\,(D-I)Q_0 + \mathrm{trace}\,(EF^\top) + \mathrm{trace}\,(E^\top F) \\ &\quad + \mathrm{trace}\,(E^\top D^{-1}E - I)Q_1 \Big) (D_0 \oplus 0_{n-\nu}). \end{aligned}$$

Using $D = D_0 + X$, $|X| < \epsilon$, we need to show that

$$\begin{aligned} &\mathrm{grad}_{[X,E]} \Big(\mathrm{trace}\,(D_0 + X - I)Q_0 + \mathrm{trace}\,(EF^\top) + \mathrm{trace}\,(E^\top F) \\ &\quad + \mathrm{trace}\,(E^\top (D_0+X)^{-1}E - I)Q_1 \Big) (0) \neq 0. \end{aligned}$$

The trace terms work out as

$$\begin{aligned} &\mathrm{grad}_X \mathrm{trace}\,(D_0 + X - I)Q_0 = \mathrm{grad}_X \mathrm{trace}\,(XQ_0) = Q_0, \\ &\mathrm{grad}_E \mathrm{trace}\,(EF^\top) = F, \\ &\mathrm{grad}_E \mathrm{trace}\,(E^\top F) = F, \\ &\mathrm{grad}_{[X,E]} \mathrm{trace}\,(E^\top (D_0+X)^{-1}E - I)Q_1 = 0. \end{aligned}$$

Summarizing, we obtain

$$\mathrm{grad}_{[D,E]} \left\langle \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1}E \end{bmatrix} - I, \begin{bmatrix} Q_0 & F \\ F^\top & Q_1 \end{bmatrix} \right\rangle (D_0 \oplus 0_{n-\nu}) = [Q_0, 2F]. \tag{27}$$

This vanishes iff $Q_0 = 0$ and $F = 0$. Thus, the only $Q \in (\mathcal{E})^\perp$ for which the gradient in (27) vanishes is of the form $Q = 0_\nu \oplus Q_1$, where $Q_1 \in S^2(\mathbf{R}^{n-\nu})$. Due to our choice of the basis, $\mathbf{R}^{n-\nu} = (\mathbf{R}^\nu)^\perp$ corresponds to $(\mathrm{im}\,(C_0 + I))^\perp = \ker\,(C_0 + I)$. Hence the gradient vanishes only for those $Q \in \mathcal{E}^\perp$ that are also elements of $S^2(\ker\,(C_0 + I))$, where the latter is considered as a linear subspace of $S^2(\mathcal{H})$ via the inclusion $\ker\,(C_0 + I) \subset \mathcal{H}$. The first statement of Theorem B follows.

To prove the second statement, let $\mathrm{rank}\,(C_0+I) = n-1$. Then $\ker\,(C_0+I) = \mathbf{R}\cdot v$, $0 \neq v \in \mathcal{H}$. Hence $S^2(\ker\,(C_0 + I)) = \mathbf{R} \cdot v \odot v$. Since $C_0 v = -v$, we have

$$\langle C_0, v \odot v\rangle = \langle C_0 v, v\rangle = -|v|^2 \neq 0.$$

Since $C_0 \in \mathcal{E}$, we see that $v \odot v \notin \mathcal{E}^\perp$, and the condition for analyticity in (2) is satisfied. Part (a) of Theorem B follows.

We now let $\mathcal{E} = S^2_0(\mathcal{H})$ so that $\mathcal{E}^\perp = S^2_0(\mathcal{H})^\perp = \mathbf{R} \cdot I$. The identity I is not contained in $S^2(\ker\,(C_0 + I))$. This is because, being traceless, $C_0 \neq I$, and $\ker\,(C_0 + I) \subset \mathcal{H}$ is a proper linear subspace. The condition of analyticity in (2) is satisfied. We obtain the first statement of (b) in Theorem B. Finally, we show (3). Given $C_0 \in \mathcal{B}^\nu_0$ as in (24), we let $s \mapsto C(s)$ be a smooth curve in $\mathcal{B}^\nu_0$ with $C(0) = C_0$ and $C'(0) = C'_0 \in \mathcal{E}$. We let

$$C(s) + I = \begin{bmatrix} D(s) & E(s) \\ E(s)^\top & E(s)^\top D(s)^{-1} E(s) \end{bmatrix}.$$

Note that $D(0) = D_0$ and $E(0) = 0$. Differentiating at $s = 0$, we obtain

$$C'_0 = \begin{bmatrix} D'(0) & E'(0) \\ E'(0)^\top & 0 \end{bmatrix}.$$

Since the lower right diagonal block corresponds to $\ker\,(C_0 + I)$, (3) follows.

