Michigan Math. J. 52 (2004)

Simplicial Intersections of a Convex Set and Moduli for Spherical Minimal Immersions

Gabor Toth

1. Introduction and Statement of Results

Let $\mathcal{H}$ be a Euclidean vector space. Let $S^2_0(\mathcal{H})$ denote the space of symmetric endomorphisms of $\mathcal{H}$ with vanishing trace; $S^2_0(\mathcal{H})$ is a Euclidean vector space with respect to the natural scalar product $\langle C, C'\rangle = \text{trace}(CC')$, $C, C' \in S^2_0(\mathcal{H})$. We define the (reduced) *moduli space* [7] as

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \{C \in S^2_0(\mathcal{H}) \mid C + I \geq 0\},$$

where $\geq$ means positive semidefinite.

We observe that $\mathcal{K}_0$ is a convex body in $S^2_0(\mathcal{H})$. The interior of $\mathcal{K}_0$ consists of those $C \in \mathcal{K}_0$ for which $C + I > 0$, and the boundary of $\mathcal{K}_0$ consists of those $C \in \mathcal{K}_0$ for which $C + I$ has nontrivial kernel. The eigenvalues of the elements in $\mathcal{K}_0$ are contained in $[-1, \dim \mathcal{H} - 1]$. Hence $\mathcal{K}_0$ is compact. Finally, an easy argument using $\text{GL}(\mathcal{H})$-invariance of $\mathcal{K}_0$ shows that the centroid of $\mathcal{K}_0$ is the origin.

Let M be a compact Riemannian manifold and $\mathcal{H} = \mathcal{H}_\lambda$ the eigenspace of the Laplacian $\triangle^M$ (acting on functions of M) corresponding to an eigenvalue λ. The DoCarmo–Wallach moduli space that parameterizes spherical minimal immersions $f : M \to S_V$ of M into the unit sphere S_V of a Euclidean vector space V, for various V, is the intersection $\mathcal{K}_0 \cap \mathcal{E}_\lambda$, where $\mathcal{E}_\lambda$ is a linear subspace of $S^2_0(\mathcal{H}_\lambda)$. Here f is an isometric minimal immersion of $\dim M/\lambda$ times the original metric of M. (For further details, see [3; 6; 8].) Intersecting $\mathcal{K}_0$ further with suitable linear subspaces of $\mathcal{E}_\lambda$, we obtain moduli that parameterize spherical minimal immersions with additional geometric properties (such as higher-order isotropy, equivariance with respect to an acting group of isometries of M, etc.).

A result of Moore [4] states that a spherical minimal immersion $f : S^m \to S^n$ with $n \leq 2m - 1$ is totally geodesic; in particular, the image of f is a great m-sphere in S^n. An important example showing that the upper bound is sharp is provided by the *tetrahedral minimal immersion* $f : S^3 \to S^6$ (see [2; 6]). Here f is $\text{SU}(2)$-equivariant and non–totally geodesic. The name comes from the fact that the invariance group of f is the binary tetrahedral group $\mathbf{T}^* \subset S^3 = \text{SU}(2)$, so that f factors through the canonical projection $S^3 \to S^3/\mathbf{T}^*$ and gives a minimal *imbedding* $\bar{f} : S^3/\mathbf{T}^* \to S^6$ of the tetrahedral manifold $S^3/\mathbf{T}^*$ into S^6.

Let $M = S^3$ and let $\mathcal{H}_{\lambda_p}$ be the pth eigenspace of the Laplacian on S^3 corresponding to the eigenvalue $\lambda_p = p(p + 2)$. According to a result in [5; 6] there

Received January 28, 2003. Revision received April 15, 2003.

exists a 2-dimensional linear subspace $\mathcal{E} \subset \mathcal{E}_{\lambda_6} \subset S^2_0(\mathcal{H}_{\lambda_6})$ containing the parameter point C_1 corresponding to the tetrahedral minimal immersion, such that the intersection $\mathcal{K}_0 \cap \mathcal{E}$ is a triangle with one vertex at C_1. The computations leading to this result are tedious. (It is relatively easy to obtain another vertex, say C_2, of the triangle, but the the main technical difficulty lies in finding the third vertex.)

Note that a similar analysis can be carried out for the octahedral minimal immersion $f\colon S^3 \to S^8$ (with invariance group $\mathbf{O}^* \subset S^3$, the binary octahedral group, and factored map $\bar{f}\colon S^3/\mathbf{O}^* \to S^8$, a minimal imbedding of the octahedral manifold $S^3/\mathbf{O}^*$ into S^8). Once again, there exists a 3-dimensional linear subspace $\mathcal{E} \subset \mathcal{E}_{\lambda_8} \subset S^2_0(\mathcal{H}_{\lambda_8})$ such that the intersection $\mathcal{K}_0 \cap \mathcal{E}$ is a tetrahedron.

A fundamental problem in the theory of moduli is to study the structure of the intersections $\mathcal{K}_0 \cap \mathcal{E}$ for various linear subspaces $\mathcal{E} \subset S^2_0(\mathcal{H})$. In view of the examples just given and since simplices are the simplest convex sets, it is natural to ask: When is the intersection $\mathcal{K}_0 \cap \mathcal{E}$ a simplex?

THEOREM A. *Let $C_1, \dots, C_n \in \partial\mathcal{K}_0$ be linearly independent with linear span $\mathcal{E}$. Then $\mathcal{K}_0 \cap \mathcal{E}$ is an n-simplex (with vertices $C_1, \dots, C_n$ and another vertex C_0) if and only if the following two conditions are satisfied*:

(i) $$\bigcap_{i=1}^{n} \ker(C_i + I) \neq \{0\};$$

(ii) $$I - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}(C_i + I) \geq 0 \text{ but } \not\geq 0,$$

where $\Lambda(C)$ is the largest eigenvalue of $C \in \partial\mathcal{K}_0$.

We will prove Theorem A in Section 4. At the end of that section we also check that conditions (i) and (ii) are satisfied in the setting for the tetrahedral minimal immersion.

As a technical tool for proving Theorem A, we introduce a sequence of invariants $\sigma_m(\mathcal{L})$, $m \geq 1$, associated to a compact convex body $\mathcal{L}$ in a Euclidean vector space. We define $\sigma_m(\mathcal{L})$ in a general setting of convex geometry.

Let $\mathcal{E}$ be a Euclidean vector space. Given a subset $\mathcal{S}$ of $\mathcal{E}$, we denote its convex hull by $[\mathcal{S}]$ and its affine hull by $\langle\mathcal{S}\rangle$. Then we have $[\mathcal{S}] \subset \langle\mathcal{S}\rangle \subset \mathcal{E}$. If $\mathcal{S}$ is finite, $\mathcal{S} = \{C_0, \dots, C_m\}$, then the convex hull and the affine hull are denoted by $[C_0, \dots, C_m]$ and $\langle C_0, \dots, C_m\rangle$, respectively. Then $[C_0, \dots, C_m]$ is a convex polytope in $\langle C_0, \dots, C_m\rangle$ (see [1]). The dimension $\dim[C_0, \dots, C_m] = \dim\langle C_0, \dots, C_m\rangle$ is maximal ($= m$) iff $[C_0, \dots, C_m]$ is an m-simplex.

A convex set $\mathcal{L}$ in $\mathcal{E}$ is called a *convex body* if $\mathcal{L}$ has nonempty interior, $\operatorname{int}\mathcal{L} \neq \emptyset$. Let $\mathcal{L} \subset \mathcal{E}$ be a compact convex body with base point $\mathcal{O} \in \operatorname{int}\mathcal{L}$. Given a boundary point $C \in \partial\mathcal{L}$, it is well known [1] that the line passing through C and $\mathcal{O}$ intersects $\partial\mathcal{L}$ at another point C^o. We call this the *opposite* of C (relative to $\mathcal{O}$). Clearly, $(C^o)^o = C$. Let d be the distance function on $\mathcal{E}$. We call the ratio $\Lambda(C) = d(\mathcal{O}, C)/d(\mathcal{O}, C^o)$ the *distortion* of $\mathcal{L}$ at C (relative to $\mathcal{O}$). We have $\Lambda(C^o) = 1/\Lambda(C)$.

For $\mathcal{E} = S_0^2(\mathcal{H})$ as before, the distortion $\Lambda(C)$ of $C \in \partial\mathcal{K}_0$ is the largest eigenvalue of C (see [6]).

In most situations $\mathcal{L}$ will contain the origin in its interior and, unless stated otherwise, we will take the origin as the base point.

