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MINIMAL IMMERSIONS OF SPHERES AND MODULI

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Eigenmaps and spherical minimal immersions

Minimal immersions of round spheres into round spheres, or spherical minimal immersions for short, or "spherical soap bubbles", belong to a fast growing and fascinating area between algebra and geometry. This theory has rich interconnections with a variety of mathematical disciplines such as invariant theory, convex geometry, harmonic maps, and orthogonal multiplications. In this survey article we browse through some of the developments of the theory in the last thirtysome years.

The theory of spherical soap bubbles studies minimal immersions of round spheres into round spheres of different dimensions. A classical example is the imbedding of the real projective plane $\mathbb{R}P^2$ into the 4-sphere S^4 as the *Veronese minimal surface*. Conveniently described by a minimal immersion $Ver : S^2 \to S^4$, it is a twofold covering projection to the image, the Veronese minimal surface, and the covering is given by the action of the antipodal map on S^2. The metric on S^2 induced from the curvature one metric on S^4 is of constant curvature $1/3$.

The general theory of spherical minimal immersions first took off in 1966 with the following result [18]:

THEOREM (TAKAHASHI, 1966). *A minimal isometric immersion $f : S^m_\kappa \to S_V$ of the m-sphere of curvature κ into the unit sphere S_V (of curvature one) of a Euclidean vector space V exists iff $\kappa = m/\lambda_p$, where $\lambda_p = p(p+m-1)$ is the p-th eigenvalue of the spherical Laplacian $\triangle^{S^m}$ on S^m. In this case, the components $\alpha \circ f$, $\alpha \in V^*$, of f are spherical harmonics of order p on S^m (with the curvature one metric), i.e. eigenfunctions of $\triangle^{S^m}$ with eigenvalue λ_p.*

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It will be convenient to keep the curvature one metric on the domain S^m. Our immersions then become homothetic (conformal with conformality factor λ_p/m):

$$\langle f_*(X), f_*(Y)\rangle = \langle X\cdot f, Y\cdot f\rangle = \frac{\lambda_p}{m}\langle X, Y\rangle, \tag{1}$$

where X, Y are vector fields on S^m.

A map $f : S^m \to S_V$ is said to be a *spherical minimal immersion of degree* p if the components of f are spherical harmonics of order p on S^m, and f is homothetic (1) (with conformality factor λ_p/m).

It will also be convenient to consider the wider class of p-eigenmaps by dropping the homothety condition. A map $f : S^m \to S_V$ is said to be a *p-eigenmap* if its components are spherical harmonics of order p on S^m. Eigenmaps (between spheres) can be characterized by being *harmonic* in the sense of Eells–Sampson [7] and having *constant energy density* ($= \lambda_p/2$). Since the energy density is constant, a p-eigenmap is a spherical minimal immersion of degree p iff it is conformal.

A fundamental problem in the theory of spherical minimal immersions is the following:

Problem. For each m and p, what is the minimum dimension of the range for a spherical minimal immersion $f : S^m \to S_V$ of degree p?

The next result [15] gives a universal lower bound independent of the degree:

Theorem (J. D. Moore, 1976). *For any spherical minimal immersion $f : S^m \to S_V$ of degree $p \geq 2$, we have* $\dim V \geq 2m + 1$.

A rich variety of spherical minimal immersions can be obtained by the "equivariant construction", first used in this context by Mashimo in 1984 [13, 14], and later exploited by DeTurck and Ziller [3, 4]. These spherical minimal immersions are given as orbit maps $f_\xi : S^3 \to S_W$ by the special unitary group $SU(2)$, where ξ is an element of unit length of a representation W of $SU(2)$:

$$f_\xi(g) = g\cdot \xi = \xi \circ L_{g^{-1}}, \quad g \in SU(2) = S^3,$$

where L stands for left quaternionic multiplication.

In the simplest case, $W = W_p$ is the complex irreducible representation of $SU(2)$ of dimension $\dim_{\mathbf{C}} W_p = p+1$. More concretely, W_p can be thought of as the linear space of degree p homogeneous complex polynomials in two variables $z, w \in \mathbf{C}$ with $SU(2)$ acting by precomposition with inverses. Setting $g = a + jb \in S^3$, $a, b \in \mathbf{C}$, we have

$$f_\xi(a+jb)(z,w) = \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad z, w \in \mathbf{C}.$$

Note that f_ξ is automatically a p-eigenmap. By $SU(2)$-equivariance:

$$f_\xi \circ L_g = g \cdot f_\xi, \quad g \in SU(2),$$

f_ξ is conformal iff it is homothetic at $1 \in S^3$. This latter condition amounts to solve a system of quadratic equations in the coefficients of ξ.

A reduction to the real (irreducible) subrepresentation $R_p \subset W_p$, for p even, $\dim R_p = p+1$, gives low codimension examples such as Mashimo's degree 6 spherical minimal immersion $f : S^3 \to S^6$ [3, 4, 5, 13]. In particular, choosing $\xi \in R_6$ the minimum degree (=6) absolute invariant [12] for the binary tetrahedral group (the double cover in $SU(2)$ of the symmetry group of a regular tetrahedron), we obtain the *tetrahedral minimal immersion* $Tet : S^3 \to S^6$. This particular example shows that Moore's lower bound cannot be improved in this generality.

