INFINITESIMAL ROTATIONS OF ISOMETRIC MINIMAL IMMERSIONS BETWEEN SPHERES

By Gabor Toth

Abstract. A fundamental problem posed by DoCarmo and Wallach is to give lower bounds for the codimension of isometric minimal immersions between round spheres. For a given domain dimension and degree, the moduli space of such immersions is a compact convex body in a representation space for a Lie group of isometries acting transitively on the domain. Infinitesimal isometric deformations of these minimal immersions give rise to a linear contraction on the moduli, and the eigenvalues of the contraction are related to the Casimir eigenvalues on the irreducible components of the ambient representation space. The study of these eigenvalues leads to new sharp lower bounds for the codimension of the immersions and gives an insight to the subtleties of the boundary of the moduli.

1. Preliminaries and statement of results. Let $f: S^m(k) \to S_V$, $m \geq 2$, be an isometric minimal immersion of the Euclidean m-sphere $S^m(k)$ of constant curvature k into the unit sphere S_V of a Euclidean vector space V. A result of Takahashi [12] asserts that f exists iff $k = k_p = m/\lambda_p$ for some $p \geq 1$, where $\lambda_p = p\,(p+m-1)$ is the pth eigenvalue of the Laplace operator $\triangle^{S^m}$ on $S^m = S^m(1)$. (λ_p and k_p also depend on m. To simplify the notation, unless important, we will suppress the dependence of various objects on m.)

Let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G} \subset so(m+1)$. Each $X \in \mathcal{G}$ defines an infinitesimal isometry on $S^m(k)$ also denoted by X. Assume that G acts on $S^m(k)$ transitively. (For the list of transitive Lie groups on spheres, see [21].) Given an isometric minimal immersion $f: S^m(k_p) \to S_V$, in [15] the author proved that the map $\hat{f}: S^m(k_p) \to V \otimes \mathcal{G}^*$ defined by

$$\hat{f}(x)(X) = \frac{1}{\sqrt{\lambda_p}} X_x \cdot f, \quad x \in S^m(k_p),\ X \in \mathcal{G}, \tag{1}$$

has image in the unit sphere of $V \otimes \mathcal{G}^*$, and gives rise to an isometric minimal immersion $\hat{f}: S^m(k_p) \to S_{V\otimes\mathcal{G}^*}$. The purpose of this paper is to show that the study of the self-map $f \mapsto \hat{f}$ on the space of all isometric minimal immersions of $S^m(k_p)$ (for fixed m and p) into S_V (for various V) gives new sharp lower bounds for the codimension of these immersions; this is a fundamental problem posed by DoCarmo and Wallach [4, 18] in the early 1970s. In addition, this self-map provides a glimpse of the subtleties of the space of minimal immersions as well.

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Let $f\colon S^m(k_p) \to S_V$, $m \geq 2$, be an isometric minimal immersion. For uniformity, we scale the metric on the domain to curvature one and call $f\colon S^m \to S_V$ a *spherical minimal immersion of degree p*. Due to the scaling, f becomes homothetic with homothety λ_p/m:

$$\langle f_*(Y), f_*(Z)\rangle = \frac{\lambda_p}{m}\langle Y, Z\rangle, \tag{2}$$

for any vector fields Y, Z on S^m. Again by Takahashi [12], each component $\phi \circ f$, $\phi \in V^*$, is a spherical harmonic of order p on S^m, an eigenfunction of Δ^{S^m} with eigenvalue λ_p. Let $\mathcal{H}^p = \mathcal{H}^p_m$ denote the space of spherical harmonics of order p on S^m. Precomposing linear functionals on V by f defines a linear map $V^* \to \mathcal{H}^p$ whose image $V_f \subset \mathcal{H}^p$ is called the *space of components* of f. This linear map establishes an isomorphism $V^* \cong V_f$ iff f is *full*, i.e., if the image of $f\colon S^m \to S_V$ spans V. Note that any map into a sphere (such as $\hat{f}$ in (1)) can be made full by restricting its image to its linear span.

The *standard minimal immersion* $f_p = f_{m,p}\colon S^m \to S_{\mathcal{H}^p}$ of degree p is defined by the requirement that its components are orthonormal relative to an orthonormal basis in $\mathcal{H}^p$. Here $\mathcal{H}^p$ is endowed with the scaled L_2-scalar product

$$\langle \chi_1, \chi_2\rangle = \frac{N(m,p)+1}{\mathrm{vol}(S^m)}\int_{S^m} \chi_1\chi_2\, v_{S^m}, \quad \chi_1, \chi_2 \in \mathcal{H}^p, \tag{3}$$

where

$$N(m,p) + 1 = \dim \mathcal{H}^p = (m + 2p - 1)\frac{(m+p-2)!}{p!(m-1)!},$$

v_{S^m} is the volume form of S^m, and $\mathrm{vol}(S^m) = \int_{S^m} v_{S^m}$ is the volume of S^m. f_p is unique up to congruence, where two maps $f_1\colon S^m \to S_{V_1}$ and $f_2\colon S^m \to S_{V_2}$ are said to be *congruent* if $f_2 = U \circ f_1$ for some isometry $U\colon V_1 \to V_2$. For any full spherical minimal immersion $f\colon S^m \to S_V$, we have $V_f \subset V_{f_p} = \mathcal{H}^p$, and there is a unique surjective linear map $A\colon \mathcal{H}^p \to V$ such that $f = A \circ f_p$. The DoCarmo-Wallach parametrization [4] of the space of all full spherical minimal immersions $f\colon S^m \to S_V$ of degree p associates to f the symmetric endomorphism $\langle f\rangle = A^\top \cdot A - I$ of $\mathcal{H}^p$. Clearly, $\langle f\rangle \in S^2(\mathcal{H}^p)$ depends only on the congruence class of f. Fixing m and p, DoCarmo and Wallach proved that the correspondence $f \mapsto \langle f\rangle$ gives rise to a parametrization of the space of congruence classes of all full spherical minimal immersions $f\colon S^m \to S_V$ of degree p. Moreover the *moduli space* $\mathcal{M}^p = \mathcal{M}^p_m$ (the image of the parametrization) is a compact convex body in a linear subspace $\mathcal{F}^p = \mathcal{F}^p_m$ of the symmetric square $S^2(\mathcal{H}^p)$. Since $A^\top A$ is always positive semidefinite, $\mathcal{M}^p$ is the linear slice of the semi-algebraic convex set $\{C \in S^2(\mathcal{H}^p) \mid C + I \text{ positive semidefinite}\}$ by the linear subspace $\mathcal{F}^p$. Since f is full, $\dim V = \dim V_f = \mathrm{rank}(\langle f\rangle + I)$ so that the interior points of $\mathcal{M}^p$

correspond to full spherical minimal immersions with maximal range dimension $\dim V = N(m,p) + 1$.

$\mathcal{H}^p$ has a natural $SO(m+1)$-module structure given by $g \in SO(m+1)$ acting as $g \cdot \chi = \chi \circ g^{-1}$, $\chi \in \mathcal{H}^p$. This action extends to the full tensor algebra over $\mathcal{H}^p$, in particular, to $S^2(\mathcal{H}^p)$. The standard minimal immersion f_p: $S^m \to S_{\mathcal{H}^p}$ is equivariant with respect to the homomorphism ρ_p: $SO(m+1) \to SO(\mathcal{H}^p)$ that defines the $SO(m+1)$-module structure on $\mathcal{H}^p$. The extension of the action of $SO(m+1)$ on $S^2(\mathcal{H}^p)$ is given by $g \cdot C = \rho_p(g) \circ C \circ \rho_p(g)^{-1}$, $C \in S^2(\mathcal{H}^p)$. This action leaves $\mathcal{M}^p$ and thereby $\mathcal{F}^p$, its linear span, invariant; in fact, the action of $g \in SO(m+1)$ on $\langle f \rangle \in \mathcal{M}^p$ is given by $g \cdot \langle f \rangle = \langle f \circ g^{-1} \rangle$. Thus $\mathcal{F}^p$ becomes an $SO(m+1)$-submodule of $S^2(\mathcal{H}^p)$.

According to a result of DoCarmo and Wallach in [4], for $p \geq q$, the (complexification of the) tensor product $\mathcal{H}^p \otimes \mathcal{H}^q$ decomposes into irreducible components as

$$(4) \qquad \mathcal{H}^p \otimes \mathcal{H}^q \cong \sum_{(a,b) \in \triangle^{p,q};\, a+b \equiv p+q \pmod 2} V^{(a,b,0,\ldots,0)}.$$

Here $\triangle^{p,q} \subset \mathbf{R}^2$ denotes the closed convex triangle with vertices $(p-q, 0)$, (p,q) and $(p+q, 0)$, and $V^{(u_1,\ldots,u_r)} = V_{m+1}^{(u_1,\ldots,u_r)}$, $r = \operatorname{rank}(SO(m+1)) = [\frac{m+1}{2}]$, stands for the complex irreducible $SO(m+1)$-module with highest weight vector $(u_1, \ldots, u_r) \in (\mathbf{Z}/2)^r$ (relative to the standard maximal torus in $SO(m+1)$). For $m = 3$, $V_4^{(u_1,u_2)}$, $u_2 > 0$, actually means $V_4^{(u_1,u_2)} \oplus V_4^{(u_1,-u_2)}$. For $p = q$, excising the skew-symmetric part, we obtain the irreducible decomposition

$$(5) \qquad S^2(\mathcal{H}_m^p) \cong \sum_{(a,b) \in \triangle_0^p;\, a,b \text{ even}} V^{(a,b,0,\ldots,0)},$$

where $\triangle_0^p = \triangle^{p,p} \subset \mathbf{R}^2$ denotes the closed convex triangle with vertices $(0,0)$, (p,p) and $(2p,0)$. (In what follows, unless stated otherwise, we denote an absolutely irreducible representation and its complexification by the same symbol.) The so called DoCarmo-Wallach problem of determining $\mathcal{F}^p$ thus reduces to finding the irreducible components of $S^2(\mathcal{H}^p)$ in (5) that belong to $\mathcal{F}^p$.

In [2], Calabi proved that every full spherical minimal immersion f: $S^2 \to S_V$ of degree p is congruent to the standard minimal immersion f_p. In [4], DoCarmo and Wallach showed that this is also the case for spherical minimal immersions f: $S^m \to S_V$ of degree $p \leq 3$. This means that, for $m = 2$ or $p \leq 3$, the moduli space $\mathcal{M}^p$ and therefore $\mathcal{F}^p$ reduces to a point. In addition, DoCarmo and Wallach also proved that in the remaining cases $m \geq 3$ and $p \geq 4$, the moduli space $\mathcal{M}^p$ is nontrivial by exhibiting a lower bound for the dimension of $\mathcal{F}^p$. In [14], sharpening the DoCarmo-Wallach result, the author proved that, for

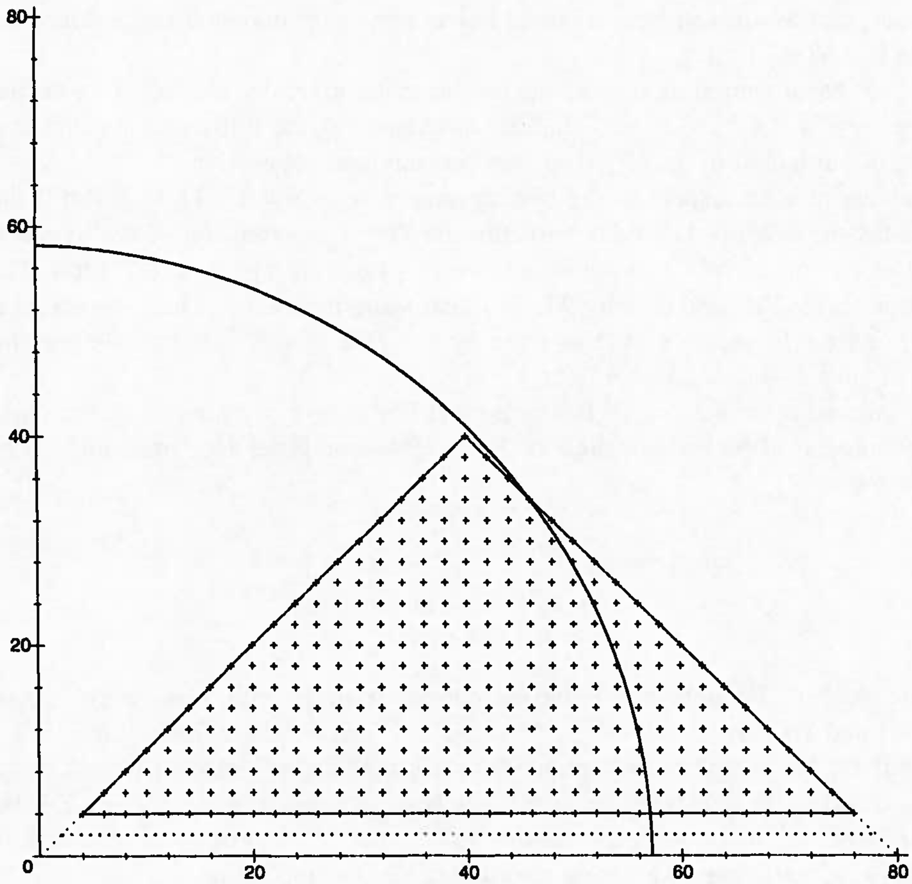

Figure 1.

$m \geq 3$ and $p \geq 4$, $\mathcal{F}^p$ decomposes as

$$\mathcal{F}^p \cong \sum_{(a,b)\in\triangle_2^p;\, a,b \text{ even}} V^{(a,b,0,\dots,0)}, \tag{6}$$

where $\triangle_2^p \subset \triangle_0^p$ denotes the closed convex triangle with vertices $(4,4)$, (p,p) and $(2(p-2),4)$. (See Figure 1 for $p = 40$ and without the circular arc.)