Let $m \geq 1$ be an integer. A finite (multi)set $\{C_0, \dots, C_m\}$ is called an *m-configuration* (*relative to* $\mathcal{O}$) if $\{C_0, \dots, C_m\} \subset \partial\mathcal{L}$ and $\mathcal{O} \in [C_0, \dots, C_m]$. Let $\mathcal{C}_m(\mathcal{L})$ denote the set of all m-configurations of $\mathcal{L}$. We define

$$\sigma_m(\mathcal{L}) = \inf_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L})} \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)}. \tag{1}$$

An m-configuration $\{C_0, \dots, C_m\}$ is called *minimal* if

$$\sigma_m(\mathcal{L}) = \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)}.$$

As shown in Section 2, minimal configurations exist.

Let $\dim \mathcal{E} = \dim \mathcal{L} = n$, $n \geq 2$. We have $\sigma_1(\mathcal{L}) = 1$ and

$$\sigma_m(\mathcal{L}) = \inf_{\mathcal{O}\in\mathcal{F}\subset\mathcal{E},\, \dim\mathcal{F}=m} \sigma_m(\mathcal{L}\cap\mathcal{F}), \quad m \leq n, \tag{2}$$

where the infimum is over affine subspaces $\mathcal{F} \subset \mathcal{E}$.

For $m \geq n$, we have

$$\sigma_m(\mathcal{L}) = \sigma_n(\mathcal{L}) + \frac{m-n}{1+\max_{\partial\mathcal{L}} \Lambda}. \tag{3}$$

Equivalently, we may say that the sequence $\{\sigma_m(\mathcal{L})\}_{m\geq n}$ is arithmetic with difference $1/(1+\max_{\partial\mathcal{L}}\Lambda)$. In view of (2) and (3), the primary invariant to study is $\sigma_n(\mathcal{L})$, where $\dim\mathcal{L} = n$. In what follows, we will suppress the index n and write $\sigma(\mathcal{L}) = \sigma_n(\mathcal{L})$ and $\mathcal{C}(\mathcal{L}) = \mathcal{C}_n(\mathcal{L})$ if $\dim\mathcal{L} = n$. (For example, $\sigma_m(\mathcal{L}\cap\mathcal{F}) = \sigma(\mathcal{L}\cap\mathcal{F})$ in (2) since $\dim(\mathcal{L}\cap\mathcal{F}) = m$.) We will also omit explicit reference to n for objects depending on n; for example, an element of $\mathcal{C}(\mathcal{L})$ will simply be called a *configuration*.

According to our first result, $\sigma_m(\mathcal{L})$ measures how distorted or symmetric $\mathcal{L}$ is (with respect to $\mathcal{O}$).

THEOREM B. *Let $\mathcal{L} \subset \mathcal{E}$ be a compact convex body in a Euclidean vector space $\mathcal{E}$ of dimension n with base point $\mathcal{O} \in \operatorname{int}\mathcal{L}$. Let $m \geq 1$. Then*

$$1 \leq \sigma_m(\mathcal{L}) \leq \frac{m+1}{2}. \tag{4}$$

If $\sigma_m(\mathcal{L}) = 1$ then $m \leq n$ and there exists an affine subspace $\mathcal{F} \subset \mathcal{E}$, $\mathcal{O} \in \mathcal{F}$, of dimension m such that $\mathcal{L}\cap\mathcal{F}$ is an m-simplex. In fact, in this case a minimal configuration $\{C_0, \dots, C_m\} \in \mathcal{C}(\mathcal{L}\cap\mathcal{F})$ is unique and is given by the set of vertices of $\mathcal{L}\cap\mathcal{F}$. Moreover, minimality

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} = 1 \tag{5}$$

implies

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} C_i = 0. \tag{6}$$

Conversely, if $\mathcal{L}$ has a simplicial intersection with an m-dimensional affine subspace $\mathcal{F} \ni \mathcal{O}$, then $\sigma_m(\mathcal{L}) = 1$.

For $m \geq 2$, $\sigma_m(\mathcal{L}) = (m+1)/2$ iff $\Lambda = 1$ on $\partial\mathcal{L}$, that is, iff $\mathcal{L}$ is symmetric.

REMARK 1. A well-known result in convex geometry [1] asserts that the distortion function $\Lambda : \partial\mathcal{L} \to \mathbf{R}$ satisfies

$$\frac{1}{n} \leq \Lambda \leq n,$$

provided that the base point is suitably chosen. (The bounds are attained for an n-simplex.) For an m-configuration $\{C_0, \dots, C_m\} \in \mathcal{C}_m(\mathcal{L})$, this gives

$$\frac{m+1}{n+1} \leq \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} \leq \frac{n}{n+1}(m+1),$$

and we obtain the (generally weaker) estimate

$$\frac{m+1}{n+1} \leq \sigma_m(\mathcal{L}) \leq \frac{n}{n+1}(m+1).$$

REMARK 2. In view of Theorem A, in the setting of the tetrahedral minimal immersion we have $\sigma_2(\mathcal{K}_0(\mathcal{H}_{\lambda_6})) = 1$. Similarly, for the octahedral minimal immersion we have $\sigma_3(\mathcal{K}_0(\mathcal{H}_{\lambda_8})) = 1$.

In the next result we indicate the dependence of $\sigma_m(\mathcal{L})$ on $\mathcal{O}$ by writing $\sigma_m(\mathcal{L}, \mathcal{O})$. It can be shown that $\sigma_m(\mathcal{L}, \mathcal{O})$ is continuous in the variable $\mathcal{O} \in \operatorname{int}\mathcal{L}$. (In fact, continuity follows from equicontinuity of the family $\{\Lambda(C, \cdot) \mid C \in \partial\mathcal{K}_0\}$ on $\operatorname{int}\mathcal{K}_0$.) Note also that Example 2 (in Section 3) shows that $\sigma_m(\mathcal{L}, \mathcal{O})$ is not smooth in $\mathcal{O} \in \operatorname{int}\mathcal{L}$. For the boundary behavior, we have the following theorem.

THEOREM C. *We have*

$$\lim_{d(\mathcal{O}, \partial\mathcal{L}) \to 0} \sigma_m(\mathcal{L}, \mathcal{O}) = 1.$$

To make $\sigma_m(\mathcal{L})$ depend only on the metric properties of $\mathcal{L}$ and not on $\mathcal{O}$, we usually choose the base point to be the centroid of $\mathcal{L}$.

Theorems B and C will be proved in Section 2.

EXAMPLE. Let $\mathcal{P}_k$ denote a regular k-sided polygon. The maximum distortion occurs at a vertex of $\mathcal{P}_k$ and the distortion is equal to $-\sec(2\pi[k/2]/k)$, where $[\cdot]$ is the greatest integer function. We obtain

$$\sigma_m(\mathcal{P}_k) = \frac{m+1}{1 - \sec(2\pi[k/2]/k)}.$$

For $k = 3$, $\mathcal{P}_3$ is a triangle and the formula gives $\sigma_m(\mathcal{P}_3) = (m+1)/3$; in particular, for $m = 2$ we have $\sigma(\mathcal{P}_3) = 1$. At the other extreme,

$$\lim_{k\to\infty} \sigma_m(\mathcal{P}_k) = \frac{m+1}{2}.$$

For the rest of the results we will be concerned with $\sigma(\mathcal{L})$ only.

Recall that a *convex polytope* $\mathcal{L}$ in a Euclidean space $\mathcal{E}$ is a compact convex body enclosed by finitely many hyperplanes [1]. To avoid redundancy, we assume that the number of participating hyperplanes is minimal. The part of the polytope that lies in one of the bounding hyperplanes is called a *cell*. (For example, a cell of a convex polygon is an edge, and a cell of a convex polyhedron is a face.) The interior of a cell relative to $\partial\mathcal{L}$ is nonempty. The part of the boundary $\partial\mathcal{L}$ that remains when we delete all relative interiors of cells is called the *skeleton* of $\mathcal{L}$. (For example, the skeleton of a polygon is the set of its vertices, and the skeleton of a polyhedron is the set of its edges and vertices.) We call a configuration *simplicial* if its elements are vertices of a simplex.

