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OPERATORS ON MODULI FOR SPHERICAL MAPS OF HOMOGENEOUS SPACES

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The DoCarmo–Wallach theory studies isometric minimal immersions $f : M \to S^n$ of a compact Riemannian homogeneous space $M = G/K$ into Euclidean n-spheres. The parameter space of such immersions is a compact convex body in a representation space for the Lie group G. In this article we give a very general definition of the moduli space and study its geometric properties such as the distortion (as a convex set). In addition, we introduce a general notion of operators, derive various criteria under which they map the moduli into one another, and finally, we show that, under general conditions, the operators are distortion decreasing.

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1. Introduction

The construction of the DoCarmo–Wallach moduli space for spherical minimal immersions [2, 10, 11] can be generalized substantially. Given a compact Lie group G and an orthogonal G-module $\mathcal{H}$, we define the *general moduli space* as

$$\mathcal{K}(\mathcal{H}) = \{C \in S^2(\mathcal{H}) \,|\, C + I \geq 0\},$$

and the *restricted moduli space* as

$$\mathcal{K}_0(\mathcal{H}) = \{C \in \mathcal{K}(\mathcal{H}) \,|\, \mathrm{trace}\,(C) = 0\}.$$

Both moduli are convex sets in $S^2(\mathcal{H})$ and $\mathcal{K}_0(\mathcal{H})$ is compact (since the eigenvalues of C in $\mathcal{K}_0(\mathcal{H})$ are bounded). In addition, $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}_0(\mathcal{H})$ have nonempty interiors in their linear span, so that they are convex bodies. All moduli for spherical minimal immersions (and their generalizations such as the moduli for eigenmaps) satisfying various geometric properties are slices of $\mathcal{K}_0(\mathcal{H})$ by affine or linear subspaces of $S^2(\mathcal{H})$ [4, 10].

In Sec. 2, we discuss the geometry of $\mathcal{K}_0(\mathcal{H})$ in general. Given a compact convex body $\mathcal{K}_0$ in a Euclidean vector space $\mathcal{V}$ and a base point $o \in \mathcal{K}_0$, any (directed) line ℓ passing through o intersects $\mathcal{K}_0$ in a finite line segment. o splits this segment in a specific ratio called the *distortion* of $\mathcal{K}_0$ with respect to ℓ. In Sec. 2, we derive a general estimate on the distortion of $\mathcal{K}_0(\mathcal{H})$ (relative to the origin). This will show, in particular, that $\mathcal{K}_0(\mathcal{H})$ is considerably less distorted than a simplex (relative to its centroid).

Given orthogonal G-modules $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{W}$, we define an *operator* as a G-module homomorphism $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$. Special cases of operators such as the degree raising and lowering operators and the operator of infinitesimal rotations have been used to determine the exact dimension of the moduli [4, 10] and to give lower estimate for the range dimension of spherical minimal immersions [9]. The operator $\mathcal{D}$ induces a G-module homomorphism $\Phi^{\mathcal{D}} : S^2(\mathcal{H}) \to S^2(\mathcal{H}')$ in a natural way. The principal question is when does $\Phi^{\mathcal{D}}$ carry the moduli into one another. In Sec. 3, we show that the answer is affirmative if $\mathcal{D}$ satisfies a conformality condition. In addition, we prove that an operator $\mathcal{D}$ in general decreases the distortion of the moduli, and give an estimate on the operator norm of $\Phi^{\mathcal{D}}$. In Sec. 4, we give a variety of explicitly computable examples of operators.

Let M be a compact Riemannian manifold with compact group G of isometries. The space $C^\infty(M)$ of smooth functions on M is a representation space for G in a natural way. If $\mathcal{H}$ is a finite dimensional G-submodule of $C^\infty(M)$ then the moduli $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}_0(\mathcal{H})$ can be interpreted as parameter spaces of certain maps as follows. A map $f : M \to V$ into a Euclidean vector space V is called an *$\mathcal{H}$-map* if the components $\alpha \circ f$, $\alpha \in V^*$, belong to $\mathcal{H}$. Then a DoCarmo–Wallach argument [2, 11] shows that the set of congruence classes of full $\mathcal{H}$-maps $f : M \to V$ can be parametrized by the general moduli space $\mathcal{K}(\mathcal{H})$. Here $f : M \to V$ is *full* if the image spans V, and two maps $f_1 : M \to V_1$ and $f_2 : M \to V_2$ are *congruent* if there is a linear isometry $U : V_1 \to V_2$ such that $f_2 = U \circ f_1$. The reduced moduli $\mathcal{K}_0(\mathcal{H})$ parametrizes the *normalized* $\mathcal{H}$-maps, i.e. maps $f : M \to V$ that satisfy

$$\int_M |f|^2 v_M = \mathrm{vol}\,(M)\,.$$

Here v_M is the Riemannian volume element and vol$\,(M)$ is the volume of M. A map $f : M \to V$ is *spherical* if the image is contained in the unit sphere S_V of V. The set of congruence classes of full spherical $\mathcal{H}$-maps $f : M \to S_V$ can be parametrized by the moduli space

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{E}(\mathcal{H})\,,$$

a slice of $\mathcal{K}_0(\mathcal{H})$ by a G-submodule $\mathcal{E}(\mathcal{H})$ (defined in Sec. 2).

Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be an operator such that $\Phi^{\mathcal{D}}$ carries $\mathcal{K}_0(\mathcal{H})$ into $\mathcal{K}_0(\mathcal{H}')$. A principal question again is whether $\mathcal{D}$ preserves sphericality in the sense that $\Phi^{\mathcal{D}}$ carries $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\mathcal{H}')$. ($\mathcal{D}$ above can be defined to carry $\mathcal{H}$-maps to $\mathcal{H}'$-maps and this induces $\Phi^{\mathcal{D}}$ on the moduli; cf. Sec. 3.) In Sec. 5, we derive a necessary and sufficient condition for an operator to preserve sphericality in case when $\mathcal{H}$, $\mathcal{H}' \subset C^\infty(M)$ are certain finite dimensional G-submodules.

2. Moduli

Let G be a compact Lie group. An orthogonal G-module $\mathcal{H}$ is a Euclidean vector space on which G acts linearly via orthogonal transformations. In other words, $\mathcal{H}$ is a representation space for G, and it is endowed with a G-invariant scalar product.

Let $\mathcal{H}$ be an orthogonal G-module. As in the introduction, we define the general moduli space for $\mathcal{H}$ by

$$\mathcal{K} = \mathcal{K}(\mathcal{H}) = \{C \in S^2(\mathcal{H}) \,|\, C + I \geq 0\},$$

where $\geq$ means positive semidefinite and I is the identity. $\mathcal{K}$ is a G-invariant set in $S^2(\mathcal{H})$, where the G-module structure on $S^2(\mathcal{H})$ is extended from that of $\mathcal{H}$.

$\mathcal{K}$ is a convex set since $C+I \geq 0$ is a convex condition. The interior of $\mathcal{K}$ consists of those C for which $C+I > 0$. Clearly $\mathcal{K}$ has a nonempty interior, and hence it is a convex body in $S^2(\mathcal{H})$. Moreover $\mathcal{K}$ is noncompact since the multiples λI, $\lambda \geq -1$, are contained in $\mathcal{K}$.

We let $S^2_0(\mathcal{H})$ denote the G-submodule of $S^2(\mathcal{H})$ comprised by the traceless symmetric endomorphisms of V. We define the reduced moduli space for $\mathcal{H}$ as

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap S^2_0(\mathcal{H}) = \{C \in S^2_0(\mathcal{H}) \,|\, C + I \geq 0\}.$$

The eigenvalues of the symmetric endomorphisms in $\mathcal{K}$ are ≥ -1. Hence the eigenvalues of the endomorphisms in $\mathcal{K}_0$ are bounded, in fact, they are contained in $[-1, \dim\mathcal{H} - 1]$. It follows that $\mathcal{K}_0$ is compact, and a convex body in $S^2_0(\mathcal{H})$.

Given $C \in \mathcal{K}_0$, $C \neq 0$, let

$$\lambda_0 > \lambda_1 > \cdots > \lambda_N > \mu_0$$

be the distinct eigenvalues of C in decreasing order. Since $\operatorname{trace} C = 0$, we have $\lambda_0 > 0$ and $\mu_0 < 0$. We define the *opposite* C^o of C by

$$C^o = \frac{\mu_0}{\lambda_0} C.$$

The distinct eigenvalues of C^o are

$$\frac{\mu_0^2}{\lambda_0} > \frac{\mu_0}{\lambda_0}\lambda_N > \cdots > \frac{\mu_0}{\lambda_0}\lambda_1 > \mu_0$$

in decreasing order. Thus

$$(C^o)^o = \frac{\mu_0}{\mu_0^2/\lambda_0}\frac{\mu_0}{\lambda_0} C = C.$$

Indicating the dependence of the eigenvalues on the respective endomorphisms, we have

$$\lambda_0(C^o) = \frac{\mu_0(C)^2}{\lambda_0(C)},$$

and hence

$$|\mu_0(C)| = \sqrt{\lambda_0(C)\lambda_0(C^o)}.$$

Recall from the definition of $\mathcal{K}_0$ that $C \in \mathcal{K}_0$ is a boundary point if and only if $C + I$ is semi-definite if and only if $\mu_0(C) = -1$. Since the minimal eigenvalues of C and C^o are the same, we see that $C \in \partial\mathcal{K}_0$ if and only if

$$C^o = -\frac{1}{\lambda_0(C)} C \in \partial\mathcal{K}_0 . \tag{2.1}$$

(Note that the multiplicities of $\mu_0(C) = -1$ and $\mu_0(C^o) = -1$ are, in general, different.) We see that the line passing through C and C^o intersects $\mathcal{K}_0$ in a line segment with endpoints C and C^o. (The open line segment is contained in the interior of $\mathcal{K}_0$ [1].) The origin splits this segment in the ratio $|C|/|C^o|$ which is called the *distortion* of $\mathcal{K}_0$ at C.