Proof of Corollary. Let $C_0 \in \mathcal{B}^\nu_0$. We use the notations in the proof of Theorem B. By (24), we have

$$\mathcal{H} = \mathbf{R}^\nu \oplus \mathbf{R}^{n-\nu}, \quad \mathbf{R}^\nu = \mathrm{im}\,(C_0 + I).$$

A typical open neighborhood $\mathcal{U}$ of $\mathbf{R}^\nu \in G_\nu(\mathcal{H})$ in the Grassmann manifold $G_\nu(\mathcal{H})$ consists of ν-dimensional linear subspaces $L \subset \mathcal{H}$ that project onto $\mathbf{R}^\nu$. Any such L is uniquely determined as the graph of a linear map $\ell_L : \mathbf{R}^\nu \to \mathbf{R}^{n-\nu}$. In this way $\mathcal{U}$ can be identified with $\mathrm{Hom}\,(\mathbf{R}^\nu, \mathbf{R}^{n-\nu}) \cong \mathbf{R}^{\nu(n-\nu)}$.

Let $C \in \mathcal{B}^\nu_0 \cap \mathcal{N}$. Using the block decomposition of $C+I$ in (25) we see that a typical element in $\mathrm{im}\,(C + I)$ is of the form $\begin{bmatrix} Dx + Ey \\ E^\top x + E^\top D^{-1} Ey \end{bmatrix}$, where $x \in \mathbf{R}^\nu$ and $y \in \mathbf{R}^{n-\nu}$. The linear map ℓ_L corresponding to $L = \mathrm{im}\,(C + I)$ thus sends $Dx + Ey$ to $E^\top x + E^\top D^{-1} Ey$. With the identifications made above we obtain $\ell_L = E^\top D^{-1}$.

Let $s \mapsto C(s)$ be a smooth curve in $\mathcal{B}^\nu_0$ with $C(0) = C_0$ and $C'(0) = C'_0$. We have

$$(\Gamma_\nu)_*(C'_0) = \frac{d}{ds}\Gamma_\nu(C(s))|_{s=0} = \frac{d}{ds}(E(s)^\top D(s)^{-1})|_{s=0} = E'(0)^\top D_0$$

since $E(0) = 0$. C_0' is tangent to the fibre of Γ^ν through C_0 iff $(\Gamma_\nu)_*(C_0') = 0$. Since $D_0 > 0$, this happens iff $E'(0) = 0$, or equivalently, $C_0' = D'(0) \oplus 0_{n-\nu}$. Comparing this with (24) we see that

$$\mathrm{im}\,(C_0 + I) = \mathrm{im}\,(C_0 + \epsilon C_0' + I)$$

for small $\epsilon > 0$. By definition, this means that, C_0' is tangent to the cell $I^\circ_{C_0}$. By the computation above it is clear that Γ_ν is a (local) submersion. Since Γ_ν is obviously G-equivariant, it is a global submersion onto $G_\nu(\mathcal{H})$ since the latter is a single G-orbit. The corollary follows.

Proof of Theorem C. The level set $\mathcal{B}_t$, $0 < t < 1$, is defined by the equations

$$\det(C + I) = t \quad \text{and} \quad \mathrm{trace}\, C = 0.$$

$C_0 \in \mathcal{B}_t \cap \mathcal{E}$ is a critical point of $\Lambda^2|_{\mathcal{B}_t \cap \mathcal{E}}$ iff $d\Lambda^2(C_0)$ vanishes on $T_{C_0}(\mathcal{B}_t) \cap \mathcal{E}$. On the other hand, by (20) and the definition of the scalar product in $S^2(\mathcal{H})$, we have

$$T_{C_0}(\mathcal{B}_t) = \mathrm{Adj}\,(C_0 + I)^\perp \cap S^2_0(\mathcal{H}).$$

