

Moduli for Spherical Maps and Minimal Immersions of Homogeneous Spaces

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Abstract. The DoCarmo-Wallach theory studies isometric minimal immersions $f: G/K \rightarrow S^n$ of a compact Riemannian homogeneous space G/K into Euclidean n -spheres for various n . For a given domain G/K , the moduli space of such immersions is a compact convex body in a representation space for the Lie group G . In 1971 DoCarmo and Wallach gave a lower bound for the (dimension of the) moduli for $G/K = S^m$, and conjectured that the lower bound was achieved. In 1997 the author proved that this was true. The DoCarmo-Wallach conjecture has a natural generalization to all compact Riemannian homogeneous domains G/K . The purpose of the present paper is to show that for G/K a nonspherical compact rank 1 symmetric space this generalized conjecture is false. The main technical tool is to consider spherical functions of subrepresentations of $C^\infty(G/K)$, express them in terms of Jacobi polynomials, and use a recent linearization formula for products of Jacobi polynomials.

1. Introduction and Statement of Results

Let $M = G/K$ be a Riemannian homogeneous space, where G is a compact Lie group and K a closed subgroup. Then G acts on the space $C^\infty(M)$ of (real valued) functions on M in a natural way: $g \cdot \xi = \xi \circ g^{-1}$, $g \in G$, $\xi \in C^\infty(M)$. This action preserves the L^2 -scalar product on $C^\infty(M)$ defined by the volume element v_M . Let $\mathcal{H} \subset C^\infty(M)$ be a G -submodule. We call a map $f: M \rightarrow S_V$ into the unit sphere S_V of a Euclidean vector space V a spherical \mathcal{H} -map if its components $\alpha \circ f$, $\alpha \in V^*$, belong to \mathcal{H} . The Dirac delta $\delta: M \rightarrow S_{\mathcal{H}^*}$ [5] defined by evaluating the elements of \mathcal{H} on points of M is the universal example of a spherical \mathcal{H}^* -map. (The scalar product on \mathcal{H}^* is induced by the L^2 -scalar product on \mathcal{H} suitably scaled.)

Remark. If $M = G/K$ is naturally reductive, and $\mathcal{H} \subset C^\infty(M)$ is irreducible then \mathcal{H} is contained in an eigenspace V_λ of the Laplacian Δ^M for some eigenvalue λ [25]. In particular, the components of an \mathcal{H} -map $f: M \rightarrow S_V$ are eigenfunctions of the Laplacian with a common eigenvalue. Thus an \mathcal{H} -map is a λ -eigenmap in the sense of Eells-Sampson [8], a harmonic map with constant energy density $\lambda/2$.

In general, a DoCarmo-Wallach type argument [6] shows that the set of (congruence classes of) full spherical \mathcal{H} -maps $f: M \rightarrow S_V$, for various V , can be parametrized by a moduli space $\mathcal{L}(\mathcal{H})$, a compact convex body in a G -submodule $\mathcal{E}(\mathcal{H})$ of the symmetric square $S^2(\mathcal{H})$ (Propositions 3.1-3.2 in Section 3 below). (The map f is full if its image is not contained in a proper great sphere of the range [3], and congruent maps differ by an isometry between the ranges.) In what follows, we identify $S^2(\mathcal{H})$ with the space of linear endomorphisms of \mathcal{H} . Then the moduli space is given by

$$\mathcal{L}(\mathcal{H}) = \{C \in \mathcal{E}(\mathcal{H}) \mid C + I \geq 0\},$$

where \geq means positive semidefinite, and I is the identity. The origin of $\mathcal{E}(\mathcal{H})$ is in the interior of $\mathcal{L}(\mathcal{H})$, and it corresponds to δ .

The G -module homomorphism

$$\Psi^0: S^2(\mathcal{H}) \rightarrow C^\infty(M)$$

given by multiplication has image $\mathcal{H} \cdot \mathcal{H} \subset C^\infty(M)$ consisting of (finite) sums of products of functions in \mathcal{H} . The DoCarmo-Wallach parametrization of $\mathcal{L}(\mathcal{H})$ implies that the kernel of Ψ^0 is $\mathcal{E}(\mathcal{H})$ (Proposition 3.3). We thus have

$$\mathcal{E}(\mathcal{H}) = S^2(\mathcal{H})/(\mathcal{H} \cdot \mathcal{H}),$$

as G -modules. To determine $\mathcal{E}(\mathcal{H})$ (and thereby to compute $\dim \mathcal{L}(\mathcal{H}) = \dim \mathcal{E}(\mathcal{H})$) amounts to decomposing $\mathcal{H} \cdot \mathcal{H}$ into irreducible components.

Let $M = G/K$ be a compact rank 1 symmetric space. Then M is the Euclidean m -sphere S^m , one of the projective spaces $\mathbf{R}P^m$, $\mathbf{C}P^m$, $\mathbf{H}P^m$, or the Cayley projective plane $\mathbf{Ca}P^2$ [2]. It is well-known that $C^\infty(M)$ has a multiplicity one decomposition into irreducible components, and each component $\mathcal{H} \subset C^\infty(M)$ is the full eigenspace V_λ of the Laplacian corresponding to an eigenvalue λ [15-17]. Our first result is the following:

Theorem. *A Let $M = G/K$ be a compact rank 1 symmetric space, $\mathcal{H} \subset C^\infty(M)$ an irreducible G -submodule. We write $\mathcal{H} = V_{\lambda_p}$, where λ_p is the p -th eigenvalue of the Laplacian on \mathcal{H} . Then we have*

$$V_{\lambda_p} \cdot V_{\lambda_p} = \begin{cases} \sum_{j=0}^p V_{\lambda_{2j}} & \text{if } M = S^m \\ \sum_{j=0}^{2p} V_{\lambda_j} & \text{otherwise.} \end{cases}$$

In particular, the dimension of the moduli space is given by

$$\dim \mathcal{L}(V_{\lambda_p}) = \begin{cases} n(\lambda_p)(n(\lambda_p) + 1)/2 - \sum_{j=0}^p n(\lambda_{2j}) & \text{if } M = S^m, \\ n(\lambda_p)(n(\lambda_p) + 1)/2 - \sum_{j=0}^{2p} n(\lambda_j) & \text{otherwise,} \end{cases}$$

where $n(\lambda_p) = \dim V_{\lambda_p}$.

Since $n(\lambda_p)$ is known for each case of M (second table in Section 2), an explicit formula can be derived for the dimension of $\mathcal{L}(V_{\lambda_p})$. If the dimension is zero then the moduli space reduces to a point, and we have rigidity. This means that the corresponding spherical V_{λ_p} -maps are rigid in the sense that any full f is congruent to the Dirac delta δ .

Corollary. *Let M and $\mathcal{H} = V_{\lambda_p}$ be as in Theorem A. Then the cases when $\dim \mathcal{L}(V_{\lambda_p})$ is trivial are summarized in the following table:*

M	m	p
S^m	$m \geq 2$	$p = 1$
$S^m, \mathbf{R}P^m$	$m = 2$	$p \geq 1$
$\mathbf{C}P^m, \mathbf{H}P^m, \mathbf{C}aP^2$	$m = 2$	$p = 1$

Remark. Rigidity of a spherical V_{λ_1} -map $f: S^m \rightarrow S_V$ is obvious since f is the restriction of a linear map, and thereby it is an isometry. Rigidity of spherical V_{λ_p} -maps $f: M \rightarrow S_V$ for $M = S^2, \mathbf{R}P^2$ is due to Calabi [3] (stated only for minimal immersions). (In general, a spherical V_{λ_p} -map $f: \mathbf{R}P^m \rightarrow S_V$ is a spherical $V_{\lambda_{2p}}$ -map $\tilde{f}: S^m \rightarrow S_V$ factored through the twofold projection $S^m \rightarrow \mathbf{R}P^m$.)

A rigidity result of DoCarmo-Wallach [6,25] asserts that a minimal immersion $f: M \rightarrow S_V$ of a compact analytic manifold M is rigid among minimal immersions, if the (geometric) degree of f is < 4 . For $M = \mathbf{C}P^m, \mathbf{H}P^m, \mathbf{C}aP^2$ as in the corollary, the degree of $\delta: M \rightarrow S_{V_{\lambda_p}}$ is $2p$ [18]. Notice however that the corollary gives rigidity among all spherical V_{λ_1} -maps not just minimal immersions.