Choosing ξ as Klein's absolute invariants [12] for finite subgroups in $SU(2)$ (cyclic, dihedral and the binary tetrahedral, octahedral, and icosahedral groups) allows the orbit maps f_ξ to be factored into spherical minimal imbeddings of the corresponding quotient lens spaces and polyhedral manifolds.

A careful enumeration of the possible cases lead DeTurck and Ziller to conclude that all homogeneous 3-dimensional spherical space forms can be isometrically and minimally imbedded into spheres [3, 4]. This is also true in any domain dimension:

THEOREM (DETURCK–ZILLER, 1992). *All homogeneous spherical space forms can be isometrically and minimally imbedded into spheres.*

It is expected that all spherical space forms (homogeneous or not) can be isometrically and minimally imbedded into spheres.

For spherical imbeddings of nonhomogeneous space forms, see the recent work of Escher [9].

A *spherical harmonic of order* p on S^m is the restriction of a homogeneous polynomial of degree p on $\mathbb{R}^{m+1}$ [28]. Our p-eigenmap $f : S^m \to S_V$ thus extends to a harmonic homogeneous polynomial map $f : \mathbb{R}^{m+1} \to V$ of degree p that is *spherical*, i.e. f maps the unit sphere S^m to the unit sphere S_V. (Looking back, the Veronese map fits perfectly into this picture; its components are given by quadratic harmonic polynomials, in particular, we see that it factors through the antipodal projection $S^2 \to \mathbb{R}P^2$.)

By definition, a p-eigenmap $f : S^m \to S_V$ can be described by the following equations on S^m:

$$\triangle^{S^m} f = \lambda_p \cdot f, \quad |f|^2 = 1, \tag{2}$$

or, as a p-homogeneous polynomial map on $\mathbb{R}^{m+1}$ satisfying:

$$\triangle f = 0, \quad |f|^2 = \rho^{2p},$$

where $\triangle$ is the Euclidean Laplacian and $\rho(x) = |x|$, $x \in \mathbb{R}^{m+1}$. Homothety imposed on a p-eigenmap $f : S^m \to S_V$ makes f a spherical minimal immersion. This condition is awkward to work with since it involves all vector fields on S^m, an infinite dimensional space. To circumvent this technical difficulty, we restrict ourselves to *conformal vector fields on* S^m. This does not loose the strength of the condition of homothety since the conformal vector fields span each tangent space of S^m. The conformal vector fields are parametrized by the ambient vector space $\mathbb{R}^{m+1}$ of the domain S^m; for $a \in \mathbb{R}^{m+1}$, the conformal vector field X^a is the projection of the uniform vector field a along $S^m \subset \mathbb{R}^{m+1}$ to the various tangent spaces of S^m:

$$X^a_x = a - a^* \cdot x = a - \langle a, x \rangle x, \quad x \in S^m.$$

Now condition (1) of homothety for f is

$$\langle f_*(X^a), f_*(X^b) \rangle = \langle X^a \cdot f, X^b \cdot f \rangle = \frac{\lambda_p}{m} \langle X^a, X^b \rangle, \quad a, b \in \mathbb{R}^{m+1}. \tag{3}$$

Continuing the analogy with f, all ingredients in this formula can be extended to $\mathbb{R}^{m+1}$ as homogeneous polynomials of various degrees. The strong algebraic flavor of these conditions is not surprising; this is due to our insistence of keeping the metric on the domain to be of constant curvature.

Moduli

By Takahashi's result, the problem of "classifying" eigenmaps and spherical minimal immersions has two natural parameters: the domain dimension m and the degree p (the range is kept arbitrary). The cases $m = 1$ and $p = 1$ are trivial and will be omitted from the considerations.

We let $\mathcal{H}^p$ denote the space of spherical harmonics of order p on S^m (we usually suppress the domain dimension). The space of components V_f of a p-eigenmap $f : S^m \to S_V$ is a linear subspace of $\mathcal{H}^p$. f is *full*, i.e. f has no zero components (geometrically, the image of f is not contained in a proper great sphere of S_V), iff $V^* \cong V_f$. In addition, *congruent maps*, i.e. maps that differ by an isometry between the ranges, have the same space of components.

The universal example of a spherical minimal immersion of degree p is the *standard minimal immersion* $f_p : S^m \to S_{\mathcal{H}^p}$. It is uniquely defined (up to congruence) by the requirement that, relative to a (suitably scaled) L_2-orthonormal basis in $\mathcal{H}^p$ the components of f_p are orthogonal with the same norm. The scaled L_2-scalar product on $\mathcal{H}^p$ is given by

$$\langle \chi_1, \chi_2 \rangle = \frac{\dim \mathcal{H}^p}{\mathrm{vol}\,(S^m)} \langle \chi_1, \chi_2 \rangle_{L^2} = \frac{\dim \mathcal{H}^p}{\mathrm{vol}\,(S^m)} \int_{S^m} \chi_1 \chi_2 \, v_{S^m}, \ \chi_1, \chi_2 \in \mathcal{H}^p.$$

The components of the Veronese map have this property so that $Ver : S^2 \to S^4$ is actually the standard minimal immersion in degree 2. The following rigidity result is due to Calabi [2]:

THEOREM (CALABI, 1967). *For $m = 2$, any full p-eigenmap $f : S^2 \to S_V$ is (up to congruence) standard.*

The natural question, posed by DoCarmo and Wallach in the early seventies, is to what extent is the standard minimal immersion unique among all spherical minimal immersions of the same degree, and if nonuniqueness occurs, what is the structure of the corresponding moduli space.