THEOREM D. *Let $\mathcal{L}$ be a convex polytope in an n-dimensional Euclidean space $\mathcal{E}$ with base point $\mathcal{O} \in \operatorname{int}\mathcal{L}$. Assume that $\{C_0, \dots, C_n\}$ is a minimal simplicial configuration. Then there exists another minimal simplicial configuration $\{C_0', \dots, C_n'\}$ such that, for $i = 0, \dots, n$, C_i' or its opposite belongs to the skeleton of $\mathcal{L}$.*

Theorem D will be proved in Section 3. As a particular case, note that, for a convex polygon $\mathcal{L}$, Theorem D reduces the determination of $\sigma(\mathcal{L})$ to a finite enumeration.

2. The Invariants $\sigma_m(\mathcal{L})$, $m \geq 1$

Let $\mathcal{L} \subset \mathcal{E}$ be a compact convex body with base point $\mathcal{O} \in \operatorname{int}\mathcal{L}$ and with $\dim\mathcal{E} = \dim\mathcal{L} = n$. Let $m \geq 1$. We first show that a sequence of m-configurations $\{C_0^k, \dots, C_m^k\} \in \mathcal{C}_m(\mathcal{L})$, $k \geq 1$, which is *minimizing* in the sense that

$$\lim_{k\to\infty} \sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i^k)} = \sigma_m(\mathcal{L}),$$

subconverges to a minimal m-configuration. Indeed, since $\partial\mathcal{L}$ is compact, by extracting suitable subsequences we may assume that $\lim_{k\to\infty} C_i^k = C_i \in \partial\mathcal{L}$ for each $i = 0, \dots, m$. We now use the well-known fact that the distance function from $\mathcal{O}$ is continuous on $\partial\mathcal{L}$ (since $\mathcal{L}$ is convex). In particular, Λ is a continuous function and we have

$$\sum_{i=0}^{m} \frac{1}{1+\Lambda(C_i)} = \sigma_m(\mathcal{L}).$$

Since $\mathcal{O} \in [C_0^k, \dots, C_m^k]$ for each $k \geq 1$, we also have $\mathcal{O} \in [C_0, \dots, C_m]$. Thus, $\{C_0, \dots, C_m\}$ is a minimal m-configuration.

As noted in Section 1, we have $\sigma_1(\mathcal{L}) = 1$. Indeed, let $\{C_0, C_1\} \in \mathcal{C}_1(\mathcal{L})$ be *any* 1-configuration. Then $\mathcal{O} \in [C_0, C_1]$ and $C_0, C_1 \in \partial\mathcal{L}$ imply that C_0 and C_1 are opposites. Thus, $\Lambda(C_1) = 1/\Lambda(C_0)$ and so we have

$$\frac{1}{1+\Lambda(C_0)} + \frac{1}{1+\Lambda(C_1)} = 1.$$

We now prove (2) and (3). First of all, (2) holds because any m-configuration $\{C_0, \dots, C_m\}$ is contained in an m-dimensional affine subspace $\mathcal{F}$ of $\mathcal{E}$. Thus, the infimum on the left-hand side of the equality in (2) can be split into the double infimum on the right-hand side.

In order to derive (3) we first claim that

$$\sigma_{m+k}(\mathcal{L}) \leq \sigma_m(\mathcal{L}) + \frac{k}{1 + \max_{\partial \mathcal{L}} \Lambda}, \quad m \geq 1, \ k \geq 0. \tag{7}$$

This inequality is obvious because a *minimal* m-configuration can always be extended to an $(m+k)$-configuration by adding k copies of a point $C \in \partial\mathcal{L}$ at which Λ attains a maximum value on $\partial\mathcal{L}$.

Note that, for $m < n$, the inequality in (7) is sharp in general. For example, if $n = 2$ and $\mathcal{L}$ is an equilateral triangle with $\mathcal{O}$ at the centroid, then $m = k = 1$ gives $\sigma_2(\mathcal{L}) = \sigma(\mathcal{L}) = 1$ (by Theorem B or inspection), $\sigma_1(\mathcal{L}) = 1$ (by the foregoing), and $\max_{\partial\mathcal{L}} \Lambda = 2$. (On the other hand, equality holds for the examples at the end of Section 3.)

Finally, to obtain (3) we need to show that equality holds in (7) for $m = n$:

$$\sigma_{n+k}(\mathcal{L}) = \sigma(\mathcal{L}) + \frac{k}{1 + \max_{\partial \mathcal{L}} \Lambda}, \quad k \geq 0.$$

Let $\{C_0, \dots, C_{n+k}\} \in \mathcal{C}_{n+k}(\mathcal{L})$ be a *minimal* $(n+k)$-configuration. The convex hull $[C_0, \dots, C_{n+k}] \ni \mathcal{O}$ is a convex polytope of dimension $\leq n$ (since it is contained in the n-dimensional linear space $\mathcal{E}$). Hence we can select a subset of $\{C_0, \dots, C_{n+k}\}$ that forms an n-configuration. Renumbering the points, we may assume that this subset is $\{C_0, \dots, C_n\} \in \mathcal{C}(\mathcal{L})$. Then we have

$$\begin{aligned} \sigma_{n+k}(\mathcal{L}) &= \sum_{i=0}^{n+k} \frac{1}{1 + \Lambda(C_i)} \\ &= \sum_{i=0}^{n} \frac{1}{1 + \Lambda(C_i)} + \sum_{i=n+1}^{n+k} \frac{1}{1 + \Lambda(C_i)} \\ &\geq \sigma(\mathcal{L}) + \frac{k}{1 + \max_{\partial \mathcal{L}} \Lambda}, \end{aligned}$$

and (3) follows.

Let $m = n$ and let $\mathcal{S}(\mathcal{L})$ denote the set of all *simplicial configurations* of $\mathcal{L}$ (relative to $\mathcal{O}$). In other words, $\{C_0, \dots, C_n\} \in \mathcal{C}(\mathcal{L})$ belongs to $\mathcal{S}(\mathcal{L})$ iff $[C_0, \dots, C_n]$ is an n-simplex. We now claim that the infimum in (1) for $\sigma(\mathcal{L}) = \sigma_n(\mathcal{L})$ can be taken over the subset $\mathcal{S}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L})$:

$$\sigma(\mathcal{L}) = \inf_{\{C_0, \dots, C_n\} \in \mathcal{S}(\mathcal{L})} \sum_{i=0}^{n} \frac{1}{1 + \Lambda(C_i)}. \tag{8}$$

Toward this end, we denote the right-hand side of (8) by $\sigma^*(\mathcal{L})$ and then show that $\sigma(\mathcal{L}) = \sigma^*(\mathcal{L})$. Clearly, we have $\sigma(\mathcal{L}) \leq \sigma^*(\mathcal{L})$. For the opposite inequality we have the following lemma.

LEMMA 1. *Let $\varepsilon > 0$. Then, for any $\{C_0, \dots, C_n\} \in \mathcal{C}(\mathcal{L})$, there exist $\{C_0', \dots, C_n'\} \in \mathcal{S}(\mathcal{L})$ such that*

$$\left| \sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i')} - \sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i)} \right| < \varepsilon. \tag{9}$$

Proof. Let $\dim\langle C_0, \dots, C_n\rangle = n_0$, $n_0 \le n$. Decomposing the convex polytope $[C_0, \dots, C_n]$ in $\langle C_0, \dots, C_n\rangle$ into a union of simplices, we can find an n_0-simplex that contains the base point $\mathcal{O}$. Renumbering, we may assume that this n_0-simplex has vertices $C_0, \dots, C_{n_0}$. For $i = 0, \dots, n_0$, let $C_i' = C_i$. For $i > n_0$, choose $C_i' \in \mathcal{E}$ such that $C_i' - C_i$ are linearly independent and have common length, say $\delta > 0$. Since the codimension of $[C_0, \dots, C_n]$ in $\mathcal{E}$ is $n - n_0$, this is possible. Because the distortion function Λ is continuous, δ can be chosen so small that (9) holds. The lemma follows. □

Finally, note that Lemma 1 implies $\sigma^*(\mathcal{L}) \le \varepsilon + \sigma(\mathcal{L})$. Letting $\varepsilon \to 0$, we obtain $\sigma^*(\mathcal{L}) \le \sigma(\mathcal{L})$. We thus have $\sigma^*(\mathcal{L}) = \sigma(\mathcal{L})$ as claimed.

REMARK. For $\sigma(\mathcal{L}) > 1$, the limit of a convergent minimizing sequence of simplices may degenerate into a nonsimplicial configuration. In Example 1 (at the end of Section 3) we will show that this degeneracy can occur.