We now return to the general setting. Let G be a compact Lie group. An orthogonal G -module \mathcal{H} is a Euclidean vector space on which G acts linearly via orthogonal transformations. In other words, \mathcal{H} is a representation space for G , and it is endowed with a G -invariant scalar product.

Let K be a closed subgroup. A class 1 representation of (G, K) is an irreducible orthogonal G -module \mathcal{H} so that there is a nonzero vector $\chi_0 \in \mathcal{H}$ fixed by K . It is well known that, for $M = G/K$ Riemannian homogeneous, the irreducible components of $C^\infty(M)$ are class 1 representations of (G, K) .

Since all components of $C^\infty(M)$ are class 1 with respect to (G, K) it is natural to ask whether $\mathcal{H} \cdot \mathcal{H} \subset C^\infty(M)$ contains all class 1 components of $S^2(\mathcal{H})$. We reformulate this by introducing $\bar{\mathcal{E}}(\mathcal{H})$ as the sum of those irreducible G -submodules of $S^2(\mathcal{H})$ that are not class 1 with respect to (G, K) . The very existence of the homomorphism Ψ^0 above implies that

$$\bar{\mathcal{E}}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}),$$

and the question is whether equality holds. For $M = S^m$, the answer is yes, and it follows from the (multiplicity one) decomposition for $S^2(V_{\lambda_p})$ derived by DoCarmo and Wallach [6]. (For a simple proof, see also [14].)

Our next result shows that the answer is negative for $M = \mathbf{C}P^m$.

Theorem. *Let $(G, K) = (U(m+1), U(m) \times U(1))$, and $M = \mathbf{C}P^m, m \geq 2$, the complex projective m -space. Let $\mathcal{H} = V_{\lambda_p}, p \geq 2$. Then $\mathcal{E}(V_{\lambda_p})$ contains class 1 submodules with respect to $(U(m+1), U(m) \times U(1))$. Equivalently*

$$\bar{\mathcal{E}}(V_{\lambda_p}) \neq \mathcal{E}(V_{\lambda_p}).$$

More precisely, we have

$$\sum_{q=2}^{2p-2} \frac{1}{2} \left(\min(q, 2p-q) + \frac{(-1)^q - 1}{2} \right) V_{\lambda_q} \subset \mathcal{E}(V_{\lambda_p}).$$

Remark. For $M = \mathbf{C}P^m$, Theorems A-B correct Theorem 4.2 of Chapter III in [22]. The formula there should give only a lower bound for $\mathcal{E}(V_{\lambda_p})$ and for the dimension of the moduli space $\mathcal{L}(V_{\lambda_p})$. This is due to the fact that the osculating spaces of $\delta: \mathbf{C}P^m \rightarrow S_{V_{\lambda_p}^*}$ are reducible as $U(m)$ -modules (Theorem 3.3 of Chapter II).

Assume now that $M = G/K$ is isotropy irreducible. This means that K acts on the tangent space $T_o(M)$, $o = \{K\}$, irreducibly by the isotropy representation. Then, for an irreducible G -submodule $\mathcal{H} \subset C^\infty(M)$, the Dirac delta $\delta: M \rightarrow S_{\mathcal{H}^*}$ is a minimal immersion inducing the $\lambda/\dim M$ -multiple of the original Riemannian metric on M [25].

DoCarmo and Wallach proved that the set of (congruence classes of) full minimal immersions $f: M \rightarrow S_V$, for various V , and with induced Riemannian metric the $\lambda/\dim M$ -multiple of the original, can be parametrized by a moduli space $\mathcal{M}(\mathcal{H})$, a compact convex body in a G -submodule $\mathcal{F}(\mathcal{H})$ of $S^2(\mathcal{H})$ (Proposition 3.4). The moduli space is given by

$$\mathcal{M}(\mathcal{H}) = \{C \in \mathcal{F}(\mathcal{H}) \mid C + I \geq 0\},$$

where \geq means positive semidefinite.

We now recall the definition of induced representations [25]. If \mathcal{W} is a K -module then $\text{Ind}_K^G(\mathcal{W})$ denotes the linear space of continuous maps $\phi: G \rightarrow \mathcal{W}$ which satisfy $\phi(kg) = k \cdot \phi(g)$, $k \in K$, $g \in G$. The action of G on $\text{Ind}_K^G(\mathcal{W})$ given by $g \cdot \phi(g') = \phi(gg')$, $g, g' \in G$, defines a G -module structure on $\text{Ind}_K^G(\mathcal{W})$. We call $\text{Ind}_K^G(\mathcal{W})$ the G -module induced from the K -module \mathcal{W} .

DoCarmo and Wallach constructed a homomorphism

$$\Psi: S^2(\mathcal{H}) \rightarrow \text{Ind}_K^G(S^2(\mathfrak{p})),$$

where K -module $S^2(\mathfrak{p})$ is the symmetric square of the isotropy representation of $M = G/K$. The kernel of Ψ is $\mathcal{F}(\mathcal{H})$.

Let $\bar{\mathcal{F}}(\mathcal{H})$ denote the sum of those components of $S^2(\mathcal{H})$ that, when restricted to K , do not contain any irreducible K -submodules of $S^2(\mathfrak{p})$. Frobenius reciprocity [25] says that

$$\bar{\mathcal{F}}(\mathcal{H}) \subset \mathcal{F}(\mathcal{H}). \tag{1}$$

Thus, once the irreducible decomposition of $S^2(\mathcal{H})$ is known, this gives a lower bound on the dimension of the moduli $\mathcal{M}(\mathcal{H})$.

DoCarmo and Wallach carried this out for $M = S^m$, and $\mathcal{H} = V_{\lambda_p}$. Identifying the irreducible components of $\bar{\mathcal{F}}(V_{\lambda_p})$, for $m \geq 3$ and $p \geq 4$, they obtained the lower estimate

$$\dim \mathcal{M}(V_{\lambda_p}) = \dim \mathcal{F}(V_{\lambda_p}) \geq \dim \bar{\mathcal{F}}(V_{\lambda_p}) \geq \dim \bar{\mathcal{F}}(V_{\lambda_4}) \geq 18.$$

They conjectured that equality holds in (1). This has been resolved by the author in [21] using different methods. (For a recent algebraic proof, see [26].) For the lowest dimensional moduli space, $\mathcal{M}(V_{\lambda_4})$ with $m = 3$, see [23].

Once again, it is natural to ask whether equality holds in (1) in general, or at least for compact rank 1 symmetric spaces $M = G/K$. Our last result is to show that the answer is negative for $M = \mathbf{C}P^m$, and $\mathcal{H} = V_{\lambda_p}$.

Theorem. *Let $m \geq 3$, $(G, K) = (U(m + 1), U(m) \times U(1))$, $M = U(m + 1)/(U(m) \times U(1)) = \mathbf{C}P^m$, and $\mathcal{H} = V_{\lambda_p}$. Then, for $p = 3$ and $m \not\equiv 1 \pmod{4}$, or for $p \geq 4$, we have*

$$\bar{\mathcal{F}}(V_{\lambda_p}) \neq \mathcal{F}(V_{\lambda_p}).$$

The striking difference between the spherical and complex projective cases is that $S^2(V_{\lambda_p})$, $V_{\lambda_p} \subset C^\infty(S^m)$, has a multiplicity one decomposition into irreducible components, but according to the multiplicity formulas developed by Barbasch [22], this fails for $S^2(V_{\lambda_p})$, $V_{\lambda_p} \subset C^\infty(\mathbf{C}P^m)$.

2. Zonal Spherical Functions and Jacobi Polynomials

In this section we describe the main idea of the proof of Theorem A as well as assemble some preliminary facts.

Let $M = G/K$ be a compact rank 1 symmetric space. As noted above, an irreducible G -submodule $\mathcal{H} \subset C^\infty(M)$ is class 1 with respect to the pair (G, K) . We call a K -fixed vector $\chi_0 \in \mathcal{H}$ a zonal spherical function [15,25]. It is well-known that a zonal spherical function is unique up to a constant multiple [2].