For eigenmaps, unicity already fails in degree 2. Indeed, the Hopf map is a quadratic eigenmap obviously incongruent to the standard minimal immersion $f_3 : S^3 \to S_{\mathcal{H}^2} = S^8$.

Since the components of the Veronese and Hopf maps are L_2-orthonormal (up to scaling) it is natural to pose the following:

PROBLEM (R.T. SMITH, 1972). Classify all eigenmaps and spherical minimal immersions whose components are L_2-orthonormal (up to suitable scaling).

This problem is unsolved; for a list of nontrivial examples, see Smith [17].

In 1971 DoCarmo and Wallach [6, 29] proved rigidity of spherical minimal immersions for $p \leq 3$. The main aim of their work, however, was to show that for the remaining ranges unicity fails:

THEOREM (DOCARMO–WALLACH, 1971). (a) *For $m \geq 3$ and $p \geq 2$, the set of (congruence classes of) full p-eigenmaps $f : S^m \to S_V$ can be parametrized by a "moduli space" $\mathcal{L}^p$, a compact convex body in a subrepresentation $\mathcal{E}^p \subset S^2(\mathcal{H}^p)$ of $SO(m+1)$ of dimension* $\dim \mathcal{E}^p \geq \mathcal{E}^4 \geq 10$.

(b) *For $m \geq 3$ and $p \geq 4$, the set of (congruence classes of) full minimal isometric immersions $f : S^m \to S_V$ of degree p can be parametrized by a moduli space $\mathcal{M}^p$, a compact convex body in a subrepresentation $\mathcal{F}^p \subset S^2(\mathcal{H}^p)$ of $SO(m+1)$ of dimension* $\dim \mathcal{F}^p \geq \mathcal{F}^4 \geq 18$.

To get a first glimpse, we briefly elaborate on the parametrization and the moduli. The parametrization given below is not the original DoCarmo–Wallach parametrization (see [19]).

The parameter point $\langle f \rangle \in \mathcal{E}^p \subset S^2(\mathcal{H}^p)$ corresponding to a full p-eigenmap $f : S^m \to S_V$ is given as follows. With respect to an orthonormal basis in V, the set of components $\{f^j\}_{j=0}^n$ of f, $\dim V = n+1$, forms a basis in the space of components V_f. This basis can be made orthonormal using the Gram–Schmidt orthonormalization process. Now $\langle f \rangle + I$ is the orthogonal projection $\mathcal{H}^p \to V_f$ followed by the Gram transformation $G(f)$ of f (given by the Gram matrix) relative to the latter orthonormal basis in V_f.

Clearly, $\langle f \rangle$ is a symmetric endomorphism of $\mathcal{H}^p$, and $\langle f \rangle + I$ is positive semidefinite. Moreover, $\langle f\rangle$ depends only on the congruence class of f. The standard minimal immersion f_p corresponds to the origin. The image of the Do Carmo–Wallach parametrization $f \mapsto \langle f\rangle$ is the moduli space $\mathcal{L}^p$ for p-eigenmaps, and $\mathcal{M}^p$ for spherical minimal immersions of degree p.

Writing the conditions of sphericality (second equation in (2)) and homothety (3) in relative forms

$$|f|^2 = |f_p|^2, \tag{4}$$

and

$$\langle X^a \cdot f, X^b \cdot f\rangle = \langle X^a \cdot f_p, X^b \cdot f_p\rangle, \quad a, b \in \mathbb{R}^{m+1}, \tag{5}$$

and in terms of the parameter $\langle f\rangle$, we obtain homogeneous linear conditions on $S^2(\mathcal{H}^p)$. We denote the corresponding linear subspaces of $S^2(\mathcal{H}^p)$ by $\mathcal{E}^p$ and $\mathcal{F}^p$. We have, $\mathcal{F}^p \subset \mathcal{E}^p$. Finally, since $\langle f\rangle + I \geq 0$, the moduli for p-eigenmaps is

$$\mathcal{L}^p = \{C \in \mathcal{E}^p \,|\, C + I \geq 0\},$$

and the moduli for minimal immersions of degree p is

$$\mathcal{M}^p = \mathcal{L}^p \cap \mathcal{F}^p = \{C \in \mathcal{F}^p \,|\, C + I \geq 0\}.$$

The defining inequality $C+I \geq 0$ being a convex condition, both moduli are convex. A simple inspection of the eigenvalues of the participating elements shows that the moduli are also compact. In fact, $\mathcal{E}^p$ consists of traceless endomorphisms. Finally, for full $f : S^m \to S_V$, we have $\operatorname{rank}(\langle f\rangle + I) = \dim V$, so that the interior points of $\mathcal{L}^p$ and $\mathcal{M}^p$ correspond to maps with maximal range ($= \mathcal{H}^p$).