LEMMA 2. *Let $[C_0, \dots, C_m]$ be an m-simplex in $\mathbf{R}^m$. For $i = 0, \dots, m$, let $\mathcal{E}_i = \langle C_0, \dots, \hat{C}_i, \dots, C_m\rangle$ be the affine hull of the ith face $[C_0, \dots, \hat{C}_i, \dots, C_m]$. If $C_i \ne 0$, define ℓ_i as the line passing through the origin and C_i. If, in addition, ℓ_i intersects $\mathcal{E}_i$ in a single point, denote this point by C_i'. Define $\lambda_i \in \mathbf{R} \cup \{\infty\}$ as follows. For $0 \in \mathcal{E}_i$, let $\lambda_i = \infty$. For $C_i = 0$ or $\ell_i \parallel \mathcal{E}_i$, let $\lambda_i = 0$. Otherwise, let λ_i be defined by the equality $C_i = -\lambda_i C_i'$. With these, we have*

$$\sum_{i=0}^{m} \frac{1}{1+\lambda_i} = 1 \tag{10}$$

and

$$\sum_{i=0}^{m} \frac{1}{1+\lambda_i} C_i = 0, \tag{11}$$

where (as usual) we set $1/\infty = 0$.

Proof. First note that $\lambda_i \ne -1$, since $[C_0, \dots, C_m]$ is an m-simplex and therefore cannot be contained in $\mathcal{E}_i$.

We may assume that $0 \notin \mathcal{E}_i$ (for all $i = 0, \dots, m$), since otherwise we can omit C_i from (10)–(11), consider the $(m-1)$-simplex $[C_0, \dots, \hat{C}_i, \dots, C_m]$, and use induction with respect to m. We may also assume that $C_i \ne 0$ for all $i = 0, \dots, m$. Indeed, if $C_i = 0$ for some $i = 0, \dots, m$ then, for all $j \ne i$, we have

$$0 \in [C_0, \dots, \hat{C}_j, \dots, C_m] \subset \mathcal{E}_j,$$

and this goes back to the previous case. (Incidentally, since $\lambda_j = \infty$ for all $j \ne i$, (10)–(11) are obviously satisfied.)

Finally, we may assume that ℓ_i is not parallel to $\mathcal{E}_i$, since otherwise we can apply a limiting argument.

With these assumptions, C_i and C_i' are distinct nonzero vectors. Letting $\delta_i = 1/\lambda_i$, the defining equation for λ_i can be written as

$$C_i' = -\delta_i C_i. \tag{12}$$

By definition, $C_i' \in \langle C_0, \dots, \hat{C}_i, \dots, C_m \rangle$ so that we have the expansion

$$C_i' = \sum_{j=0;\ j\neq i}^{m} \lambda_j^i C_j, \tag{13}$$

where the coefficients λ_j^i satisfy

$$\sum_{j=0;\ j\neq i}^{m} \lambda_j^i = 1. \tag{14}$$

Combining (12) and (13), we obtain the system

$$\sum_{j=0;\ j\neq i}^{m} \lambda_j^i C_j + \delta_i C_i = 0, \quad i = 0, \dots, m. \tag{15}$$

Since $[C_0, \dots, C_m]$ is an m-simplex, the vectors $C_0, \dots, \hat{C}_i, \dots, C_m$ are linearly independent. This implies that the coefficient matrix of the system (15) has rank 1 (since all the 2×2 subdeterminants vanish). We generalize this in the following lemma.

LEMMA 3. *Assume that the matrix*

$$\begin{bmatrix} \delta_0 & \lambda_1^0 & \dots & \lambda_m^0 \\ \lambda_0^1 & \delta_1 & \dots & \lambda_m^1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^m & \lambda_1^m & \dots & \delta_m \end{bmatrix}, \quad \delta_0, \dots, \delta_m \neq -1,$$

has rank 1, *and assume that* (14) *holds. Then we have*

$$\lambda_j^i = \frac{\delta_j}{1+\delta_j}(1+\delta_i). \tag{16}$$

In particular,

$$\sum_{j=0}^{m} \frac{\delta_j}{1+\delta_j} = 1. \tag{17}$$

Proof of Lemma 3. Let $i \neq j$ and consider all 2×2 subdeterminants in the ith and jth rows that contain the ith column. We have

$$\lambda_k^i \lambda_i^j = \delta_i \lambda_k^j, \quad k = 0, \dots, \hat{\imath}, \dots, \hat{\jmath}, \dots, m,$$

and

$$\lambda_j^i \lambda_i^j = \delta_i \delta_j.$$

Adding these and using (14), we obtain

$$\lambda_i^j = \delta_i(\lambda_0^j + \cdots + \hat{\lambda}_i^j + \cdots + \lambda_{j-1}^j + \delta_j + \lambda_{j+1}^j + \cdots + \lambda_m^j).$$

Again by (14), the sum in the parentheses is $\delta_j + 1 - \lambda_i^j$, and (16) follows. Finally, substituting (16) into (13) yields (17). Lemma 3 follows. □

Lemma 2 is an immediate consequence of Lemma 3. Indeed, substituting $\delta_i = 1/\lambda_i$ into (17), we have (10). Finally, using (16) in (15) yields (11). □

Proof of Theorem B. We may assume that the base point is the origin. We first show that the lower bound in (4) holds. Let $\{C_0, \ldots, C_m\} \in \mathcal{C}_m(\mathcal{L})$ be a minimal configuration:

$$\sigma_m(\mathcal{L}) = \sum_{i=0}^{m} \frac{1}{1 + \Lambda(C_i)}.$$

The convex hull $[C_0, \ldots, C_m]$ is a convex polytope in the affine hull $\mathcal{F} = \langle C_0, \ldots, C_m \rangle$. Since the origin is contained in $[C_0, \ldots, C_m]$, $\mathcal{F} \subset \mathcal{E}$ is a linear subspace. Observe that $\mathcal{L} \cap \mathcal{F}$ is a compact convex body in $\mathcal{F}$ that contains the origin in its interior. Let $m_0 = \dim \mathcal{F}$. We have $m_0 \leq m$. Decomposing $[C_0, \ldots, C_m]$ into a union of simplices, we can find an m_0-simplex that also contains the origin. Renumbering the points, we may assume that this m_0-simplex has vertices $C_0, \ldots, C_{m_0}$. Clearly, $\mathcal{F} = \langle C_0, \ldots, C_{m_0} \rangle$ and, by definition, we have $\{C_0, \ldots, C_{m_0}\} \in \mathcal{S}(\mathcal{L} \cap \mathcal{F})$. We now use Lemma 2 with m replaced by m_0. Since the origin is in the interior of $\mathcal{L} \cap \mathcal{F}$, we have $C_i \neq 0$ for all $i = 0, \ldots, m_0$. Moreover, since $0 \in [C_0, \ldots, C_{m_0}]$, we also have $\ell_i \not\parallel \mathcal{E}_i$ for all $i = 0, \ldots, m_0$. Thus we obtain that $\lambda_i > 0$ or $\lambda_i = \infty$. In the first case, $C_i' = -1/\lambda_i C_i$, so $\lambda_i = |C_i|/|C_i'|$ is the distortion of the simplex $[C_0, \ldots, C_{m_0}]$ at the vertex C_i. In the second case, the origin is contained in the ith face of $[C_0, \ldots, C_{m_0}]$ and $C_i' = 0$.

Let C_i^o be the opposite of $C_i \in \partial\mathcal{L}$ relative to $\mathcal{L}$. The vectors C_i, C_i', and C_i^o are collinear. Since $[C_i, \ldots, C_{m_0}] \subset \mathcal{L} \cap \mathcal{F}$, we have $|C_i^o| \geq |C_i'|$. Hence, for $\lambda_i > 0$,

$$\lambda_i = \frac{|C_i|}{|C_i'|} \geq \frac{|C_i|}{|C_i^o|} = \Lambda(C_i). \tag{18}$$

For $\lambda_i = \infty$, we automatically have $\lambda_i > \Lambda(C_i)$. Because the function $x \mapsto 1/(1+x)$, $x > 0$, is strictly decreasing, (10) (for $m = m_0$) implies

$$\sum_{i=0}^{m_0} \frac{1}{1 + \Lambda(C_i)} \geq 1. \tag{19}$$

Comparing this with our foregoing condition of minimality of $\{C_0, \ldots, C_m\}$ shows that $\sigma_m(\mathcal{L}) \geq 1$.

If $\sigma_m(\mathcal{L}) = 1$ then, by (3), $m \leq n$; the comparison argument used previously gives $m_0 = m$, so that $[C_0, \ldots, C_m]$ is an m-simplex and $\lambda_i = \Lambda(C_i), i = 0, \ldots, m$. In particular, we obtain (5).