Let χ_0 be a zonal spherical function of \mathcal{H} . Its square $\chi_0^2 \in \mathcal{H} \cdot \mathcal{H}$ is also fixed by K . Since $C^\infty(M)$ has a multiplicity one decomposition into irreducible components, as an element of $C^\infty(M)$, χ_0^2 decomposes into a sum

$$\chi_0^2 = \sum_{j=1}^n \chi_j,$$

where each χ_j belongs to a unique irreducible component $\mathcal{H}_j \subset C^\infty(M)$. Clearly, χ_j is a zonal spherical function of \mathcal{H}_j . Since $\chi_j \in \mathcal{H}_j$ is a component of $\chi_0^2 \in \mathcal{H} \cdot \mathcal{H}$, by Schur's lemma, \mathcal{H}_j projects nontrivially to $\mathcal{H} \cdot \mathcal{H}$, and we obtain

$$\sum_{j=1}^n \mathcal{H}_j \subset \mathcal{H} \cdot \mathcal{H}.$$

In Section 4 we will prove Theorem A by showing that equality holds here. We now illustrate this in a different setting by a simple example.

Example. Let G be a compact Lie group viewed as a symmetric space $G \times G/G^*$ of compact type, where $G^* \subset G \times G$ is the diagonal [15,16,24]. (The map $(g_1, g_2)G^* \mapsto g_1g_2^{-1}$, $g_1, g_2 \in G$, identifies $G \times G/G^*$ with G .) The space $C^\infty(G \times G/G^*, \mathbf{C})$ of complex valued smooth functions on $G \times G/G^*$ has a multiplicity one decomposition into irreducible components. A component, a complex irreducible $G \times G$ -submodule of $C^\infty(G \times G/G^*, \mathbf{C})$, has the form $\mathcal{H}^* \otimes \mathcal{H}$, where \mathcal{H} is a complex irreducible G -module. The G^* -fixed vectors in $\mathcal{H}^* \otimes \mathcal{H}$ can be identified with the (multiples of a normalized) character χ_0 of \mathcal{H} [24]. Given χ_0 , according to our procedure, we need to decompose the square χ_0^2 into a sum of (nonzero) characters

$$\chi_0^2 = \sum_{j=1}^n c_j \chi_j.$$

By elementary character theory, this decomposition corresponds to the decomposition of the tensor product

$$\mathcal{H} \otimes \mathcal{H} = \sum_{j=1}^n c_j \mathcal{H}_j$$

as a G -module, where χ_j is the character of \mathcal{H}_j and $c_j \in \mathbf{N}$ is the multiplicity of \mathcal{H}_j in $\mathcal{H} \otimes \mathcal{H}$. Since $\chi_0^2 \in (\mathcal{H}^* \otimes \mathcal{H}) \cdot (\mathcal{H}^* \otimes \mathcal{H})$, Schur's lemma tells us that

$$\sum_{j=1}^n \mathcal{H}_j^* \otimes \mathcal{H}_j \subset (\mathcal{H}^* \otimes \mathcal{H}) \cdot (\mathcal{H}^* \otimes \mathcal{H})$$

as $G \times G$ -modules. We claim that equality holds here. Indeed, consider the natural extension of Ψ^0 above

$$\Psi^0: (\mathcal{H}^* \otimes \mathcal{H}) \otimes (\mathcal{H}^* \otimes \mathcal{H}) \rightarrow (\mathcal{H}^* \otimes \mathcal{H}) \cdot (\mathcal{H}^* \otimes \mathcal{H})$$

given by multiplication. The domain of Ψ^0 , as a $G \times G$ -module, can be decomposed as

$$\begin{aligned} (\mathcal{H}^* \otimes \mathcal{H}) \otimes (\mathcal{H}^* \otimes \mathcal{H}) &= (\mathcal{H}^* \otimes \mathcal{H}^*) \otimes (\mathcal{H} \otimes \mathcal{H}) \\ &= \left(\sum_{j=1}^n c_j \mathcal{H}_j^* \right) \otimes \left(\sum_{l=1}^n c_l \mathcal{H}_l \right) = \sum_{j,l=1}^n c_j c_l (\mathcal{H}_j^* \otimes \mathcal{H}_l). \end{aligned}$$

Finally, by Schur's lemma again $\mathcal{H}_j^* \otimes \mathcal{H}_l$ contains a G^* -fixed vector if and only if $j = l$ [24]. The claim follows.

We now return to our compact rank 1 symmetric space $M = G/K$. As noted in Section 1, the full eigenspace V_λ of the Laplacian Δ^M corresponding to an eigenvalue λ is an irreducible G -module. Moreover, if $\{\lambda_p\}_{p=0}^\infty$ denotes the sequence of eigenvalues of Δ^M in increasing order, then we have

$$C^\infty(M) = \sum_{p=0}^\infty V_{\lambda_p}.$$

By the above, $V_{\lambda_p} \subset C^\infty(M)$ contains a zonal spherical function χ_0 , unique up to a constant multiple.

We now recall that, for fixed $\alpha, \beta > -1$, the Jacobi polynomials $P_n^{(\alpha, \beta)}$, $n \geq 0$, form an orthogonal series on $[-1, 1]$ with respect to the weight function $(1 - x)^\alpha(1 + x)^\beta$ [1]. The polynomial $P_n^{(\alpha, \beta)}$ can be defined by

$$(1 - x)^\alpha(1 + x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x)^{n+\alpha}(1 + x)^{n+\beta}].$$

With a suitable choice of parameters on M , the zonal function χ_0 of a component V_{λ_p} of $C^\infty(M)$ is a constant multiple of $P_n^{(\alpha, \beta)}$ with α, β, n depending on V_{λ_p} [4,10,24]. (For example, $n = p$ in all cases but $M = \mathbf{R}P^m$ for which $n = 2p$.) The classification of compact rank 1 symmetric spaces M , the eigenvalues of Δ^M [2], and the Jacobi polynomials corresponding to the zonal spherical harmonics χ_0 are summarized in the following table:

(G, K)	$M = G/K$	λ_p	χ_0
$(SO(m + 1), SO(m))$	S^m	$p(p + m - 1)$	$P_p^{(m/2-1, m/2-1)}$
$(SO(m + 1), O(m))$	$\mathbf{R}P^m$	$2p(2p + m - 1)$	$P_{2p}^{(m/2-1, m/2-1)}$
$(U(m + 1), U(m) \times U(1))$	$\mathbf{C}P^m$	$4p(p + m)$	$P_p^{(m-1, 0)}$
$(Sp(n + 1), (Sp(n) \times Sp(1)))$	$\mathbf{H}P^m$	$4p(p + 2m + 1)$	$P_p^{(2m-1, 1)}$
$(F_4, Spin(9))$	$\mathbf{Ca}P^2$	$4p(p + 11)$	$P_p^{(7, 3)}$

The multiplicities $n(\lambda_p) = \dim V_{\lambda_p}$ are given as follows:

M	$n(\lambda_p)$
S^m	$\binom{p+m}{m} - \binom{p+m-2}{m}$
$\mathbf{R}P^m$	$\binom{2p+m}{m} - \binom{2p+m-2}{m}$
$\mathbf{C}P^m$	$\binom{p+m}{m}^2 - \binom{p+m-1}{m}^2$
$\mathbf{H}P^m$	$\frac{2p+2m+1}{2m(2m+1)} \binom{p+2m}{2m-1} \binom{p+2m-1}{2m-1}$
$\mathbf{Ca}P^2$	$\frac{2p+11}{1320} \binom{p+10}{7} \binom{p+7}{7}$

To simplify the treatment and to avoid some overlapping cases, we will assume that $m \geq 2$.

Since, up to parametrization, the zonals are Jacobi polynomials, we need to obtain a decomposition of the square $(P_p^{(\alpha, \beta)})^2$ into a sum of Jacobi polynomials:

$$(P_p^{(\alpha, \beta)})^2 = \sum_{j=0}^{2p} c(j, p; \alpha, \beta) P_j^{(\alpha, \beta)}. \tag{2}$$

More generally, a formula of the type

$$P_p^{(\alpha, \beta)} P_q^{(\alpha, \beta)} = \sum_{j=|p-q|}^{p+q} c(j, p, q; \alpha, \beta) P_j^{(\alpha, \beta)}. \tag{3}$$

is usually called “linearization of the product.”