Problem (DoCarmo–Wallach, 1971). Determine $\dim \mathcal{L}^p$ and $\dim \mathcal{M}^p$.

In [6], DoCarmo and Wallach computed $\dim \mathcal{L}^p$, and, as noted above, gave a lower bound for $\dim \mathcal{M}^p$. They conjectured that the lower bound was the actual dimension of $\mathcal{M}^p$. This so-called "exact dimension conjecture" was resolved by the author in 1994 [22]. In what follows, we give a brief account on the solution. Note also that, for $m = 3$, $\dim \mathcal{M}^4 = 18$ was determined by Muto in 1984 [16], and recently, Weingart [30] gave an algebraic computation for $\dim \mathcal{M}^p$.

The standard minimal immersion $f_p : S^m \to S_{\mathcal{H}^p}$ is equivariant with respect to the homomorphism $\rho_p : SO(m+1) \to SO(\mathcal{H}^p)$ that defines $\mathcal{H}^p$ as an irreducible representation space for $SO(m+1)$, where the latter is given by $\chi \mapsto g\cdot\chi = \chi \circ g^{-1}$, $\chi \in \mathcal{H}^p$, $g \in SO(m+1)$. Equivariance means that

$$f_p \circ g = \rho_p(g) \circ f_p, \quad g \in SO(m+1).$$

The DoCarmo–Wallach parametrization is itself equivariant; $g \cdot \langle f\rangle = \langle f \circ g^{-1}\rangle$, where the representation of $SO(m+1)$ on $\mathcal{H}^p$ is extended to the full tensor algebra over $\mathcal{H}^p$, in particular, to $S^2(\mathcal{H}^p)$. Since the moduli $\mathcal{L}^p$ and $\mathcal{M}^p$ are $SO(m+1)$-invariant, their linear span $\mathcal{E}^p$ and $\mathcal{F}^p$ are subrepresentations of $S^2(\mathcal{H}^p)$. The problem of determining the dimensions $\dim \mathcal{L}^p = \dim \mathcal{E}^p$ and $\dim \mathcal{M}^p = \dim \mathcal{F}^p$ now becomes more tractable since all we need to do is to decompose $\mathcal{E}^p$ and $\mathcal{F}^p$ into irreducible components, a standard problem in representation theory (save the problem of translating sphericality and homothety into representation theoretical data).

Exact dimension of the moduli

Cartan's theory tells us that a complex irreducible representation of $SO(n)$ is determined by its highest weight vector υ. With respect to the standard maximal torus in $SO(n)$ (providing a coordinate system for the Cartan subalgebra), υ becomes an $[n/2]$-tuple $\upsilon = (\upsilon_1, \dots, \upsilon_{[n/2]}) \in \mathbf{Z}^{[n/2]}$, $[n/2] = \operatorname{rank}(SO(n))$. We denote this representation by V^υ. As an example, we have

$$\mathcal{H}^q = V^{(q,0,\dots,0)} \tag{6}$$

as complex representations of $SO(n)$. With this, our problem becomes twofold:

Problem I. What are the highest weights of the irreducible representations of $SO(m+1)$ that occur (with possible multiplicity) in $S^2(\mathcal{H}^p)$ or, more generally, in $\mathcal{H}^p \otimes \mathcal{H}^q$?

Problem II. Which of the highest weights in $S^2(\mathcal{H}^p)$ occur in the subrepresentations $\mathcal{E}^p$ and $\mathcal{F}^p$?

The Weyl dimension formula gives $\dim V^\upsilon$ in terms of the highest weight υ. Thus, once Problems I–II are solved, the Weyl dimension formula will give $\dim \mathcal{E}^p = \dim \mathcal{L}^p$ and $\dim \mathcal{F}^p = \dim \mathcal{M}^p$.

In 1971 Wallach gave an affirmative answer for Problem I [29]. In fact, with the notations above, we have

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{\substack{(u,v)\in \triangle_0^{p,q};\\ u+v\equiv p+q \pmod 2}} V^{(u,v,0,\dots,0)}, \quad p \geq q \geq 1,\ m \geq 3, \tag{7}$$

where $\triangle_0^{p,q}$ is the closed convex triangle in $\mathbb{R}^2$ with vertices $(p-q,0)$, (p,q) and $(p+q,0)$.

For $p=q$, $\mathcal{H}^p \otimes \mathcal{H}^p = S^2(\mathcal{H}^p) \oplus \wedge^2(\mathcal{H}^p)$, and the irreducible subrepresentations in the skew-symmetric part $\wedge^2(\mathcal{H}^p)$ have highest weights with odd integral components. We obtain

$$S^2(\mathcal{H}^p) = \sum_{\substack{(u,v)\in\triangle_0^p;\\ u,v\,\text{even}}} V^{(u,v,0,\dots,0)}, \tag{8}$$

where we set $\triangle_0^p = \triangle_0^{p,p}$.