If we replace in (2) the differential f_* of a spherical minimal immersion $f\colon S^m \to S_V$ of degree p by the higher fundamental forms $\beta_l(f)$, $l = 1,\dots,k$, of f, $k \leq p$, we arrive at the concept of isotropic minimal immersion of degree p and order of isotropy k. A spherical minimal immersion $f\colon S^m \to S_V$ of degree p is *isotropic with order of isotropy* k, $1 \leq k \leq p$, if, for $l = 1,\dots,k$, we have

$$\langle \beta_l(f)(Y_1,\dots,Y_l), \beta_l(f)(Z_1,\dots,Z_l)\rangle = \langle \beta_l(f_p)(Y_1,\dots,Y_l), \beta_l(f_p)(Z_1,\dots,Z_l)\rangle,$$

for all vector fields $Y_1, \ldots, Y_l, Z_1, \ldots, Z_l$ on S^m. (For details, see [8, 18].) Isotropy of order 1 is just homothety. In [8], Gauchman and the author proved that if $p \leq 2k+1$ then a full isotropic minimal immersion of degree p and order of isotropy k is congruent to the standard minimal immersion. In addition, for $p \geq 2(k+1)$, the space $\mathcal{M}^{p;k}$ of congruence classes of full isotropic minimal immersions of degree p and order of isotropy k corresponds, under the DoCarmo-Wallach parametrization, to the linear slice of $\mathcal{M}^p$ by the $SO(m+1)$-submodule $\mathcal{F}^{p;k}$ of $S^2(\mathcal{H}^p)$ whose complexification decomposes as

$$\mathcal{F}^{p;k} \cong \sum_{(a,b)\in\triangle^p_{k+1};\, a,b \text{ even}} V^{(a,b,0,\ldots,0)}, \tag{7}$$

where $\triangle^p_{k+1} \subset \mathbf{R}^2$ is the closed convex triangle with vertices $(2(k+1), 2(k+1))$, (p,p) and $(2(p-k-1), 2(k+1))$. (This was proved in [8] for $m \geq 4$. The proof can be extended to cover the case $m = 3$ as well. Recently, Weingart [20] gave an independent algebraic proof for all $m \geq 3$.) In particular, for $m \geq 4$, p even, and $k = p/2 - 1$ (maximal), $\mathcal{F}^{p,p/2-1} \cong V^{(p,p,0,\ldots,0)}$ is irreducible.

A map $f\colon S^m \to S_V$ is said to be a *p-eigenmap* if the components of f are spherical harmonics of order p on S^m. The DoCarmo-Wallach parametrization extends to eigenmaps. The congruence classes of all full p-eigenmaps $f\colon S^m \to S_V$ are parametrized by a moduli space $\mathcal{L}^p = \mathcal{L}^p_m$, a compact convex body in a linear subspace $\mathcal{E}^p = \mathcal{E}^p_m$ of $S^2(\mathcal{H}^p)$. (We have $\mathcal{F}^p \subset \mathcal{E}^p$; the linear slice $\mathcal{M}^p$ of $\mathcal{L}^p$ is obtained by imposing (2) on eigenmaps.) This parametrization of eigenmaps is implicitly contained in the work of DoCarmo and Wallach. For $m = 2$ or $p = 1$, $\mathcal{L}^p$ is trivial (the former is just a reformulation of the result of Calabi cited above), and for $m \geq 3$ and $p \geq 2$, the complexification of $\mathcal{E}^p$ decomposes as

$$\mathcal{E}^p \cong \sum_{(a,b)\in\triangle^p_1;\, a,b \text{ even}} V^{(a,b,0,\ldots,0)}. \tag{8}$$

We thus have the chain of moduli

$$\mathcal{L}^p = \mathcal{M}^{p;0} \supset \mathcal{M}^p = \mathcal{M}^{p;1} \supset \cdots \supset \mathcal{M}^{p;[p/2]-1},$$

each of which is a compact convex body obtained by intersecting $\mathcal{L}^p$ with the corresponding $SO(m+1)$-submodule in the chain of $SO(m+1)$-submodules of $S^2(\mathcal{H}^p)$:

$$\mathcal{E}^p = \mathcal{F}^{p;0} \supset \mathcal{F}^p = \mathcal{F}^{p;1} \supset \cdots \supset \mathcal{F}^{p;[p/2]-1}$$

given by restricting the summation in (8) to the triangles

$$\triangle^p_1 \supset \triangle^p_2 \supset \cdots \supset \triangle^p_{[p/2]-1}.$$

For a uniform treatment, we agree that isotropy of order 0 means that $f\colon S^m \to S_V$ is a p-eigenmap.

We now return to the initial setting. Let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G} \subset so(m+1)$, and assume that G acts on S^m transitively. Given a spherical minimal immersion $f\colon S^m \to S_V$ of degree p, formula (1) defines the spherical minimal immersion $\hat{f}\colon S^m \to S_{V\otimes\mathcal{G}^*}$. Since congruence is preserved under the correspondence $f \mapsto \hat{f}$, we obtain a self-map $\mathcal{A}_p = \mathcal{A}_{m,p}$ of the moduli $\mathcal{M}^p$ by setting $\mathcal{A}_p(\langle f\rangle) = \langle \hat{f}\rangle$. In Section 3 we derive another formula for $\mathcal{A}_p$ which shows that $\mathcal{A}_p$ is the restriction of a linear self-map of $S^2(\mathcal{H}^p)$ to $\mathcal{M}^p$. In fact, according to a result in [15], this map is a symmetric endomorphism of the $SO(m+1)$-module $S^2(\mathcal{H}^p)$. We denote this extension by the same symbol. $\mathcal{A}_p$ leaves the compact convex body $\mathcal{L}^p$ invariant, and hence the eigenvalues of $\mathcal{A}_p$ are contained in $[-1,1]$. Finally, the eigenspace of $\mathcal{A}_p$ corresponding to the eigenvalue +1 is $(\mathcal{E}^p)^G = \mathrm{Fix}_G(\mathcal{E}^p)$, and the eigenspace corresponding to the eigenvalue -1 is contained in the orthogonal complement of $(\mathcal{E}^p)^G$ in $(\mathcal{E}^p)^{[G,G]}$. In particular, for $m \geq 4$ and $G = SO(m+1)$, the eigenvalues of $\mathcal{A}_p$ on $\mathcal{E}^p$ are contained in $(-1,1)$ so that $\mathcal{A}_p$ is a contraction on $\mathcal{L}^p$.

Our first result is the following:

THEOREM 1. (a) *Let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G}$, and assume that G acts transitively on S^m. Then $\mathcal{A}_p$ maps $\mathcal{F}^{p;k}$ into itself, $k = 0,\ldots,[p/2]-1$, or equivalently, if $f\colon S^m \to S_V$ is an isotropic minimal immersion of degree p and order of isotropy k then so is $\hat{f}\colon S^m \to S_{V\otimes\mathcal{G}^*}$. On $\mathcal{E}^p$, we have*

$$\mathcal{A}_p = I - \frac{1}{2\lambda_p}\, Cas, \tag{9}$$

where $Cas = -\mathrm{trace}\,\{(X,Y) \to [X,[Y,.]]\}$ is the Casimir operator of G acting on $\mathcal{E}^p \subset S^2(\mathcal{H}^p)$.

(b) *For $G = SO(m+1)$ and $m \geq 4$, the eigenvalue $\Lambda_p^{a,b} = \Lambda_{m,p}^{a,b}$ of $\mathcal{A}_p$ on the irreducible component $V^{(a,b,0,\ldots,0)} \subset \mathcal{E}^p$, $(a,b) \in \triangle_1^p$, a, b even, is*

$$\Lambda_p^{a,b} = 1 - \frac{\mu^{a,b}}{2\lambda_p}, \tag{10}$$

where

$$\mu^{a,b} = \mu_m^{a,b} = a^2 + b^2 + a(m-1) + b(m-3) \tag{11}$$

is the eigenvalue of the Casimir operator on $V^{(a,b,0,\ldots,0)}$.

Remark 1. As noted above, for $k = 0$ and $k = 1$, the first statement in part (a) of Theorem 1 was proved in [15] covering the cases of eigenmaps and spherical minimal immersions.

Remark 2. The Casimir eigenvalue for the basic representations can be computed in terms of the highest weight and the Cartan matrix (for a comprehensive account, see the article by Wang and Ziller [19], especially formula (2.5) therein; note that their conventions are somewhat different from the ones we adopted here (essentially taken from Wallach), as they write a representation in terms of its dominant weight as an integral linear combination of dominant fundamental weights). Thus (11) can be derived using only representation theory. In our proof of Theorem 1 given in Section 3, we derive (10)–(11) as a byproduct of the (fairly technical) proof of the first statement, the preservance of isotropy under $\mathcal{A}_p$. We thus obtain an independent computation for the Casimir eigenvalues (11).

Regarding a and b as variables, the equation $\Lambda_p^{a,b} = 0$ defines the circle

$$\left(a + \frac{m-1}{2}\right)^2 + \left(b + \frac{m-3}{2}\right)^2 = 2\lambda_p + \left(\frac{m-1}{2}\right)^2 + \left(\frac{m-3}{2}\right)^2. \tag{12}$$

$\Lambda_p^{a,b} > 0$ iff the even lattice point $(a, b) \in \triangle_1^p$ is inside this circle. Figures 1–2 depict the situations for $m = 4$, $p = 40$, and $m = 1000$, $p = 40$.

THEOREM 2. *Let $f\colon S^m \to S_V$, $m \geq 4$, be a full spherical minimal immersion of degree p. Assume that $\langle f \rangle$ is contained in an irreducible component $V^{(a,b,0,\ldots,0)}$ of $\mathcal{F}^p$ with $\Lambda_p^{a,b} \geq 0$. Then, we have*

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim so(m+1)} = \frac{2(m+2p-1)(m+p-2)!}{p!(m+1)!}. \tag{13}$$

Remark 1. (a) If $\Lambda_p^{a,b} < 0$ in Theorem 2, then either $\langle \hat{f} \rangle$ is an interior point of $\mathcal{M}^p$, in which case the lower estimate (13) applies, or $\langle \hat{f} \rangle$ is a boundary point of $\mathcal{M}^p$. In the latter case the Connecting Lemma in [16] gives the weaker estimate

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim so(m+1) + 1} = \frac{2(m+2p-1)(m+p-2)!}{p!(m-1)!(m(m+1)+2)}$$

(cf. [15]).

(b) If $\langle f \rangle$ is *not* contained in an irreducible component of $\mathcal{F}^p$ then the dth power of $\mathcal{A}_p$ sends $\langle f \rangle$ into the interior of $\mathcal{M}^p$, where d is the number of distinct eigenvalues of $\mathcal{A}_p$ on the submodule of $\mathcal{F}^p$ that contains $\langle f \rangle$. This follows from Theorem 4 in [15]. Since the components of the minimal immersion corresponding to $(\mathcal{A}_p)^d(\langle f \rangle)$ are obtained by applying monomials of degree $\leq d$ of infinitesimal isometries to the components of f, we obtain

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim \mathcal{U}^d(so(m+1)) - 1} = \frac{\dim \mathcal{H}^p}{\begin{pmatrix} \frac{m(m+1)}{2} + d \\ d \end{pmatrix} - 1},$$

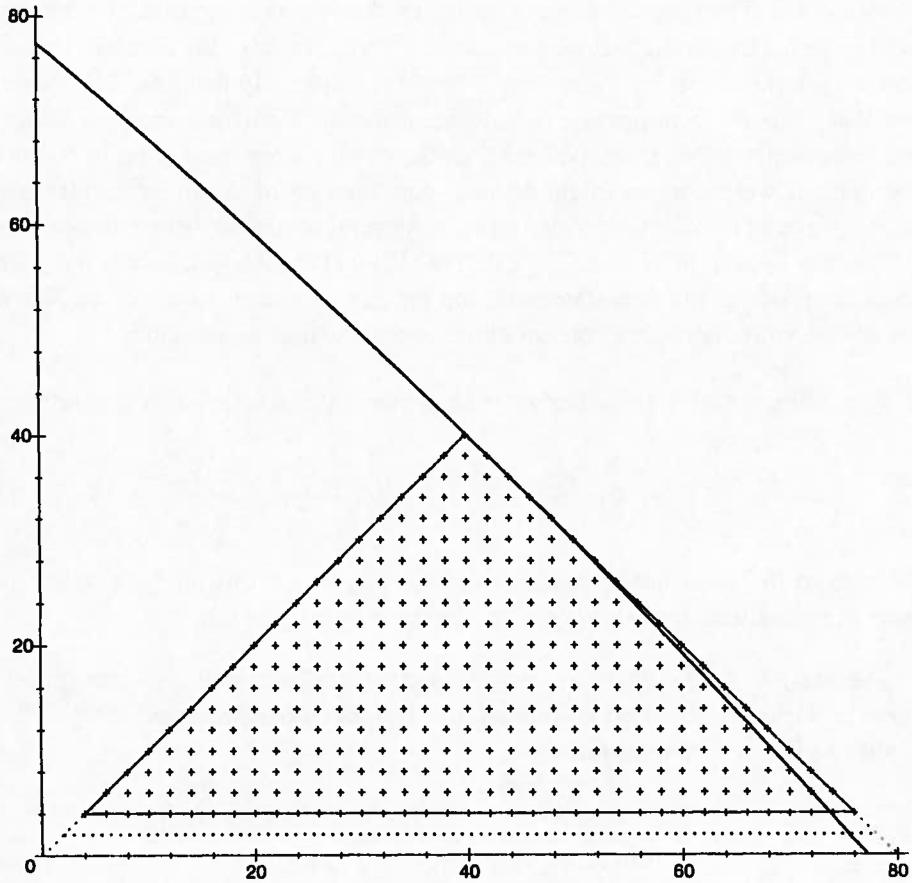

Figure 2.

where $\mathcal{U}(so(m+1))$ is the universal enveloping algebra of $so(m+1)$, and $\mathcal{U}^d(so(m+1))$ is the linear subspace of elements of degree $\leq d$.