For $\alpha = \beta$, the Jacobi polynomial $P_p^{(\alpha, \beta)}$ is, up to normalization, the ultraspherical (or Gegenbauer) polynomial C_p^ν , where $\nu - 1/2 = \alpha = \beta$. (The precise formula is given in (20) below.) Linearization of the product of ultraspherical polynomials dates back to the early twentieth century, and the coefficients $c(j, p, q; \lambda - 1/2, \lambda - 1/2)$ have been calculated explicitly [1,7,24]. For our purposes, we need only that $c(j, p, q; \lambda - 1/2, \lambda - 1/2)$ is positive if and only if $|p - q| \leq j \leq p + q$ and $j \equiv p + q \pmod{2}$.

For Jacobi polynomials in general linearization proved to be much more difficult and the exact decomposition formula is fairly recent [19]. A general and sharp positivity result for the coefficients $c(j, p, q; \alpha, \beta)$ (covering the remaining cases in the table above for $m \geq 2$) is due to Gasper [11,12]. It states that if $\alpha, \beta > -1$, $a = \alpha + \beta + 1$, $b = \alpha - \beta$, then $c(j, p, q; \alpha, \beta) > 0$ provided that (α, β) is in the interior of the set

$$V = \{(\alpha, \beta) \mid \alpha \geq \beta, a(a + 5)(a + 3)^2 \geq (a^2 - 7a - 24)b^2\}.$$

Note that Theorem 1 in [12] states nonnegativity of the coefficients for $(\alpha, \beta) \in V$. As Professor Gasper communicated to the author [13], a closer inspection of his proof of Theorem 1 in [12], pp. 585-591, shows strict positivity of the coefficients if (α, β) is in the interior of V . Another proof of the positivity follows by using the $\{ \}_9 F_8$ series representations for the linearization coefficients in [19] (formula (3.9)).

3. Generalities on the Moduli

Let G be a compact Lie group and \mathcal{H} an orthogonal G -module. We define

$$\mathcal{K}(\mathcal{H}) = \{C \in S^2(\mathcal{H}) \mid C + I \geq 0\}.$$

We write $\mathcal{K} = \mathcal{K}(\mathcal{H})$ if there is no danger of confusion. \mathcal{K} is a G -invariant set in $S^2(\mathcal{H})$, where the G -module structure on $S^2(\mathcal{H})$ is extended from that of \mathcal{H} .

Since $C + I \geq 0$ is a convex condition, \mathcal{K} is a convex set. The interior of \mathcal{K} consists of those endomorphisms C that satisfy $C + I > 0$. It follows that \mathcal{K} has a nonempty interior, and hence it is a convex body in $S^2(\mathcal{H})$. Notice that \mathcal{K} is noncompact since the multiples λI , $\lambda \geq -1$, are contained in \mathcal{K} . We call $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the general moduli space for \mathcal{H} .

We let $S_0^2(\mathcal{H})$ denote the G -submodule of $S^2(\mathcal{H})$ comprised of the traceless symmetric endomorphisms of V . We define

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap S_0^2(\mathcal{H}) = \{C \in S_0^2(\mathcal{H}) \mid C + I \geq 0\}.$$

The eigenvalues of the symmetric endomorphisms in \mathcal{K} are greater or equal to -1 . Hence the eigenvalues of the endomorphisms in \mathcal{K}_0 are contained in $[-1, \dim \mathcal{H} - 1]$. It follows that \mathcal{K}_0 is compact, and a convex body in $S_0^2(\mathcal{H})$. We call $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H})$ the reduced moduli space for \mathcal{H} .

We now give an interpretation of the moduli as parameter spaces for certain maps. We let M be a compact Riemannian manifold, and G a compact Lie group of isometries of M . (G is a closed subgroup of the full isometry group of M .) The space $C^\infty(M)$ of smooth functions on M is a representation space for G , where $g \in G$ acts on $\xi \in C^\infty(M)$ by $g \cdot \xi = \xi \circ g^{-1}$. We fix a finite dimensional G -submodule $\mathcal{H} \subset C^\infty(M)$. We endow \mathcal{H} with the scaled L^2 -scalar product

$$\langle \chi_1, \chi_2 \rangle = \frac{\dim \mathcal{H}}{\text{vol}(M)} \int_M \chi_1 \chi_2 v_M, \quad \chi_1, \chi_2 \in \mathcal{H}, \tag{4}$$

where v_M is the Riemannian volume form on M , and $\text{vol}(M) = \int_M v_M$ is the volume of M . With this scalar product \mathcal{H} becomes an orthogonal G -module.

A smooth map $f: M \rightarrow V$ into a Euclidean vector space V is said to be full if the image of f spans V . A component of f is $\alpha \circ f \in C^\infty(M)$, where $\alpha \in V^*$. The space of components of f is defined as

$$V_f = \{\alpha \circ f \mid \alpha \in V^*\} \subset C^\infty(M).$$

The map f is full if and only if the linear map $f^*: V^* \rightarrow V_f$, given by precomposition with f , is an isomorphism. Since V is Euclidean we also have $V \cong V^* \cong V_f$. Note that any map can be made full by restricting its range to the linear span of the image.

Two maps $f_1: M \rightarrow V_1$ and $f_2: M \rightarrow V_2$ are said to be congruent if there is a linear isometry $U: V_1 \rightarrow V_2$ such that $f_2 = U \circ f_1$.

With \mathcal{H} as above, $f: M \rightarrow V$ is said to be an \mathcal{H} -map if $V_f \subset \mathcal{H}$. Note that any smooth map $f: M \rightarrow V$ is an \mathcal{H} -map for \mathcal{H} the smallest G -invariant linear subspace in $C^\infty(M)$ that contains V_f .

The Dirac delta as a map $\delta_{\mathcal{H}}: M \rightarrow \mathcal{H}^*$ is defined in the usual way

$$\delta_{\mathcal{H}}(x)(\chi) = \chi(x), \quad x \in M, \chi \in \mathcal{H}.$$

The component of $\delta_{\mathcal{H}}$ corresponding to $\chi \in \mathcal{H} = \mathcal{H}^{**}$ is $\langle \delta_{\mathcal{H}}, \chi \rangle = \chi$. Hence, $V_{\delta_{\mathcal{H}}} = \mathcal{H}$ and $\delta_{\mathcal{H}}$ is a full \mathcal{H} -map.

In what follows we will identify \mathcal{H} with its dual \mathcal{H}^* via the scalar product on \mathcal{H} . With respect to an orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$, $\dim \mathcal{H} = N + 1$, the Dirac delta as a map $\delta_{\mathcal{H}}: M \rightarrow \mathcal{H}$ can be written as

$$\delta_{\mathcal{H}}(x) = \sum_{j=0}^N \chi^j(x) \chi^j, \quad x \in M. \tag{5}$$

Indeed, for $\chi \in \mathcal{H}$, we have

$$\langle \delta_{\mathcal{H}}(x), \chi \rangle = \chi(x) = \sum_{j=0}^N \langle \chi, \chi^j \rangle \chi^j(x) = \left\langle \sum_{j=0}^N \chi^j(x) \chi^j, \chi \right\rangle.$$

The Dirac delta $\delta_{\mathcal{H}}$ is equivariant with respect to the homomorphism $\rho_{\mathcal{H}}: G \rightarrow O(\mathcal{H})$ that defines the orthogonal G -module structure on $\mathcal{H} \cong \mathcal{H}^*$.

For a full \mathcal{H} -map $f: M \rightarrow V$, we have $f = A \circ \delta_{\mathcal{H}}$, where $A: \mathcal{H} \rightarrow V$ is a surjective linear map. We associate to f the symmetric linear endomorphism

$$\langle f \rangle = A^*A - I \in S^2(\mathcal{H}).$$

It is clear that $\langle f \rangle$ depends only on the congruence class of f . Since A^*A is always positive semidefinite, we also have $\langle f \rangle \in \mathcal{K}(\mathcal{H})$. A DoCarmo-Wallach type argument shows that $f \mapsto \langle f \rangle$ gives rise to a one-to-one correspondence between the set of congruence classes of full \mathcal{H} -maps and the general moduli space $\mathcal{K}(\mathcal{H})$ [6,25].