For Problem II we need to make a slight detour. The space of homogeneous polynomials $\mathcal{P}^q$ of degree q is a representation space for $SO(m+1)$ with the same action as for the irreducible subrepresentation $\mathcal{H}^q \subset \mathcal{P}^q$. It is a classical fact that any homogeneous polynomial ξ of degree q can be written as $\xi = \chi + \rho^2 \cdot \xi_0$, where $\chi \in \mathcal{H}^q$, and $\xi_0 \in \mathcal{P}^{q-2}$ are uniquely determined. The linear map $H : \mathcal{P}^q \to \mathcal{H}^q$, $H(\xi) = \chi$, is called the *harmonic projection operator*. Iterating this procedure, we obtain

$$\mathcal{P}^q = \sum_{j=0}^{[q/2]} \mathcal{H}^{q-2j} \cdot \rho^{2j}.$$

Thus, by (6), as complex representations of $SO(m+1)$, we have

$$\mathcal{P}^q = \sum_{j=0}^{[q/2]} V^{(q-2j,0,\dots,0)}.$$

The key to encode sphericality of eigemaps into our representation theoretical framework is to consider the difference

$$\Psi_p^0(f) = |f|^2 - |f_p|^2 = \langle \langle f \rangle \cdot f_p, f_p \rangle$$

of the two sides of (4) as a linear map

$$\Psi_p^0 : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p} = \sum_{l=0}^{p} V^{(2l,0,\dots,0)}$$

defined by $\Psi_p^0(C) = \langle C \cdot f_p, f_p \rangle$, $C \in S^2(\mathcal{H}^p)$. Since the entire construction is $SO(m+1)$-equivariant, Ψ_p^0 is a homomorphism of $SO(m+1)$-representations. By definition, the kernel of Ψ_p^0 is $\mathcal{E}^p$. A little analysis shows that Ψ_p^0 is onto. Thus, we see that the irreducible subrepresentations of $S^2(\mathcal{H}^p)$ that do not occur in $\mathcal{E}^p$ correspond to the even lattice points along the base of the triangle $\triangle_0^p$ in (8):

$$\mathcal{E}^p = \sum_{\substack{(u,v)\in\triangle_1^p;\\ u,v\,\text{even}}} V^{(u,v,0,\dots,0)}, \tag{9}$$

where $\triangle_1^p$ is the closed convex triangle in $\mathbb{R}^2$ with vertices $(2,2)$, (p,p) and $(2(p-1),2)$.

We obtain the following result (contained in [6] with a different proof):

THEOREM (DoCarmo–Wallach, 1971). *The symmetric square $S^2(\mathcal{H}^p)$ contains $\mathcal{P}^{2p}$ as an $SO(m+1)$-submodule and*

$$\mathcal{E}^p \cong S^2(\mathcal{H}^p)/\mathcal{P}^{2p}. \tag{10}$$

In particular, we have

$$\begin{aligned}\dim \mathcal{L}^p &= \dim \mathcal{E}^p \\ &= \begin{pmatrix} \binom{p+m}{m} - \binom{p+m-2}{m} + 1 \\ 2 \end{pmatrix} - \binom{2p+m}{m}.\end{aligned} \tag{11}$$

Although technically more involved, encoding the condition of homothety into this framework follows the same lines. The difference

$$\begin{aligned}\Psi_p(f)(a,b) &= \langle X^a \cdot f, X^b \cdot f\rangle - \langle X^a \cdot f_p, X^b \cdot f_p\rangle \\ &= \langle X^a\langle f\rangle \cdot f_p, X^b \cdot f_p\rangle\end{aligned}$$

of the two sides of (5) can be viewed as a linear map

$$\Psi_p : \mathcal{E}^p \to \mathcal{P}^{2(p-1)} \otimes (\mathcal{H}^1 \times \mathcal{H}^1),$$

defined by

$$\Psi_p(C)(a,b) = \langle X^a C \cdot f_p, X^b \cdot f_p\rangle, \quad a, b \in \mathbb{R}^{m+1}.$$

Here we extended all data to homogeneous polynomials in $\mathbb{R}^{m+1}$ and identified $\mathcal{H}^1$ with the dual of $\mathbb{R}^{m+1}$. Ψ_p is bilinear, symmetric and traceless in the arguments $(a,b) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$. Since the traceless part of $S^2(\mathcal{H}^1) \subset \mathcal{H}^1 \otimes \mathcal{H}^1$ is isomorphic to $\mathcal{H}^2$ as representations of $SO(m+1)$, we arrive at

$$\Psi_p : \mathcal{E}^p \to \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2$$

As before, Ψ_p is a homomorphism of $SO(m+1)$-representations. By definition, the kernel of Ψ_p is $\mathcal{F}^p$. The range of Ψ_p decomposes as

$$\mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2 \cong \sum_{l=1}^{p-1} \mathcal{H}^{2l} \otimes \mathcal{H}^2$$

as representations of $SO(m+1)$. A quick comparison of this in the use of (7) with the decomposition of the domain $\mathcal{E}^p$ in (9) shows that the only common components of the domain and the range of Ψ_p are the representations

$$V^{(2l,2,\dots,0)}, \quad l = 1, \dots, p-1. \tag{12}$$

These correspond to the base of the triangle $\triangle_1^p$. We obtain

$$\mathcal{F}^p \supset \sum_{\substack{(u,v)\in\triangle_2^p;\\ u,v \text{ even}}} V^{(u,v,0,\dots,0)} \tag{13}$$

where $\triangle_2^p$ is the closed convex triangle in $\mathbb{R}^2$ with vertices $(4,4)$, (p,p) and $(2(p-2),4)$.