It remains to show that $\mathcal{L} \cap \mathcal{F}$ is an m-simplex. Since $\lambda_i = \Lambda(C_i)$, we also have $C_i' = C_i^o \in \partial\mathcal{L}$ *for all* $i = 0, \ldots, m$. On the other hand, C_i' (being in the interior of the ith face) is a boundary point of $\mathcal{L}\cap\mathcal{F}$ iff the entire ith face $[C_0,\dots,\hat{C}_i,\dots,C_m]$ is contained in $\partial\mathcal{L}\cap\mathcal{F}$. We conclude that $\mathcal{L}\cap\mathcal{F}=[C_0,\dots,C_m]$ and that $\mathcal{L}\cap\mathcal{F}$ is an m-simplex. The rest of the statements in Theorem B concerning the case $\sigma_m(\mathcal{L})=1$ follow from Lemma 2.

In order to derive the upper bound in (4) for $\sigma_m(\mathcal{L})$, we use (7) for $m=1$ and $k=m-1$. We obtain

$$\sigma_m(\mathcal{L})\le\sigma_1(\mathcal{L})+\frac{m-1}{1+\max_{\partial\mathcal{L}}\Lambda}\le 1+\frac{m-1}{2}=\frac{m+1}{2}. \tag{20}$$

The last inequality follows because $\max_{\partial\mathcal{L}}\Lambda\ge 1$ (since $\Lambda(C^o)=1/\Lambda(C)$, $C\in\partial\mathcal{L}$).

If $\sigma_m(\mathcal{L})=(m+1)/2$, $m\ge 2$, then (20) gives $\max_{\partial\mathcal{L}}\Lambda=1$. This implies not only $\Lambda=1$ on $\partial\mathcal{L}$ but also the symmetry of $\mathcal{L}$. □

REMARK. We give here another proof of the upper bound in (4) as follows. Assume that the base point is the origin, and let $\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L})$. By (1), we have

$$\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i)}\ge\sigma_m(\mathcal{L}). \tag{21}$$

Consider the opposite points $C_0^o,\dots,C_m^o\in\partial\mathcal{L}$. We claim that $\{C_0^o,\dots,C_m^o\}\in\mathcal{C}_m(\mathcal{L})$. In order to prove this we need to show that $0\in[C_0,\dots,C_m]$ implies $0\in[C_0^o,\dots,C_m^o]$. Indeed, let $\sum_{i=0}^m\lambda_iC_i=0$ for some $0\le\lambda_i\le 1$ with $\sum_{i=0}^m\lambda_i=1$. Since $C_i=-\Lambda(C_i)C_i^o$, by substituting we obtain $\sum_{i=0}^m\lambda_i\Lambda(C_i)C_i^o$, where $\sum_{i=0}^m\lambda_i\Lambda(C_i)>0$. Normalizing, the claim follows.

Once again by the definition of $\sigma_m(\mathcal{L})$, we have

$$\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i^o)}\ge\sigma_m(\mathcal{L}). \tag{22}$$

Since

$$\frac{1}{1+\Lambda(C_i^o)}=\frac{1}{1+1/\Lambda(C_i)}=\frac{\Lambda(C_i)}{1+\Lambda(C_i)}=1-\frac{1}{1+\Lambda(C_i)}, \tag{23}$$

(22) and (23) together give

$$\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i^o)}=m+1-\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i)}\ge\sigma_m(\mathcal{L}).$$

This, combined with (21), yields $m+1\ge 2\sigma_m(\mathcal{L})$. The upper bound for $\sigma_m(\mathcal{L})$ follows.

In this argument we used an involution ${}^o\colon\mathcal{C}_m(\mathcal{L})\to\mathcal{C}_m(\mathcal{L})$, $\{C_0,\dots,C_m\}^o=\{C_0^o,\dots,C_m^o\}$. As a further application, we define

$$\Sigma_m(\mathcal{L})=\sup_{\{C_0,\dots,C_m\}\in\mathcal{C}_m(\mathcal{L})}\sum_{i=0}^{m}\frac{1}{1+\Lambda(C_i)}.$$

We then have

$$\Sigma_m(\mathcal{L}) = m + 1 - \sigma_m(\mathcal{L}).$$

Indeed, using (23) we compute

$$\begin{aligned}\Sigma_m(\mathcal{L}) &= \sup_{\{C_0,\dots,C_m\}^o \in \mathcal{C}_m(\mathcal{L})} \sum_{i=0}^m \frac{1}{1+\Lambda(C_i)} \\ &= \sup_{\{C_0,\dots,C_m\} \in \mathcal{C}_m(\mathcal{L})} \sum_{i=0}^m \frac{1}{1+\Lambda(C_i^o)} \\ &= m+1- \inf_{\{C_0,\dots,C_m\} \in \mathcal{C}_m(\mathcal{L})} \sum_{i=0}^m \frac{1}{1+\Lambda(C_i)} \\ &= m+1-\sigma_m(\mathcal{L}).\end{aligned}$$

Proof of Theorem C. Let $B \in \operatorname{int}\mathcal{L}$ be a fixed base point. Let $\varepsilon > 0$, and let $\mathcal{O} \in \operatorname{int}\mathcal{L}$ be such that

$$d(\mathcal{O}, \partial\mathcal{L}) = \min_{X \in \partial\mathcal{L}} d(\mathcal{O}, X) < \varepsilon.$$

By choosing ε small enough, we may assume that $\mathcal{O}$ is different from B. Let $\mathcal{O}^* \in \partial\mathcal{L}$ be such that $d(\mathcal{O}, \mathcal{O}^*) < \varepsilon$. Finally, let $C \in \partial\mathcal{L}$ be on the line passing through B and $\mathcal{O}$ on the same side as $\mathcal{O}$ relative to B. Since $\Lambda(C^o) \leq \max_{\partial\mathcal{L}} \Lambda$, by (20) we have

$$\sigma_m(\mathcal{L}, \mathcal{O}) \leq 1 + \frac{m-1}{1+\Lambda(C^o)} = 1 + (m-1)\frac{\Lambda(C)}{1+\Lambda(C)}.$$

Using the definition of Λ, we arrive at the estimate

$$\sigma_m(\mathcal{L}, \mathcal{O}) \leq 1 + (m-1)\frac{d(\mathcal{O}, C)}{d(C, C^o)}.$$

In the remaining part of the proof, we give an upper bound for the ratio $d(\mathcal{O}, C)/d(C, C^o)$ in terms of ε. Toward this end, we let

$$\delta = \min_{X \in \partial\mathcal{L}} d(B, X) \quad \text{and} \quad \Delta = \max_{X \in \partial\mathcal{L}} d(B, X).$$

Since $\partial\mathcal{L}$ is compact, we have $0 < \delta \leq \Delta < \infty$. By construction, B, C, and C^o are collinear. Thus

$$d(C, C^o) = d(B, C) + d(B, C^o) \geq 2\delta.$$

It remains to give an upper estimate for $d(\mathcal{O}, C)$. If $C = \mathcal{O}^*$, then $d(\mathcal{O}, C) = d(\mathcal{O}, \mathcal{O}^*) < \varepsilon$. We then obtain

$$\sigma(\mathcal{L}, \mathcal{O}) < 1 + (m-1)\frac{\varepsilon}{2\delta}.$$

From now on we may assume that $C \neq \mathcal{O}^*$. Let Π denote the affine span of B, C, and $\mathcal{O}^*$. By assumption, Π is a 2-dimensional plane and $\mathcal{O} \in \Pi$. From now on we will work in Π. The line passing through B and parallel to the line $\overline{\mathcal{O}\mathcal{O}^*}$ intersects $\partial\mathcal{L}$ in two points, B^* and its opposite. We can choose B^* on the same side as $\mathcal{O}^*$ relative to the line $\overline{\mathcal{O}B}$. It is easy to see that the line segment $[C, B^*]$ intersects the line segment $[\mathcal{O}, \mathcal{O}^*]$. Denote this intersection point by $\mathcal{O}'$. We thus have

$$\frac{d(C,\mathcal{O})}{d(C,B)} = \frac{d(\mathcal{O},\mathcal{O}')}{d(B,B^*)} \le \frac{d(\mathcal{O},\mathcal{O}^*)}{d(B,B^*)}.$$

Rearranging, we find

$$d(\mathcal{O},C) \le d(\mathcal{O},\mathcal{O}^*)\frac{d(B,C)}{d(B,B^*)} < \varepsilon\frac{\Delta}{\delta}.$$

We finally obtain

$$\sigma(\mathcal{L},\mathcal{O}) < 1 + (m-1)\frac{\varepsilon\Delta}{2\delta^2}.$$

In both cases, if $\varepsilon \to 0$ then $\sigma(\mathcal{L},\mathcal{O}) \to 1$. Theorem C follows. □

3. Computation of $\sigma(\mathcal{L})$

Before giving the proof of Theorem D, we derive several lemmas. We state Lemma 1 and Lemma 3 in a slightly more general setting than necessary.