Remark 2. Some eigenvalues of $\mathcal{A}_p$ may well be zero on $\mathcal{F}^p$. For example, $\Lambda^{a,b}_{m,p} = 0$ ($(a,b) \in \Delta^p_2$, a, b even) *for all* $m \geq 4$, iff

$$a = e(e+2), \quad b = e^2, \text{ and } p = e(e+1) \tag{14}$$

for some e even. Since the origin in $\mathcal{F}^p$ corresponds to the standard minimal immersion, we obtain that for a full spherical minimal immersion $f\colon S^m \to S_V$, $m \geq 4$, of degree $e(e+1)$, e even, with $\langle f \rangle \in V^{(e(e+2),e^2,0,\ldots,0)}$, the minimal immersion $\hat{f}\colon S^m \to S_{V\otimes so(m+1)^*}$ (made full) is standard, i.e. congruent to $f_{e(e+1)}$. The points in (14) correspond to the northeast edge of the triangle Δ^p_2 since $a+b = e(e+2)+e^2 = 2e(e+1) = 2p$. Finally, note that $\Lambda^{a,b}_{m,p}$ may vanish *for some* $m \geq 4$ for (a,b) inside Δ^p_2, for example $\Lambda^{18,4}_{23,12} = 0$.

The circle (12) intersects the line $a + b = 2p$ in two points

$$\left(p - \frac{1}{2} \mp \sqrt{p + \frac{1}{4}}, p + \frac{1}{2} \pm \sqrt{p + \frac{1}{4}}\right),$$

where the lower sign gives the intersection point on the northeast side of the triangle Δ_1^p. In particular, the northern vertex (p, p) is always inside the circle (12), or equivalently, for p even, $\Lambda_p^{p,p}$ is positive. Isotropy of order $k = p/2 - 1$, p even, imposed on a full spherical minimal immersion $f: S^m \to S_V$ of degree p, guarantees that $\langle f \rangle$ lies in the irreducible component $\mathcal{F}^{p,p/2-1} \cong V^{(p,p,0,\dots,0)}$. Theorem 2 now gives:

COROLLARY 1. *Let $m \geq 4$ and p even. If $f: S^m \to S_V$ is an isotropic minimal immersion of degree p and order of isotropy $p/2 - 1$ then the lower estimate* (13) *holds.*

For $p = 4$ isotropy becomes redundant, and we obtain:

COROLLARY 2. *Let $m \geq 4$. For a quartic spherical minimal immersion $f: S^m \to S_V$, we have*

$$\dim V \geq \frac{(m+2)(m+7)}{12}. \tag{15}$$

A result of Moore [11] asserts that for any spherical minimal immersion $f: S^m \to S_V$, we have $\dim V \geq 2m$. (15) replaces this linear estimate with a quadratic lower bound in the quartic case as well as improves the first result of the author [15] in this direction.

Remark. Rescaling the original metric on S^m, Corollary 2 states that the sphere $S^m(\frac{m}{4(m+3)})$ of curvature $\frac{m}{4(m+3)}$ (the first nontrivial admissible case) cannot be isometrically immersed into S^n for $n < \frac{(m+2)(m+7)}{12} - 1$.

Theorem 2 automatically extends to the case when $\langle f \rangle$ lies in the convex hull of slices of $\mathcal{M}^p$ with irreducible components on which $\mathcal{A}_p$ acts with nonnegative eigenvalues. The problem of giving suitable lower bounds on the codimension of spherical minimal immersions thus amounts to studying how far the moduli $\mathcal{M}^p$ is from this convex hull.

The case $m = 3$ deserves special attention since it is not covered by part (b) of Theorem 1, and Theorem 2. This case is also unique since S^3 is itself a Lie group. The orthogonal group $SO(4)$ splits as

$$SO(4) = SU(2) \cdot SU(2)', \tag{16}$$

where $SU(2) \cap SU(2)' = \{\pm I\}$ and $SU(2)'$ is a conjugate of $SU(2)$ in $SO(4)$. The following notation will be useful in the sequel: If W is an $SU(2)$-module then W' denotes the $SU(2)'$-module obtained from W by conjugating first $SU(2)'$ back to $SU(2)$ in $SO(4)$, and then applying the $SU(2)$-module structure W. In addition, if $-I$ acts on W' trivially, then W' is also an $SO(4)$-module with trivial action of $SU(2)$ on W'. The notation is analogous when the roles of $SU(2)$ and $SU(2)'$ are switched.

The complex irreducible $SU(2)$-modules are parametrized by the dimension; we denote by W_p the irreducible $SU(2)$-module with $\dim W_p = p+1$. We realize W_p as the $SU(2)$-module of homogeneous polynomials of degree p in two complex variables. The tensor product $W_r \otimes W_{r'}$, $r \geq r' \geq 0$, decomposes as

$$W_r \otimes W_{r'} = \sum_{t=0}^{r'} W_{r+r'-2t}. \tag{17}$$

In view of the splitting (16), the $SO(4)$-module of (complex valued) spherical harmonics $\mathcal{H}_3^p$ can be written as

$$\mathcal{H}_3^p = W_p \otimes W_p'. \tag{18}$$

As $SO(4)$-modules:

$$V_4^{(u,u)} = W_{2u} \oplus W_{2u}'.$$

For p even, W_p is the complexification of a real $SO(m+1)$-submodule R_p. For p odd, W_p is irreducible as a real $SO(m+1)$-module. For p even, we have (with obvious notations):

$$\mathcal{H}_3^p \cong R_p \otimes R_p', \quad \mathcal{H}_3^p|_{SU(2)} = (p+1)R_p, \quad V_4^{(u,u)} = R_{2u} \oplus R_{2u}' \tag{19}$$

as real modules. (For details, see Section 1.5 in [16].)

The $SU(2)$-equivariant eigenmaps and spherical minimal immersions are parametrized by the "$SU(2)$-equivariant moduli"

$$(\mathcal{L}_3^p)^{SU(2)} = \mathcal{L}_3^p \cap (\mathcal{E}_3^p)^{SU(2)} \quad \text{and} \quad (\mathcal{M}_3^p)^{SU(2)} = \mathcal{M}_3^p \cap (\mathcal{F}_3^p)^{SU(2)}.$$

By (6), (8) and (19), as real modules

$$(\mathcal{E}_3^p)^{SU(2)} = \sum_{k=1}^{[p/2]} R'_{4k} \quad \text{and} \quad (\mathcal{F}_3^p)^{SU(2)} = \sum_{k=2}^{[p/2]} R'_{4k}.$$

In [16] Ziller and the author proved that the linear slice of $\mathcal{L}_3^p$ with the $SO(4)$-submodule

$$\sum_{k=1}^{[p/2]} V_4^{(2k,2k)} = (\mathcal{E}_3^p)^{SU(2)} \oplus (\mathcal{E}_3^p)^{SU(2)'} = \sum_{k=1}^{[p/2]} (R'_{4k} \oplus R_{4k})$$

corresponding to the even lattice points along the northwest edge of $\triangle_1^p$ is the *convex hull of the two equivariant moduli* $(\mathcal{L}_3^p)^{SU(2)}$ and $(\mathcal{L}_3^p)^{SU(2)'}$. (The proof in [16] is given only for the slice of the moduli with $V_4^{(2k,2k)}$, but it immediately extends to this more general situation.) In particular, in the lowest nonrigid range $m = 3$ and $p = 2$, $\triangle_1^p$ reduces to the single point $(2,2)$: $\mathcal{E}_3^2 = R'_4 \oplus R_4$. The moduli $\mathcal{L}_3^2$ is the convex hull of the linear slices $(\mathcal{L}_3^2)^{SU(2)}$ and $(\mathcal{L}_3^2)^{SU(2)'}$. Based on this, a complete geometric description of $\mathcal{L}_3^2$ is given in [13].

In the lowest nonrigid range $p = 4$ for spherical minimal immersions, we have $\mathcal{F}_3^4 = R'_8 \oplus R_8$, and $\mathcal{M}_3^4$ is the convex hull of the equivariant moduli $(\mathcal{M}_3^4)^{SU(2)}$ and $(\mathcal{M}_3^4)^{SU(2)'}$. The structure of $\mathcal{M}_3^4$ is subtle; a full analysis of $\mathcal{M}_3^4$ is given in [16].

The "equivariant construction," first used by Mashimo [10] in the context of spherical minimal immersions, and subsequently exploited by DeTurck and Ziller [3] provides a general method to manufacture $SU(2)$-equivariant spherical minimal immersions $f: S^3 \to S_V$. If V is a multiple of W_p then the orbit map $f_\xi: S^3 \to S_V$ of a polynomial $\xi \in V$ of unit length automatically gives a p-eigenmap. Homothety imposed on f_ξ gives finitely many quadratic conditions on the coefficients of ξ. Any solution of this system provides an example of an $SU(2)$-equivariant spherical minimal immersion f_ξ. For p even, replacing W_p by R_p allows one to reduce the range dimension of f_ξ. If, in addition, ξ is an invariant of a finite subgroup K of S^3 then one obtains a minimal imbedding of a 3-dimensional spherical space form into a sphere. (K is either cyclic, or the binary group of a regular spherical tessellation [21]. ξ is what Klein called an absolute invariant of K, see [9] for a classical exposition.) Solving the quadratic system for homothety within the invariants, in [3], DeTurck and Ziller showed that every homogeneous spherical space form (such as the homogeneous lens spaces, the dihedral, tetrahedral, octahedral, and icosahedral manifolds) can be minimally imbedded into spheres. Explicit examples given by Escher [6] show that homogeneity for lens spaces can in some cases be dispensed with. The tetrahedral, octahedral, and icosahedral invariants are given by

$$\xi_{Tet} = c_0 ab(a^4 - b^4), \quad \xi_{Oct} = c_1(a^8 + 14a^4b^4 + b^8), \quad \xi_{Ico} = c_2(a^{11}b + 11a^6b^6 - ab^{11}),$$

where we used the complex variables $a, b \in \mathbf{C}$ and

$$c_0 = \frac{1}{4\sqrt{15}}, \quad c_1 = \frac{1}{96\sqrt{21}}, \quad c_2 = \frac{1}{3600\sqrt{11}}.$$

The orbit maps of ξ_{Tet}, ξ_{Oct} and ξ_{Ico} define the tetrahedral, octahedral, and icosahedral minimal immersions

$$Tet\colon S^3 \to S_{R_6}, \quad Oct\colon S^3 \to S_{R_8}, \quad Ico\colon S^3 \to S_{R_{12}}.$$

Factored through the respective binary groups, these minimal immersions give minimal imbeddings of the corresponding space forms into spheres. The minimal imbeddings obtained this way have the smallest codimension within the class of $SU(2)$-equivariant minimal immersions.

The tetrahedral immersion *Tet*: $S^3 \to S_{R_6}$ discovered by Mashimo [10] (its mapping properties recognized later by DeTurck and Ziller [3]) provides an example for a spherical minimal immersion of degree 6 where Moore's linear estimate is sharp. DeTurck and Ziller showed that *Tet* is rigid among all $SU(2)$-equivariant minimal immersions. It is an open question whether it is actually rigid in the class of *all* minimal immersions. Most recently, Escher [7] proved that *Tet* is rigid among all minimal immersions in the infinitesimal sense, i.e., if *Tet* is subjected to an infinitesimal deformation through minimal immersions then the deformation is obtained from infinitesimal isometric deformations on both the domain and the range.

For $G = SU(2)$, the Casimir eigenvalue on W_p is $p(p+2)$ [19]. By Theorem 1, for p even and $k = 1, \ldots, p/2$, the eigenvalue of $\mathcal{A}_{3,p}$ on $R'_{4k} \subset (\mathcal{E}_3^p)^{SU(2)}$ is $1 - \frac{4k(2k+1)}{p(p+2)}$. (For $p = 2, 4$ and $k = 1, 2$, these values have been obtained in [15] by explicit computations.) Let $f\colon S^3 \to S_V$ be a full $SU(2)$-equivariant minimal immersion of degree p. Since the eigenvalues of $\mathcal{A}_{3,p}$ on the $SU(2)$-irreducible components of $(\mathcal{F}_3^p)^{SU(2)}$ are distinct, the smallest $\mathcal{A}_{3,p}$-invariant linear subspace $\mathcal{A}_f$ that contains $\langle f \rangle$ is at most $(p/2 - 1)$-dimensional. The intersection $\mathcal{S}_f = \mathcal{A}_f \cap \mathcal{M}_3^p$ is of importance since it admits a simpler geometric description than the whole moduli. In our next result we will describe $\mathcal{S}_f$ for $f = Tet$ and $f = Oct$, the tetrahedral and the octahedral minimal immersions. If $\langle f \rangle$ is contained in one of the $SU(2)$-irreducible components of $(\mathcal{F}_3^p)^{SU(2)}$, say R'_{4k}, $k = 2, \ldots, p/2$, then this slice collapses to a segment. In this case, if the eigenvalue of $\mathcal{A}_{3,p}$ is positive on R'_{4k} then the proof of Theorem 2 gives $\dim V \geq (p+1)^2/3$. Otherwise, the weaker estimate $\dim V \geq (p+1)^2/4$ holds. In both cases V is a multiple of R_p so that $\dim V$ is divisible by $p+1$. In particular, for a full $SU(2)$-equivariant