Theorem 2.1. *The distortion of $C \in \partial\mathcal{K}_0(\mathcal{H})$ (with respect to the origin) is the maximal eigenvalue $\lambda_0(C)$ of C as a symmetric endomorphism of $\mathcal{H}$. The distortion function λ_0 on $\partial\mathcal{K}_0(\mathcal{H})$ extends as a convex function to $\mathcal{K}_0(\mathcal{H})$. In particular, maximum distortion occurs at an extremal point (in the sense of convex geometry). The distortion satisfies*

$$\frac{1}{\dim\mathcal{H} - 1} \le \lambda_0 \le \dim\mathcal{H} - 1 . \tag{2.2}$$

Proof. By (2.1), we have

$$\lambda_0(C) = \frac{|C|}{|C^o|} ,$$

and the first statement follows. The maximal eigenvalue λ_0 as a function on $\mathcal{K}_0$ is convex since

$$\lambda_0(C) = \max\{\langle C\chi, \chi\rangle \,|\, |\chi| \le 1 , \chi \in \mathcal{H}\} .$$

The right-hand side of this equation is the convex extension of λ_0 to $\mathcal{K}_0$. Finally, for $C \in \partial\mathcal{K}_0$, we have $\lambda_0(C^o) = 1/\lambda_0(C)$. Since $\lambda_0 \le \dim\mathcal{H} - 1$, we obtain (2.2). □

Remark 2.2. According to a result in convex geometry [1, 3], the distortion function λ_0 of a compact convex set $\mathcal{K}_0 \subset \mathcal{V}$ in a Euclidean vector space $\mathcal{V}$ satisfies

$$\frac{1}{\dim\mathcal{V}} \le \lambda_0 \le \dim\mathcal{V} ,$$

provided that the base point is chosen suitably. (This estimate is sharp, e.g. consider a simplex in $\mathcal{V}$ with base point the centroid.) Since

$$\dim\mathcal{K}_0(\mathcal{H}) = \dim\mathcal{K}(\mathcal{H}) - 1 = \dim S^2(\mathcal{H}) - 1 = \frac{\dim\mathcal{H}(\dim\mathcal{H} + 1)}{2} - 1$$

we see that the estimate in (2.2) is significantly better than this general estimate. In other words, the moduli $\mathcal{K}_0(\mathcal{H})$ is significantly less distorted than a simplex.

As in the introduction, let $\mathcal{H} \subset C^\infty(M)$ be a finite dimensional G-module, where M is a compact Riemannian manifold with compact Lie group G of isometries. $\mathcal{H}$ is an orthogonal G-module with respect to the scaled L^2-scalar product

$$\langle \chi_1, \chi_2 \rangle = \frac{\dim \mathcal{H}}{\mathrm{vol}\,(M)} \int_M \chi_1 \chi_2 v_M\,, \quad \chi_1, \chi_2 \in \mathcal{H}\,.$$

We define the *Dirac delta* as a map $\delta_{\mathcal{H}} : M \to \mathcal{H}^*$ by

$$\delta_{\mathcal{H}}(x)(\chi) = \chi(x)\,, \quad x \in M, \chi \in \mathcal{H}\,.$$

We identify $\mathcal{H}^{**}$ and $\mathcal{H}$ in the usual way. The component of $\delta_{\mathcal{H}}$ corresponding to $\chi \in \mathcal{H} = \mathcal{H}^{**}$ is $\langle \delta_{\mathcal{H}}, \chi \rangle = \chi$. In particular, $\delta_{\mathcal{H}}$ is a full $\mathcal{H}$-map.

We identify $\mathcal{H}$ with its dual $\mathcal{H}^*$ via the scalar product on $\mathcal{H}$. With this, and with respect to an orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$, $\dim \mathcal{H} = N+1$, the Dirac delta as a map $\delta_{\mathcal{H}} : M \to \mathcal{H}$ can be written as

$$\delta_{\mathcal{H}}(x) = \sum_{j=0}^N \chi^j(x)\chi^j\,, \quad x \in M\,.$$

In fact, for $\chi \in \mathcal{H}$, we have

$$\langle \delta_{\mathcal{H}}(x), \chi \rangle = \chi(x) = \sum_{j=0}^N \langle \chi, \chi^j \rangle \chi^j(x) = \left\langle \sum_{j=0}^N \chi^j(x)\chi^j, \chi \right\rangle.$$

(Note that $\delta_{\mathcal{H}}$ has been introduced in [2] as the standard minimal immersion.) The Dirac delta $\delta_{\mathcal{H}}$ is equivariant with respect to the homomorphism $\rho_{\mathcal{H}} : G \to O(\mathcal{H})$ that defines the orthogonal G-module structure on $\mathcal{H} \cong \mathcal{H}^*$.

Given a full $\mathcal{H}$-map $f : M \to V$, we have $f = A \circ \delta_{\mathcal{H}}$, where $A : \mathcal{H} \to V$ is a surjective linear map. We associate to f the symmetric linear endomorphism

$$\langle f \rangle = A^*A - I \in S^2(\mathcal{H})\,.$$

Clearly $\langle f \rangle$ depends only on the congruence class of f. Since A^*A is always positive semidefinite, we also have $\langle f \rangle \in \mathcal{K}(\mathcal{H})$. A DoCarmo–Wallach type argument shows that $f \mapsto \langle f \rangle$ gives rise to a one-to-one correspondence between the set of congruence classes of full $\mathcal{H}$-maps and the general moduli space $\mathcal{K}(\mathcal{H})$ [2, 7, 11].

For a full $\mathcal{H}$-map $f : M \to V$ with $f = A \circ \delta_{\mathcal{H}}$ as above, we have

$$\mathrm{trace}\,(\langle f \rangle + I) = \mathrm{trace}\, A^*A$$

so that $\langle f \rangle$ is traceless if and only if

$$\int_M |f|^2 v_M = \mathrm{vol}\,(M)\,.$$

We call $f : M \to V$ *normalized* if this is satisfied. We obtain that the reduced moduli $\mathcal{K}_0(\mathcal{H})$ parametrizes the set of congruence classes of full normalized $\mathcal{H}$-maps.

An $\mathcal{H}$-map $f : M \to V$ is called *spherical* if the image of f is contained in the unit sphere S_V of V. A finite dimensional G-module $\mathcal{H} \subset C^\infty(M)$ is called *δ-spherical* if $\delta_{\mathcal{H}}$ is spherical. $\mathcal{H}$ is δ-spherical if and only if

$$\sum_{j=0}^{N} (\chi^j)^2 = 1$$

on M for an(y) orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$.

If $M = G/K$ is homogeneous then any $\mathcal{H} \subset C^\infty(M)$ is δ-spherical. Indeed, $\delta_{\mathcal{H}}$ is equivariant, and hence its image is a G-orbit in $\mathcal{H}$ which must be contained in $S_{\mathcal{H}}$.

Let $\mathcal{H}$ be a δ-spherical G-module. A full $\mathcal{H}$-map $f : M \to V$ is spherical if and only if

$$|f(x)|^2 - |\delta_{\mathcal{H}}(x)|^2 = \langle (A^*A - I)\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle = \langle \langle f \rangle, \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x)\rangle = 0\,,$$

for all $x \in M$. We define

$$\mathcal{E}(\mathcal{H}) = \{\delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x) \,|\, x \in M\}^\perp \subset S^2(\mathcal{H})\,. \tag{2.3}$$

The computation above implies that an $\mathcal{H}$-map $f : M \to V$ is spherical if and only if $\langle f \rangle \in \mathcal{E}(\mathcal{H})$. Again by the equivariance of $\delta_{\mathcal{H}}$, $\mathcal{E}(\mathcal{H}) \subset S^2(\mathcal{H})$ is a G-submodule.

We obtain that the set of congruence classes of full spherical $\mathcal{H}$-maps $f : M \to S_V$ can be parametrized by the moduli space

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap \mathcal{E}(\mathcal{H})\,.$$

$\mathcal{L}(\mathcal{H})$ is a compact convex body in $\mathcal{E}(\mathcal{H})$. Note that $\mathcal{L}(\mathcal{H})$ is compact since spherical maps are automatically normalized

$$\mathcal{L}(\mathcal{H}) \subset \mathcal{K}_0(\mathcal{H}) \Rightarrow \mathcal{E}(\mathcal{H}) \subset S_0^2(\mathcal{H})\,,$$

and thus

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{E}(\mathcal{H})\,.$$

Given a δ-spherical G-module $\mathcal{H} \subset C^\infty(M)$, we define

$$\Psi^0 = \Psi^0_{\mathcal{H}} : S^2(\mathcal{H}) \to C^\infty(M) \tag{2.4}$$

by

$$\Psi^0(C)(x) = \langle C\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle = \langle C, \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x)\rangle\,, \quad x \in M\,.$$

By (2.3), Ψ^0 is a homomorphism of G-modules (since $\delta_{\mathcal{H}}$ is equivariant). By definition, we have

$$\ker \Psi^0 = \mathcal{E}(\mathcal{H})\,. \tag{2.5}$$

The image of Ψ^0 is the G-submodule

$$\mathcal{H} \cdot \mathcal{H} = \{\chi_1\chi_2 \,|\, \chi_1, \chi_2 \in \mathcal{H}\} \subset C^\infty(M)\,.$$

This follows immediately writing Ψ^0 in coordinates

$$\Psi^0(C) = \sum_{j,l=0}^{N} c_{jl}\chi^j\chi^l\,,$$

where $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ is an orthonormal basis, and the c_{jl}'s are the matrix entries of $C \in S^2(\mathcal{H})$.