Let $f: M \rightarrow V$ be a full \mathcal{H} -map. With respect to an orthonormal basis in V , f can be written in components as $f = (f^0, \dots, f^n)$, $\dim V = n + 1$. With the orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ as above, $A: \mathcal{H} \rightarrow V$ becomes an $(n+1) \times (N+1)$ -matrix with entries a_{kj} , $k = 0, \dots, n$, $j = 0, \dots, N$. In components, $f = A \circ \delta_{\mathcal{H}}$ can be written as

$$f^k = \sum_{j=0}^N a_{kj} \chi^j, \quad k = 0, \dots, n.$$

We now calculate

$$\text{trace}(\langle f \rangle + I) = \text{trace} A^*A = \sum_{k=0}^n \sum_{j=0}^N a_{kj}^2 = \sum_{k=0}^n |f^k|^2.$$

We conclude that, in terms of the scaled L^2 -scalar product (4) on \mathcal{H} , the parameter point $\langle f \rangle \in \mathcal{K}(\mathcal{H})$ is traceless if and only if

$$\int_M \sum_{k=0}^n (f^k)^2 v_M = \text{vol}(M). \tag{6}$$

We call f normalized if (6) is satisfied. Clearly, $\delta_{\mathcal{H}}$ is normalized.

It is also clear that, by suitable scaling, any nontrivial map can be normalized.

Summarizing, we obtain the following:

Proposition 3.1. *Let M be a compact Riemannian manifold with compact group G of isometries. Given a finite dimensional G -submodule \mathcal{H} of $C^\infty(M)$, the set of congruence classes of full \mathcal{H} -maps $f: M \rightarrow V$ can be parametrized by the general moduli space $\mathcal{K}(\mathcal{H})$. The reduced moduli $\mathcal{K}_0(\mathcal{H})$ parametrizes the normalized \mathcal{H} -maps.*

An \mathcal{H} -map $f: M \rightarrow V$ is called spherical if the image of f is contained in the unit sphere S_V of V . A finite dimensional G -module $\mathcal{H} \subset C^\infty(M)$ is called δ -spherical if $\delta_{\mathcal{H}}$ is spherical. Due to the scaling of the L^2 -scalar product in (4), \mathcal{H} is δ -spherical if and only if

$$\sum_{j=0}^N (\chi^j)^2 = 1$$

on M , where $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ is an orthonormal basis.

If $M = G/K$ is homogeneous then any $\mathcal{H} \subset C^\infty(M)$ is δ -spherical. This is because $\delta_{\mathcal{H}}$ is equivariant, and thereby its image is a G -orbit in \mathcal{H} necessarily contained in $S_{\mathcal{H}}$.

Let \mathcal{H} be a δ -spherical G -module. A full \mathcal{H} -map $f: M \rightarrow V$ is spherical if and only if

$$|f(x)|^2 - |\delta_{\mathcal{H}}(x)|^2 = \langle (A^*A - I)\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x) \rangle = \langle \langle f \rangle, \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x) \rangle = 0,$$

for all $x \in M$. Here \odot denotes the symmetric tensor product. We define

$$\mathcal{E}(\mathcal{H}) = \{ \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x) \mid x \in M \}^\perp \subset S^2(\mathcal{H}). \tag{7}$$

The previous computation shows that an \mathcal{H} -map $f: M \rightarrow V$ is spherical if and only if $\langle f \rangle \in \mathcal{E}(\mathcal{H})$.

Once again, since $\delta_{\mathcal{H}}$ is equivariant, $\mathcal{E}(\mathcal{H}) \subset S^2(\mathcal{H})$ is a G -submodule.

We obtain the following:

Proposition 3.2. *Let M be a compact Riemannian manifold with compact group G of isometries, and $\mathcal{H} \subset C^\infty(M)$ a δ -spherical G -submodule. Then the set of congruence classes of full spherical \mathcal{H} -maps $f: M \rightarrow S_V$ can be parametrized by the moduli space*

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap \mathcal{E}(\mathcal{H}).$$

Moreover $\mathcal{L}(\mathcal{H})$ is a compact convex body in $\mathcal{E}(\mathcal{H})$.

Compactness follows since spherical maps are automatically normalized:

$$\mathcal{L}(\mathcal{H}) \subset \mathcal{K}_0(\mathcal{H}) \Rightarrow \mathcal{E}(\mathcal{H}) \subset S_0^2(\mathcal{H}),$$

so that

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{E}(\mathcal{H}).$$

Remark. Let $M = G/K$ be a compact naturally reductive Riemannian homogeneous space, and $V_\lambda \subset C^\infty(M)$ the eigenspace of Δ^M corresponding to an eigenvalue λ . Recall from Section 1 that a λ -eigenmap $f: M \rightarrow S_V$ is a spherical V_λ -map.

Let $\mathcal{H} \subset C^\infty(M)$ be a finite dimensional G -submodule. Then $\mathcal{H} \subset V_\lambda$ for some λ . Proposition 3.2 asserts that $\mathcal{L}(\mathcal{H})$ parametrizes the congruence classes of full λ -eigenmaps $f: M \rightarrow S_V$ with components in $\mathcal{H} \subset V_\lambda$. In particular, $\mathcal{L}(V_\lambda)$ parametrizes the congruence classes of all full λ -eigenmaps $f: M \rightarrow S_V$.

Returning to the general situation, let $\mathcal{H} \subset C^\infty(M)$ be a δ -spherical G -module. We define

$$\Psi^0 = \Psi_{\mathcal{H}}^0: S^2(\mathcal{H}) \rightarrow C^\infty(M) \tag{8}$$

by

$$\Psi^0(C)(x) = \langle C\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x) \rangle = \langle C, \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x) \rangle, \quad x \in M.$$

Since $\delta_{\mathcal{H}}$ is equivariant, Ψ^0 is a homomorphism of G -modules. By (7), we have

$$\ker \Psi^0 = \mathcal{E}(\mathcal{H}). \tag{9}$$

We claim that the image of Ψ^0 is the G -submodule

$$\mathcal{H} \cdot \mathcal{H} = \text{span} \{ \chi_1 \chi_2 \mid \chi_1, \chi_2 \in \mathcal{H} \} \subset C^\infty(M).$$

Indeed, using (5) in the definition of Ψ^0 , we obtain

$$\Psi^0(C) = \sum_{j,l=0}^N c_{jl} \chi^j \chi^l,$$

where $\{ \chi^j \}_{j=0}^N \subset \mathcal{H}$ is an orthonormal basis, and the c_{jl} 's are the matrix entries of $C \in S^2(\mathcal{H})$. The claim follows.

Note that $\mathcal{H} \cdot \mathcal{H}$ always contains the trivial G -module, a consequence of δ -sphericity.

We obtain the following:

Proposition 3.3. *Let $\mathcal{H} \subset C^\infty(M)$ be a δ -spherical G -module. Then the G -module homomorphism*

$$\Psi^0: S^2(\mathcal{H}) \rightarrow \mathcal{H} \cdot \mathcal{H}$$

is onto, and has kernel $\mathcal{E}(\mathcal{H})$. In particular, $\mathcal{H} \cdot \mathcal{H}$ is (isomorphic to) a G -submodule of $S^2(\mathcal{H})$ and we have

$$\mathcal{E}(\mathcal{H}) \cong S^2(\mathcal{H}) / (\mathcal{H} \cdot \mathcal{H})$$

as G -modules.

Let $K \subset G$ be a closed subgroup. Recall that an irreducible orthogonal G -module \mathcal{V} is called class 1 with respect to the pair (G, K) if \mathcal{V} contains a nonzero K -fixed vector, or equivalently, if $\mathcal{V}|_K$ contains the trivial representation.

We now assume that $M = G/K$ is Riemannian homogeneous. As noted in Section 1, any irreducible G -submodule of $C^\infty(M)$ is class 1 with respect to (G, K) . Conversely, any class 1 G -module \mathcal{V} with respect to (G, K) is isomorphic to an irreducible G -submodule of $C^\infty(M)$ [25].

Let $\mathcal{H} \subset C^\infty(M)$ be a δ -spherical G -submodule. We define $\bar{\mathcal{E}}(\mathcal{H}) \subset S^2(\mathcal{H})$ as the

sum of those irreducible G -submodules in $S^2(\mathcal{H})$ that are not class 1 with respect to (G, K) . By the description of class 1 modules above and (8)-(9), we see that

$$\bar{\mathcal{E}}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}). \quad (10)$$

Equality holds if and only if the sum of all irreducible G -submodules in $S^2(\mathcal{H})$ that are class 1 with respect to (G, K) , is isomorphic to $\mathcal{H} \cdot \mathcal{H}$.