Indeed, let $s \mapsto C(s)$ be a smooth curve in $\mathcal{B}_t$ with $C(0) = C_0$ and $C'(0) = C_0' \in \mathcal{E}$. Differentiating $\det(C(s) + I) = t$ at $s = 0$, we obtain

$$\frac{d}{ds}\det(C(s) + I)|_{s=0} = \sum_{j,l=1}^{n} \frac{\partial}{\partial c_{jl}} \det(C + I)\,|_{C=C_0} \left.\frac{\partial c_{jl}}{\partial s}\right|_{s=0} = \langle\, \mathrm{Adj}\,(C_0 + I), C_0' \rangle = 0.$$

Given $\mathcal{E} \subset S^2_0(\mathcal{H})$, the differential $d\Lambda^2(C_0)$ vanishes on

$$T_{C_0}(\mathcal{B}_t) \cap \mathcal{E} = \mathrm{Adj}\,(C_0 + I)^\perp \cap \mathcal{E}$$

iff

$$C_0 \in (\,\mathrm{Adj}\,(C_0 + I)^\perp \cap \mathcal{E})^\perp = \mathbf{R} \cdot \mathrm{Adj}\,(C_0 + I) + \mathcal{E}^\perp.$$

The first statement of Theorem C follows. For the critical value, using $C_0 \in \mathcal{E}$, (5) and (20), we compute

$$|C_0|^2 = \lambda \langle C_0, \mathrm{Adj}\,(C_0 + I)\rangle = \lambda t \langle C_0, (C_0 + I)^{-1}\rangle.$$

The critical value stated in Theorem C now follows from (22).
For $\mathcal{E} = S^2_0(\mathcal{H})$, we have $\mathcal{E}^\perp = \mathbf{R} \cdot I$. By (5), C_0 is a critical point of $\Lambda^2|_{\mathcal{B}_t}$ iff

$$C_0 = \lambda\, \mathrm{Adj}\,(C_0 + I) + \mu I$$

for some $\lambda, \mu \in \mathbf{R}$. By (20), we can write this as

$$C_0 - \mu I = \lambda t (C_0 + I)^{-1},$$

or equivalently,

$$(C_0 - \mu I)\,(C_0 + I) = \lambda t I. \tag{28}$$

Let $v \in \mathcal{H}$ be an eigenvector of C_0 with eigenvalue α. (28) then gives

$$(\alpha - \mu)\,(\alpha + 1) = \lambda t. \tag{29}$$

This is a quadratic equation in α with two solutions. Since trace $(C_0) = 0$ and $C_0 \neq 0$, the two roots have different signs. We denote these roots by $\alpha_+ > 0$ and $\alpha_- < 0$. Since $0 < t < 1$, we have $C_0 + I > 0$ so that

$$\alpha_\pm > -1. \tag{30}$$

By (28), we have

$$\alpha_+ + \alpha_- = \mu - 1 \quad \text{and} \quad \alpha_+\alpha_- = -(\mu + \lambda t)\,. \tag{31}$$

Conversely, let $C_0 \in \mathcal{B}_t$, $0 < t < 1$, have two eigenvalues $\alpha_\pm$. Setting $\mu = \alpha_+ + \alpha_- + 1$ and $\lambda = -(\alpha_+\alpha_- + \mu)/t$, (31) says that $\alpha_\pm$ are the roots of equation (29). Since C_0 is diagonalizable, it not only satisfies its characteristic equation but it also satisfies its minimal polynomial. We obtain that (28) holds. This, however, is equivalent to (5) so that C_0 is a critical point of Λ^2 on $\mathcal{B}_t$.

Let $\nu_\pm$ denote the corresponding multiplicities of $\alpha_\pm$ as eigenvalues of C_0. Since trace $C_0 = 0$,

$$\nu_+\alpha_+ + \nu_-\alpha_- = 0 \tag{32}$$

and $\nu_+ + \nu_- = n$. Combining this with (29), (6) follows. Finally, since det $(C_0 + I) = t$, we also have

$$(\alpha_+ + 1)^{\nu_+}(\alpha_- + 1)^{\nu_-} = t. \tag{33}$$

Expressing α_- in terms of α_+ via (32) and using (33), we obtain (7). For the stated unicity of α satisfying (6)–(7), note that the left-hand side of (7) is strictly decreasing in α on $[0, n/\nu - 1]$.