As a byproduct, we obtain the isomorphism of G-modules:

$$S^2(\mathcal{H})/\mathcal{E}(\mathcal{H}) \cong \mathcal{H}\cdot\mathcal{H}\,.$$

If $M = G/K$ is a compact rank one symmetric space then an irreducible G-submodule $\mathcal{H} \subset C^\infty(M)$ is the full eigenspace V_λ of the Laplacian $\triangle^M$ of M for some eigenvalue λ [5]. According to a result in [8] if λ_p denotes the pth eigenvalue then we have

$$V_{\lambda_p}\cdot V_{\lambda_p} = \begin{cases} \displaystyle\sum_{j=0}^{p} V_{\lambda_{2j}} & \text{if } M \text{ is a Euclidean sphere} \\ \displaystyle\sum_{j=0}^{2p} V_{\lambda_j} & \text{otherwise}. \end{cases}$$

This gives the dimension $\dim\mathcal{L}(V_{\lambda_p}) = \dim\mathcal{E}(V_{\lambda_p})$. In Sec. 5, we give another computation for $V_{\lambda_p}\cdot V_{\lambda_p}$ for the case when M is a Euclidean sphere.

3. Operators

Let $\mathcal{W}, \mathcal{H}, \mathcal{H}'$ be orthogonal G-modules. An *operator* is a G-module homomorphism $\mathcal{D} : \mathcal{W} \to \mathcal{H}^*\otimes\mathcal{H}'$. We also write $\mathcal{D} : \mathcal{H}\otimes\mathcal{W} \to \mathcal{H}'$. For $\xi \in \mathcal{W}$, we set $\mathcal{D}(\xi) = \mathcal{D}_\xi : \mathcal{H} \to \mathcal{H}'$. With this, the homomorphism property can be written as

$$\mathcal{D}_{g\cdot\xi} = g\cdot\mathcal{D}_\xi = g\circ\mathcal{D}_\xi\circ g^{-1}\,,\quad g \in G\,.$$

We think of $\mathcal{D}$ as a family of linear maps $\mathcal{D}_\xi : \mathcal{H} \to \mathcal{H}'$ parametrized by $\xi \in \mathcal{W}$.

If $\mathcal{W} = \mathbf{R}$ is the trivial G-module then $\mathcal{D}_1 : \mathcal{H} \to \mathcal{H}'$ is a G-module homomorphism and it uniquely determines $\mathcal{D}$ since $\mathcal{D}_t = t\mathcal{D}_1$, $t \in \mathbf{R}$. There is thus a one-to-one correspondence between G-module homomorphisms and operators with $\mathcal{W} = \mathbf{R}$.

The adjoint of $\mathcal{D}$ is the operator $\mathcal{D}^* : \mathcal{W} \to \mathcal{H}'^*\otimes\mathcal{H}$, defined by $\mathcal{D}^*_\xi = (\mathcal{D}_\xi)^*$, $\xi \in \mathcal{W}$.

Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^*\otimes\mathcal{H}'$ and $\mathcal{D}' : \mathcal{W}' \to \mathcal{H}'^*\otimes\mathcal{H}''$ be operators, where all modules are over G. We define the composition $\mathcal{D}'\circ\mathcal{D} : \mathcal{W}'\otimes\mathcal{W} \to \mathcal{H}^*\otimes\mathcal{H}''$ by

$$(\mathcal{D}'\circ\mathcal{D})_{\xi'\otimes\xi} = \mathcal{D}'_{\xi'}\circ\mathcal{D}_\xi\,,\quad \xi \in \mathcal{W}, \xi' \in \mathcal{W}'\,. \tag{3.1}$$

(On $\mathcal{W}'\otimes\mathcal{W}$, we take the orthogonal G-module structure defined by $\mathcal{W}$ and $\mathcal{W}'$.) We have

$$(\mathcal{D}'\circ\mathcal{D})^* = \mathcal{D}^*\circ\mathcal{D}'^*\,.$$

The restriction of an operator $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ to a G-submodule $\mathcal{W}_0 \subset \mathcal{W}$ is defined in an obvious way.

Example 3.1. As an application, given an operator $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$, we can consider the restriction of the composition $\mathcal{D} \circ \mathcal{D}^* : \mathcal{W} \otimes \mathcal{W} \to \mathcal{H}'^* \otimes \mathcal{H}'$ to any of the three summands in

$$\mathcal{W} \otimes \mathcal{W} = \mathbf{R} \oplus S_0^2(\mathcal{W}) \oplus \wedge^2(\mathcal{W}),$$

where $S_0^2(\mathcal{W}) \subset S^2(\mathcal{W})$ is the G-module of traceless symmetric endomorphisms of $\mathcal{W}$.

For example, as noted above, the restriction $\mathcal{D} \circ \mathcal{D}^*|_{\mathbf{R}}$ (to the trivial summand) is defined by the G-module endomorphism $(\mathcal{D} \circ \mathcal{D}^*)_1$ of $\mathcal{H}'$. With respect to an orthonormal basis $\{\xi_i\}_{i=1}^d \subset \mathcal{W}$, $1 \in \mathbf{R}$ corresponds to the tensor $\sum_{i=1}^d \xi_i \otimes \xi_i$ so that $(\mathcal{D} \circ \mathcal{D}^*)_1$ can be written as

$$(\mathcal{D} \circ \mathcal{D}^*)_1 = \sum_{i=1}^d \mathcal{D}_{\xi_i} \circ \mathcal{D}^*_{\xi_i}.$$

Given an operator $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$, we define $\mathcal{D}^\dagger : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{W}$ as the adjoint of $\mathcal{D}$ viewed as a G-module homomorphism $\mathcal{D} : \mathcal{H} \otimes \mathcal{W} \to \mathcal{H}'$. $\mathcal{D}^\dagger$ is a homomorphism of G-modules. $\mathcal{D}^\dagger$ is the trace of the bilinear form $(\xi, \eta) \mapsto \mathcal{D}^*_\xi \otimes \eta$, i.e. with respect to an orthonormal basis $\{\xi_i\}_{i=1}^d \subset \mathcal{W}$, we have

$$\mathcal{D}^\dagger = \sum_{i=1}^d \mathcal{D}^*_{\xi_i} \otimes \xi_i. \tag{3.2}$$

Indeed, for $\chi \in \mathcal{H}$, $\chi' \in \mathcal{H}'$, $\xi \in \mathcal{W}$, we have

$$\begin{aligned} \langle \mathcal{D}^\dagger(\chi'), \chi \otimes \xi \rangle &= \langle \chi', \mathcal{D}(\chi \otimes \xi) \rangle = \langle \chi', \mathcal{D}_\xi \chi \rangle \\ &= \sum_{i=1}^d \langle \mathcal{D}^*_{\xi_i} \chi', \chi \rangle \langle \xi_i, \xi \rangle \\ &= \left\langle \sum_{i=1}^d (\mathcal{D}^*_{\xi_i} \chi') \otimes \xi_i, \chi \otimes \xi \right\rangle. \end{aligned}$$

Using (3.2), we obtain

$$\mathcal{D} \circ \mathcal{D}^\dagger = \sum_{i=1}^d \mathcal{D}_{\xi_i} \circ \mathcal{D}^*_{\xi_i} = (\mathcal{D} \circ \mathcal{D}^*)_1. \tag{3.3}$$

Given an operator $\mathcal{D}$, we define the G-module homomorphism

$$\Phi^{\mathcal{D}} : S^2(\mathcal{H}) \to S^2(\mathcal{H}')$$

(between the symmetric squares of $\mathcal{H}$ and $\mathcal{H}'$) by

$$\Phi^{\mathcal{D}}(C) = \mathcal{D} \circ (C \otimes I) \circ \mathcal{D}^\dagger, \quad C \in S^2(\mathcal{H}). \tag{3.4}$$

Using (3.2) again, with respect to the orthonormal basis above, we have

$$\Phi^{\mathcal{D}}(C) = \sum_{i=1}^{d} \mathcal{D}_{\xi_i} \circ C \circ \mathcal{D}^*_{\xi_i} . \tag{3.5}$$

Substituting $C = I$ in (3.4), we obtain

$$\Phi^{\mathcal{D}}(I) = \mathcal{D} \circ \mathcal{D}^\dagger . \tag{3.6}$$

(3.1) and (3.5) give

$$\Phi^{\mathcal{D}'\circ\mathcal{D}} = \Phi^{\mathcal{D}'} \circ \Phi^{\mathcal{D}} . \tag{3.7}$$

We define the operator norm of $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ in the usual way as

$$\|\mathcal{D}\| = \max\{|\mathcal{D}_\xi(\chi)| \mid |\xi| \leq 1, |\chi| \leq 1\} .$$

Proposition 3.2. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be an operator. Then the following are equivalent*:

(i) $\|\mathcal{D}\| \leq 1$;
(ii) $\|\mathcal{D}^\dagger\| \leq 1$;
(iii) $\Phi^{\mathcal{D}}(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}')$.

Proof. By a standard result

$$\|\mathcal{D}\| = \|\mathcal{D}^\dagger\|$$

so that (i) and (ii) are equivalent. (ii) is clearly equivalent to

$$I - \mathcal{D} \circ \mathcal{D}^\dagger = I - \Phi^{\mathcal{D}}(I) \geq 0 , \tag{3.8}$$

where we also used (3.6). It remains to show that this is equivalent to (iii). Assume that (3.8) holds. Let $C \in \mathcal{K}(\mathcal{H})$ so that $C + I \geq 0$. By (3.5), $\Phi^{\mathcal{D}}(C + I) \geq 0$. Adding this and $I - \Phi^{\mathcal{D}}(I) \geq 0$, we obtain $\Phi^{\mathcal{D}}(C) + I \geq 0$. This means $\Phi^{\mathcal{D}}(C) \in \mathcal{K}(\mathcal{H}')$. (iii) follows.