A map $f: M \rightarrow V$ is said to be conformal if

$$\langle f_*(X), f_*(Y) \rangle = c \langle X, Y \rangle, \quad X, Y \in T(M),$$

where $c > 0$ is a constant. Then c is called the conformality constant of f . We say that a finite dimensional G -module $\mathcal{H} \subset C^\infty(M)$ is δ -conformal if $\delta_{\mathcal{H}}$ is conformal.

Using (5), we have

$$(\delta_{\mathcal{H}})_*(X) = X\delta_{\mathcal{H}} = \sum_{j=1}^N X(\chi^j)\chi^j, \quad X \in T(M). \quad (11)$$

Thus, \mathcal{H} is δ -conformal if and only if

$$\sum_{j=0}^N X(\chi^j)Y(\chi^j) = c \langle X, Y \rangle, \quad X, Y \in T(M), \quad (12)$$

holds for any orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$.

Let $f: M \rightarrow V$ be a conformal map as above, and assume that $V_f \subset V_\lambda$ for some eigenvalue λ of Δ^M . Then $f: M \rightarrow V$ is an isometric immersion with respect to c times the original metric on M . By Takahashi's theorem [20] f maps into a sphere rS_V for some r . Calculating $\Delta^M(|f|^2)$, we obtain $c = r^2\lambda/\dim M$. If f is normalized then $r = 1$ and we get $c = \lambda/\dim M$. Again by Takahashi, we obtain that $f: M \rightarrow S_V$ is an isometric minimal immersion of the $\lambda/\dim M$ -multiple of the metric on M .

Let $\mathcal{H} \subset V_\lambda$ be a δ -conformal G -submodule. By definition, $\delta_{\mathcal{H}}: M \rightarrow \mathcal{H}$ is conformal with $V_{\delta_{\mathcal{H}}} = \mathcal{H} \subset V_\lambda$ so that the argument above applies. Since $\delta_{\mathcal{H}}$ is automatically normalized, we obtain that $\delta_{\mathcal{H}}: M \rightarrow S_{\mathcal{H}}$ is an isometric minimal immersion of the $\lambda/\dim M$ -multiple of the metric on M . In particular, \mathcal{H} is δ -spherical.

Remark. Let $M = G/K$ be isotropy irreducible. Then any irreducible G -submodule $\mathcal{H} \subset C^\infty(M)$ is δ -conformal. Indeed, (12) holds because $\sum_{j=0}^N d\chi^j \odot d\chi^j$ is a G -invariant bilinear form on \mathcal{H} . Its coordinate representation in (5) shows that $\delta_{\mathcal{H}}: M \rightarrow S_{\mathcal{H}^*}$ is the standard minimal immersion [6,25].

A DoCarmo-Wallach type argument gives the following:

Proposition 3.4. *Let $\mathcal{H} \subset V_\lambda \subset C^\infty(M)$ be a δ -conformal G -submodule. Then the congruence classes of isometric minimal \mathcal{H} -immersions $f: M \rightarrow S_V$ (with respect to the $\lambda/\dim M$ -multiple of the metric on M) are parametrized by the compact convex body*

$$\mathcal{M}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}), \tag{13}$$

in the G -module

$$\mathcal{F}(\mathcal{H}) = \{X\delta_{\mathcal{H}} \odot Y\delta_{\mathcal{H}} \mid X, Y \in T(M)\}^\perp \subset S^2(\mathcal{H}). \tag{14}$$

We also have

$$\mathcal{F}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}), \tag{15}$$

so that

$$\mathcal{M}(\mathcal{H}) = \mathcal{L}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}).$$

Let $M = G/K$ be a naturally reductive homogeneous space with orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{g} and \mathfrak{k} are the Lie algebras of G and K , and \mathfrak{p} is identified with the tangent space $T_o(M)$, $o = \{K\}$. The subgroup K acts on its Lie algebra \mathfrak{k} by the adjoint representation, and, under the identification $\mathfrak{p} \cong T_o(M)$, this action corresponds to the action of K on $T_o(M)$ via the isotropy representation.

As noted in Section 1, if \mathcal{W} is any (finite dimensional) orthogonal K -module then the induced G -module $\text{Ind}_K^G(\mathcal{W})$ is comprised of continuous maps $f: G \rightarrow \mathcal{W}$ that satisfy $f(kg) = k \cdot f(g)$, $g \in G$, $k \in K$. Precomposition of these maps by right multiplication on G defines the G -module structure on $\text{Ind}_K^G(\mathcal{W})$. In addition, integration with respect to the Haar measure on G and the scalar product on \mathcal{W} define a G -invariant scalar product on $\text{Ind}_K^G(\mathcal{W})$.

By Frobenius reciprocity, we have

$$\text{Hom}_G(\mathcal{V}, \text{Ind}_K^G(\mathcal{W})) = \text{Hom}_K(\mathcal{V}|_K, \mathcal{W}),$$

where \mathcal{V} is an orthogonal G -module and \mathcal{W} is an orthogonal K -module.

Let $\mathcal{H} \subset C^\infty(M)$ be a δ -conformal G -module. Restricting the differential of $\delta_{\mathcal{H}}$ to $\mathfrak{p} = T_o(M)$ gives a K -equivariant linear imbedding $(\delta_{\mathcal{H}})_*: \mathfrak{p} \rightarrow \mathcal{H}$. We identify the K -module \mathfrak{p} with the image, and think of \mathfrak{p} as a K -submodule of $\mathcal{H}|_K$. Notice that this can also be thought of as the inclusion $\mathfrak{p} \subset \mathcal{H}^* \cong \mathcal{H}$ given by the action of the tangent vectors at o to M on the elements of \mathcal{H} by directional differentiation. We define

$$\Psi: S^2(\mathcal{H}) \rightarrow \text{Ind}_K^G(S^2(\mathfrak{p})) \tag{16}$$

as follows. For $C \in S^2(\mathcal{H})$, we let $\Psi(C): G \rightarrow S^2(\mathfrak{p})$ be the map defined by $\Psi(C)(g) = \pi(g \cdot C)$, where $\pi: S^2(\mathcal{H}) \rightarrow S^2(\mathfrak{p})$ is the orthogonal projection, a homomorphism of K -modules.

We have

$$\ker \Psi = \mathcal{F}(\mathcal{H}).$$

Indeed, for $C \in S^2(\mathcal{H})$, $\Psi(C) = 0$ if and only if $\langle g \cdot C, S^2(\mathfrak{p}) \rangle = 0$ for all $g \in G$, if and only if $\langle C, g \cdot S^2(\mathfrak{p}) \rangle = 0$ for all $g \in G$. In view of the identification $\mathfrak{p} \subset \mathcal{H}|_K$, this holds if and only if

$$\langle C, S^2((\delta_{\mathcal{H}})_*(T_x(M))) \rangle = 0, \quad x \in M.$$

This is equivalent to $C \in \mathcal{F}(\mathcal{H})$.

4. Proofs of Theorems A-C.

PROOF OF THEOREM A. We first let $(G, K) = (SO(m + 1), S(m))$, $SO(m) = SO(m) \oplus [1] \subset SO(m + 1)$, with $M = G/K = S^m$ the Euclidean m -sphere, and $\mathcal{H} = V_{\lambda_p}$. The eigenspace V_{λ_p} corresponding to $\lambda_p = p(p + m - 1)$ is \mathcal{H}^p , the irreducible $SO(m + 1)$ -module of spherical harmonics of order p on S^m .

We let \mathcal{P}^p denote the $SO(m + 1)$ -module of homogeneous polynomials on \mathbf{R}^{m+1} of degree p (with the usual action $g \cdot \xi = \xi \circ g^{-1}$, $g \in SO(m + 1)$, $\xi \in \mathcal{P}^p$). By homogeneity, a polynomial in \mathcal{P}^p is uniquely determined by its restriction to $S^m \subset \mathbf{R}^{m+1}$.

We also think of a spherical harmonic χ of order p on S^m as a harmonic homogeneous polynomial on \mathbf{R}^{m+1} of degree p . (The equivalence of these two representations is given by restriction from \mathbf{R}^{m+1} to S^m , and comparison of the Euclidean and spherical Laplacians. We suppress the restriction if there is no danger of confusion.) This way \mathcal{H}^p becomes an $SO(m + 1)$ -submodule of \mathcal{P}^p . We have the orthogonal decomposition

$$\mathcal{P}^p = \mathcal{H}^p \oplus \mathcal{P}^{p-2} \cdot \rho^2 = \sum_{k=0}^{\lfloor p/2 \rfloor} \mathcal{H}^{p-2k} \cdot \rho^{2k}, \tag{17}$$

where $\rho(x) = |x|$, $x = (x_0, \dots, x_m) \in \mathbf{R}^{m+1}$ [15,24].