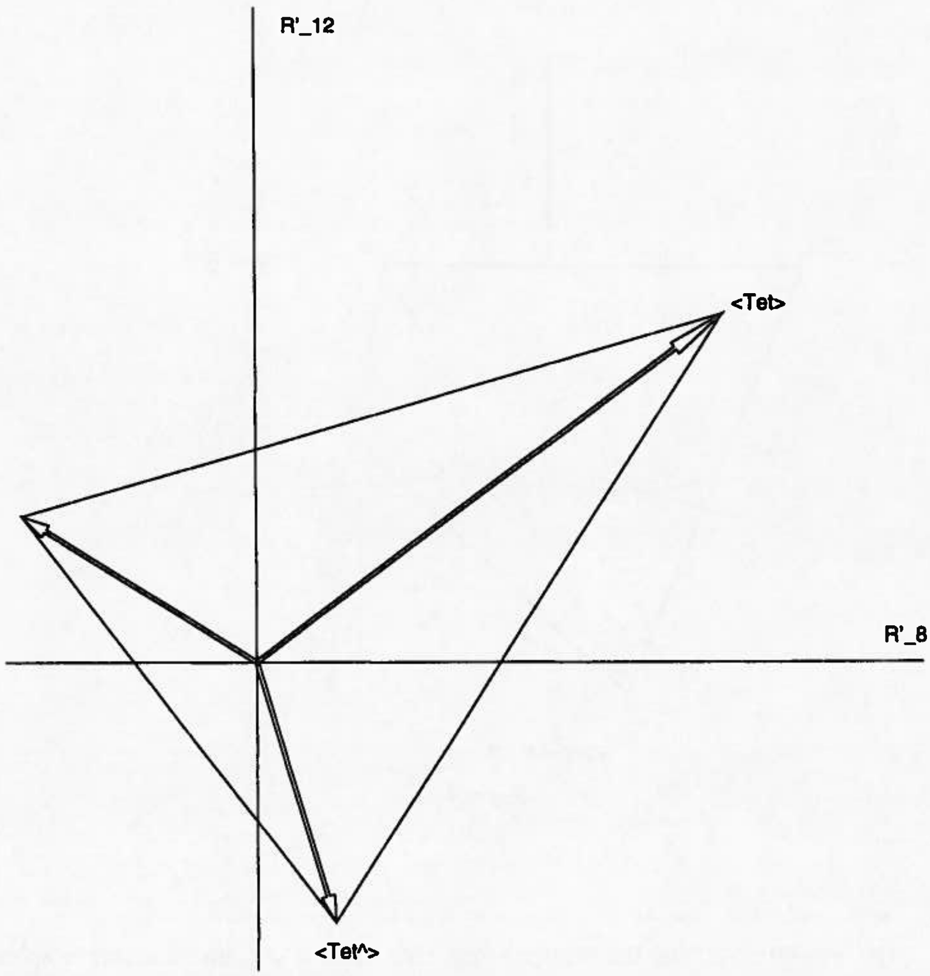

Figure 3.

minimal immersion f: $S^3 \to S_V$ of degree 6, if $\langle f \rangle \in R'_8$ then $\dim V \geq 14$, and if $\langle f \rangle \in R'_{12}$ then $\dim V \geq 21$. In any case, it is then clear that the weaker lower bound 14 applies to any point in the convex hull of the slices of $\mathcal{M}_3^6$ by R'_8 and R'_{12} corresponding to a full $SU(2)$-equivariant minimal immersion of degree 6. We conclude that the tetrahedral minimal immersion *Tet*: $S^3 \to S_{R_6}$ is *not* in the convex hull of the slices of the moduli $(\mathcal{M}_3^6)^{SU(2)}$ by irreducible $SU(2)'$-components.

THEOREM 3. (a) $\dim \mathcal{A}_{Tet} = 2$, *and* $\mathcal{S}_{Tet}$ *is an isosceles triangle with vertex at* $\langle Tet \rangle$ *and one endpoint of the base at* $\langle \widehat{Tet} \rangle$ *(cf. Figure 3). The other endpoint on the base corresponds to a minimal immersion with range* $3R_6$.

(b) $\dim \mathcal{A}_{Oct} = 3$, *and* $\mathcal{S}_{Oct}$ *is a tetrahedron with one vertex at* $\langle Oct \rangle$ *and another at* $\langle \widehat{Oct} \rangle$ *with range* $3R_8$ *(cf. Figure 4). The other two vertices correspond to ranges* $2R_8$ *and* $3R_8$.

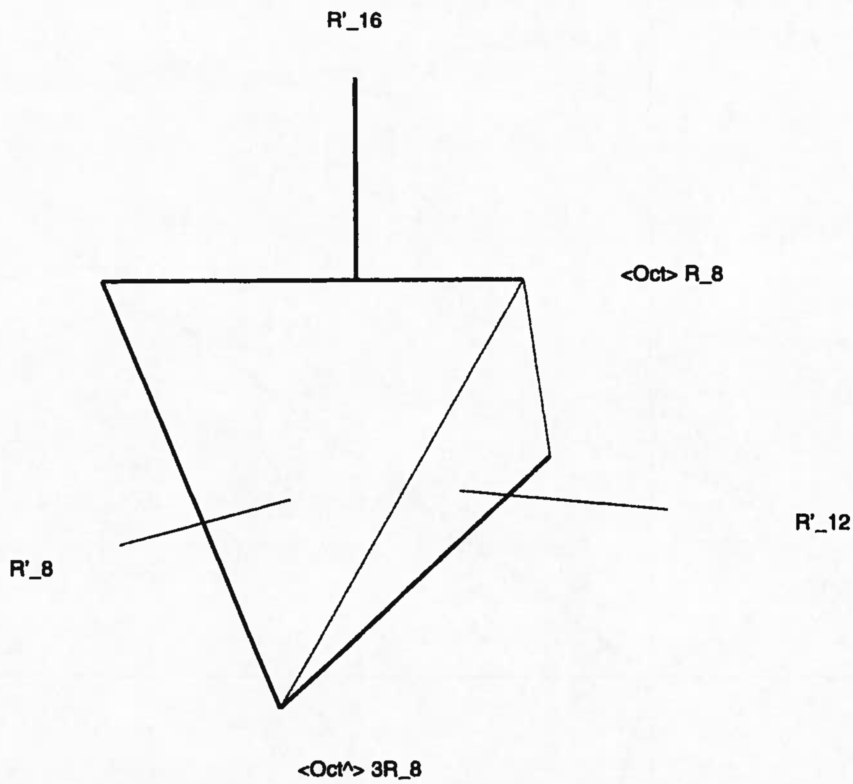

Figure 4.

The triangle $\mathcal{S}_{Tet}$ has the property that side/base = $\sqrt{2}$, and the origin splits the altitude from $\langle Tet\rangle$ in the ratio 1:6. The interior points of the sides correspond to minimal immersions with range $4R_6$, while the points at the interior of the base to range $6R_6$. In $\mathcal{S}_{Oct}$ the ranges of minimal immersions corresponding to the interior points of the edges and the faces are contained in the last two tables at the end of Section 4.

COROLLARY. *There exists a full $SU(2)$-equivariant isotropic minimal immersion $S^3 \to S^{27}$ of degree 6 and order of isotropy 2. Also there exist full $SU(2)$-equivariant isotropic minimal immersions $S^3 \to S^{53}$ of degree 8 and order of isotropy 2 and 3.*

An explicit description of the tetrahedron in Theorem 3 as well as the isotropic minimal immersions in the corollary will be given in Section 4.

Based on analogy, it may be reasonable to conjecture that $\mathcal{A}_{Ico}$ is 4-dimensional and $\mathcal{S}_{Ico}$ is a pentatope. Most recently Weingart proved, however, that $\dim \mathcal{A}_{Ico} = 3$ and $\mathcal{S}_{Ico}$ is a tetrahedron. The computations are facilitated by his observation that all matrices in $\mathcal{A}_{Ico}$ (and also in $\mathcal{A}_{Tet}$ and $\mathcal{A}_{Oct}$) commute.

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2. Operators on spherical minimal immersions. We motivate the concept of operator by a simple example. Let $f\colon S^m \to S_V$ be a p-eigenmap. We define a map $f^-\colon \mathbf{R}^{m+1} \to V \otimes \mathcal{H}^1$ as follows. Since $\mathcal{H}^1 = (\mathbf{R}^{m+1})^*$, for $a \in \mathbf{R}^{m+1}$, $f^-(a)$ should be a vector valued function on S^m with values in V. We set

$$f^-(a) = \sqrt{\frac{2}{\lambda_{2p}}}\,\partial_a f, \quad a \in \mathbf{R}^{m+1},$$

where the directional derivative ∂_a acts on f componentwise: $\phi \circ \partial_a f = \partial_a(\phi \circ f)$, $\phi \in V^*$. (As usual, we identify a spherical harmonic of order p on S^m with its extension to $\mathbf{R}^{m+1}$ as a harmonic polynomial of degree p.) With respect to the standard basis $\{e_r\}_{r=0}^m \subset \mathbf{R}^{m+1}$, we have

$$f^- = \sqrt{\frac{2}{\lambda_{2p}}}\sum_{r=0}^m \partial_r f \otimes y_r,$$

where $\{y_r\}_{r=0}^m \subset \mathcal{H}^1$ is the dual basis. Clearly, the components of f^- are spherical harmonics of order $p-1$ on S^m. f^- is thus a $(p-1)$-eigenmap iff it maps into the unit sphere $S_{V\otimes\mathcal{H}^1}$. Actually, according to a result in [8], f^- is an isotropic minimal immersion of degree $p-1$ and order of isotropy k iff f is an isotropic minimal immersion of degree p and order of isotropy k.

The correspondence $f \mapsto f^-$ can be conveniently expressed in terms of the $SO(m+1)$-module homomorphism $D^-\colon \mathbf{R}^{m+1} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p-1}$ defined by

$$D^-(a) = \sqrt{\frac{2}{\lambda_{2p}}}\,\partial_a, \quad a \in \mathbf{R}^{m+1}.$$

We call D^- the *degree lowering operator*.

Let $G \subset SO(m+1)$ be a closed subgroup, $\mathcal{G} \subset so(m+1)$ the Lie algebra of G, and assume that G acts on S^m transitively. Let W be an orthogonal G-module (a representation space for G with a G-invariant scalar product; irreducibility is *not* assumed). We call a homomorphism $D\colon W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ of G-modules (by restriction) an *operator*. For $a \in W$, $D(a)\colon \mathcal{H}^p \to \mathcal{H}^q$ is a linear map. For simplicity, we write $D_a = D(a)$, $a \in W$. Since D is a homomorphism we have

$$D_{g\cdot a} = g \cdot D_a = \rho_p(g) \circ D_a \circ \rho_p(g)^{-1}, \quad g \in G,$$

where $\rho_p\colon SO(m+1) \to SO(\mathcal{H}^p)$ is the $SO(m+1)$-module structure on $\mathcal{H}^p$.

Taking transposes, D determines (and is determined by) a homomorphism $\iota = \iota^D \colon \mathcal{H}^q \to \mathcal{H}^p \otimes W$ of G-modules, where

$$\iota^D(\chi') = \sum_{r=0}^{n} D_{e_r}^{\top} \chi' \otimes e_r, \quad \chi' \in \mathcal{H}^q,$$

and

$$(\iota^D)^{\top}(\chi \otimes e) = D_e \chi, \quad \chi \in \mathcal{H}^p,$$

with $\{e_r\}_{r=0}^n \subset W$ an orthonormal basis.

An operator D is called *metric* if ι^D is an isometric imbedding, or equivalently, if $(\iota^D)^{\top} \circ \iota^D$ is the identity on $\mathcal{H}^q$. If $G = SO(m+1)$ then, up to a constant multiple, any nontrivial operator is metric; this is because $\mathcal{H}^q$ is irreducible as an $SO(m+1)$-module.

The degree lowering operator $D^- \colon \mathbf{R}^{m+1} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p-1}$ is an operator with $G = SO(m+1)$, and $W = \mathbf{R}^{m+1}$ with its standard $SO(m+1)$-module structure given by matrix multiplication. We claim that D^- is metric. We first note that the transpose of ∂_a is

$$\partial_a^{\top} \chi' = p \frac{2p+m-1}{p+m-2} \delta_a \chi', \quad \chi' \in \mathcal{H}_m^{p-1}, \tag{20}$$

where

$$\delta_a \chi' = H(a^* \chi') = a^* \chi' - \frac{\rho^2}{2p+m-3} \partial_a \chi'. \tag{21}$$

Here $a^* \in \mathcal{H}_m^1$ is defined by $a^*(x) = \langle a, x \rangle$, $x \in \mathbf{R}^{m+1}$, H is the harmonic projection operator [17], and (20) follows by integration. Letting $\iota^- = \iota_{D^-}$, we have

$$\iota^-(\chi') = \frac{2p+m-3}{p+m-2} \sqrt{\frac{p}{p+m-2}} \sum_{r=0}^{m} \delta_r \chi' \otimes y_r,$$

where $\{y_r\}_{r=0}^m \subset \mathcal{H}^1$ is the standard orthonormal basis. Differentiating (21), $(\iota^-)^{\top} \circ \iota^-$ can easily be verified to be the identity on $\mathcal{H}^{p-1}$.

Let $D \colon W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ be a metric operator. Given a p-eigenmap $f \colon S^m \to S_V$, we define $f^D \colon S^m \to V \otimes W^*$ as follows. For $a \in W$, $f^D(a)$ should be a vector valued function on S^m with values in V. We set $f^D(a) = D_a f$. Here, $D_a \colon \mathcal{H}^p \to \mathcal{H}^q$ is applied to the vector valued spherical harmonic f of order p, and gives a vector valued spherical harmonic of order q in a natural way: $\phi \circ D_a f = D_a(\phi \circ f)$, $\phi \in V^*$. In terms of an orthonormal basis $\{e_r\}_{r=0}^n \subset W$ and its dual basis $\{\phi_r\}_{r=0}^n \subset W^*$, we have $f^D = \sum_{r=0}^n D_{e_r} f \otimes \phi_r$. Our present problem is to study under what circumstances will f^D map into the unit sphere $S_{V \otimes W^*}$, and when will f^D be a spherical minimal immersion assuming that f is. Our first lemma asserts that, for metric D, $(f_p)^D$: $S^m \to \mathcal{H}^p \otimes W^*$ (made full) is congruent to f_q: $S^m \to S_{\mathcal{H}^q}$.