Conversely, if (iii) holds then $\Phi^{\mathcal{D}}(-I) \in \mathcal{K}(\mathcal{H}')$, so that

$$\Phi^{\mathcal{D}}(-I) + I = I - \Phi^{\mathcal{D}}(I) \geq 0 ,$$

and (3.8) follows. □

A (nontrivial) operator $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ is called *conformal* if, up to a constant multiple κ, $\mathcal{D}^\dagger : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{W}$ is a linear isometry. κ is called the conformality constant. This holds if and only if

$$\Phi^{\mathcal{D}}(I) = \mathcal{D} \circ \mathcal{D}^\dagger = \kappa I , \tag{3.9}$$

or equivalently

$$\langle \mathcal{D}^\dagger \chi_1, \mathcal{D}^\dagger \chi_2 \rangle = \kappa \langle \chi_1, \chi_2 \rangle , \quad \chi_1, \chi_2 \in \mathcal{H}' .$$

In terms of an orthonormal basis $\{\xi_i\}_{i=1}^d \subset \mathcal{W}$, this rewrites as

$$\sum_{i=1}^{d} \langle \mathcal{D}^*_{\xi_i}\chi_1, \mathcal{D}^*_{\xi_i}\chi_2\rangle = \lambda\langle \chi_1, \chi_2\rangle\,, \quad \chi_1, \chi_2 \in \mathcal{H}'.$$

Note that if $\mathcal{H}'$ is irreducible then any operator $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ is conformal. Using (3.7), we see that if $\mathcal{D}$ and $\mathcal{D}'$ are conformal (with conformality constants κ and κ') then $\mathcal{D}' \circ \mathcal{D}$ is also conformal (with conformality constant $\kappa\kappa'$).

A conformal operator $\mathcal{D}$ with conformality $\kappa = 1$ is called metric. In other words, $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ is metric if $\mathcal{D}^\dagger : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{W}$ is a linear isometric imbedding:

$$\Phi^{\mathcal{D}}(I) = \mathcal{D} \circ \mathcal{D}^\dagger = I\,. \tag{3.10}$$

If $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ is a conformal operator with conformality κ then $\kappa > 0$, and $\mathcal{D}/\sqrt{\kappa}$ is metric.

Theorem 3.3. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a conformal operator with conformality κ. Then, for $C \in S^2(\mathcal{H})$, we have*

$$\lambda_0(\Phi^{\mathcal{D}}(C)) \leq \kappa\lambda_0(C)\,, \tag{3.11}$$

where λ_0 stands for the maximal eigenvalue. Equality holds in (3.11) *if and only if*

$$\mathcal{D}(\mathrm{im}\,(C - \lambda_0(C)I) \otimes \mathcal{W}) \neq \mathcal{H}'\,. \tag{3.12}$$

Similarly, we have

$$\mu_0(\Phi^{\mathcal{D}}(C)) \geq \kappa\mu_0(C)\,, \tag{3.13}$$

where μ_0 stands for the minimal eigenvalue, and equality holds if and only if

$$\mathcal{D}(\mathrm{im}\,(C - \mu_0(C)I) \otimes \mathcal{W}) \neq \mathcal{H}'\,. \tag{3.14}$$

Proof. Let $C \in S^2(\mathcal{H})$. Using that $(1/\sqrt{\kappa})\mathcal{D}^\dagger : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{W}$ is a linear isometry, we compute

$$\begin{aligned}
\lambda_0(\Phi^{\mathcal{D}}(C)) &= \max\{\langle \Phi^{\mathcal{D}}(C)\chi, \chi\rangle \,|\, |\chi| \leq 1, \chi \in \mathcal{H}'\} \\
&= \max\{\langle (C \otimes I)\mathcal{D}^\dagger\chi, \mathcal{D}^\dagger\chi\rangle \,|\, |\chi| \leq 1, \chi \in \mathcal{H}'\} \\
&= \kappa \max\{\langle (C \otimes I)(\tau), \tau\rangle \,|\, |\tau| \leq 1, \tau \in \mathrm{im}\,(\mathcal{D}^\dagger)\} \\
&\leq \kappa \max\{\langle (C \otimes I)(\sigma), \sigma\rangle \,|\, |\sigma| \leq 1, \sigma \in \mathcal{H} \otimes \mathcal{W}\} \\
&= \kappa\lambda_0(C \otimes I) = \kappa\lambda_0(C)\,.
\end{aligned} \tag{3.15}$$

Equality holds in (3.16) if and only if equality holds in (3.15). The eigenvalues of $C \otimes I$ are the same as the eigenvalues of C (with the multiplicities multiplied my $\dim \mathcal{H}$). Hence, equality holds in (3.11) if and only if

$$\mathrm{im}\,(\mathcal{D}^\dagger) \cap \ker(C - \lambda_0(C)I) \otimes \mathcal{W} \neq 0\,.$$

Since C is symmetric, this holds if and only if

$$\mathrm{im}\,(\mathcal{D}^\dagger) \cap \mathrm{im}\,(C - \lambda_0(C)I)^\perp \otimes \mathcal{W} \neq 0\,.$$

This is equivalent to (3.12). □

The proof of the last statement is analogous.

Corollary 3.4. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a conformal operator with conformality $\kappa \leq 1$. Then, we have*

$$\Phi^{\mathcal{D}}(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}')\,. \tag{3.16}$$

Proof. Let $C \in S^2(\mathcal{H})$. Since C is a symmetric endomorphism of $\mathcal{H}$, it is diagonalizable, and $C + I \geq 0$ holds if and only if the eigenvalues of C are ≥ -1. To show that $\Phi^{\mathcal{D}}(C) + I \geq 0$, we thus need to prove that the eigenvalues of $\Phi^{\mathcal{D}}(C)$ are ≥ -1. This follows from (3.11) since $\kappa \leq 1$. The corollary follows. □

Proposition 3.5. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be an operator. Then the adjoint of $\Phi^{\mathcal{D}}$ is $\Phi^{\mathcal{D}^*}$, i.e. for $C \in S^2(\mathcal{H})$ and $C' \in S^2(\mathcal{H}')$, we have*

$$\langle \Phi^{\mathcal{D}}(C), C' \rangle = \langle C, \Phi^{\mathcal{D}^*}(C') \rangle\,. \tag{3.17}$$

Proof. Using (3.5), we compute

$$\begin{aligned}
\langle \Phi^{\mathcal{D}}(C), C' \rangle &= \mathrm{trace}\,(C' \circ \Phi^{\mathcal{D}}(C)) \\
&= \mathrm{trace}\left(C' \circ \sum_{i=1}^{n} \mathcal{D}_{\xi_i} \circ C \circ \mathcal{D}^*_{\xi_i}\right) \\
&= \mathrm{trace}\left(C \circ \sum_{i=1}^{n} \mathcal{D}^*_{\xi_i} \circ C' \circ \mathcal{D}_{\xi_i}\right) \\
&= \mathrm{trace}\,(C \circ \Phi^{\mathcal{D}^*}(C')) \\
&= \langle C, \Phi^{\mathcal{D}^*}(C') \rangle\,.
\end{aligned}$$

The proposition follows. □

Corollary 3.6. *We have*

$$\Phi^{\mathcal{D}}(S^2_0(\mathcal{H})) \subset S^2_0(\mathcal{H}') \tag{3.18}$$

if and only if $\mathcal{D}^*$ is conformal. In particular, this holds if $\mathcal{H}$ is irreducible.

Proof. Substituting $C = I$ in (3.17), we obtain

$$\mathrm{trace}\,(\Phi^{\mathcal{D}}(C)) = \langle \Phi^{\mathcal{D}^*}(I), C \rangle\,. \tag{3.19}$$

We have the orthogonal decomposition

$$S^2(\mathcal{H}) = \mathbf{R} \cdot I \oplus S^2_0(\mathcal{H})\,,$$

and similarly for $S^2(\mathcal{H}')$. By (3.19), (3.18) holds if and only if $\Phi^{\mathcal{D}^*}$ is a multiple of the identity, say $\Phi^{\mathcal{D}^*}(I) = \kappa I$. Replacing $\mathcal{D}$ by $\mathcal{D}^*$ in (3.9), this means that $(\mathcal{D}^*)^\dagger$ is conformal with conformality constant κ. □

Corollary 3.7. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a metric operator with $\mathcal{D}^*$ conformal. Then, we have*

$$\Phi^{\mathcal{D}}(\mathcal{K}_0(\mathcal{H})) \subset \mathcal{K}_0(\mathcal{H}').$$

Proof. We have $\mathcal{K}_0(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap S_0^2(\mathcal{H})$, and similarly for $\mathcal{H}'$. The first statement follows from the corollary to Theorem 3.3 and Corollary 3.6 above.

Let $C \in \mathcal{K}_0(\mathcal{H})$, $C \neq 0$. By the definition of the antipodal, we have

$$\begin{aligned} \operatorname{im}(C - \lambda_0(C)I) &= \operatorname{im}\left(\frac{\lambda_0(C)}{\mu_0(C)}C^o - \lambda_0(C)I\right) \\ &= \operatorname{im}(C^o - \mu_0(C)I) = \operatorname{im}(C^o - \mu_0(C^o)I). \end{aligned}$$

We obtain that (3.12) (for C) is equivalent to (3.14) for C^o. In particular, equality holds in (3.12) (for C) if and only if equality holds in (3.14) for C^o. Since $\mu_0 = -1$ on the boundary of $\mathcal{K}_0$, we obtain the following: □

Corollary 3.8. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a metric operator with $\mathcal{D}^*$ conformal. Then $\Phi^{\mathcal{D}}|_{\partial\mathcal{K}_0(\mathcal{H})}$ is distortion decreasing, i.e. if $C \in \partial\mathcal{K}_0(\mathcal{H})$ and $\Phi^{\mathcal{D}}(C) \in \partial\mathcal{K}_0(\mathcal{H}')$, then we have*

$$\lambda_0(\Phi^{\mathcal{D}}(C)) \leq \lambda_0(C).$$

Equality holds if and only if $\Phi^{\mathcal{D}}(C^o) \in \partial\mathcal{K}_0(\mathcal{H}')$.

