Eigenmaps and the Space of Minimal Immersions between Spheres

Gabor Toth

Abstract. In 1971 DoCarmo and Wallach gave a lower bound for the dimension of the space of minimal immersions between spheres and they believed that the lower estimate was sharp. We give here a different approach using conformal fields and eigenmaps; determine the exact dimension of this space and conclude that their conjecture is true.

1. Introduction and preliminaries. Let V be a Euclidean vector space. An isometric immersion $f : S^m_k \to S_V$ of the m-sphere S^m_k of constant curvature k into the unit sphere S_V of curvature 1 of V is *minimal* if the mean curvature of f vanishes [12]. Let $\mathcal{S}(m, k)$ denote the space of full minimal isometric immersions $f : S^m_k \to S_V$, for various V. (Fullness means that the image is not contained in any great hypersphere.) Composing isometric immersions with isometries between the ranges gives rise to an equivalence relation $\cong$ on $\mathcal{S}(m, k)$.

A theorem of Takahashi [11] implies that, for fixed m, the set of $k > 0$ such that $\mathcal{S}(m, k) \neq \emptyset$ is infinite discrete: $\{k_p\}_{p=1}^{\infty}$. In 1967, Calabi [1] proved that any isometric immersion $f : S^2_{k_p} \to S_V$ is equivalent to the (generalized) Veronese map, implying that $\mathcal{S}(2, k_p)/\cong$ is a single point. In 1971, DoCarmo and Wallach [3] showed that $\mathcal{S}(m, k_p)/\cong$ can be parametrized by a compact convex body $\mathcal{M}^p_m$ contained in a finite dimensional vector space $\mathcal{F}^p_m$. The parametrization is continuous on $\mathcal{M}^p_m$ and smooth in the interior of $\mathcal{M}^p_m$. They derived a positive lower estimate $d(m, k_p)$ on the dimension of $\mathcal{M}^p_m$ and conjectured that it is sharp, i.e. actually $d(m, k_p) = \dim \mathcal{M}^p_m$. The main result of this paper is a proof of this conjecture.

The structure of the boundary of $\mathcal{M}^p_m$ is subtle. In 1992, DeTurck and Ziller [2] gave many interesting examples of minimal isometric immersions $f : S^m_{k_p} \to S_V$, which correspond to boundary points of $\mathcal{M}^p_m$. These 'boundary minimal immersions' possess rich geometry as they are equivariant with respect to proper subgroups of $SO(m+1)$ that act transitively on S^m. For further work in this direction, cf. [5].

Let $f : S^m_k \to S_V$ be an isometric minimal immersion. As noted above, $k = k_p$ for some $p = 1, 2, \ldots$. More precisely, the components $\varphi \circ f$, $\varphi \in V^*$, of f are eigenfunctions of the Laplacian on S^m_k with eigenvalue m. In particular, $k = k_p = m/\lambda_p$, where $\lambda_p = p(p+m-1)$ is the p^{th} eigenvalue of the Laplacian on $S^m = S^m_1$. We now scale the metric on $S^m_{k_p}$ to curvature 1 so that the isometric immersion $f : S^m_{k_p} \to S_V$ becomes *homothetic*, i.e. $f : S^m \to S_V$ satisfies

$$\langle f_*(X), f_*(Y)\rangle = \left(\frac{\lambda_p}{m}\right)\langle X, Y\rangle \tag{1.1}$$

for any vector fields X and Y on S^m. Moreover, since f is minimal, its components become spherical harmonics of order p on S^m, i.e. eigenfunctions of the Laplacian on S^m with eigenvalue λ_p. Note that a spherical harmonic of order p on S^m is nothing but the restriction (to S^m) of a harmonic homogeneous polynomial of degree p in the variables $x_0, \ldots, x_m$, $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$.

The key to pin down the structure of the space $\mathcal{S}(m, k_p)$ of isometric minimal immersions $f : S^m_{k_p} \to S_V$, or what is the same, the space $\mathcal{M}^p_k$ parametrizing the homothetic minimal immersions $f : S^m \to S_V$ with homothety λ_p/m is to introduce a wider class of maps, called eigenmaps, as follows.