Lemma 2.1. *We have*

$$(f_p)^D(x) = \iota^D(f_q(x)), \quad x \in S^m. \tag{22}$$

Proof. Let $\{f_p^j\}_{j=0}^{N(m,p)} \subset \mathcal{H}^p$ be an orthonormal basis. With the previous notations, we compute

$$\begin{aligned}
(f_p)^D(x) &= \sum_{r=0}^{n} \sum_{j=0}^{N(m,p)} (D_{e_r} f_p^j)(x) \cdot f_p^j \otimes \phi_r \\
&= \sum_{r=0}^{n} \sum_{j,l=0}^{N(m,p)} f_q^l(x) \langle D_{e_r} f_p^j, f_q^l \rangle f_p^j \otimes \phi_r \\
&= \sum_{r=0}^{n} \sum_{l=0}^{N(m,p)} f_q^l(x) D_{e_r}^\top (f_q^l) \otimes \phi_r \\
&= \sum_{r=0}^{n} D_{e_r}^\top (f_q(x)) \otimes \phi_r \\
&= \iota^D(f_q(x)).
\end{aligned}$$

The lemma follows.

Let f: $S^m \to S_V$ be a full p-eigenmap with $\langle f \rangle = A^\top A - I \in S^2(\mathcal{H}^p)$, where A: $\mathcal{H}^p \to V$ is the unique surjective linear map satisfying $f = Af_p$. Applying a metric operator D: $W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ to both sides of the equality $f = Af_p$, for $x \in S^m$, we have

$$f^D(x) = (A \otimes I)(f_p)^D(x) = (A \otimes I)\iota^D(f_q(x)),$$

where, in the last equality, we used (22). Since D is metric, we thus obtain

$$\langle f^D \rangle = (\iota^D)^\top \circ (A^\top A - I) \circ \iota^D + I = (\iota^D)^\top \circ (\langle f \rangle \otimes I) \circ \iota^D.$$

This motivates us to define Φ^D: $S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^q)$ by

$$\Phi^D(C) = (\iota^D)^\top \circ (C \otimes I) \circ \iota^D, \quad C \in S^2(\mathcal{H}^p). \tag{23}$$

By the previous computation, we have

$$\Phi^D(\langle f\rangle) = \langle f^D\rangle. \tag{24}$$

Φ^D: $S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^q)$ is a homomorphism of G-modules. By (24), given a p-eigenmap f: $S^m \to S_V$, f^D will map into the unit sphere $S_{V\otimes W^*}$ and thereby f^D will become a q-eigenmap, if $\Phi^D(\mathcal{E}^p) \subset \mathcal{E}^q$. In a similar vein, if f is a spherical minimal immersion, then so is f^D if $\Phi^D(\mathcal{F}^p) \subset \mathcal{F}^q$. We arrive at the following:

PROPOSITION 2.2. *Let $G \subset SO(m+1)$ be a closed subgroup acting transitively on S^m. Let D: $W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ be a metric operator, and $r = \max(p, q)$. Assume that $S^2(\mathcal{H}^r)|G$ has multiplicity 1 decomposition into irreducible G-modules. Then, for any spherical minimal immersion f: $S^m \to S_V$ of degree p, f^D: $S^m \to S_{V\otimes W^*}$ is a spherical minimal immersion of degree q. The statement is also true for isotropic minimal immersions, including eigenmaps.*

Remark. In view of (5), for $G = SO(m+1)$, the conditions of Proposition 2.2 are satisfied.

The degree lowering operator applied to a full p-eigenmap f: $S^m \to S_V$, gives the map $f^- = f^{D^-}$: $S^m \to S_{V\otimes\mathcal{H}^1}$ defined at the beginning of this section. Proposition 2.2 asserts that f^- is a p-eigenmap; in fact, degree lowering preserves minimality and isotropy. As usual, we have the extension $\Phi^- = \Phi^{D^-}$: $S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p-1})$ that is a homomorphism of $SO(m+1)$-modules. One of the main results in [14] asserts that Φ^- is surjective, so that its kernel must consist of the sum of the $SO(m+1)$-modules

$$V^{(2(p-k),2k,0,\ldots,0)}, \quad k = 1, \ldots, [p/2]$$

corresponding to the even lattice points along the northeast edge of $\triangle_1^p$. Note that the restriction $\Phi^-|_{\mathcal{L}^p}$: $\mathcal{L}^p \to \mathcal{L}^{p-1}$, $\Phi^-(\langle f\rangle) = \langle f^-\rangle$ is, in general, not surjective.

The *degree raising operator* D^+: $\mathbf{R}^{m+1} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p+1}$ is defined analogously by

$$D_a^+ = \sqrt{\frac{2p+m-1}{p+m-1}}\,\delta_a, \quad a \in \mathbf{R}^{m+1},$$

where δ_a is defined in (21). A simple computation shows that D^+ is metric. As before, degree raising preserves minimality and isotropy. Setting $\iota^+ = \iota^{D^+}$, we have

$$\iota^+(\chi') = \frac{1}{p+1}\sqrt{\frac{2p+m-1}{p+m-1}}\sum_{r=0}^{m} \partial_r\chi' \otimes y_r, \quad \chi' \in \mathcal{H}^{p+1}.$$

$\Phi^+ = \Phi^{D^+}$: $S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p+1})$ is injective [14] (in fact, up to a nonzero constant multiple, $\Phi^\pm$ are transposes of each other), and it thus restricts to an $SO(m+1)$-equivariant imbedding Φ^+: $\mathcal{L}^p \to \mathcal{L}^{p+1}$, $\Phi^+(\langle f \rangle) = \langle f^+ \rangle$, where f: $S^m \to S_V$ is a full p-eigenmap, and $f^+ = f^{D^+}$: $S^m \to S_{V \otimes \mathcal{H}^1}$.

In degree raising and lowering we have set $G = SO(m+1)$. In our next example we let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G}$, and assume that G acts on S^m transitively. We let $W = \mathcal{G}$ be the G-module with the adjoint representation, and $\mathcal{A}_p$ the induced action of $\mathcal{G}$ on $\mathcal{H}^p$. Then $\mathcal{A}_p$: $\mathcal{G} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^p$ is an operator; inserting a factor of $1/\lambda_p$, it becomes metric [15]. By definition, for a full p-eigenmap f: $S^m \to S_V$, we have $\mathcal{A}_p(\langle f \rangle) = \langle \hat{f} \rangle$, where $\hat{f}$: $S^m \to S_{V \otimes \mathcal{G}^*}$ is given in (1). We call $\mathcal{A}_p$ the *operator of infinitesimal rotations*. (For $G = SO(m+1)$, an orthonormal basis of $so(m+1)$ is given by $\{E_{rr'}\}_{0 \le r < r' \le m}$, where $E_{rr'} = x_{r'}\partial_r - x_r\partial_{r'}$ is infinitesimal rotation on the $x_r x_{r'}$-plane.) We denote $\alpha_p = \iota^{\mathcal{A}_p}$. With this, for $C \in S^2(\mathcal{H}^p)$, (23) specializes to $\mathcal{A}_p(C) = \alpha_p^\top \circ (C \otimes I) \circ \alpha_p$. As noted in Section 1, $\mathcal{A}_p$ is a symmetric endomorphism of $S^2(\mathcal{H}^p)$ that may well vanish on some irreducible components. As shown in [15], we have $\mathcal{A}_p(\mathcal{E}^p) \subset \mathcal{E}^p$, and $\mathcal{A}_p(\mathcal{F}^p) \subset \mathcal{F}^p$, in particular, $\mathcal{A}_p$ restricts to self-maps on the moduli $\mathcal{L}^p$ and $\mathcal{M}^p$. Note also that Proposition 2.2 implies the first statement in part (a) of Theorem 1 for $G = SO(m+1)$.

3. Isotropy and the eigenvalues of $\mathcal{A}_p$ on $\mathcal{E}^p$. Let $G \subset SO(m+1)$ be a closed subgroup acting transitively on S^m, and let $\mathcal{G}$ denote the Lie algebra of G. Let $\{E_i\}_{i=1}^s \subset \mathcal{G}$ be an orthonormal basis, and $\{\phi_i\}_{i=1}^s \subset \mathcal{G}^*$ its dual basis. For $C \in S^2(\mathcal{H}^p)$ and $\chi \in \mathcal{H}^p$, we have

$$\begin{aligned}\mathcal{A}_p(C)\chi &= \alpha_p^\top (C \otimes I)\alpha_p(\chi) \\ &= \alpha_p^\top (C \otimes I)\left(\frac{1}{\sqrt{\lambda_p}} \sum_{i=1}^s E_i\chi \otimes \phi_i\right) \\ &= \frac{1}{\sqrt{\lambda_p}} \sum_{i=1}^s \alpha_p^\top (CE_i\chi \otimes \phi_i) \\ &= -\frac{1}{\lambda_p} \sum_{i=1}^s E_i C E_i \chi.\end{aligned}$$

On the other hand, for the Casimir operator, we have

$$\begin{aligned}Cas(C) &= -\sum_{i=1}^s [E_i, [E_i, C]] \\ &= -\sum_{i=1}^s E_i^2 \circ C - C \circ \sum_{i=1}^s E_i^2 + 2\sum_{i=1}^s E_i \circ C \circ E_i.\end{aligned}$$

On $\mathcal{H}^p$, we have $\sum_{i=1}^{s} E_i^2 = -\lambda_p I$ so that the first two terms on the right-hand side give $2\lambda_p I$. By the previous computation, $\sum_{i=1}^{s} E_i C E_i = -\lambda_p \mathcal{A}_p(C)$. Putting these together, (9) follows. We also see that (11) implies (10). We will prove (11) at the end of this section.

To prove the first assertion in part (a) of Theorem 1, we need a sketch proof of the decompositions (7)–(8) (for more details, see [8, 14]) as well as some preparations. A vector valued map $f\colon S^m \to V$ with spherical harmonic components (that is $V_f \subset \mathcal{H}^p$) is a spherical minimal immersion iff f maps into the unit sphere S_V and f is homothetic. This is because harmonicity of the components guarantees that f is harmonic in the sense of Eells-Sampson, and a homothetic immersion is minimal iff it is harmonic [5]. To pin down the two constraints, we now recall that a spherical harmonic of order p on S^m is the restriction of a harmonic homogeneous polynomial of degree p in $m+1$ variables. Thus f automatically extends to a map $f\colon \mathbf{R}^{m+1} \to V$. The image of the unit sphere S^m under f is contained in S_V iff the homogeneous polynomial

$$\Psi_p^0(f) = |f|^2 - \rho^{2p}, \quad \rho^2(x) = |x|^2,\ x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1},$$

of degree $2p$ vanishes. Setting $f = Af_p$ and $C = \langle f \rangle = A^\top A - I$, we have $\Psi_p^0(f) = \langle Cf_p, f_p \rangle$. We now redefine Ψ_p^0 by the right-hand side of this formula. We obtain the linear map

$$\Psi_p^0\colon S^2(\mathcal{H}^p) \to \mathcal{P}^{2p},$$

where $\mathcal{P}^q$ denotes the $SO(m+1)$-module of homogeneous polynomials of degree q on $\mathbf{R}^{m+1}$ (with the obvious action of $SO(m+1)$ on $\mathcal{P}^q$; an extension of the action on $\mathcal{H}^q \subset \mathcal{P}^q$). Ψ_p^0 is a homomorphism of $SO(m+1)$-modules. An easy induction [8] in the use of the canonical decomposition

$$\mathcal{P}^{2p} = \sum_{l=0}^{p} \mathcal{H}^{2l} \cdot \rho^{2(p-l)}$$

shows that Ψ_p^0 is surjective. Comparing this with (5) and (8), we see that the irreducible $SO(m+1)$-modules

$$\mathcal{H}^{2l} \cdot \rho^{2(p-l)} \cong V^{(2l,0,\ldots,0)}, \quad l = 0, \ldots, p, \tag{25}$$

are exactly the ones that need to be deleted from $S^2(\mathcal{H}^p)$ to satisfy the condition that our map f sends S^m into S_V. The modules in (25) are parametrized by the even lattice points along the base of the triangle $\triangle_0^p$. We now define $\triangle_1^p \subset \mathbf{R}^2$ as the triangle with vertices $(2,2)$, (p,p) and $(2(p-1),2)$. A DoCarmo-Wallach type argument then gives the moduli space $\mathcal{L}^p$ as a compact convex body in $\mathcal{E}^p$ parametrizing the p-eigenmaps.