Let $\mathcal{L}$ be a compact convex body in a Euclidean vector space $\mathcal{E}$. Recall that a boundary point C of $\mathcal{L}$ is called *extremal* if C is not contained in the interior of a line segment in $\mathcal{L}$. (For example, the extremal points of a polytope are its vertices.) By the Krein–Milman theorem, $\mathcal{L}$ is the convex hull of its extremal points [1].

LEMMA 1. *Let* $\dim\mathcal{E} = 2$ *and let* $\mathcal{L} \subset \mathcal{E}$ *be a compact convex body with base point* $\mathcal{O} \in \operatorname{int}\mathcal{L}$. *Assume that the distortion function* $\Lambda\colon \partial\mathcal{L} \to \mathbf{R}$ *has a critical point at a nonextremal point* C. *If the opposite* C^o *is also nonextremal then* Λ *is constant in a neighborhood of* C *in* $\partial\mathcal{L}$.

Proof. We may assume that $\mathcal{O}$ is the origin. Let $\mathcal{I} \subset \partial\mathcal{L}$ and $\mathcal{I}^o \subset \partial\mathcal{L}$ be open line segments with $C \in \mathcal{I}$ and $C^o \in \mathcal{I}^o$. We parameterize $\mathcal{I}$ by $t \mapsto C + tV$ (for small t), where V is parallel to $\mathcal{I}$. By assumption, $(C+tV)^o \in \mathcal{I}^o$ (again for small t) and so we can write $(C+tV)^o = C^o + sV^o$, where V^o is parallel to $\mathcal{I}^o$ and s is a smooth function of t. ($\mathcal{I}$ and $\mathcal{I}^o$ define a projectivity so that s is a linear fractional transformation of t, but we do not need this fact.)

By the definition of distortion,

$$(C+tV)^o = -\frac{1}{\Lambda(C+tV)}(C+tV) = C^o + sV^o. \tag{24}$$

Since Λ is critical at C, we have $(d/dt)\Lambda(C+tV)|_{t=0} = 0$. Differentiating (24) at $t = 0$ then yields

$$-\frac{1}{\Lambda(C)}V = s'(0)V^o;$$

in particular, V and V^o and hence $\mathcal{I}$ and $\mathcal{I}^o$ are parallel.

Using this in (24) to eliminate V^o, after rearranging we obtain

$$\left(\frac{1}{\Lambda(C+tV)} - \frac{1}{\Lambda(C)}\right)C + \left(\frac{t}{\Lambda(C+tV)} - \frac{1}{\Lambda(C)}\frac{s}{s'(0)}\right)V = 0.$$

Since the origin is in the interior of $\mathcal{L}$, we know that C and V are linearly independent. We obtain $\Lambda(C+tV) = \Lambda(C)$, and the lemma follows. (Vanishing of the second coefficient also gives $s(t) = s'(0)t$.) □

REMARK. As a by-product, we also see that the line segment neighborhoods $\mathcal{I}$ and $\mathcal{I}^o$ of C and C^o are parallel.

The next lemma follows from Lemma 1 and the previous remark by taking plane sections of the polytope.

LEMMA 2. *Let $\mathcal{L} \subset \mathcal{E}$ be a convex polytope with base point $\mathcal{O} \in \operatorname{int} \mathcal{L}$, and assume that $\Lambda \colon \partial\mathcal{L} \to \mathbf{R}$ has a critical point C in the interior $\mathcal{I}$ of a cell of $\mathcal{L}$. If C^o is also contained in the interior $\mathcal{I}^o$ of a cell then Λ is constant on $\mathcal{I}$, and $\mathcal{I}$ and $\mathcal{I}^o$ are parallel.*

Theorem D will be proved by induction with respect to $\dim \mathcal{E} = n$. The next lemma provides the basic step of the induction. In addition, for a plane polygon, the lemma reduces the computation of $\sigma(\mathcal{L})$ to a finite enumeration.

LEMMA 3. *Let $\dim \mathcal{E} = 2$, and let $\mathcal{L} \subset \mathcal{E}$ be a compact convex body with base point $\mathcal{O} \in \operatorname{int} \mathcal{L}$. Let $\{C_0, C_1, C_2\}$ be a minimal triangular configuration of $\mathcal{L}$. Then there exists another minimal triangular configuration $\{C_0', C_1', C_2'\}$ of $\mathcal{L}$ such that, for each $i = 0, 1, 2$, C_i' or its opposite is extremal.*

Proof. By minimality,

$$\sigma(\mathcal{L}) = \sum_{i=0}^{2} \frac{1}{1 + \Lambda(C_i)}.$$

We first assume that $\mathcal{O} \in \partial[C_0, C_1, C_2]$, say $\mathcal{O} \in [C_1, C_2]$. This means that C_1 and C_2 are opposites. Therefore, their contribution to the sum just displayed is 1. We can move C_1 and C_2 simultaneously along $\partial\mathcal{L}$, keeping them opposites and away from C_0, until either the moved C_1 (say, C_1') or its opposite (C_2') hits an extremal point. (The Krein–Milman theorem guarantees that this is possible.) If C_0 or its opposite happens to be extremal, we set $C_0' = C_0$ and the lemma follows. Otherwise, as in the proof of Lemma 1, let $\mathcal{I}$ and $\mathcal{I}^o$ be maximal neighborhoods of C_0 and C_0^o. By minimality of $\{C_0, C_1', C_2'\}$, C_0 must be a critical point of Λ. Then C_0 can be moved to one of the endpoints of $\mathcal{I}$, say C_0' (which is not C_1' or C_2'), where it becomes extremal. By Lemma 1, $\Lambda(C_0') = \Lambda(C_0)$. We arrive at $\{C_0', C_1', C_2'\}$ and the lemma follows.

Next we assume that $\mathcal{O}$ is in the interior of $[C_0, C_1, C_2]$. If C_0 and its opposite are not extremal then, by minimality of $\{C_0, C_1, C_2\}$, C_0 must be critical. By Lemma 1, C_0 can be moved along $\partial\mathcal{L}$ (keeping it away from C_1 and C_2) without changing Λ until it hits an extremal point C_0', unless one of the edges emanating from the moved C_0 (and terminating in C_1 or C_2) hits $\mathcal{O}$. If the latter happens then we go back to the first case, already discussed.

The same procedure works for modifying C_1 and C_2, and the lemma follows. □

REMARK. An inspection of the preceding proof reveals that, for the resulting minimal configuration $\{C_0', C_1', C_2'\}$, either all the points are extremal or two of them are extremal and the third is an opposite.

Proof of Theorem D. As noted previously, the proof proceeds by induction with respect to $\dim \mathcal{E} = n$. By Lemma 3, we need only perform the general induction step $n-1 \Rightarrow n$, where $n \geq 3$. The proof that follows is patterned after the proof of Lemma 3.

Assume first that $\mathcal{O} \in \partial[C_0, \dots, C_n]$, say $\mathcal{O} \in [C_1, \dots, C_n]$. Consider the compact convex body $\mathcal{L} \cap \langle C_1, \dots, C_n \rangle$ in $\langle C_1, \dots, C_n \rangle$. By assumption, $\mathcal{O}$ is contained in the interior of $\mathcal{L} \cap \langle C_1, \dots, C_n \rangle$; in addition, $\{C_1, \dots, C_n\}$ is a simplicial configuration of $\mathcal{L} \cap \langle C_1, \dots, C_n \rangle$. Since $\{C_0, \dots, C_n\}$ is minimal in $\mathcal{L}$, it follows that $\{C_1, \dots, C_n\}$ is also minimal in $\mathcal{L} \cap \langle C_1, \dots, C_n \rangle$. Since $\dim(\mathcal{L} \cap \langle C_1, \dots, C_n \rangle) = n-1$, the induction hypothesis applies. Thus, there exists a minimal simplicial configuration $\{C_1', \dots, C_n'\} \in \mathcal{S}(\mathcal{L} \cap \langle C_1, \dots, C_n \rangle)$ such that, for each $i = 1, \dots, n$, C_i' or its opposite is in the skeleton of the convex polytope $\mathcal{L} \cap \langle C_1, \dots, C_n \rangle$. Because $\mathcal{O}$ is in the interior of this polytope, any relative interior of a cell in $\mathcal{L}$ intersects $\langle C_1, \dots, C_n \rangle$ transversally. Therefore, the skeleton of $\mathcal{L} \cap \langle C_1, \dots, C_n \rangle$ is contained in the skeleton of $\mathcal{L}$. We obtain that, for each $i = 1, \dots, n$, C_i' or its opposite is in the skeleton of $\mathcal{L}$.