Since $\mathcal{H}^p \subset \mathcal{P}^p$, we have

$$\mathcal{H}^p \cdot \mathcal{H}^p \subset \mathcal{P}^{2p} = \sum_{j=0}^p \mathcal{H}^{2j}$$

as $SO(m + 1)$ -modules. Theorem A for $M = S^m$ states that equality holds, and this is what we need to show.

We define the harmonic projection operator as the orthogonal projection $H: \mathcal{P}^p \rightarrow \mathcal{H}^p$ with kernel $\ker H = \mathcal{P}^{p-2} \cdot \rho^2$ [24]. It is given explicitly by

$$H(\xi) = \xi + \sum_{j=1}^{\lfloor p/2 \rfloor} \frac{(-1)^j (p-1) \dots (p-j)}{j! \lambda_{2(p-1)} \dots \lambda_{2(p-j)}} \Delta^j \xi \cdot \rho^{2j}, \quad \xi \in \mathcal{P}^p. \tag{18}$$

Since $SO(m)$ fixes x_m , a zonal spherical harmonic in \mathcal{H}^p is $H(x_m^p)$. By (18), it is given by

$$H(x_m^p) = x_m^p + \sum_{j=1}^{\lfloor p/2 \rfloor} \frac{(-1)^j (p-1) \dots (p-j) p(p-1) \dots (p-2j+1)}{j! \lambda_{2(p-1)} \dots \lambda_{2(p-j)}} x_m^{p-2j} \rho^{2j}.$$

Rewriting the coefficients in terms of the Gamma function, we obtain

$$H(x_m^p) = \frac{p!}{2^p \Gamma(p + \frac{m-1}{2})} \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{(-1)^j \Gamma(p + \frac{m-1}{2} - j)}{j!(p-2j)!} (2x_m)^{p-2j} \rho^{2j}.$$

Up to a normalizing factor, this is the ultraspherical polynomial C_p^ν with $\nu = (m-1)/2$ [1,24]:

$$H(x_m^p) = \frac{p! \Gamma(\frac{m-1}{2})}{2^p \Gamma(p + \frac{m-1}{2})} \rho^p C_p^{(m-1)/2}(\cos \theta), \tag{19}$$

where $x_m/\rho = \cos \theta$. In terms of the Jacobi polynomials, we have

$$C_p^\nu = \frac{(2\nu)_p}{(\nu + 1/2)_p} P_p^{\nu-1/2, \nu-1/2}, \tag{20}$$

where $(a)_p = \Gamma(a+p)/\Gamma(a)$. The choice of the zonal spherical harmonic χ_0 for $M = S^m$ specified in the first table of Section 2 follows. The linearization of the product formula for ultraspherical polynomials [7] reads as

$$C_p^\nu C_q^\nu = \sum_{k=0}^{\min(p,q)} \frac{(p+q+\nu-2k)}{(p+q+\nu-k)} \times \frac{(\nu)_k (\nu)_{p-k} (\nu)_{q-k} (2\nu)_{p+q-k} (p+q-2k)!}{k!(p-k)!(q-k)!(\nu)_{p+q-k} (2\nu)_{p+q-2k}} C_{p+q-2k}^\nu.$$

We now let $p = q$ and $\nu = (m-1)/2$. In view of (20), the linearization formula above reduces to (2) with $\alpha = \beta = m/2 - 1$, and we also obtain an explicit formula for the linearization coefficients. This immediately shows that $c(j, p; m/2 - 1, m/2 - 1)$ is nonzero if and only if $0 \leq j \leq 2p$ is even. By (19), evaluating (2) on $\cos \theta$, the Jacobi polynomials become zonal spherical harmonics. Suppressing the argument $\cos \theta$, by definition, $(P_p^{(m/2-1, m/2-1)})^2 \in \mathcal{H}^p \cdot \mathcal{H}^p$. The restriction of the orthogonal projection $\mathcal{P}^{2p} \rightarrow \mathcal{H}^{2j}$, $j = 0, \dots, p$, to $\mathcal{H}^p \cdot \mathcal{H}^p \subset \mathcal{P}^{2p}$ maps $(P_p^{(m/2-1, m/2-1)})^2$ to a nonzero constant multiple of $P_{2j}^{(m/2-1, m/2-1)}$ since $c(2j, p; m/2 - 1, m/2 - 1)$ is nonzero. Schur's lemma implies that \mathcal{H}^{2j} must be a component of $\mathcal{H}^p \cdot \mathcal{H}^p$ for $j = 0, \dots, p$. Theorem A follows for $M = S^m$.

For $M = \mathbf{R}P^m$, the real projective m -space, the eigenspace V_{λ_p} corresponding to the p -th eigenvalue $\lambda_p = 2p(2p+m-1)$ of the Laplacian $\Delta^{\mathbf{R}P^m}$ can be identified with \mathcal{H}^{2p} . Theorem A follows from the spherical case above.

Next we let $(G, K) = (U(m+1), U(m) \times U(1))$ with $\mathbf{C}P^m = U(m+1)/(U(m) \times U(1))$, the complex projective m -space. Let $\mathcal{P}^{p,q}$ denote the space of complex homogeneous polynomials of bidegree (p, q) on \mathbf{C}^{m+1} . An element $\xi \in \mathcal{P}^{p,q}$ is a complex valued homogeneous polynomial that has degree p in the variables $z_0, \dots, z_m \in \mathbf{C}$ and degree q in the variables $\bar{z}_0, \dots, \bar{z}_m \in \mathbf{C}$. By homogeneity, ξ can be thought of as a function on the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$.

The space $\mathcal{P}^{p,p}$ is the complexification of a real $U(m+1)$ -submodule, and this real submodule is also denoted by the same symbol. An element in $\mathcal{P}^{p,p}$ can be thought of as a function on $\mathbf{C}P^m$.

The decomposition in (17) gives

$$\mathcal{P}^{p,q} = \mathcal{H}^{p,q} \oplus \mathcal{P}^{p-1, q-1} \cdot \rho^2 = \sum_{k=0}^{\min(p,q)} \mathcal{H}^{p-k, q-k} \cdot \rho^{2k},$$

where $\rho = |z|$, $z = (z_0, \dots, z_m) \in \mathbf{C}^{m+1}$, and $\mathcal{H}^{p,q}$ is the space of complex harmonic homogeneous polynomials of bidegree (p, q) on \mathbf{C}^{m+1} . Then $\mathcal{H}^{p,q}$ is a complex irreducible $U(m+1)$ -module. For real valued polynomials we also have

$$\mathcal{P}^{p,p} = \sum_{k=0}^p \mathcal{H}^{p-k,p-k} \cdot \rho^{2k},$$

as real $U(m+1)$ -modules. Here $\mathcal{P}^{p,p}$ is the space of real valued homogeneous polynomials of bidegree (p, p) on \mathbf{C}^{m+1} , and $\mathcal{H}^{j,j}$ is the eigenspace V_{λ_j} corresponding to the j -th eigenvalue $\lambda_j = 4j(j+m)$.

Since $\mathcal{H}^{p,p} \subset \mathcal{P}^{p,p}$, we have

$$\mathcal{H}^{p,p} \cdot \mathcal{H}^{p,p} \subset \mathcal{P}^{2p,2p} = \sum_{j=0}^{2p} \mathcal{H}^{j,j} \tag{21}$$

as real $U(m+1)$ -modules. To prove Theorem A for $M = \mathbf{C}P^m$, it remains to show that equality holds.