For the condition of homothety, we first note that (2) needs to be satisfied only for conformal fields on S^m since they span each tangent space of S^m. Given $a \in \mathbf{R}^{m+1}$, we define the conformal field X^a on $\mathbf{R}^{m+1}$ by

$$X^a_x = a - \frac{\langle a, x\rangle}{|x|^2}x, \quad x \in \mathbf{R}^{m+1}.$$

Here we identify tangent vectors on $\mathbf{R}^{m+1}$ with their translates at the origin; note also that the factor $1/|x|^2$ is inserted to preserve homogeneity. In analogy with the previous condition, for a map $f\colon S^m \to S_V$ with spherical harmonic components, we define

$$\begin{aligned}\Psi^1_p(f)(a,b) &= \langle f_*(X^a), f_*(X^b)\rangle - \frac{\lambda_p}{m}\langle X^a, X^b\rangle \\ &= \langle f_*(X^a), f_*(X^b)\rangle - \langle f_{p*}(X^a), f_{p*}(X^b)\rangle,\end{aligned}$$

where $a, b \in \mathbf{R}^{m+1}$. Clearly, f is homothetic (and thereby minimal) iff $\Psi^1_p(f)$ vanishes. Writing X^a as a differential operator (in terms of directional derivatives):

$$X^a = \partial_a - \frac{\langle a, x\rangle}{|x|^2}\partial_x,$$

with the previous notation $C = \langle f\rangle \in \mathcal{E}^p$, we obtain

$$\Psi^1_p(f)(a,b) = \langle \partial_a C f_p, \partial_b f_p\rangle. \tag{26}$$

As before, the right-hand side of (26) defines $\Psi^1_p(C)(a,b)$, with $C \in \mathcal{E}^p$. As $SO(m+1)$-modules, $(\mathbf{R}^{m+1})^* \cong \mathcal{H}^1$, so that the symmetric bilinear map $\Psi^1_p(C)$ becomes a tensor $\Psi^1_p(C) \in \mathcal{P}^{2(p-1)} \otimes S^2(\mathcal{H}^1)$. Ψ^1_p is zero on the trivial summand of $S^2(\mathcal{H}^1) \cong \mathcal{H}^0 \oplus \mathcal{H}^2$ since it corresponds to the trace. By restriction, we end up with an element $\Psi^1_p(C) \in \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2$. Varying C within $\mathcal{E}^p$ finally gives the $SO(m+1)$-module homomorphism

$$\Psi^1_p\colon \mathcal{E}^p \to \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2.$$

By the canonical decomposition applied to $\mathcal{P}^{2(p-1)}$, as $SO(m+1)$-modules

$$\mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2 \cong \sum_{l=0}^{p-1} \mathcal{H}^{2l} \otimes \mathcal{H}^2. \tag{27}$$

A quick comparison reveals that the only common irreducible components in (8) and in (27) are

$$V^{(2l,2,\ldots,0)}, \quad l = 1, \ldots, p-1,$$

and these correspond to the even lattice points along the base of $\triangle_p^1$. Now Ψ_p^1 is nonzero on these components [14] so that these are exactly the components that are to be deleted from (8) to satisfy the homothety condition for our maps to become spherical minimal immersions. Formula (6) follows.

The proof of (7) patterns that of (6). Proceeding inductively, for $f\colon S^m \to S_V$ a full isotropic minimal immersion of degree p and order of isotropy $k-1$, $k \geq 2$, we define

$$\begin{aligned}(28)\qquad \Psi_p^k(f)(a_1,\dots,a_k,b_1,\dots,b_k) &= \langle \beta_k(f)(X^{a_1},\dots,X^{a_k}), \beta_k(f)(X^{b_1},\dots,X^{b_k})\rangle \\ &\quad - \langle \beta_k(f_p)(X^{a_1},\dots,X^{a_k}), \beta_k(f_p)(X^{b_1},\dots,X^{b_k})\rangle \\ &= \langle \partial_{a_1}\dots\partial_{a_k} f, \partial_{b_1}\dots\partial_{b_k} f\rangle \\ &\quad - \langle \partial_{a_1}\dots\partial_{a_k} f_p, \partial_{b_1}\dots\partial_{b_k} f_p\rangle \\ &= \langle \partial_{a_1}\dots\partial_{a_k}\langle f\rangle f_p, \partial_{b_1}\dots\partial_{b_k} f_p\rangle,\end{aligned}$$

where $\langle f\rangle \in \mathcal{F}^{p,k-1}$. As before, $\Psi_p^k(f)$ vanishes iff f is isotropic of order k. Ψ_p^k extends to a homomorphism

$$\Psi_p^k\colon \mathcal{F}^{p;k-1} \to \mathcal{P}^{2(p-k)} \otimes \mathcal{H}^{2k}$$

of $SO(m+1)$-modules. The canonical decomposition gives

$$(29)\qquad \mathcal{P}^{2(p-k)} \otimes \mathcal{H}^{2k} \cong \sum_{l=0}^{p-k} \mathcal{H}^{2l} \otimes \mathcal{H}^{2k},$$

and Ψ_p^k is nonzero on the components

$$V^{(2l,2k,0,\dots,0)}, \quad l = k,\dots,p-k.$$

Degree raising and lowering interact with Ψ_p^k in a particularly beautiful way (and this provides the main induction step in proving (7)). Given a full isotropic minimal immersion $f\colon S^m \to S_V$ of degree p and order of isotropy $k-1$, $k \geq 1$; for $c_1,\dots,c_{2k} \in \mathbf{R}^{m+1}$, we have

$$\begin{aligned}\Psi_{p+1}^k(f^+)(c_1,\dots,c_{2k}) &= \frac{\lambda_{4k}}{4k}\Psi_p^k(f)(c_1,\dots,c_{2k})\rho^2 \\ &\quad + \frac{p^2}{\lambda_p\lambda_{2p}}\triangle(\Psi_p^k(f)(c_1,\dots,c_{2k}))\rho^4,\end{aligned}$$

and

$$\triangle(\Psi_p^k(f)(c_1,\ldots,c_{2k}))=\lambda_{2p}\Psi_{p-1}^k(f^-)(c_1,\ldots,c_{2k}). \tag{30}$$

In particular, we see that if f is isotropic of order k then so are $f^{\pm}$.

Finally, we will need a technical tool called the Inductive Lemma [8]; this will help to rearrange the partial derivatives inside various scalar products in $\Psi_p^k(f)(c_1,\ldots,c_{2k})$, $c_1,\ldots,c_{2k}\in\mathbf{R}^{m+1}$, where $f\colon S^m\to S_V$ is a full isotropic minimal immersion of degree p and order of isotropy $k-1$, $k\geq 1$. To simplify matters, we use multiindex notation $\partial_{c_I}=\partial_{c_{i_1}}\ldots\partial_{c_{i_l}}$, for $I=\{i_1,\ldots,i_l\}$. We have

$$\langle\partial_{c_I}\langle f\rangle f_p,\partial_{c_J}f_p\rangle=0 \tag{31}$$

for all I and J with $|I|+|J|\leq 2k-1$, $I,J\subset\{1,\ldots,2k\}$; and

$$\Psi_p^k(f)(c_1,\ldots,c_{2k})=(-1)^{\frac{|I|-|J|}{2}}\langle\partial_{c_I}\langle f\rangle f_p,\partial_{c_J}f_p\rangle \tag{32}$$

for all I and J disjoint with $I\cup J=\{1,\ldots,2k\}$.

With $f\colon S^m\to S_V$ as above, let $\Xi_p^k(f)(c_1,\ldots,c_{2k})$, $c_1,\ldots,c_{2k}\in\mathbf{R}^{m+1}$, be the trace of the bilinear form

$$(X,Y)\mapsto X\cdot\sum_{j=1}^{2k}\Psi_p^k(f)(c_1,\ldots,c_{j-1},Yc_j,c_{j+1},\ldots,c_{2k}) \tag{33}$$

on $\mathcal{G}$ (with values in $\mathcal{P}^{2(p-k)}$).

Proposition 3.1. *Let $f\colon S^m\to S_V$ be a full isotropic minimal immersion of degree p and order of isotropy $k-1$, $k\geq 2$. Then, for $c_1,\ldots,c_{2k}\in\mathbf{R}^{m+1}$, we have*

$$\begin{aligned}2\lambda_p\Psi_p^k(\hat{f})&(c_1,\ldots,c_{2k})\\&=(2\lambda_p-\mu^{2(p-k),2k}+2km-4k)\Psi_p^k(f)(c_1,\ldots,c_{2k})\\&\quad-\sum_{j=1}^{2k}\Psi_p^k(f)(c_1,\ldots,c_{j-1},Cas(c_j),c_{j+1},\ldots,c_{2k})\\&\quad+2\Xi_p^k(f)(c_1,\ldots,c_{2k}).\\&\quad+\lambda_{2p}\Psi_{p-1}^k(f^-)(c_1,\ldots,c_{2k}),\end{aligned} \tag{34}$$

where $\mu^{2(p-k),2k}$ is given in (11). *For $G=SO(m+1)$, we have*

$$\Xi_p^k(f)(c_1,\ldots,c_{2k})=2k\Psi_p^k(f)(c_1,\ldots,c_{2k}), \tag{35}$$

in particular

$$\Lambda_p^{2(p-k),2k} = 1 - \frac{\mu^{2(p-k),2k}}{2\lambda_p}. \tag{36}$$

If $f\colon S^m \to S_V$ is a full isotropic minimal immersion of degree p and order of isotropy k, $k \geq 2$, then $\Psi_p^k(f) = 0$ and hence $\Xi_p^k(f) = 0$, so that (34) along with (30) imply that $\Psi_p^k(\hat{f}) = 0$. This is the first statement in part (a) of Theorem 1. Also, for $G = SO(m+1)$, substituting (35) into (34), we obtain that $\Psi_p^k(\hat{f})$ is a linear combination of $\Psi_p^k(f)$ and $\Psi_{p-1}^k(f^-)$. (The Casimir operator on $\mathbf{R}^{m+1} \cong \mathcal{H}^1$ is multiplication by $m = \lambda_1$ so that the contribution from the second term on the right-hand side of (34) is $-2mk\Psi_p^k(f)$. By (35), the contribution from the third term is $4k\Psi_p^k(f)$.) If, in addition, $\langle f \rangle \in V^{(2(p-k),2k,0,\ldots,0)}$, then $\Psi_{p-1}^k(f^-)$ vanishes and $\Psi_p^k(\hat{f})$ becomes a constant multiple of $\Psi_p^k(f)$. The constant then must be $\Lambda_p^{2(p-k),2k}$. Thus (36) follows. To prove Theorem 1, it remains to derive (34) and (35), as well as to extend (36) to all even lattice points (a, b) in $\triangle_1^p$.

We need some preparatory lemmas.

LEMMA 3.2. *For $a \in \mathbf{R}^{m+1}$ and $X \in \mathcal{G}$, we have*

$$[\partial_a, X] = \partial_{X\cdot a}. \tag{37}$$

Proof. With obvious notations

$$X_x = \partial_{Xx} = \sum_{r=0}^{m} x_r \partial_{Xe_r}.$$

Here X is first the vector field induced by the action of G on $\mathbf{R}^{m+1}$, and then the skew-symmetric matrix in $\mathcal{G} \subset so(m+1)$ acting on vectors in $\mathbf{R}^{m+1}$ by matrix multiplication. With this, we compute

$$\begin{aligned}[\partial_a, X] &= \sum_{r=0}^{m} [\partial_a, x_r \partial_{Xe_r}] = \sum_{r=0}^{m} \partial_a(x_r)\partial_{Xe_r} \\ &= \sum_{r=0}^{m} a_r \partial_{Xe_r} = \partial_{Xa}.\end{aligned}$$

LEMMA 3.3. *For $a \in \mathbf{R}^{m+1}$ and $X, Y \in \mathcal{G}$, we have*

$$X\partial_{Ya} + Y\partial_{Xa} = -\partial_{XYa} + \partial_a YX - YX\partial_a. \tag{38}$$

Proof. Replacing a by Ya in (37) we have

$$[X, \partial_{Ya}] = -\partial_{XYa}.$$

Using this, we compute

$$\begin{aligned} X\partial_{Ya} + \partial_{XYa} &= \partial_{Ya}X = [\partial_a, Y]X \\ &= \partial_a YX - Y\partial_a X \\ &= \partial_a YX - Y([\partial_a, X] + X\partial_a) \\ &= \partial_a YX - Y\partial_{Xa} - YX\partial_a. \end{aligned}$$

The lemma follows.

Taking traces of both sides of (38), we obtain that on $\mathcal{H}^p$:

$$\text{(39)} \qquad 2\,\text{trace}\,\{(X, Y) \mapsto X\partial_{Ya}\} = \partial_{Cas\,(a)} - (\lambda_p - \lambda_{p-1})\partial_a,$$

where $Cas\,(a)$ is the Casimir operator of $a \in \mathcal{G}$ on $\mathbf{R}^{m+1}$: the trace of the bilinear map $(X, Y) \to -XYa$. Formula (39) follows because the trace of the bilinear map $(X, Y) \mapsto XY$ on $\mathcal{H}^p$ is $-\lambda_p \cdot I$.

Proof of Proposition 3.1. Let $\{E_i\}_{i=1}^s \subset \mathcal{G}$ be an orthonormal basis. For notational convenience, we set $a_j = c_j$, and $b_j = c_{k+j}$, $j = 1, \ldots, k$. In view of (28) and (1), we need to work out

$$\text{(40)} \qquad \begin{aligned} &\langle \partial_{a_1} \ldots \partial_{a_k} \hat{f}, \partial_{b_1} \ldots \partial_{b_k} \hat{f} \rangle \\ &= \frac{1}{\lambda_p} \sum_{i=1}^{s} \langle \partial_{a_1} \ldots \partial_{a_k} E_i f, \partial_{b_1} \ldots \partial_{b_k} E_i f \rangle. \end{aligned}$$

By Lemma 3.2, we have

$$\text{(41)} \qquad \begin{aligned} \partial_{a_1} \ldots \partial_{a_k} E_i f &= \partial_{a_1} \ldots \partial_{a_{k-1}} [\partial_{a_k}, E_i] f + \partial_{a_1} \ldots \partial_{a_{k-1}} E_i \partial_{a_k} f \\ &= \partial_{E_i a_k} \partial_{a_1} \ldots \partial_{a_{k-1}} f + \partial_{a_1} \ldots \partial_{a_{k-1}} E_i \partial_{a_k} f \\ &= \sum_{j=1}^{k} \partial_{E_i a_j} \partial_{a_1} \ldots \widehat{\partial_{a_j}} \ldots \partial_{a_k} f \\ &\quad + E_i \partial_{a_1} \ldots \partial_{a_k} f, \end{aligned}$$

where we also used that directional derivatives commute. As usual, $\widehat{\ }$ means that the corresponding factor is absent. We write the result as $A_1 + A_2$, and as $B_1 + B_2$ when $a_1, \ldots, a_k$ are replaced by $b_1, \ldots, b_k$. To work out (40) now amounts to determining $\sum_{i=1}^s \sum_{\alpha,\beta=1}^2 \langle A_\alpha, B_\beta \rangle$. To simplify the computations we will use the notation

$$F(a_1, \ldots, a_l) = \partial_{a_1} \ldots \partial_{a_l} f.$$

Since f is harmonic, we have

$$(42)\quad \triangle\langle F(a_1,\ldots,a_l),F(b_1,\ldots,b_l)\rangle = 2\sum_{r=0}^{m}\langle F(e_r,a_1,\ldots,a_l),F(e_r,b_1,\ldots,b_l)\rangle,$$

where $\triangle$ is the Euclidean Laplacian and $\{e_r\}_{r=0}^{m}\subset \mathbf{R}^{m+1}$ is the standard basis. We now simplify each scalar product $\langle A_\alpha, B_\beta\rangle$. Using (41), we have

$$(43)\quad \sum_{i=1}^{s}\langle A_1,B_1\rangle=\sum_{i=1}^{s}\sum_{j,l=1}^{k}\langle F(E_ia_j,a_1,\ldots,\widehat{a}_j,\ldots,a_k),F(E_ib_l,b_1,\ldots,\widehat{b}_l,\ldots,b_k)\rangle.$$