Consider now C_0. If C_0 or its opposite is in the skeleton of $\mathcal{L}$ then we are done. Otherwise, C_0 and C_0^o are in the interior $\mathcal{I}$ and $\mathcal{I}^o$ of cells of $\mathcal{L}$. By minimality, C_0 must be a critical point of Λ. By Lemma 2, Λ must be constant on $\mathcal{I}$. Hence C_0 can be moved to a boundary point C_0' of $\mathcal{I}$ that is part of the skeleton of $\mathcal{L}$. In addition, we may also require that $C_0' \notin \langle C_1', \dots, C_n' \rangle$. Since $\Lambda(C_0') = \Lambda(C_0)$, $\{C_0', \dots, C_n'\}$ remains a minimal simplicial configuration.

Next we assume that $\mathcal{O}$ is in the interior of $[C_0, \dots, C_n]$. We may also assume that C_0 and C_0^o are not contained in the skeleton of $\mathcal{L}$ (since otherwise we set $C_0' = C_0$). As before, let $\mathcal{I}$ and $\mathcal{I}^o$ denote the corresponding interiors of cells that contain C_0 and C_0^o. Again by minimality, Λ is constant on $\mathcal{I}$. Moving C_0 to the boundary of $\mathcal{I}$, either we hit the skeleton of $\mathcal{L}$ or the boundary of $[C_0, \dots, C_n]$ hits $\mathcal{O}$. In the latter case, the previous discussion applies; in the former, we can make sure that the moved C_0 is away from $\langle C_1, \dots, C_n \rangle$. The same procedure works for $C_1, \dots, C_n$, and Theorem D follows. □

EXAMPLE 1. Let $\mathcal{P}$ be the pentagon in $\mathbf{R}^2$ with vertices $(1,-1)$, $(1,1)$, $(0,2)$, $(-1,1)$ and $(-1,-1)$. For the opposite points, we have

$$(1,a)^o = (-1,-a) \quad \text{and} \quad (a,-1)^o = \left(\frac{2a}{a+1}, \frac{2}{a+1}\right), \quad -1 \leq a \leq 1.$$

The distortions are:

$$\Lambda(a,-1) = \frac{|a|+1}{2}, \quad -1 \leq a \leq 1;$$
$$\Lambda(\pm 1, a) = 1, \qquad -1 \leq a \leq 1;$$
$$\Lambda\left(\pm\frac{2a}{a+1}, \frac{2}{a+1}\right) = \frac{2}{a+1}, \qquad 0 \leq a \leq 1.$$

A case-by-case analysis in the use of Lemma 3 shows that $\sigma(\mathcal{P}) = 4/3$ and that the minimal configurations are of two types. The first type is triangular, with one vertex the topmost vertex $(0, 2)$ of $\mathcal{P}$ and with the other two vertices on the vertical sides of $\mathcal{P}$. The second type is triangular or degenerate, with one vertex the topmost vertex of $\mathcal{P}$, another vertex C on the horizontal side of $\mathcal{P}$, and a third vertex C^o. If $C = (0, -1)$ then the triangle degenerates to a vertical line segment. We see that all possible scenarios in the proof of Lemma 3 arise.

A minimizing sequence for $\sigma(\mathcal{P})$ may consist of triangles with vertices $(0, -1)$ and $(\pm 2/(n+1), 2n/(n+1))$, and these triangles shrink to the minimal vertical line segment. Since $\max_{\partial\mathcal{P}} \Lambda = 2$, we also see that $\sigma_m(\mathcal{P}) = (m+2)/3$ for $m \geq 1$.

Example 2. Let $0 < \varepsilon \leq 1$ and let $\mathcal{L}_\varepsilon$ be the square (of side length 2) with vertices $(1, 2-\varepsilon)$, $(-1, 2-\varepsilon)$, $(-1, -\varepsilon)$, and $(1, -\varepsilon)$. The distortions of the horizontal top and base sides are as follows:

$$\Lambda(a, 2-\varepsilon) = \frac{2-\varepsilon}{\varepsilon}, \quad -1 \leq a \leq 1;$$

$$\Lambda(a, -\varepsilon) = \begin{cases} \frac{\varepsilon}{2-\varepsilon}, & |a| \leq \frac{\varepsilon}{2-\varepsilon}, \\ |a|, & \frac{\varepsilon}{2-\varepsilon} < |a| \leq 1. \end{cases}$$

The other distortions can be obtained by taking opposite points and using $\Lambda(C^o) = 1/\Lambda(C)$. A case-by-case analysis in the use of Lemma 3 shows that

$$\sigma(\mathcal{L}_\varepsilon) = 1 + \frac{\varepsilon}{2},$$

with many triangles realizing the infimum in $\sigma(\mathcal{L}_\varepsilon)$. In particular, in agreement with Theorem C we have

$$\lim_{\varepsilon\to 0} \sigma(\mathcal{L}_\varepsilon) = 1.$$

Since $\max_{\partial\mathcal{L}_\varepsilon} \Lambda = (2-\varepsilon)/\varepsilon$, we also see that $\sigma_m(\mathcal{L}_\varepsilon) = 1 + (m-1)\varepsilon/2$ for $m \geq 1$.

4. Proof of Theorem A

Let $\mathcal{H}$ be a Euclidean vector space and $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H})$ the associated reduced moduli space. As noted in Section 1, the distortion at a boundary point $C \in \partial\mathcal{K}_0$ is the *largest eigenvalue* of C, also denoted by $\Lambda(C)$ (see [6]). The opposite of C is therefore given by

$$C^o = -\frac{1}{\Lambda(C)} C.$$

Remark. According to a result in [6], the distortion function $\Lambda \colon \partial\mathcal{K}_0 \to \mathbf{R}$ satisfies

$$\frac{1}{h-1} \leq \Lambda \leq h-1,$$

where $\dim \mathcal{H} = h$. Thus we have

$$\frac{n+1}{h} \leq \sigma(\mathcal{K}_0 \cap \mathcal{E}) \leq (n+1)\left(1 - \frac{1}{h}\right).$$

Comparing this with (4), we see that the lower estimate here is stronger while the upper estimate is weaker. Combining the stronger estimates, we obtain

$$\frac{n+1}{h} \leq \sigma(\mathcal{K}_0 \cap \mathcal{E}) \leq \frac{n+1}{2}. \tag{25}$$

Note that the estimates are sharp for $h = 2$. In fact, identifying $S_0^2(\mathbf{R}^2)$ with $\mathbf{R}^2$ by associating to the matrix $\left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$ the point $(a,b) \in \mathbf{R}^2$, we see that $\mathcal{K}_0$ is identified with the unit disk in $\mathbf{R}^2$. For $h = 2$ we have $\mathcal{E} = S_0^2(\mathcal{H})$ and so obtain $\sigma(\mathcal{K}_0) = 3/2$; for $h = 1$, we have $\sigma(\mathcal{K}_0 \cap \mathcal{E}) = 1$ because $\mathcal{K}_0 \cap \mathcal{E}$ is a line segment. Finally, if $\mathcal{E} = S_0^2(\mathcal{H})$ then (25) reduces to

$$\frac{h+1}{2} \leq \sigma(\mathcal{K}_0) \leq \frac{h(h+1)}{4}.$$

Returning to our problem of simplicial intersections of $\mathcal{K}_0$, let $\mathcal{E} \subset S_0^2(\mathcal{H})$ be a linear subspace (of dimension n) and assume that $\mathcal{K}_0 \cap \mathcal{E}$ is an n-simplex, $\sigma(\mathcal{K}_0 \cap \mathcal{E}) = 1$, with $\mathcal{K}_0 \cap \mathcal{E} = [C_0, \dots, C_n]$. By (10) and (11) we have $\lambda_i = \Lambda(C_i)$, so

$$\sum_{i=0}^{n} \frac{1}{1+\Lambda(C_i)}(C_i + I) = I; \tag{26}$$

we rewrite this as

$$\sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}(C_i + I) = -\frac{1}{1+\Lambda(C_0)}(C_0 - \Lambda(C_0)I). \tag{27}$$

Since $C_i + I \geq 0$ for all $i = 0, \dots, n$, we obtain

$$\ker(C_0 - \Lambda(C_0)I) = \bigcap_{i=1}^{n} \ker(C_i + I). \tag{28}$$

Before proceeding with the proof of Theorem A, we show the following lemma.