Coupled with the Weyl dimension formula, this gives (a different proof for) the DoCarmo–Wallach lower bound for $\dim \mathcal{M}^p$ noted above [6]. Notice that our proof gives the main result of DoCarmo–Wallach almost immediately from the setup.

To prove that the lower bound is sharp, one needs to show that Ψ_p is nonzero on the irreducible representations (12). This is technical, and it is accomplished by a double induction with respect to the domain dimension m and the degree p. The main tool in the general induction step is to use operators that associate to a p-eigenmap $f : S^m \to S_V$, a p-eigenmap with domain dimension $m+1$ (domain dimension raising operator), and $(p \pm 1)$-eigenmaps (degree raising and lowering operators). We give some details of the latter below.

Once we proved nonvanishing of Ψ on the components (12), we arrive at the decomposition

$$\mathcal{F}^p = \sum_{\substack{(u,v)\in\triangle_2^p;\\ u,v \text{ even}}} V^{(u,v,0,\dots,0)}. \tag{14}$$

As noted above, this decomposition now gives the dimension of the moduli $\mathcal{M}^p$, and resolves the exact dimension conjecture of DoCarmo–Wallach.

The clear analogy between eigenmaps and spherical minimal immersions suggests to consider the sum

$$\mathcal{F}^{p;k} = \sum_{\substack{(u,v)\in\triangle_{k+1}^p;\\ u,v \text{ even}}} V^{(u,v,0,\dots,0)}, \quad k = 1, \dots, [p/2]-1, \tag{15}$$

where $\triangle_{k+1}^p$ is the closed convex triangle in $\mathbb{R}^2$ with vertices $(2(k+1), 2(k+1))$, (p,p) and $(2(p-k-1), 2(k+1))$. The moduli $\mathcal{M}^{p;k} = \mathcal{M}^p \cap \mathcal{F}^{p;k}$, should parametrize "isotropic" spherical minimal immersions, where isotropy means that the k-th fundamental form of the immersions are "intrinsically the same" as the k-th fundamental form of the standard minimal immersion. This is indeed the case [11, 23].

For $m = 3$, the "equivariant moduli" $(\mathcal{L}^p)^{SU(2)}$ and $(\mathcal{M}^p)^{SU(2)}$ of $SU(2)$-equivariant eigenmaps and minimal immersions $f : S^3 \to S_V$ are distributed along the northwestern edge of the triangles $\triangle_1^p$ and $\triangle_2^p$. These moduli are perhaps the least subtle to analyze. We have

$$\dim(\mathcal{M}^p)^{SU(2)} = \dim(\mathcal{F}^p)^{SU(2)} = \left(2\left[\frac{p}{2}\right] + 5\right)\left(\left[\frac{p}{2}\right] - 1\right).$$

This formula was first derived by DeTurck–Ziller [4] using a "partially heuristic argument". The exact computation above is due to Tóth–Ziller [27].

In the case $m = 3$ and $p = 2$, the 5-dimensional equivariant moduli space $(\mathcal{L}^2)^{SU(2)}$ is the convex hull of a Veronese surface minimally imbedded into S^4.

For $m = 3$ and $p = 4$, the 9-dimensional equivariant moduli space $(\mathcal{M}^4)^{SU(2)}$ is the convex hull of an octahedral manifold minimally imbedded into S^8 plus some more complicated structure. In fact, for $m = 3$, the entire moduli $\mathcal{L}^2$ and $\mathcal{M}^4$ can be completely described (see Tóth [26] and Tóth–Ziller [27]).

The finer technical details of the proof of the exact dimension conjecture require a novel approach that uses *operators* defined on eigenmaps and spherical minimal immersions. The simplest of these are the *degree raising and lowering operators* that associate to a p-eigenmap $f : S^m \to S_V$ $(p \pm 1)$-eigenmaps $f^{\pm} : S^m \to S_{V \otimes \mathcal{H}^1}$ [25]. $f^{\pm}$ are defined by

$$f^+(a) = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}}\delta_a f \quad \text{and} \quad f^-(a) = \sqrt{\frac{2}{\lambda_{2p}}}\partial_a f, \quad a \in \mathbb{R}^{m+1},$$

where, as usual, $\mathcal{H}^1 = (\mathbb{R}^{m+1})^*$, and f is considered as a vector-valued spherical harmonic with values in V, and, in $\delta_a f = H(a^* f)$, $a^*(x) = \langle a, x\rangle$, $x \in \mathbb{R}^{m+1}$, the harmonic projection operator H acts on f componentwise. With respect to the standard basis $\{e_r\}_{r=0}^m \subset \mathbb{R}^{m+1}$, we have

$$f^+ = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}} \sum_{r=0}^m \delta_r f \otimes y_r \quad \text{and} \quad f^- = \sqrt{\frac{2}{\lambda_{2p}}} \sum_{r=0}^m \partial_r f \otimes y_r, \tag{16}$$

where, $e_r^* = y_r$, $r = 0, \ldots, m$, are the elements of the dual basis in $\mathcal{H}^1$. (We use here the variable $y \in \mathbb{R}^{m+1}$ to distinguish it from the natural variable x of f.) Note that $f^{\pm}$ may not be full even if f is.