Recall from (32) that in the difference of the right-hand side of (43) and the analogous terms when f is replaced by the standard minimal immersion f_p, the partial derivatives can be permuted. Thus, up to sign, it is enough to consider

$$\sum_{i=1}^{s}\sum_{j,l=1}^{k}\langle F(E_ia_j,E_ib_l,a_1,\ldots,\widehat{a}_j,\ldots,a_k),F(b_1,\ldots,\widehat{b}_l,\ldots,b_k)\rangle$$
$$=\sum_{j,l=1}^{k}\langle\sum_{i=1}^{s}\left(\partial_{E_ia_j}\partial_{E_ib_l}\right)F(a_1,\ldots,\widehat{a}_j,\ldots,a_k),F(b_1,\ldots,\widehat{b}_l,\ldots,b_k)\rangle.$$

We need to work out the differential operator $\sum_{i=1}^{s}\partial_{E_ia}\partial_{E_ib}$, $a,b\in\mathbf{R}^{m+1}$, *acting on* $\mathcal{H}^{p-k+1}$ (since $F(a_1,\ldots,\widehat{a}_j,\ldots,a_k)$ is of degree $p-k+1$). Using Lemma 3.2 and (39), we compute

$$\begin{aligned}\sum_{i=1}^{s}\partial_{E_ia}\partial_{E_ib} &= \sum_{i=1}^{s}[\partial_a,E_i]\partial_{E_ib}\\ &= \partial_a\sum_{i=1}^{s}E_i\partial_{E_ib}-\sum_{i=1}^{s}E_i\partial_{E_ib}\partial_a\\ &= \frac{1}{2}\partial_a\left(\partial_{Cas\,(b)}-(\lambda_{p-k+1}-\lambda_{p-k})\partial_b\right)\\ &\quad -\frac{1}{2}\left(\partial_{Cas\,(b)}-(\lambda_{p-k}-\lambda_{p-k-1})\partial_b\right)\partial_a\\ &= -\frac{1}{2}(\lambda_{p-k+1}+\lambda_{p-k-1}-2\lambda_{p-k})\partial_a\partial_b\\ &= -\partial_a\partial_b.\end{aligned}$$

Summarizing, we see that the terms coming from $\langle A_1,B_1\rangle$ contribute to $2\lambda_p\Psi_p^k(\hat{f})$ the term $-2k^2\Psi_p^k(f)$ (where we suppressed the arguments $(a_1,\ldots,a_k,b_1,\ldots,b_k)$).

Second, we have

$$\begin{aligned}\sum_{i=1}^{s}\langle A_1, B_2\rangle &= \sum_{i=1}^{s}\sum_{j=1}^{k}\langle \partial_{E_i a_j}F(a_1,\ldots,\widehat{a}_j,\ldots,a_k), E_iF(b_1,\ldots,b_k)\rangle \\ &= \sum_{i=1}^{s}\sum_{j=1}^{k} E_i\langle \partial_{E_i a_j}F(a_1,\ldots,\widehat{a}_j,\ldots,a_k), F(b_1,\ldots,b_k)\rangle \\ &\quad - \sum_{j=1}^{k}\langle \sum_{i=1}^{s} E_i\partial_{E_i a_j}F(a_1,\ldots,\widehat{a}_j,\ldots,a_k), F(b_1,\ldots,b_k)\rangle.\end{aligned}$$

The resulting first sum along with the analogous term from $\sum_{i=1}^{s}\langle A_2, B_1\rangle$ contribute to $2\lambda_p\Psi_p^k(\hat{f})$ the term $2\Xi_p^k(f)$. Using (39), the second sum rewrites as

$$\begin{aligned}&-\frac{1}{2}\sum_{j=1}^{k}\langle F(a_1,\ldots,a_{j-1}, Cas(a_j), a_{j+1},\ldots,a_k), F(b_1,\ldots,b_k)\rangle \\ &\quad + \frac{k}{2}(\lambda_{p-k+1}-\lambda_{p-k})\langle F(a_1,\ldots,a_k), F(b_1,\ldots,b_k)\rangle.\end{aligned}$$

These contribute to $2\lambda_p\Psi_p^k(\hat{f})$ the term

$$\begin{aligned}&-\sum_{j=1}^{k}\Psi_p^k(f)(a_1,\ldots,a_{j-1}, Cas(a_j), a_{j+1},\ldots,a_k, b_1,\ldots,b_k) \\ &\quad + k(\lambda_{p-k+1}-\lambda_{p-k})\Psi_p^k(f)(a_1,\ldots,a_k,b_1,\ldots,b_k).\end{aligned}$$

The computation for $\sum_{i=1}^{s}\langle A_1, B_2\rangle$ is analogous.

Finally, using the fact that the components of f are spherical harmonics on S^m of order p, the connection between the Euclidean and spherical Laplacians, and $\Delta^{S^m} = -\sum_{i=1}^{s} E_i^2$, we compute

$$\begin{aligned}2\sum_{i=1}^{s}\langle A_2, B_2\rangle &= 2\sum_{i=1}^{s}\langle E_iF(a_1,\ldots,a_k), E_iF(b_1,\ldots,b_k)\rangle \\ &= -\Delta^{S^m}\langle F(a_1,\ldots,a_k), F(b_1,\ldots,b_k)\rangle \\ &\quad + \langle \Delta^{S^m}F(a_1,\ldots,a_k), F(b_1,\ldots,b_k)\rangle \\ &\quad + \langle F(a_1,\ldots,a_k), \Delta^{S^m}F(b_1,\ldots,b_k)\rangle \\ &= \Delta\langle F(a_1,\ldots,a_k), F(b_1,\ldots,b_k)\rangle \\ &\quad + (2\lambda_{p-k}-\lambda_{2(p-k)})\langle F(a_1,\ldots,a_k), F(b_1,\ldots,b_k)\rangle\end{aligned}$$

(where we doubled for convenience). These contribute to $2\lambda_p \Psi_p^k(\hat{f})$ the terms $\lambda_{2p}\Psi_{p-1}^k(f^-)$ (by (30)) and $(2\lambda_{p-k} - \lambda_{2(p-k)})\Psi_p^k(f)$. We now put all the contributions together and obtain (34).

To prove (35) we assume that $G = SO(m+1)$. As usual, we choose the orthonormal basis $\{E_{rr'}\}_{0\le r<r'\le m} \subset so(m+1)$, where $E_{rr'} = x_{r'}\partial_r - x_r\partial_{r'}$, $r, r' = 0, \ldots, m$. To work out the trace of the bilinear form (33), we compute

$$\begin{aligned}
&\sum_{0\le r<r'\le m} \sum_{j=1}^k E_{rr'} \langle \partial_{E_{rr'}} F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&= \frac{1}{2} \sum_{r,r'=0}^m \sum_{j=1}^k (x_{r'}\partial_r - x_r\partial_{r'}) \langle (a_{j,r'}\partial_r - a_{j,r}\partial_{r'}) F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&= \sum_{r=0}^m \sum_{j=1}^k \langle a_j, x\rangle \partial_r \langle \partial_r F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&\quad - \sum_{r=0}^m \sum_{j=1}^k x_r \partial_{a_j} \langle \partial_r F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&= \frac{1}{2} \sum_{j=1}^k \langle a_j, x\rangle \Delta \langle F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&\quad + \sum_{r=0}^m \sum_{j=1}^k a_{j,r} \langle F(e_r, a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&\quad - \sum_{j=1}^k \partial_{a_j} \langle \sum_{r=0}^m x_r \partial_r F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&= \frac{1}{2} \sum_{j=1}^k \langle a_j, x\rangle \Delta \langle F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&\quad + k \langle F(a_1, \ldots, a_k), F(b_1, \ldots, b_k)\rangle \\
&\quad - (p-k+1) \sum_{j=1}^k \partial_{a_j} \langle F(a_1, \ldots, \hat{a}_j, \ldots, a_k), F(b_1, \ldots, b_k)\rangle .
\end{aligned}$$

The first and third terms in the resulting sum contribute zero by (31), and the second term contributes $k\Psi_p^k(f)(a_1, \ldots, a_k, b_1, \ldots, b_k)$. Switching the roles of the a's and the b's, (35) now follows.

We finally show that (36) implies the eigenvalue formula (10)–(11) for all even lattice points (a, b) in Δ_1^p. To do this, we need to see how degree raising interacts with $\mathcal{A}_p$. A formula that relates these two operators was derived in [15];

in fact, for $C \in S^2(\mathcal{H}^p)$, we have

$$\lambda_p(\Phi_p^+(C) - \Phi_p^+(\mathcal{A}_p(C))) = \lambda_{p+1}(\Phi_p^+(C) - \mathcal{A}_{p+1}(\Phi_p^+(C))). \tag{44}$$

Restricting C to an irreducible component, we obtain

$$\lambda_p(1 - \Lambda_p^{a,b}) = \lambda_q(1 - \Lambda_q^{a,b}). \tag{45}$$

Indeed, for $q = p+1$, this follows immediately from (44); in general, by induction.

Formula (36) combined with (45) now gives all eigenvalues of $\mathcal{A}_p$ on $\mathcal{E}^p$. Indeed, for $(a,b) \in \Delta_1^p$, a, b even, (36) rewrites as

$$\Lambda^{a,b}_{\frac{a+b}{2}} = 1 - \frac{\mu^{a,b}}{2\lambda_{\frac{a+b}{2}}},$$

and (45) with $q = \frac{a+b}{2}$ now gives (11). Theorem 1 follows.

Proof of Theorem 2. Let $f\colon S^m \to S_V$ be a full spherical minimal immersion of degree p, and assume that $\langle f \rangle \in V^{(a,b,0,\dots,0)} \subset \mathcal{F}^p$ with $\Lambda_p^{a,b} \geq 0$. Then $\hat{f}\colon S^m \to S_V$ corresponds to an interior point of $\mathcal{M}^p$ since the contraction $\mathcal{A}_p$ is multiplication by the nonnegative constant $\Lambda_p^{a,b} < 1$ on $V^{(a,b,0,\dots,0)}$. Since the interior points of $\mathcal{M}^p$ correspond to spherical minimal immersions with maximal range dimension, we have

$$\dim(V \otimes so(m+1)^*) \geq \dim \mathcal{H}^p.$$

4. $SU(2)$-equivariant minimal immersions. As in Section 1, we let W_p denote the irreducible $SU(2)$-module of homogeneous polynomials of degree p in two complex variables. Given a polynomial $\xi \in W_p$ of unit length, the equivariant construction [3] applied to ξ gives a p-eigenmap $f_\xi\colon S^3 \to S_{W_p}$ defined by

$$f_\xi(a) = \xi \circ L_{a^{-1}}, \quad a \in SU(2) = S^3.$$

Here L is left quaternionic multiplication, the identification $SU(2) = S^3$ is given by

$$\begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} = z + jw, \quad z, w \in \mathbf{C},$$

and left quaternionic multiplication corresponds to left matrix multiplication. In coordinates, we have

$$f_\xi(z + jw)(a, b) = \xi(a\bar{z} + b\bar{w}, -aw + bz),$$

where we used $a, b \in \mathbf{C}$ as variables for ξ. For $p = 2d$ even, W_{2d} has a real $SU(2)$-submodule R_{2d} given by the fixed point set of the antilinear map $a^q b^{2d-q} \mapsto (-1)^q a^{2d-q} b^q$, $q = 0, \ldots, 2d$. If $\xi \in R_{2d}$ then the image of f_ξ lies automatically in R_{2d} so that we have $f_\xi \colon S^3 \to S_{R_{2d}}$. Finally, note that the equivariant construction extends naturally to any multiples of W_p and R_{2d}.

Throughout this section, we define the operator $\mathcal{A}_{3,p}$ of infinitesimal rotations with respect to the group $G = SU(2)'$. We let X_L, Y_L, Z_L the left-invariant extensions of the standard basis elements $X, Y, Z \in su(2)$, where

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

We have, as differential operators:

$$\begin{aligned} X_L &= -\bar{w}\partial_z - w\partial_{\bar{z}} + \bar{z}\partial_w + z\partial_{\bar{w}}, \\ Y_L &= i(-\bar{w}\partial_z + w\partial_{\bar{z}} + \bar{z}\partial_w - z\partial_{\bar{w}}), \\ Z_L &= i(z\partial_z - \bar{z}\partial_{\bar{z}} + w\partial_w - \bar{w}\partial_{\bar{w}}). \end{aligned}$$

LEMMA 4.1. *For any $U \in su(2)$, we have*

$$U_L \cdot f_\xi = -f_{U_R \cdot \xi},$$

where U_R is the right-invariant extension of U.