LEMMA. *Let $C_1, \dots, C_n \in \partial\mathcal{K}_0$ be linearly independent. Then $[C_1, \dots, C_n] \subset \partial\mathcal{K}_0$ iff* (i) *of Theorem A holds.*

Proof. Let $C \in [C_1, \dots, C_n]$ be such that $C = \sum_{i=1}^{n} \lambda_i C_i$ with $\sum_{i=1}^{n} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$. Then

$$C + I = \sum_{i=1}^{n} \lambda_i (C_i + I).$$

Since $C + I \geq 0$ and $C_i + I \geq 0$ for all $i = 1, \dots, n$, we obtain

$$\ker(C + I) \supset \bigcap_{i=1}^{n} \ker(C_i + I)$$

(with equality if $\lambda_i > 0$ for all $i = 0, \dots, n$) iff C is in the interior of $[C_1, \dots, C_n]$. The lemma follows. □

Proof of Theorem A. Assume first that $\mathcal{K}_0 \cap \mathcal{E}$ is an n-simplex $[C_0, \dots, C_n]$ with extra vertex C_0. The zeroth face $[C_1, \dots, C_n]$ is on the boundary of $\mathcal{K}_0$. By the lemma just proved, (i) follows. Rearranging the terms in (27), we obtain

$$\frac{1}{1+\Lambda(C_0)}(C_0+I) = I - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}(C_i+I).$$

Since $C_0 \in \partial\mathcal{K}_0$, we know that $C_0 + I$ is positive semidefinite but not positive definite; (ii) follows.

Conversely, assume that (i) and (ii) hold. Taking traces of both sides of (ii) (and dividing by n) then yields

$$1 - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} \geq 0, \tag{29}$$

where we have used the fact that all C_i have zero trace. We first claim that strict inequality holds in (29). Indeed, if the left-hand side of (29) were zero then in (ii) we would have a positive semidefinite endomorphism with zero trace. We would then have

$$I - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}(C_i+I) = 0$$

or, equivalently,

$$\left(1 - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}\right) I = \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} C_i.$$

By assumption, the left-hand side vanishes, and this contradicts to the linear independence of $C_1, \ldots, C_n$. The claim follows and we obtain

$$\sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} < 1. \tag{30}$$

We now define

$$\tilde{C} = -\sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} C_i \in \mathcal{E}.$$

We calculate the maximal eigenvalue $\Lambda(\tilde{C})$:

$$\Lambda(\tilde{C}) = \max_{|x|=1} \langle \tilde{C}x, x\rangle = -\min_{|x|=1}\left(\sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} \langle C_i x, x\rangle\right).$$

Since $C_i + I \geq 0$, by (i) the minimum is attained at a simultaneous eigenvector $x = x_0$ of C_i with eigenvalue -1. We obtain

$$\Lambda(\tilde{C}) = \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}.$$

By (30) we have $\Lambda(\tilde{C}) < 1$, so there exists a $\Lambda > 0$ satisfying

$$\Lambda(\tilde{C}) = \frac{\Lambda}{1+\Lambda}.$$

Next we define

$$C_0 = (1+\Lambda)\tilde{C} \in \mathcal{E}.$$

The maximal eigenvalue of C_0 is

$$\Lambda(C_0) = (1+\Lambda)\Lambda(\tilde{C}) = \Lambda.$$

With this, we have

$$\Lambda(\tilde{C}) = \frac{\Lambda(C_0)}{1+\Lambda(C_0)} = \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}.$$

The last equality gives (5). Thus Theorem B applies, completing the proof, *once we show that* $C_0 \in \partial\mathcal{K}_0$. Equivalently, we need to show that $C_0 + I$ is positive semidefinite but not positive definite. To do this, we first note that

$$\tilde{C} = -\sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} C_i = \frac{1}{1+\Lambda(C_0)} C_0,$$

where the last equality gives (6). Moreover, we have

$$\begin{aligned} \frac{1}{1+\Lambda(C_0)}(C_0+I) &= \frac{1}{1+\Lambda(C_0)} I - \frac{1}{1+\Lambda(C_0)} C_0 \\ &= \left(1 - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)}\right) I - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} C_i \\ &= I - \sum_{i=1}^{n} \frac{1}{1+\Lambda(C_i)} (C_i + I). \end{aligned}$$

By (ii) this is positive semidefinite but not positive definite. Theorem A follows. □

As an application, consider now the tetrahedral minimal immersion. Relative to an orthonormal basis, we write $\mathcal{H}_{\lambda_6} = \mathbf{R}^7 \otimes \mathbf{R}^7 = \mathbf{R}^{49}$ (see [6]). We view an endomorphism of $\mathcal{H}_{\lambda_6}$ as a matrix with 7×7 blocks, each block being a 7×7 matrix. Using the computations in [6] yields

$$C_1 + I = \operatorname{diag}[0, 0, 7, 0, 0, 0, 0].$$

This is a diagonal 7×7 block matrix, and each number c represents a diagonal 7×7 matrix with diagonal entry c. The distortion at C_1 is $\Lambda(C_1) = 6$.

In a similar vein, we have

$$C_2 + I = \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 & -\frac{\sqrt{15}}{24} & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{\sqrt{15}}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{\sqrt{15}}{24} & 0 & 0 & 0 & \frac{5}{24} & 0 & 0 \\ 0 & \frac{\sqrt{15}}{24} & 0 & 0 & 0 & \frac{5}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with distortion $\Lambda(C_2) = 4/3$.

We are now in the position to apply Theorem A. Condition (i) is obviously satisfied, since the last copy of $\mathbf{R}^7$ in $\mathbf{R}^7 \otimes \mathbf{R}^7$ is in the common kernel of $C_1 + I$ and $C_2 + I$. The matrix on the left-hand side in (ii) is

$$I - \frac{1}{7}(C_1 + I) - \frac{3}{7}(C_2 + I) = \begin{bmatrix} \frac{53}{56} & 0 & 0 & 0 & -\frac{\sqrt{15}}{42} & 0 & 0 \\ 0 & \frac{53}{56} & 0 & 0 & 0 & \frac{\sqrt{15}}{42} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{7} & 0 & 0 & 0 \\ -\frac{\sqrt{15}}{42} & 0 & 0 & 0 & \frac{37}{42} & 0 & 0 \\ 0 & \frac{\sqrt{15}}{42} & 0 & 0 & 0 & \frac{37}{42} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A simple computation shows that this matrix is positive semidefinite. Theorem A now asserts that the intersection $\mathcal{K}_0 \cap \mathcal{E}$ is a triangle. Note that the proof actually constructs the third vertex C_0.

References

[1] M. Berger, *Geometry I–II,* Springer-Verlag, New York, 1994.

[2] D. DeTurck and W. Ziller, *Minimal isometric immersions of spherical space forms in spheres,* Comment. Math. Helv. 67 (1992), 428–458.

[3] M. DoCarmo and N. Wallach, *Minimal immersions of spheres into spheres,* Ann. of Math. (2) 93 (1971), 43–62.

[4] J. D. Moore, *Isometric immersions of space forms into space forms,* Pacific J. Math. 40 (1972), 157–166.

[5] G. Toth, *Infinitesimal rotations of isometric minimal immersions between spheres,* Amer. J. Math. 122 (2000), 117–152.

[6] ———, *Finite Möbius groups, minimal immersions of spheres, and moduli*, Springer-Verlag, New York, 2002.

[7] ———, *Moduli for spherical maps and minimal immersions of homogeneous spaces,* J. Lie Theory 12 (2002), 551–570.

[8] N. Wallach, *Minimal immersions of symmetric spaces into spheres,* Symmetric spaces (St. Louis, 1969–1970), Pure Appl. Math., 8, pp. 1–40, Dekker, New York, 1972.

Department of Mathematics
Rutgers University
Camden, NJ 08102

gtoth@camden.rutgers.edu