Corollary 3.9. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a conformal operator with conformality $\kappa \leq 1$. Then, with respect to the operator norms on $S^2(\mathcal{H})$ and $S^2(\mathcal{H}')$, we have*

$$\|\Phi^{\mathcal{D}}\| \leq 1. \tag{3.20}$$

Proof. By the definition of the operator norm, for $C \in S^2(\mathcal{H})$, we have

$$\begin{aligned} \|C\|^2 &= \max\{|C\chi|^2 \mid |\chi| \leq 1, \chi \in \mathcal{H}\} = \max\{\langle C^2\chi, \chi\rangle \mid |\chi| \leq 1, \chi \in \mathcal{H}\} \\ &= \lambda_0(C^2) = \max(\lambda_0(C)^2, \mu_0(C)^2). \end{aligned}$$

Similarly

$$\|\Phi^{\mathcal{D}}(C)\|^2 = \max(\lambda_0(\Phi^{\mathcal{D}}(C))^2, \mu_0(\Phi^{\mathcal{D}}(C))^2).$$

Hence

$$\|\Phi^{\mathcal{D}}(C)\| \leq \|C\|.$$

By the definition of the operator norm, this is equivalent to (3.20). □

Assume that M is a compact Riemannian manifold on which a compact Lie group G acts via isometries. Let $\mathcal{H} \subset C^\infty(M)$ be a finite dimensional G-submodule. As noted in Sec. 2, the general moduli space $\mathcal{K}(\mathcal{H})$ can be interpreted as a parameter space of the congruence classes of full $\mathcal{H}$-maps $f : M \to V$.

Now let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a metric operator, where $\mathcal{H}$, $\mathcal{H}' \subset C^\infty(M)$ are finite dimensional G-submodules. By the corollary to Theorem 3.3, the induced G-module homomorphism $\Phi^{\mathcal{D}} : S^2(\mathcal{H}) \to S^2(\mathcal{H}')$ carries $\mathcal{K}(\mathcal{H})$ into $\mathcal{K}(\mathcal{H}')$. In view of the interpretation of the general moduli as parameter spaces of maps, it is natural to ask whether the operator $\mathcal{D}$ can be used to carry $\mathcal{H}$-maps to $\mathcal{H}'$-maps inducing $\Phi^{\mathcal{D}}$ on the congruence classes.

To do this, we let $f : M \to V$ be a $\mathcal{H}$-map, and define $f^{\mathcal{D}} : M \to V \otimes \mathcal{W}^*$ by $f^{\mathcal{D}}(\xi) = \mathcal{D}_\xi f$, where f is viewed as a vector valued function on which $\mathcal{D}_\xi$ acts componentwise; $\alpha \circ (\mathcal{D}_\xi f) = \mathcal{D}_\xi(\alpha \circ f)$, $\alpha \in V^*$. Clearly, $f^{\mathcal{D}}$ is an $\mathcal{H}'$-map, and $V_{f^{\mathcal{D}}} \subset \mathcal{D}(V_f \otimes \mathcal{W})$, where $\mathcal{D}$ is viewed as $\mathcal{D} : \mathcal{H} \otimes \mathcal{W} \to \mathcal{H}'$.

With respect to an orthonormal basis $\{\xi_i\}_{i=1}^n \subset \mathcal{W}$, we have

$$f^{\mathcal{D}} = \sum_{i=1}^n \mathcal{D}_{\xi_i} f \otimes \xi_i ,$$

where, as usual, we identify $\mathcal{W}$ with its dual $\mathcal{W}^*$.

Note that, in general, $f^{\mathcal{D}}$ is not full even if f is.

Proposition 3.10. *Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be a metric operator where $\mathcal{H}$, $\mathcal{H}' \subset C^\infty(M)$ are finite dimensional G-submodules. Then we have*

$$\delta_{\mathcal{H}}^{\mathcal{D}}(x) = \mathcal{D}^\dagger(\delta_{\mathcal{H}'}(x)), \quad x \in M . \tag{3.21}$$

For a full $\mathcal{H}$-map $f : M \to V$, we have

$$\langle f^{\mathcal{D}} \rangle = \Phi^{\mathcal{D}}(\langle f \rangle) . \tag{3.22}$$

Proof. Choosing orthonormal bases in $\mathcal{H}$ and $\mathcal{H}'$, we write the Dirac delta maps as

$$\delta_{\mathcal{H}}(x) = \sum_{j=0}^N \chi^j(x)\chi^j \quad \text{and } \delta_{\mathcal{H}'}(x) = \sum_{l=0}^{N'} \chi'^l(x)\chi'^l .$$

We compute

$$\begin{aligned} \delta_{\mathcal{H}}^{\mathcal{D}}(x) &= \sum_{i=1}^n \sum_{j=0}^N (\mathcal{D}_{\xi_i}\chi^j)(x)\chi^j \otimes \xi_i \\ &= \sum_{i=1}^n \sum_{j=0}^N \sum_{l=0}^{N'} \langle \mathcal{D}_{\xi_i}\chi^j, \chi'^l \rangle \chi'^l(x)\chi^j \otimes \xi_i \\ &= \sum_{i=1}^n \sum_{j=0}^N \sum_{l=0}^{N'} \chi'^l(x) \langle \chi^j, \mathcal{D}_{\xi_i}^* \chi'^l \rangle \chi^j \otimes \xi_i \end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{l=0}^{N'} \chi'^{l}(x) \mathcal{D}^*_{\xi_i} \chi'^{l} \otimes \xi_i$$
$$= \sum_{i=1}^{n} \mathcal{D}^*_{\xi_i}(\delta_{\mathcal{H}'}(x)) \otimes \xi_i$$
$$= \mathcal{D}^{\dagger}(\delta_{\mathcal{H}'}(x)),$$

where we used (3.2). (3.21) follows. For (3.22), we let $f : M \to V$ be a full $\mathcal{H}$-map, and $f = A \circ \delta_{\mathcal{H}}$ where $A : \mathcal{H} \to V$ is linear and onto. The parameter point corresponding to f is $\langle f \rangle = A^*A - I \in S^2(\mathcal{H})$. For $f^{\mathcal{D}} : M \to V \otimes \mathcal{W}$, we have

$$\begin{aligned} f^{\mathcal{D}} = \sum_{i=1}^{n} \mathcal{D}_{\xi_i} f \otimes \xi_i &= \sum_{i=1}^{n} A \circ \mathcal{D}_{\xi_i} \delta_{\mathcal{H}} \otimes \xi_i \\ &= (A \otimes I) \sum_{i=1}^{n} \mathcal{D}_{\xi_i} \delta_{\mathcal{H}} \otimes \xi_i \\ &= (A \otimes I) \delta_{\mathcal{H}}^{\mathcal{D}} \\ &= (A \otimes I) \circ \mathcal{D}^{\dagger} \circ \delta_{\mathcal{H}'} . \end{aligned}$$

Hence, we have

$$\begin{aligned} \langle f^{\mathcal{D}} \rangle &= \mathcal{D} \circ (A \otimes I)^* \circ (A \otimes I) \circ \mathcal{D}^{\dagger} - I \\ &= \mathcal{D} \circ ((A^*A) \otimes I) \circ \mathcal{D}^{\dagger} - I \\ &= \mathcal{D} \circ ((A^*A - I) \otimes I) \circ \mathcal{D}^{\dagger} \\ &= \mathcal{D} \circ (\langle f \rangle \otimes I) \circ \mathcal{D}^{\dagger} = \Phi^{\mathcal{D}}(\langle f \rangle) . \end{aligned}$$

In the last equality we used (3.4). (3.22) follows. □

Remark 3.11. Let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{D}$, and f be as in the proposition above. Then, for $C = \langle f \rangle \in \partial \mathcal{K}_0(\mathcal{H})$, we have $\mathrm{im}\,(C - \mu_0(C)I) = \mathrm{im}\,(C + I) = V_f$, the space of components of f. Thus (3.11) in Theorem 3.3 is equivalent to $\mathcal{D}(V_f \otimes \mathcal{W}) \neq \mathcal{H}'$. In a similar vein, (3.13) is equivalent to $\mathcal{D}(V_{f^o} \otimes \mathcal{W}) \neq \mathcal{H}'$, where $f^o : M \to V^o$ is a representative of the congruence class C^o of C.

4. Examples

Example 4.1. Let G be a compact Lie group with Lie algebra $\mathbf{g}$, and $\mathcal{H}$ an irreducible orthogonal G-module. $\mathcal{H}$ is also a module over the Lie algebra $\mathbf{g}$ in a natural way. The elements of $\mathbf{g}$ act as skew-symmetric endomorphisms on $\mathcal{H}$:

$$\langle \xi \cdot \chi_1, \chi_2 \rangle = -\langle \chi_1, \xi \cdot \chi_2 \rangle , \quad \chi_1, \chi_2 \in \mathcal{H}, \xi \in \mathbf{g} .$$

The Casimir operator $\mathrm{Cas}_{\mathcal{H}}$ of g on $\mathcal{H}$ is the trace of the symmetric bilinear form $(\xi, \eta) \mapsto -\xi \circ \eta$. Since it commutes with the elements of g, $\mathrm{Cas}_{\mathcal{H}}$ is a constant multiple of the identity:

$$\mathrm{Cas}_{\mathcal{H}} = -\sum_{i=1}^{d} \xi_i^2 = \lambda I. \tag{4.1}$$

(The Casimir eigenvalues are known for all modules over semisimple Lie algebras [12].)