Proof. Using standard notation, we compute

$$\begin{aligned} (U_L f_\xi)(z + jw)(a, b) &= \frac{d}{dt}\Big|_{t=0} f_\xi((z + jw)\exp(tU))(a, b) \\ &= \frac{d}{dt}\Big|_{t=0} \xi(L_{\exp(-tU)(z+jw)^{-1}}(a, b)) \\ &= \frac{d}{dt}\Big|_{t=0} \xi(L_{\exp(-tU)} L_{(z+jw)^{-1}}(a, b)) \\ &= -(U_R \xi)(L_{(z+jw)^{-1}}(a, b)) \\ &= -f_{U_R \xi}(z + jw)(a, b). \end{aligned}$$

Lemma 4.1 provides a particularly simple way to express $\widehat{f}_\xi$ in terms of ξ:

COROLLARY. *With respect to the standard basis $X, Y, Z \in su(2)$, we have*

$$\widehat{f}_\xi = -\frac{1}{\sqrt{\lambda_p}} \left(f_{X_R \xi}, f_{Y_R \xi}, f_{Z_R \xi} \right), \tag{46}$$

where the right-invariant extensions X_R, Y_R, Z_R acting on W_p are

$$X_R = b\partial_a - a\partial_b$$
$$Y_R = i(b\partial_a + a\partial_b)$$
$$Z_R = i(a\partial_a - b\partial_b).$$

In view of this corollary, we write $\widehat{f}_\xi = f_{\widehat{\xi}}$, where

$$\widehat{\xi} = -\frac{1}{\sqrt{\lambda_p}}(X_R\xi, Y_R\xi, Z_R\xi).$$

This we will also apply when ξ has several components, that is, when ξ is an element of some multiples of W_p or R_{2d}.

From now on let $p = 2d$ be even. We fix the orthonormal basis $\{\xi_j\}_{j=1}^{2d+1}$ in $R_{2d} \subset W_{2d}$ with elements

$$\xi_{2l+1} = \frac{1}{\sqrt{2(2d-l)!l!}}(a^{2d-l}b^l + (-1)^l a^l b^{2d-l}), \quad l = 0, \ldots, d-1$$
$$\xi_{2l+2} = \frac{i}{\sqrt{2(2d-l)!l!}}(a^{2d-l}b^l - (-1)^l a^l b^{2d-l}), \quad l = 0, \ldots, d-1$$
$$\xi_{2d+1} = \frac{i^d}{d!}a^d b^d.$$

LEMMA 4.2. *Let $p = 2d$ be even. The $2d$-eigenmap f_{2d}: $S^3 \to S_{(2d+1)R_{2d}}$ defined by*

$$f_{2d} = \frac{1}{\sqrt{2d+1}}(f_{\xi_1}, f_{\xi_2} \ldots, f_{\xi_{2d+1}}), \tag{47}$$

is congruent to the standard minimal immersion $f_{3,2d}$.

Proof. We use the general fact that a full p-eigenmap f: $S^m \to S_V$ that is equivariant with respect to the entire group $SO(m+1)$ must be standard. (This is because $\mathcal{E}^p$ has no trivial component.) In our case, we have $SO(4) = SU(2)\cdot SU(2)'$, and f_{2d} is clearly $SU(2)$-equivariant since its components are. It remains to show that f_{2d} is also $SU(2)'$-equivariant. $SU(2)'$ is obtained from $SU(2)$ by conjugation with the diagonal matrix γ with diagonal elements $1, 1, 1, -1$. In terms of complex variables, γ: $(z, w) \mapsto (z, \bar{w})$. The stated $SU(2)'$-equivariance follows from the (easily verifiable) formula

$$f_\xi \circ (\gamma \circ L_{a^{-1}} \circ \gamma) = f_{\xi \circ L_{\gamma(a)}}, \quad a \in SU(2).$$

(Notice that $\gamma \circ L_a \circ \gamma = R_{\gamma(a)}$, $a \in SU(2)$.) We have

$$\xi_j \circ L_{\gamma(a)} = \sum_{j'=1}^{2d+1} \rho_{jj'} \xi_{j'},$$

where the matrix $(\rho_{jj'})_{j,j'=1}^{2d+1}$ is orthogonal. The lemma follows.

With respect to the orthonormal basis in R_{2d} above, any $\xi \in R_{2d}$ can be written as $\xi = \sum_{j=1}^{2d+1} a_j \xi_j$, $a_j \in \mathbf{R}$, $j = 1, \ldots, 2d+1$. Applying the equivariant construction to all the polynomials involved, we obtain

$$f_\xi = \sum_{j=1}^{2d+1} a_j f_{\xi_j} = \sqrt{2d+1} \sum_{j=1}^{2d+1} \frac{a_j}{\sqrt{2d+1}} f_{\xi_j}.$$

Comparing this with (47), we see that $f_\xi = A f_{2d}$, where A: $(2d+1)R_{2d} \to R_{2d}$ is a linear map. In terms of the orthonormal basis in R_{2d}, A can be written in the block form

$$A = \sqrt{2d+1}\,[a_1, a_2, \ldots, a_{2d+1}],$$

where the jth block is a diagonal $(2d+1) \times (2d+1)$-matrix with diagonal element a_j, $j = 1, \ldots, 2d+1$. Using the DoCarmo-Wallach parametrization, we obtain $\langle f_\xi \rangle = C = A^\top A - I \in S^2((2d+1)R_{2d})$ where the jj'th block of C is a diagonal $(2d+1) \times (2d+1)$-matrix with diagonal element $c_{jj'} = (2d+1)a_j a_{j'} - \delta_{jj'}$, $j, j' = 1, \ldots, 2d+1$. In what follows, we always represent our points in $\mathcal{L}_3^{2d}$ in this form. This notation also naturally extends to the case when ξ is vector valued. In particular, it also applies to $\widehat{f}_\xi = f_{\widehat{\xi}}$. Letting $f_{\widehat{\xi}} = \widehat{A} f_{2d}$, the matrix $\widehat{A}$ and therefore $\langle \widehat{f} \rangle = \widehat{C} = \widehat{A}^\top \widehat{A} - I$ can be computed in terms of A.

To begin with a simple example, we first consider the case of quadratic eigenmaps, i.e., $d = 1$. The equivariant construction applied to the polynomial $\xi_3 = iab$ gives the quadratic eigenmap f_{ξ_3}: $S^3 \to S_{R_2}$ which is congruent to the Hopf map. Simple computation in the use of (46) now gives $\widehat{f}_{\xi_3} = \frac{1}{\sqrt{2}} \left(f_{-\xi_2}, f_{\xi_1} \right) : S^3 \to S_{2R_2}$ (with a component vanishing), and this is congruent to the complex Veronese map (the lift of the holomorphic imbedding $\mathbf{C}P \to \mathbf{C}P^2$). On the moduli, we have $\widehat{C} = -\frac{1}{2} C$ in agreement with the fact that $\mathcal{A}_{3,2}$ has eigenvalue $-\frac{1}{2}$ on $(\mathcal{E}_3^2)^{SU(2)} = R'_4$.

We skip the case of quartic minimal immersions as it has been treated thoroughly in [16]. We now put $d = 3$ and study the effect of $\mathcal{A}_{3,6}$ on $(\mathcal{F}_3^6)^{SU(2)} = R'_8 \oplus R'_{12}$. As noted in Section 1, the eigenvalues of $\mathcal{A}_{3,6}$ on R'_8 and on R'_{12} are $\frac{1}{6}$

and $-\frac{3}{4}$. We also have $\xi_{Tet} = \xi_3$. As before, using (46), we obtain

$$\hat{\xi}_{Tet} = \left(-\frac{1}{2\sqrt{2}}\xi_1 + \frac{\sqrt{30}}{12}\xi_5, \frac{1}{2\sqrt{2}}\xi_2 + \frac{\sqrt{30}}{12}\xi_6, \frac{1}{\sqrt{3}}\xi_4\right).$$

The corresponding matrices are

$$A = \sqrt{7}[0,0,1,0,0,0,0]$$

and

$$\hat{A} = \sqrt{7}\begin{bmatrix} -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{\sqrt{30}}{12} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{\sqrt{30}}{12} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \end{bmatrix}.$$

The matrices C and $\hat{C}$ are linearly independent, and they span the $\mathcal{A}_{3,6}$-invariant plane $\mathcal{A}_{Tet}$. Using C and $\hat{C}$ as a basis, we see that the intersection of $\mathcal{A}_{Tet}$ with $\mathcal{M}_3^6$ consists of those linear combinations $uC + v\hat{C}$ for which $uC + v\hat{C} + I$ is positive semidefinite. This is easily resolved, and we obtain the isosceles triangle with vertices C, $\hat{C}$ and $C' = -\frac{1}{3}C - \hat{C}$. (The three sides of the triangle are determined by $\det(uC + v\hat{C} + I) = 0$.) The matrix $\hat{\hat{C}}$ is in the interior of the triangle. The assertion about the range of the corresponding minimal immersions follows by working out the rank of $uC + v\hat{C} + I$ for the vertices and (open) edges of the triangle. As for the first statement in the corollary to Theorem 3, we see that the point $\frac{1}{9}C - \frac{2}{3}\hat{C}$ is on the side of the triangle with vertices C and C', and it is an eigenvector of $\mathcal{A}_{3,6}$ with eigenvalue $-\frac{3}{4}$, thereby it belongs to R'_{12}. The corresponding minimal immersion $f\colon S^3 \to S_{4R_6}$ is therefore isotropic of order 2. The explicit form of f can be obtained by inverting the DoCarmo-Wallach parametrization, i.e., by working out $\sqrt{\frac{1}{9}C - \frac{2}{3}\hat{C} + I}$ and precomposing it with the standard minimal immersion f_6.

Finally, we treat the case $d = 4$. We have

$$(\mathcal{F}_3^8)^{SU(2)} = R'_8 \oplus R'_{12} \oplus R'_{16}$$

and the eigenvalues of $\mathcal{A}_{3,8}$ on the terms of the right-hand side are $\frac{1}{2}$, $-\frac{1}{20}$, and $-\frac{4}{5}$. As in Section 1, in terms of the chosen orthonormal basis in R_8, the octahedral minimal immersion Oct corresponds to

$$\xi_{Oct} = \frac{\sqrt{5}}{2\sqrt{3}}\xi_1 + \frac{\sqrt{7}}{2\sqrt{3}}\xi_9.$$

Using (46), we have

$$\hat{\xi}_{Oct} = \left(\frac{\sqrt{2}}{4\sqrt{3}}\xi_3 - \frac{\sqrt{14}}{4\sqrt{3}}\xi_7, -\frac{\sqrt{2}}{4\sqrt{3}}\xi_4 - \frac{\sqrt{14}}{4\sqrt{3}}\xi_8, -\frac{1}{\sqrt{3}}\xi_2\right).$$

Applying once more (46) to this, we obtain

$$\begin{aligned}\hat{\hat{\xi}}_{Oct} = &\frac{1}{960\sqrt{21}}\\ &\times \Big(-336\sqrt{5}\xi_5 - 48\sqrt{35}\xi_1 + 1680\xi_9, -48\sqrt{35}\xi_2 + 336\sqrt{5}\xi_6,\\ &- 72\sqrt{70}\xi_4 + 168\sqrt{10}\xi_8, 48\sqrt{35}\xi_2 + 336\sqrt{5}\xi_6,\\ &- 48\sqrt{35}\xi_1 - 672\sqrt{5}\xi_5 - 1680\xi_9, -72\sqrt{70}\xi_3 - 168\sqrt{10}\xi_7,\\ &- 96\sqrt{70}\xi_4, -96\sqrt{70}\xi_3, -384\sqrt{35}\xi_1\Big).\end{aligned}$$

The matrices corresponding to ξ_{Oct} and $\hat{\xi}_{Oct}$ are

$$A = 3\left[\frac{\sqrt{5}}{2\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0, \frac{\sqrt{7}}{2\sqrt{3}}\right]$$

and

$$\hat{A} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{4\sqrt{3}} & 0 & 0 & 0 & -\frac{\sqrt{14}}{4\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{4\sqrt{3}} & 0 & 0 & 0 & -\frac{\sqrt{14}}{4\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices C, $\hat{C}$ and $\hat{\hat{C}}$ are linearly independent, and they span the $\mathcal{A}_{3,8}$-invariant 3-space $\mathcal{A}_{Oct}$. The intersection of $\mathcal{A}_{Oct}$ with $\mathcal{M}_3^8$ is a tetrahedron with vertices C, $\hat{C}$, and

$$\begin{aligned} C' &= \frac{4}{21}C - \frac{9}{7}\hat{C} - \frac{40}{21}\hat{\hat{C}} \\ C'' &= -\frac{11}{14}C + \frac{3}{7}\hat{C} + \frac{20}{7}\hat{\hat{C}}. \end{aligned}$$

(The computations are tedious but elementary; the use of a computer algebra system is recommended.) $\hat{\hat{C}}$ is in the interior of the tetrahedron. The metric properties of the tetrahedron can be easily derived using the orthogonal decomposition

$C = C_1 + C_2 + C_3$, $C_l \in R'_{4l+4}$, $l = 1, 2, 3$, and solving the system

$$C = C_1 + C_2 + C_3$$
$$\widehat{C} = \frac{1}{2}C_1 - \frac{1}{20}C_2 - \frac{4}{5}C_3$$
$$\widehat{\widehat{C}} = \frac{1}{4}C_1 + \frac{1}{400}C_2 + \frac{16}{25}C_3.$$

The range (in multiples of R_8) of the minimal immersions corresponding to the vertices, (open) edges, and (open) faces are summarized in the following tables:

Vertex	C	$\widehat{C}$	C'	C''
Range	1	3	2	3

Edge	$[C\widehat{C}]$	$[CC']$	$[CC'']$	$[\widehat{C}C']$	$[\widehat{C}C'']$	$[C'C'']$
Range	4	3	4	5	6	5

Face	$[C\widehat{C}C']$	$[C\widehat{C}C'']$	$[CC'C'']$	$[\widehat{C}C'C'']$
Range	6	7	6	8

Theorem 3 follows. (Notice additivity of the ranges.) Finally, note that (7) implies the second statement in Corollary to Theorem 3.

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