It turns out that $f^{\pm}$ are spherical so that the definitions above make sense.

Between the moduli spaces, the correspondences $f \mapsto f^{\pm}$ give rise to homomorphisms $\Phi^{\pm} : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p\pm 1})$ of $SO(m+1)$-representations. They satisfy

$\Phi^\pm(\mathcal{L}^p) \subset \mathcal{L}^{p\pm 1}$ and $\Phi^\pm(\mathcal{M}^p) \subset \mathcal{M}^{p\pm 1}$. In terms of the decomposition (8), Φ^+ is injective and corresponds to the inclusion $\triangle_0^p \subset \triangle_0^{p+1}$, and Φ^- is surjective with kernel distributed along the northeastern edge of $\triangle_0^p$. Injectivity of Φ^+ [22] gives the folowing:

Equivariant Imbedding Theorem (Tóth, 1997). *The correspondence $f \mapsto f^+$ gives rise to $SO(m+1)$-equivariant imbeddings of the moduli spaces: $\mathcal{L}^p$ into $\mathcal{L}^{p+1}$, and $\mathcal{M}^p$ into $\mathcal{M}^{p+1}$.*

Notice that $\mathcal{F}^4$ is the whole kernel of Φ^-. This means that f^-, for any quartic spherical minimal immersion $f : S^m \to S_V$, is congruent to the standard minimal immersion, in particular

$$\dim V \otimes \mathcal{H}^1 \geq \dim \mathcal{H}^3.$$

Hence we have the following [22]:

Theorem (Tóth, 1997). *Let $f : S^m \to S_V$ be a quartic ($p = 4$) spherical minimal immersion. Then we have*

$$\dim V \geq \frac{m(m+5)}{6}.$$

Notice that the lower bound is quadratic in m, a significant improvement of Moore's linear bound. It is not known whether the quadratic lower bound is sharp. In fact, very few examples of spherical minimal immersions are known for domain dimension $m \geq 4$.

Most of the developments here can be naturally extended to isometric minimal immersions $f : M \to S_V$ of a compact isotropy irreducible Riemannian homogeneous manifold M.

Problem. Derive the dimension formula for the moduli of isometric minimal immersions $f : M \to S_V$, where M is a compact rank one symmetric space.

For lower bounds on the dimension of certain moduli of isometric minimal immersions $f : \mathbf{C}P^m \to S_V$, see [26]. Note that the representation theory for the case $M = \mathbf{C}P^m$ is simpler than the spherical case due to the Littlewood-Richardson rule.

The operator of infinitesimal rotations

A general operator associating to a p-eigenmap a q-eigemap is determined by an irreducible component W of $\mathcal{H}^p \otimes \mathcal{H}^q$, and is described by polynomials in the two sets of commuting variables $\partial_0, \ldots, \partial_m$ and $\delta_0, \ldots, \delta_m$. The operator reflects the symmetries of the Young tableau of W [24].

A prominent example is the *operator of infinitesimal rotations* that associates to a p-eigenmap $f : S^m \to S_V$ a p-eigenmap $\hat{f} : S^m \to S_{V \otimes so(m+1)^*}$ whose components are obtained by infinitesimally rotating the components of f in the coordinate planes [20, 21]. More generally, given a closed subgroup $G \subset SO(m+1)$ that acts transitively on S^m, for a p-eigenmap $f : S^m \to S_V$ we can define a p-eigenmap $\hat{f} : S^m \to S_{V \otimes \mathcal{G}^*}$, where $\mathcal{G}$ is the Lie algebra of G, as follows:

$$\hat{f}(x)(X) = \frac{1}{\sqrt{\lambda_p}} X_x(f), \quad X \in \mathcal{G}, \qquad x \in S^m.$$

Given an orthonormal basis $\{X_s\}_{s=1}^n \subset \mathcal{G}$ with dual basis $\{\phi_s\}_{s=1}^n \subset \mathcal{G}^*$, we have

$$\hat{f} = \frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^n X_s(f) \otimes \phi_s. \tag{17}$$

Note that $\hat{f}$ may not be full even if f is.

The following result [21] describes the correspondence $f \mapsto \hat{f}$ on the moduli:

Theorem (Tóth, 1999). (i) *For a p-eigenmap $f : S^m \to S_V$, $\hat{f} : S^m \to S_{V \otimes \mathcal{G}^*}$ is spherical, and hence a p-eigenmap.*

(ii) *On the congruence classes, the correspondence $f \mapsto \hat{f}$ gives rise to a self-map of $\mathcal{L}^p$ which is the restriction of a symmetric linear map $\mathcal{A}_p : \mathcal{E}^p \to \mathcal{E}^p$.*

(iii) *All eigenvalues of $\mathcal{A}_p$ are real and contained in $[-1, 1]$.*

(iv) *$\mathcal{A}_p$ is an endomorphism of the G-representation $\mathcal{E}^p|G$. The eigenspace of $\mathcal{A}_p$ corresponding to the eigenvalue $+1$ is the fixed point set $(\mathcal{E}^p)^G$. The eigenspace of $\mathcal{A}_p$ corresponding to the eigenvalue -1 is contained in the orthogonal complement of $(\mathcal{E}^p_m)^G$ in $(\mathcal{E}^p_m)^{[G,G]}$. In particular, -1 is not an eigenvalue if G is semisimple.*