The g-module structure on $\mathcal{H}$ defines an operator $\mathcal{D}^\wedge : \mathrm{g} \to \mathcal{H}^* \otimes \mathcal{H}$, by

$$\mathcal{D}^\wedge_\xi(\chi) = \frac{1}{\sqrt{\lambda}} \xi \cdot \chi, \quad \xi \in \mathrm{g}, \chi \in \mathcal{H}.$$

This operator is metric. Indeed, by (3.3), we have

$$\mathcal{D}^\wedge \circ (\mathcal{D}^\wedge)^\dagger(\chi) = -\frac{1}{\lambda} \sum_{i=1}^{d} \xi_i^2(\chi) = \chi.$$

For the induced map between the moduli, we have

$$\Phi^{\mathcal{D}^\wedge} = I - \frac{1}{2\lambda} \mathrm{Cas}_{S^2(\mathcal{H})},$$

where $\mathrm{Cas}_{S^2(\mathcal{H})}$ is the Casimir operator on $S^2(\mathcal{H})$, the trace of the bilinear form $(\xi, \eta) \mapsto -[\xi, [\eta, \cdot]]$. In fact, with respect to an orthonormal basis $\{\xi_i\}_{i=1}^n \subset \mathrm{g}$, for $C \in S^2(\mathcal{H})$, we compute

$$\begin{aligned}
\mathrm{Cas}_{S^2(\mathcal{H})}(C) &= -\sum_{i=1}^{d} [\xi_i, [\xi_i, C]] \\
&= -\sum_{i=1}^{d} \xi_i^2 \circ C - C \circ \sum_{i=1}^{d} \xi_i^2 + 2 \sum_{i=1}^{n} \xi_i \circ C \circ \xi_i \\
&= 2\lambda C + 2\lambda \sum_{i=1}^{d} \mathcal{D}^\wedge_{\xi_i} \circ C \circ \mathcal{D}^\wedge_{\xi_i} \\
&= 2\lambda - 2\lambda \Phi^{\mathcal{D}^\wedge}(C),
\end{aligned}$$

where we used (3.5) and (4.1). The formula for $\Phi^{\mathcal{D}^\wedge}$ above follows.

Finally, note that $\mathcal{D}$ is skew-adjoint: $\mathcal{D}^* = -\mathcal{D}$. In particular, since $\mathcal{D}$ is metric, $\mathcal{D}^*$ is conformal. Corollary 3.7 applies, and we obtain that $\Phi^{\mathcal{D}}$ carries $\mathcal{K}_0(\mathcal{H})$ into itself. $\mathcal{D}^\wedge$ is introduced in [9] in the special case when $G \subset SO(m+1)$ is a closed subgroup transitive on the Euclidean m-sphere S^m, and $\mathcal{H} = \mathcal{H}^p$, is the G-module of spherical harmonics of order p on S^m. $\mathcal{D}^\wedge$ is called the *operator of infinitesimal rotations* since $\xi \cdot \chi$, for a canonical basis element ξ in $so(m+1)$, is the spherical harmonic $\chi \in \mathcal{H}^p$ infinitesimally rotated in the plane singled out by ξ. The induced

G-module homomorphism $\Phi^{\mathcal{D}^\wedge}$ is a self-map of $\mathcal{K}_0^p = \mathcal{K}_0^p(\mathcal{H}^p)$, and its contraction properties play a crucial role in giving lower estimates on the codimension of minimal isometric immersions between spheres.

We now take a look at the case when $m = 3$ and $p = 6$. Let $Tet : S^3 \to S^6$ be the tetrahedral minimal immersion [7, 9]. Let $C = \langle Tet \rangle$ be the corresponding point on the moduli. We have $C \in \partial\mathcal{K}_0^6$. For reasons of dimension, we also have $\Phi^{\mathcal{D}^\wedge}(C) \in \partial\mathcal{K}_0^6$. By the computations in [9], it follows that $\lambda_0(C) = 6$, $\lambda_0(\Phi^{\mathcal{D}^\wedge}(C)) = 4/3$, and these are the distortions at C and $\Phi^{\mathcal{D}^\wedge}(C)$. For the antipodal, we have $C^o = -(1/6)C$, and $\Phi^{\mathcal{D}^\wedge}(C^o) = (1/8)\Phi^{\mathcal{D}^\wedge}(C)^o$, in particular, $\Phi^{\mathcal{D}^\wedge}(C^o)$ is in the interior of $\mathcal{K}_0^6$.

The minimal $\Phi^{\mathcal{D}^\wedge}$-invariant slice of $\mathcal{K}_0^6$ is 2-dimensional, and it is an isosceles triangle with vertices C and $\Phi^{\mathcal{D}^\wedge}(C)$, and $-(1/3)C - \Phi^{\mathcal{D}^\wedge}(C)$.

Example 4.2. As above, let $\mathcal{H}^p$ denote the irreducible $SO(m+1)$-module of spherical harmonics of order p on S^m. There are two natural operators acting on spherical harmonics. First, for $a \in \mathbf{R}^{m+1}$, directional derivative at the direction a defines a linear map

$$\partial_a : \mathcal{H}^p \to \mathcal{H}^{p-1} .$$

Second, multiplication by the linear functional $a^* \in \mathcal{H}^1$ corresponding to $a \in \mathbf{R}^{m+1}$ ($a^*(x) = \langle a, x\rangle$, $x \in \mathbf{R}^{m+1}$), followed by harmonic projection H defines

$$\delta_a : \mathcal{H}^p \to \mathcal{H}^{p+1} .$$

We have

$$\delta_a \chi = H(a^* \cdot \chi) = a^* \cdot \chi - \frac{2p}{\lambda_{2p}} \partial_a \chi \cdot \rho^2 , \quad a \in \mathbf{R}^{m+1} , \tag{4.2}$$

where $\rho^2 = |x|^2$, $x \in \mathbf{R}^{m+1}$.

Up to constant multiples, ∂_a and δ_a are adjoints of each other. In fact [10], on $\mathcal{H}^p$, we have

$$\partial_a^* = \mu_p \delta_a , \quad \text{and } \delta_a^* = \frac{1}{\mu_{p-1}} \partial_a , \tag{4.3}$$

where

$$\mu_p = (p+1)\frac{\lambda_{2p}}{2\lambda_p} . \tag{4.4}$$

In what follows we will need the derivative of the harmonic projection formula (4.2). Using $\partial_a b^* = \langle a, b\rangle$ and $\partial_a \rho^2 = 2a^*$, a, $b \in \mathbf{R}^{m+1}$, we obtain

$$\partial_a \delta_b \chi = \langle a, b\rangle \chi + b^* \partial_a \chi - \frac{4p}{\lambda_{2p}} a^* \partial_b \chi - \frac{2p}{\lambda_{2p}} \rho^2 \partial_a \partial_b \chi .$$

Taking traces (in (a,b) with respect to the standard orthonormal basis in $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$), and using homogeneity, we obtain

$$\sum_{i=0}^{m} \partial_i \delta_i \chi = \frac{2\lambda_p}{\lambda_{2p}} \frac{\lambda_{2(p+1)}}{2(p+1)} \chi \,, \quad \chi \in \mathcal{H}^p \,, \tag{4.5}$$

where we set $\partial_i = \partial_{e_i}$ and $\delta_i = \delta_{e_i}$.

We now introduce the operators $\mathcal{D}^\pm = \mathcal{D}_p^\pm : \mathbf{R}^{m+1} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p\pm1}$ defined by

$$\mathcal{D}_a^- = \sqrt{\frac{2}{\lambda_{2p}}} \partial_a \quad \text{and } \mathcal{D}_a^+ = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}} \delta_a \,.$$

We claim that $\mathcal{D}^\pm$ are both metric. To do this, we first work out the adjoints $(\mathcal{D}^\pm)^\dagger : \mathcal{H}^{p\pm1} \to \mathcal{H}^p \otimes \mathcal{H}^1$. Using (4.3), we find

$$(\mathcal{D}^-)^\dagger(\chi) = c_p^- \sum_{i=0}^{m} \delta_i \chi \otimes y_i \,, \quad \chi \in \mathcal{H}^{p-1} \,,$$

and

$$(\mathcal{D}^+)^\dagger(\chi) = c_p^+ \sum_{i=0}^{m} \partial_i \chi \otimes y_i \,, \quad \chi \in \mathcal{H}^{p+1} \,,$$

where

$$c_p^- = \frac{p}{\sqrt{\lambda_{2p}/2}} \frac{\lambda_{2(p-1)}}{2\lambda_{p-1}} \,,$$

$$c_p^+ = \frac{1}{p+1} \sqrt{\frac{2\lambda_p}{\lambda_{2p}}} \,.$$

With this, we have

$$\mathcal{D}^- \circ (\mathcal{D}^-)^\dagger(\chi) = c_p^- \sqrt{\frac{2}{\lambda_{2p}}} \sum_{i=0}^{m} \partial_i \delta_i \chi = \chi \,, \quad \chi \in \mathcal{H}^{p-1} \,.$$

This shows that $\mathcal{D}^-$ is metric. The computation for $\mathcal{D}^+$ is analogous (but simpler). Due to (4.3), we also know that, up to constant multiples, $\mathcal{D}^\pm$ are adjoints of each other. Indeed, we have

$$(\mathcal{D}_p^-)^* = p \sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{p-1}\lambda_{2p}}} \mathcal{D}_{p-1}^+ \,,$$

where we indicated the dependence of $\mathcal{D}^\pm$ on the degree by subscripts.

In particular, it follows that $(\mathcal{D}^\pm)^*$ are both conformal so that Corollary 3.7 applies. Thus

$$\Phi^{\mathcal{D}^\pm}(\mathcal{K}_0^p) \subset \mathcal{K}_0^{p\pm1} \,.$$

By Corollary 3.8, $\Phi^{\mathcal{D}^{\pm}}$ are both distortion decreasing. $\mathcal{D}^{\pm}$ are called the *degree raising and lowering operators.* They play a crucial role in determining the exact dimension of the moduli for spherical minimal immersions [10], a problem posed by DoCarmo and Wallach in [2].

Example 4.3. Applying the construction in the example in Sec. 3 to $G = SO(m+1)$, $\mathcal{H} = \mathcal{H}^p$, $\mathcal{H}' = \mathcal{H}^{p-1}$, and $\mathcal{D} = \mathcal{D}_p^-$, we obtain three operators. Since, up to a constant multiple, $(\mathcal{D}_p^-)^*$ is $\mathcal{D}_{p-1}^+$ (Example 4.2), we will consider the restrictions of $\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-$ to the three summands in

$$\mathbf{R}^{m+1} \otimes \mathbf{R}^{m+1} = \mathbf{R} \oplus \mathcal{H}^2 \oplus so(m+1)\,.$$

Here $S_0^2(\mathbf{R}^{m+1})$ is identified with its dual $\mathcal{H}^2$, and $so(m+1)$ is identified with $\wedge^2(\mathbf{R}^{m+1})$.