Since symmetric endomorphisms are diagonalizable, the eigenvalues (counted with multiplicity) determine the endomorphism up to conjugation of an orthogonal transformation. Hence, for each value of the multiplicity of the positive eigenvalue, there is a unique $O(\mathcal{H})$-orbit of critical points as stated in Theorem C. The critical value of Λ^2 on this orbit is

$$\operatorname{trace} C_0^2 = \nu_+\alpha_+^2 + \nu_-\alpha_-^2 = \nu_+\left(1 + \frac{\nu_+}{\nu_-}\right)\alpha_+^2,$$

where we used (32). Theorem C follows.

Proof of Theorem D. $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}$ is a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}}$ iff $d\Lambda^2(C_0)$ vanishes on $T_{C_0}(\mathcal{B}_0^\nu) \cap \mathcal{E}$, By (3), this happens iff $d\Lambda^2(C_0)$ vanishes on

$$S^2(\ker(C_0 + I))^\perp \cap \mathcal{E},$$

or by (32),

$$C_0 \in \left(S^2(\ker(C_0 + I))^\perp \cap \mathcal{E}\right)^\perp = S^2(\ker(C_0 + I)) + \mathcal{E}^\perp.$$

The first statement of Theorem D follows.

Let $C_0 \in \mathcal{B}_0^\nu \cap \mathcal{E}$ be a critical point of $\Lambda^2|_{\mathcal{B}_0^\nu \cap \mathcal{E}}$. Since the cell $I_{C_0}^\circ$ is contained in $\mathcal{B}_0^\nu$ we see that C_0 is orthogonal to $I_{C_0}^\circ \cap \mathcal{E}$. Thus, if $C_1 \in I_{C_0}^\circ \cap \mathcal{E}$ were another critical point then the triangle $\Delta 0 C_0 C_1$ would have angle sum $> \pi$. Unicity of C_0 follows.

For $\mathcal{E} = S_0^2(\mathcal{H})$, we have $\mathcal{E}^\perp = \mathbf{R} \cdot I$. Condition (8) gives $C_0 - \mu I \in S^2(\ker(C_0 + I))$ for some $\mu \in \mathbf{R}$. Hence $(C_0 + I)(C_0 - \mu I) = 0$. In particular, C_0 has eigenvalues μ and -1 with

correponding multiplicities ν and $n-\nu$. On the other hand, trace $C_0 = \nu\mu - (n-\nu) = 0$ so that $\mu = n\nu - 1$. Hence C_0 has the stated eigenvalues in Theorem D.
By the unicity of a critical point in an open cell, it remains to show that on each open cell in $\mathcal{B}_0^\nu$ the centroid is a critical point. Let C_0 be the centroid of $I_{C_0}^\circ$. We claim that

$$O(\mathcal{H})(C_0) \cap I_{C_0}^\circ = \{C_0\}.$$

Assume that $g \cdot C_0 \in I_{C_0}^\circ$ for some $g \in O(\mathcal{H})$. Since the action of $O(\mathcal{H})$ respects the cell-decomposition, g maps $I_{C_0}^\circ$ onto itself isometrically. Thus, g must fix the centroid C_0. Hence $g \cdot C_0 = C_0$, and the claim follows.
Let $C \in I_{C_0}^\circ$, and choose a basis such that $C + I = D \oplus 0_{n-\nu}$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_\nu)$, $\lambda_1, \ldots, \lambda_\nu > 0$. Then $g \in O(\mathcal{H})$ leaves $\operatorname{im}(C+I) = \mathbf{R}^\nu$ invariant iff $g \in O(\nu) \times O(n-\nu)$. Thus the action of $O(\nu) \times O(n-\nu)$ on C establishes a one-to-one correspondence between the sets $O(\mathcal{H})(C) \cap I_{C_0}^\circ$ and $(O(\nu) \times O(n-\nu))/\mathcal{C}$, where $\mathcal{C}$ is the centralizer of C in $O(\nu) \times O(n-\nu)$. We now claim that if $C \in I_{C_0}^\circ$ is not critical then $O(\mathcal{H})(C) \cap I_{C_0}^\circ$ consists of at least two points. Indeed, if C is not critical then, by what we proved above, C has at least three eigenvalues, or equivalently, $C + I$ has at least two nonzero eigenvalues. It is now clear that $\mathcal{C}$ is a proper subgroup of $O(\nu) \times O(n-\nu)$. The claim follows. Combining the two claims, we obtain that the centroid is always critical. The last statement of Theorem D is clear.

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