$\mathcal{A}_p$ is also a self-map of $\mathcal{M}^p$. The eigenvalues of $\mathcal{A}_p$ on the irreducible components of $\mathcal{E}^p$ can be computed since, up to a suitable affine transformation, $\mathcal{A}_p$ is essentially given by the Casimir operator [20]. In fact, we have

$$\mathcal{A}_p|\mathcal{E}^p = I - \frac{1}{2\lambda_p} \, Cas, \tag{18}$$

where $Cas = -\mathrm{trace}\,\{(X, Y) \to [X, [Y, .]]\}$ is the Casimir operator of G acting on $\mathcal{E}^p$. In fact, the eigenvalue of $\mathcal{A}_p$ on the irreducible component $V^{(u,v,0,\ldots,0)} \subset \mathcal{F}^p$ is

$$\Lambda_p^{u,v} = 1 - \frac{\mu^{u,v}}{2\lambda_p},$$

where

$$\mu^{u,v} = u^2 + v^2 + u(m-1) + v(m-3)$$

is the eigenvalue of the Casimir operator on $V^{(u,v,0,\dots,0)}$.

$\mathcal{A}_p$ is a contraction on $\mathcal{L}^p$, and suitable (and computable) iterates of $\mathcal{A}_p$ bring all boundary points of $\mathcal{L}^p$ into the interior of $\mathcal{L}^p$. Since the interior corresponds to maximal range dimension, we obtain a variety of lower bounds for the range of the original spherical minimal immersions [20]:

THEOREM (TÓTH, 2000). *Let $f : S^m \to S_V$ be a full spherical minimal immersion of degree p, and assume that the corresponding point in the moduli $\mathcal{M}^p$ is contained in a sum of irreducible components of $\mathcal{F}^p|G$ with $d \geq 2$ distinct eigenvalues of $\mathcal{A}_p$. Then, we have*

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim \mathcal{U}^d(\mathcal{G})} = \frac{\dim \mathcal{H}^p}{\binom{\dim G + d}{d}},$$

where $\mathcal{U}(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$, and $\mathcal{U}^d(\mathcal{G})$ is the linear subspace of elements of degree $\leq d$.

Once again, for quartic minimal immersions $f : S^m \to S_V$, $m \geq 4$, we obtain $\dim V \geq (m+2)(m+7)/12$, a quadratic lower bound that is slightly weaker than the one we obtained above.

The structure of the moduli

The inclusion relation among the space of components defines a cell-decomposition of the moduli. Each cell is a compact convex set that has nonempty interior in its affine span. The cell containing the point corresponding to an eigenmap (resp. spherical minimal immersion) $f : S^m \to S_V$ is called the *relative moduli of f*, and it is denoted by $\mathcal{L}_f$ (resp. $\mathcal{M}_f$). Clearly, $\mathcal{L}_{f_p} = \mathcal{L}^p$ and $\mathcal{M}_{f_p} = \mathcal{M}^p$. The interior points of each relative moduli $\mathcal{L}_f$ and $\mathcal{M}_f$ (within the affine span) correspond to maps whose space of components are maximal ($= V_f$) within the relative moduli.

The cell-decomposition is $SO(m+1)$-equivariant. Apart from the cases $m = 3$ and $p = 2$ (eigenmaps), or $m = 3$ and $p = 4$ (spherical minimal immersions), very little is known about the cell-structures of $\mathcal{L}^p$ and $\mathcal{M}^p$.

PROBLEM (HARD). Describe the cell-structure of the boundary of the moduli $\mathcal{L}^p$ and $\mathcal{M}^p$ in general.

A p-eigenmap $f : S^m \to S_V$ is called *linearly rigid* if $\mathcal{L}_f$ is one point. This notion is due to Wallach [29]. Similarly, a spherical minimal immersion $f : S^m \to S_V$ of degree p is linearly rigid if $\mathcal{M}_f$ is trivial. The extremal points of the moduli (as a convex set) in $\mathcal{L}^p$ and in $\mathcal{M}^p$ correspond to linearly rigid p-eigenmaps and linearly rigid spherical minimal immersions of degree p. By the Krein-Milman theorem [1], the convex hull of the extremal points gives the whole moduli. Thus, the linearly rigid eigenmaps and spherical minimal immersions determine the structure of the moduli. For example, for $m = 3$ and $p = 2$, the extremal points in $\mathcal{L}^2$ fill two Veronese surfaces minimally imbedded into a pair of orthogonal 4-spheres of common radius $2\sqrt{3}$. For $m = 3$ and $p = 4$, the minimum range dimension (linearly rigid) spherical minimal immersions correspond to points in two octahedral manifolds minimally imbedded into a pair of orthogonal 8-spheres of common radius $5\sqrt{3/2}$, but there are other linearly rigid spherical minimal immersions whose range dimension is not minimal.

CONJECTURE. The points in $\mathcal{M}^p$ furthermost from the origin correspond to spherical minimal immersions with least range dimension. (This is true for $m = 3$ and $p = 4$ by explicit computation, cf. Weingart [30].)

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