Using the definitions of $\mathcal{D}^{\pm}$, for $a, b \in \mathbf{R}^{m+1}$, we obtain

$$(\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-)_{a\otimes b} = \sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}\,\delta_a \circ \partial_b\,. \tag{4.6}$$

To obtain $(\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-)|_{\mathbf{R}}$ we first note that $1 \in \mathbf{R}$ corresponds to $\sum_{i=0}^m e_i \otimes e_i$, where $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$ is the standard orthonormal basis. For $\chi \in \mathcal{H}^p$, we have

$$(\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-)|_{\sum_{i=0}^m e_i\otimes e_i}\chi = \sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}\sum_{i=0}^m \delta_i\partial_i\chi = p\sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}\chi$$

where, in the last equality we used homogeneity of χ as a polynomial. We thus have

$$(\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-)|_{\mathbf{R}} = p\sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}\,I\,.$$

For the restriction to $so(m+1)$, we note that

$$(\delta_a \circ \partial_b - \delta_b \circ \partial_a)\chi = (a^*\partial_b - b^*\partial_a)\chi\,, \quad \chi \in \mathcal{H}^p\,,$$

since the right-hand side is harmonic. Moreover, for a, b orthonormal, this is χ infinitesimally rotated in the plane spanned by a and b, a typical element in $so(m+1)$. Hence

$$(\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-)|_{a\wedge b} = \frac{1}{2}\sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}\,\mathcal{D}_{a\wedge b}^{\wedge}\,.$$

We obtain

$$(\mathcal{D}_{p-1}^+ \circ \mathcal{D}_p^-)|_{\wedge^2(\mathbf{R}^{m+1})} = \sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}\,\mathcal{D}^{\wedge}\,,$$

where $\mathcal{D}^{\wedge}$ is defined in Example 4.1.

Finally, we have

$$(\mathcal{D}^+_{p-1} \circ \mathcal{D}^-_p)|_{a\odot b}\chi = \frac{1}{2}\sqrt{\frac{\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}}}H(a^*\partial_b\chi + b^*\partial_a\chi)\,, \quad \chi \in \mathcal{H}^p\,.$$

Discarding the trace, this defines the restriction $(\mathcal{D}^+_{p-1} \circ \mathcal{D}^-_p)|_{\mathcal{H}^2}$.

Example 4.4. The composition

$$\mathcal{D}^-_{p-q+1} \circ \cdots \mathcal{D}^-_{p-1} \circ \mathcal{D}^-_p : (\mathbf{R}^{m+1})^{\otimes q} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p-q+1} \tag{4.7}$$

is given by

$$(\mathcal{D}^-_{p-q+1} \circ \cdots \mathcal{D}^-_{p-1} \circ \mathcal{D}^-_p)_{e_{i_q}\otimes\cdots\otimes e_{i_1}}\chi = \frac{2^{q/2}}{\sqrt{\lambda_{2p}\lambda_{2(p-1)}\cdots\lambda_{2(p-q+1)}}}\partial_{i_1}\ldots\partial_{i_q}\chi\,. \tag{4.8}$$

Since this is symmetric in the indices $i_1, \ldots, i_q$, we restrict (4.7) to $S^q(\mathbf{R}^{m+1})$. The dual $S^q(\mathbf{R}^{m+1})^* = S^q(\mathcal{H}^1)$ is the space of degree q homogeneous polynomials in $\mathbf{R}^{m+1}$. We have

$$S^q(\mathcal{H}^1) = \mathcal{H}^q \oplus \rho^2 \cdot S^{q-2}(\mathcal{H}^1)\,.$$

Since χ is harmonic, (4.8) vanishes on the component $\rho^2 \cdot S^{q-2}(\mathcal{H}^1)$. We obtain the operator

$$(\mathcal{D}^-_{p-q+1} \circ \cdots \mathcal{D}^-_{p-1} \circ \mathcal{D}^-_p)|_{\mathcal{H}^q} : \mathcal{H}^q \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p-q+1}\,.$$

5. Operators on Moduli for Spherical Maps

In this section we first give a necessary and sufficient condition for an operator to preserve sphericality. Let M be a compact Riemannian manifold with a compact Lie group G of isometries, and $\mathcal{H} \subset C^\infty(M)$ a finite dimensional G-module. Recall from Sec. 2, the G-module epimomorphism

$$\Psi^0_{\mathcal{H}} : S^2(\mathcal{H}) \to \mathcal{H}\cdot\mathcal{H}$$

with kernel $\mathcal{E}(\mathcal{H})$.

Theorem 5.1. *Let M be a compact Riemannian manifold with a compact Lie group G of isometries. Let $\mathcal{H}$, $\mathcal{H}' \subset C^\infty(M)$ be spherical G-submodules. Let $\mathcal{D} : \mathcal{W} \to \mathcal{H}^* \otimes \mathcal{H}'$ be an operator. Then $\Phi^{\mathcal{D}} : S^2(\mathcal{H}) \to S^2(\mathcal{H}')$ carries $\mathcal{E}(\mathcal{H})$ into $\mathcal{E}(\mathcal{H}')$ if and only if there exists a G-module homomorphism $\bar{\Phi}^{\mathcal{D}} : \mathcal{H}\cdot\mathcal{H} \to \mathcal{H}'\cdot\mathcal{H}'$ that makes the diagram*

$$\begin{array}{ccc} S^2(\mathcal{H}) & \xrightarrow{\Phi^{\mathcal{D}}} & S^2(\mathcal{H}') \\ \Psi^0_{\mathcal{H}}\downarrow & & \downarrow\Psi^0_{\mathcal{H}'} \\ \mathcal{H}\cdot\mathcal{H} & \xrightarrow{\bar{\Phi}^{\mathcal{D}}} & \mathcal{H}'\cdot\mathcal{H}' \end{array}$$

commutative.

Proof. If $\bar{\Phi}^{\mathcal{D}}$ exists then

$$\Phi^{\mathcal{D}}(\mathcal{E}(\mathcal{H})) = \Phi^{\mathcal{D}}(\ker \Psi^0_{\mathcal{H}}) \subset \ker \Psi^0_{\mathcal{H}'} = \mathcal{E}(\mathcal{H}').$$

Conversely, if $\Phi^{\mathcal{D}}(\mathcal{E}(\mathcal{H})) \subset \mathcal{E}(\mathcal{H}')$ then $\Phi^{\mathcal{D}}$ defines a homomorphism of G-modules $S^2(\mathcal{H})/\mathcal{E}(\mathcal{H}) \to S^2(\mathcal{H}')/\mathcal{E}(\mathcal{H}')$. Since $S^2(\mathcal{H})/\mathcal{E}(\mathcal{H}) \cong \mathcal{H}\cdot\mathcal{H}$ and $S^2(\mathcal{H}')/\mathcal{E}(\mathcal{H}') \cong \mathcal{H}'\cdot\mathcal{H}'$, this also defines $\bar{\Phi}^{\mathcal{D}}$. □

$\Phi^{\mathcal{D}}$ naturally extends to a G-module homomorphism $\Phi^{\mathcal{D}} : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}'\otimes\mathcal{H}'$ by setting

$$\Phi^{\mathcal{D}} = \sum_{i=1}^{d} \mathcal{D}_{\xi_i} \otimes \mathcal{D}_{\xi_i},$$

where $\{\xi_i\}_{i=1}^d \subset \mathcal{W}$ is an orthonormal basis. The fact that this is an extension follows from (3.5). With this, the commutative diagram above becomes

$$\begin{array}{ccc} \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\Phi^{\mathcal{D}}} & \mathcal{H}'\otimes\mathcal{H}' \\ \Pi_{\mathcal{H}}\downarrow & & \downarrow\Pi_{\mathcal{H}'} \\ \mathcal{H}\cdot\mathcal{H} & \xrightarrow{\bar{\Phi}^{\mathcal{D}}} & \mathcal{H}'\cdot\mathcal{H}' \end{array}$$

where the vertical arrows are given by multiplications. (The extended diagram commutes since $\Pi_{\mathcal{H}}|_{\wedge^2(\mathcal{H})} = 0$, and similarly for $\mathcal{H}'$.) Hence, commutativity of the diagram amounts to the existence of a G-module homomorphism $\bar{\Phi}^{\mathcal{D}}$ such that

$$(\Pi_{\mathcal{H}'} \circ \Phi^{\mathcal{D}})(\chi^1 \otimes \chi^2) = \sum_{i=1}^{d} \mathcal{D}_{\xi_i}\chi^1 \mathcal{D}_{\xi_i}\chi^2 = \bar{\Phi}^{\mathcal{D}}(\chi^1\chi^2), \quad \chi^1, \chi^2 \in \mathcal{H}. \tag{5.1}$$

Example 5.2. Let M be a Riemannian manifold with a compact Lie group G of isometries, and $\mathcal{H} \subset C^\infty(M)$ an irreducible δ-spherical G-submodule. Let $\mathcal{D}^\wedge : \mathbf{g} \to \mathcal{H}^* \otimes \mathcal{H}$ be the operator defined in Example 4.1. Using the notations there, for χ^1, $\chi^2 \in \mathcal{H}$, we have

$$(\Pi_{\mathcal{H}} \circ \Phi^{\mathcal{H}^\wedge})(\chi^1 \otimes \chi^2) = \frac{1}{\lambda}\sum_{i=1}^{d} \xi_i\chi^1\xi_i\chi^2.$$

On the other hand, since λ is the Casimir eigenvalue of $\mathcal{H}$, we have

$$\begin{aligned} \mathrm{Cas}_{C^\infty(M)}(\chi^1\chi^2) &= -\sum_{i=1}^{d} \xi_i^2(\chi^1\chi^2) \\ &= \mathrm{Cas}_{\mathcal{H}}\chi^1 \cdot \chi^2 - 2\sum_{i=1}^{d} \xi_i\chi^1\xi_i\chi^2 + \chi^1 \cdot \mathrm{Cas}_{\mathcal{H}}\chi^2 \\ &= 2\lambda\chi^1\chi^2 - 2\sum_{i=1}^{d} \xi_i\chi^1\xi_i\chi_2, \end{aligned}$$

where we used the fact that $\xi \in \mathbf{g}$ acts as a vector field on $C^\infty(M)$.