Since $U(m)$ fixes z_m and the center $U(1)$ acts on $\mathcal{H}^{p,p}$ trivially, a zonal spherical harmonic in $\mathcal{H}^{p,p}$ is $H(|z_m|^{2p})$. Here the harmonic projection operator H is the restriction of the harmonic projection for the spherical case above. We have

$$H(|z_m|^{2p}) = \frac{p!(p+m-1)!}{(2p+m-1)!} \rho^{2p} P_p^{(m-1,0)}(\cos(2\theta)),$$

where $|z_m|/\rho = \cos \theta$. (See also [24], formula (5') in Chapter 11.3.2, Vol.2.) In the linearization of the square $(P_p^{(m-1,0)})^2$ all coefficients $c(j, p; m-1, 0)$, $j = 0, \dots, 2p$, are positive for $m \geq 2$. As in the spherical case it follows that $\mathcal{H}^{j,j}$, $j = 0, \dots, 2p$, are $U(m+1)$ -submodules of $\mathcal{H}^{p,p} \cdot \mathcal{H}^{p,p}$. The equality in (21) follows.

The cases of the quaternionic projective space $\mathbf{H}P^m$ and the Cayley projective plane $\mathbf{Ca}P^2$ are entirely analogous [10]. The zonal spherical functions for $\mathbf{H}P^m$ are explicitly derived in [24] (cf. formula (14) in Chapter 11.7.4, Vol. 2). Another approach for the Cayley projective plane is to determine the highest weights of the class 1 modules with respect to the pair $(F_4, Spin(9))$ and to use the Weyl dimension formula for the multiplicities.

PROOF OF THEOREM B. One of the principal results of [22] (Theorem 4.1, p. 136) gives the multiplicity m of $\mathcal{H}^{q,q}$ in $S^2(\mathcal{H}^{p,p})$ as follows:

$$m [\mathcal{H}^{q,q}: S^2(\mathcal{H}^{p,p})] = \frac{1}{2} \left[\min(q, 2p-q) + 1 + \frac{1+(-1)^q}{2} \right].$$

By the definition of $\bar{\mathcal{E}}(\mathcal{H}^{p,p})$, we thus have

$$S^2(\mathcal{H}^{p,p}) = \sum_{q=0}^{2p} \frac{1}{2} \left(\min(q, 2p-q) + 1 + \frac{1+(-1)^q}{2} \right) \mathcal{H}^{q,q} \oplus \bar{\mathcal{E}}(\mathcal{H}^{p,p}).$$

On the other hand, by Proposition 3.3 and Theorem A, we have

$$\mathcal{E}(V_{\lambda_p}) = S^2(V_{\lambda_p})/(V_{\lambda_p} \cdot V_{\lambda_p}) = S^2(V_{\lambda_p}) / \left(\sum_{q=0}^{2p} V_{\lambda_q} \right).$$

Since $V_{\lambda_q} = \mathcal{H}^{q,q}$, combining these two formulas, Theorem B follows.

PROOF OF THEOREM C. The proof is based on comparing the multiplicities of some irreducible components of the domain and the image of Ψ in (16). To do this, we first complexify, and consider

$$\Psi : S^2(\mathcal{H}^{p,p}) \otimes_{\mathbf{R}} \mathbf{C} \rightarrow \text{Ind}_{U(m) \times U(1)}^{U(m+1)}(S^2(\mathbf{C}^m) \otimes_{\mathbf{R}} \mathbf{C}), \tag{22}$$

where $\mathbf{C}^m = T_o(\mathbf{C}P^m)$ with $U(m) \times U(1)$ -module structure given by $U(m)$ acting on \mathbf{C}^m by matrix multiplication, and the center $U(1) \subset U(m+1)$ acting trivially. For the irreducible decompositions, recall that a complex irreducible $U(m+1)$ -module \mathcal{V} is given by its highest weight which, with respect to the standard maximal torus in $U(m+1)$, is an $(m+1)$ -tuple with integral coefficients. We write $V^\rho = V_{U(m+1)}^\rho$, where $\rho = (\rho_1, \dots, \rho_{m+1}) \in \mathbf{Z}^{m+1}$ with

$$\rho_1 \geq \rho_2 \geq \dots \geq \rho_{m+1}.$$

The center $U(1) \subset U(m+1)$ acts by the weight $\sum_{j=1}^{m+1} \rho_j$. For example, we have

$$\mathcal{H}^{p,q} = V^{(p,0,\dots,0,-q)}.$$

The branching rule for restrictions from $U(m+1)$ to $U(m)$ takes the form

$$V_{U(m+1)}^\rho|_{U(m)} = \sum_{\sigma} V_{U(m)}^\sigma,$$

where the summation runs over all $\sigma \in \mathbf{Z}^m$ for which

$$\rho_1 \geq \sigma_1 \geq \rho_2 \geq \dots \geq \rho_m \geq \sigma_m \geq \rho_{m+1}.$$

The decomposition of the domain in (22) into irreducible components is one of the technical results in [22] (Theorem 4.1 on p. 136). For $m \geq 3$, we have

$$S^2(\mathcal{H}^{p,p}) \otimes_{\mathbf{R}} \mathbf{C} \cong \sum_{b=0}^{2p} \sum_{c=0}^{\min(b,2p-b)} \sum_{d=0}^{\min(b,p,e)} \frac{n_0(b,c,d) + m_0(b,c,d)}{2} \times V^{(b,c,0,\dots,0,-d,d-b-c)}. \tag{23}$$

Here $e = \lfloor \frac{b+c}{2} \rfloor$, and

$$n_0(b,c,d) = \min(b-c, b-d, p-c, p-d, b+c-2d, 2p-b-c) + 1.$$

$m_0(b,c,d) = 0$ for $b \not\equiv c \pmod{2}$, and for $b \equiv c \pmod{2}$, we have

$$m_0(b,c,d) = \begin{cases} -1 & \text{if } b, d \text{ are odd and } m \equiv 1 \pmod{4} \\ 1 & \text{otherwise.} \end{cases}$$

We now fix a component $V^{(b,c,0,\dots,0,-d,d-b-c)}$ in $S^2(\mathcal{H}^{p,p}) \otimes_{\mathbf{R}} \mathbf{C}$. We need to determine the multiplicity

$$m \left[V^{(b,c,0,\dots,0,-d,d-b-c)} : \text{Ind}_{U(m) \times U(1)}^{U(m+1)}(S^2(\mathbf{C}^m) \otimes_{\mathbf{R}} \mathbf{C}) \right]. \tag{24}$$

First note that the multiplicity in (24) is the dimension of the module

$$\mathrm{Hom}_{U(m)} \left(V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}, S^2(\mathbf{C}^m) \otimes_{\mathbf{R}} \mathbf{C} \right). \quad (25)$$

This follows by Frobenius reciprocity along with the fact that $U(1)$ acts trivially. In particular, the multiplicity in (24) is nonzero if and only if $V^{(b,c,0,\dots,0,-d,d-b-c)}$ is disjoint from $\bar{\mathcal{F}}(\mathcal{H}^{p,p})$.

As a real $SO(2m)$ -module

$$S^2(\mathbf{C}^m) = S^2(\mathbf{R}^{2m}) = \mathcal{H}^0 \oplus \mathcal{H}^2.$$

Complexifying, and restricting to $U(m) \subset SO(2m)$, we obtain

$$S^2(\mathbf{C}^m) \otimes_{\mathbf{R}} \mathbf{C} = \mathcal{H}^0|_{U(m)} \oplus \mathcal{H}^2|_{U(m)} = \mathcal{H}^{0,0} \oplus \sum_{j=0}^2 \mathcal{H}^{2-j,j}.$$

Thus (25) can be written as

$$\mathrm{Hom}_{U(m)} \left(V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}, \mathcal{H}^{0,0} \oplus \sum_{j=0}^2 \mathcal{H}^{2-j,j} \right).$$

The dimension of this module is equal to

$$m [\mathcal{H}^{0,0} : V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}] + \sum_{j=0}^2 m [\mathcal{H}^{2-j,j} : V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}].$$

By the branching rule, the first multiplicity is 1 if and only if $c = d = 0$ and zero otherwise. The remaining multiplicities can be obtained similarly using the branching rule. For $0 \leq j \leq 2$, we obtain

$$m [\mathcal{H}^{2-j,j} : V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}] = \begin{cases} 1 & \text{if } b \geq 2 - j \geq c \text{ and } -d \geq -j \geq d - b - c \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this with (23), we see that, for $m \not\equiv 1 \pmod{4}$, $p = 3$, $b = 3$, $c = d = 1$, the multiplicity of the component $V^{(3,1,0,\dots,0,-1,-3)}$ is 2 in the domain of Ψ and 1 in the image of Ψ . The same holds for $p = 4$, $b = 4$, $c = d = 1$. Theorem C follows.

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