Combining these, we obtain that $\bar{\Phi}^{\mathcal{D}^\wedge} : S^2(\mathcal{H}) \to S^2(\mathcal{H})$ exists, and it is given by

$$\bar{\Phi}^{\mathcal{D}^\wedge} = I - \frac{1}{2\lambda}\mathrm{Cas}_{C^\infty(M)} .$$

By the corollary to Theorems 3.3 and 5.1, $\bar{\Phi}^{\mathcal{D}^\wedge}$ is a self-map of $\mathcal{L}(\mathcal{H})$. Note finally that if $M = G/K$ is naturally reductive then $\mathrm{Cas}_{C^\infty(M)}$ is the Laplacian Δ^M.

Example 5.3. Consider again the degree raising and lowering operators $\mathcal{D}^\pm$: $\mathcal{H}^1 \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p\pm1}$ with induced homomorphisms

$$\Phi^{\mathcal{D}^\pm} : \mathcal{H}^p \otimes \mathcal{H}^p \to \mathcal{H}^{p\pm1} \otimes \mathcal{H}^{p\pm1} .$$

We claim that $\bar{\Phi}^{\mathcal{D}^\pm}$ on $\chi_1 \otimes \chi_2 \in \mathcal{H}^p \otimes \mathcal{H}^p$ exists and determine it explicitly. We let $\chi^1, \chi^2 \in \mathcal{H}^p$. We work out (5.1) with respect to the standard orthonormal basis $\{\xi_i = e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$. Using (4.2), we have

$$\begin{aligned}
\sum_{i=0}^m \mathcal{D}^+_{e_i}\chi^1\mathcal{D}^+_{e_i}\chi^2 &= \frac{\lambda_{2p}}{2\lambda_p}\sum_{i=0}^m \delta_i\chi^1\delta_i\chi^2 \\
&= \frac{\lambda_{2p}}{2\lambda_p}\sum_{i=0}^m H(x_i\chi^1)H(x_i\chi^2) \\
&= \frac{\lambda_{2p}}{2\lambda_p}\sum_{i=0}^m \left(x_i\chi^1 - \frac{2p}{\lambda_{2p}}\partial_i\chi^1\cdot\rho^2\right)\left(x_i\chi^2 - \frac{2p}{\lambda_{2p}}\partial_i\chi^2\cdot\rho^2\right) \\
&= \left(1 - \frac{p^2}{\lambda_p}\right)\chi^1\chi^2\cdot\rho^2 + \frac{p^2}{\lambda_p\lambda_{2p}}\Delta(\chi^1\chi^2)\cdot\rho^4 ,
\end{aligned}$$

where Δ in the Euclidean Laplacian. In a similar vein

$$\sum_{i=0}^m \mathcal{D}^-_{e_i}\chi^1\mathcal{D}^-_{e_i}\chi^2 = \frac{2}{\lambda_{2p}}\sum_{i=0}^m \partial_i\chi^1\partial_i\chi^2 = \frac{1}{\lambda_{2p}}\Delta(\chi^1\chi^2) .$$

If we define $\bar{\Phi}^{\mathcal{D}^\pm} : \mathcal{P}^{2p} \to \mathcal{P}^{2(p\pm1)}$ by

$$\bar{\Phi}^{\mathcal{D}^+} = \left(1 - \frac{p^2}{\lambda_p}\right)\rho^2 I + \frac{p^2}{\lambda_p\lambda_{2p}}\rho^4\Delta , \tag{5.2}$$

and

$$\bar{\Phi}^{\mathcal{D}^-} = \frac{1}{\lambda_{2p}}\Delta \tag{5.3}$$

then the diagrams

$$\begin{array}{ccc}
S^2(\mathcal{H}^p) & \xrightarrow{\Phi^{\mathcal{D}^\pm}} & S^2(\mathcal{H}^{p\pm1}) \\
\Psi^0_{\mathcal{H}^p}\downarrow & & \downarrow\Psi^0_{\mathcal{H}^{p\pm1}} \\
\mathcal{P}^{2p} \supset \mathcal{H}^p\cdot\mathcal{H}^p & \xrightarrow{\bar{\Phi}^{\mathcal{D}^\pm}} & \mathcal{H}^{p\pm1}\cdot\mathcal{H}^{p\pm1} \subset \mathcal{P}^{2(p\pm1)}
\end{array}$$

commute. We obtain that

$$\Phi^{\mathcal{D}^{\pm}}(\mathcal{E}(\mathcal{H}^p) \subset (\mathcal{E}(\mathcal{H}^{p\pm 1}))$$

and hence $\Phi^{\mathcal{D}^{\pm}}$ carries the moduli $\mathcal{M}(\mathcal{H}^p)$ into $\mathcal{M}(\mathcal{H}^{p\pm 1})$.

As an application, now use the operator $\mathcal{D}^+$ to prove that

$$\mathcal{H}^p \cdot \mathcal{H}^p = \mathcal{P}^{2p}. \tag{5.4}$$

(Another proof using linearlization of ultraspherical polynomials is in [8].) We proceed by induction with respect to p. For $p = 0, 1$ this is obvious. For the general induction step, $p \Rightarrow p+1$, we first note that, by definition, $\mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1} \subset \mathcal{P}^{2(p+1)}$. Since $\mathcal{P}^{2(p+1)} = \mathcal{H}^{2(p+1)} \oplus \mathcal{P}^{2p}$, it remains to show that

$$\mathcal{H}^{2(p+1)} \subset \mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1} \quad \text{and } \mathcal{P}^{2p} \cdot \rho^2 \subset \mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1}. \tag{5.5}$$

First, by definition, $H(x_0^{p+1})^2 \in \mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1}$. Its harmonic projection to $\mathcal{H}^{2(p+1)}$ within $\mathcal{P}^{2(p+1)}$ is

$$H(H(x_0^{p+1})^2) = H(x_0^{2(p+1)}),$$

since H annihilates multiples of ρ^2. This latter polynomial is nonzero, and thus irreducibility of $\mathcal{H}^{2(p+1)}$ gives

$$\mathcal{H}^{2(p+1)} \subset \mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1}.$$

This is the first inclusion in (5.5). It remains to show the second inclusion in (5.5). Comparison of the Euclidean and spherical Laplacians shows that, on $\mathcal{P}^{2p}$, we have

$$\Delta = -\Delta^{S^m} + \lambda_{2p} I.$$

Thus (5.2) and (5.3) are rewriten,

$$\bar{\Phi}^{\mathcal{D}^+} = I - \frac{p^2}{\lambda_p \lambda_{2p}} \Delta^{S^m}, \tag{5.6}$$

and

$$\bar{\Phi}^{\mathcal{D}^-} = I - \frac{1}{\lambda_{2p}} \Delta^{S^m}. \tag{5.7}$$

By the induction hypothesis, we have (5.4). We claim that

$$\bar{\Phi}^{\mathcal{D}^+} : \mathcal{P}^{2p} \to \mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1}$$

is injective. Indeed, if $\xi \in \mathcal{P}^{2p}$ is in the kernel of $\bar{\Phi}^{\mathcal{D}^+}$ then, writing $\xi = \sum_{j=0}^{p} \chi_j$, $\chi_j \in \mathcal{H}^{2j}$, we have

$$\chi_j = \frac{p^2}{\lambda_p} \frac{\lambda_{2j}}{\lambda_{2p}} \chi_j$$

and this is possible only for $\chi_j = 0$.

By injectivity, $\mathcal{P}^{2p}$ is an $SO(m+1)$-submodule of $\mathcal{H}^{p+1} \cdot \mathcal{H}^{p+1}$. The second inclusion in (5.5) follows. (Also, (5.7) implies that $\bar{\Phi}^{\mathcal{D}^-} : \mathcal{P}^{2p} \to \mathcal{P}^{2(p-1)}$ is onto with kernel $\mathcal{H}^{2p}$.)

References

1. M. Berger, *Geometry I-II*, Springer, New York, 1977.
2. M. DoCarmo and N. Wallach, Minimal immersions of spheres into spheres, *Ann. Math.* **93** (1971), 43–62.
3. A. Dvoretzky, Some results on convex bodies in Banach spaces, in *Int. Symp. Linear Spaces*, Jerusalem Academy Press, 1961, pp. 123–160.
4. H. Gauchman and G. Toth, Fine structure of the space of spherical minimal immersions, *Trans. Amer. Math. Soc.* **348**(6) (1996), 2441–2463.
5. S. Helgason, *Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Mathematical Surveys and Monographs 83, Amer. Math. Soc., Providence, 2000.
6. S. Helgason, *Geometric Analysis on Symmetric Spaces*, Mathematical Surveys and Monographs 39, Amer. Math. Soc., Providence, 1994.
7. G. Toth, *Finite Möbius Groups, Minimal Immersions of Spheres, and Moduli*, Universitext, Springer, New York, 2001.
8. G. Toth, Moduli for spherical maps and minimal immersions of homogeneous spaces, *J. Lie Theory* **12**(2) (2002), 551–570.
9. G. Toth, Infinitesimal rotations of isometric minimal immersions between spheres, *Amer. J. Math.* **122** (2000), 117–152.
10. G. Toth, Eigenmaps and the space of minimal immersions between spheres, *Indiana Univ. Math. J.* **46**(2) (1997), 637–658.
11. N. Wallach, Minimal immersions of symmetric spaces into spheres, *Symmetric Spaces*, Dekker, New York, 1972, pp. 1-40.
12. M. Wang and W. Ziller, On normal homogeneous Einstein manifolds, *Ann. Sci. École Norm. Sup.* **18**(4) (1985), 563–633.