A map $f : \mathbf{R}^{m+1} \to V$ into a Euclidean vector space V is a *p-form* if the components $\varphi \circ f$, $\varphi \in V^*$, of f are homogeneous polynomials of degree p in the variables $x_0, \ldots, x_m$. f is *spherical* if it maps the unit sphere S^m to the unit sphere S_V of V. In this case, we say that (the restriction) $f : S^m \to S_V$ is also a *p-form*.

A p-form f is *harmonic* if the components of $f : \mathbf{R}^{m+1} \to V$ are harmonic functions in the variables $x_0, \ldots, x_m$. If, in addition, f is spherical then the components of $f : S^m \to S_V \subset V$ are spherical harmonics of order p, i.e. eigenfunctions of the spherical Laplacian on S^m with eigenvalue $\lambda_p = p(p+m-1)$. In this case, we say that $f : S^m \to S_V$ is an *eigenmap with eigenvalue* λ_p. By the above, a homothetic immersion $f : S^m \to S_V$ is minimal iff it is an eigenmap with eigenvalue λ_p for some p. In this case the homothety constant is λ_p/m so that (1) is satisfied. Note also that eigenmaps are harmonic in the sense of Eells-Sampson [4], in fact, an eigenmap with eigenvalue λ_p is nothing but a harmonic map with constant energy density $\lambda_p/2$.

A p-form $f : S^m \to S_V$ is *full* if its image is not contained in any proper great sphere. Two p-forms $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ are *equivalent*, written as $f_1 \cong f_2$, if there exists an isometry $U : V_1 \to V_2$ such that $f_2 = U \circ f_1$.

For fixed m and p, the equivalence classes of full eigenmaps $f : S^m \to S_V$ (for various V) with eigenvalue λ_p can be parametrized by a compact convex body $\mathcal{L}^p_m$ in a finite dimensional representation space of $SO(m+1)$. We now briefly recall the construction of the parameter space $\mathcal{L}^p = \mathcal{L}^p_m$; for details cf. [7].

(Since we will mostly work over a fixed domain S^m, the subscript will often be suppressed.) Let $\mathcal{H}^p = \mathcal{H}^p_m$ denote the space of spherical harmonics of order p on S^m. Let $\{f_p^j\}_{j=0}^{n(p)} \subset \mathcal{H}^p$ be an orthonormal basis with respect to the normalized L_2-scalar product

$$\langle h, h' \rangle = \frac{n(p)+1}{\mathrm{vol}(S^m)} \int_{S^m} h\, h'\, v,$$

where v is the volume form on S^m, $\mathrm{vol}(S^m) = \int_{S^m} v$ is the volume of S^m and

$$n(p) + 1 = \dim \mathcal{H}^p = (m + 2p - 1)\frac{(m+p-2)!}{p!(m-1)!}.$$

The standard minimal immersion $f_p : S^m \to S_{\mathcal{H}^p}$ is the full eigenmap with eigenvalue λ_p defined by

$$f_p(x) = \sum_{j=0}^{n(p)} f_p^j(x) f_p^j, \quad x \in S^m.$$

f_p clearly does not depend on the orthonormal basis chosen.

Given a full eigenmap $f : S^m \to S_V$ with eigenvalue λ_p, there exists a linear map $A : \mathcal{H}^p \to V$ such that $f = A \circ f_p$. We associate to f the symmetric linear endomorphism

$$\langle f \rangle = A^\top A - I \in S^2(\mathcal{H}^p), \qquad (I = \text{ identity }).$$

The correspondence $f \mapsto \langle f \rangle$ gives a parametrization of the space of equivalence classes of full eigenmaps $f : S^m \to S_V$ with eigenvalue λ_p by the compact convex body

$$\mathcal{L}^p_m = \{C \in \mathcal{E}^p_m \mid C + I \geq 0\}$$

in the linear subspace

$$\mathcal{E}^p_m = \text{ span}\{f_p(x) \odot f_p(x) \mid x \in S^m\}^\perp \subset S^2(\mathcal{H}^p).$$

Here '$\geq$' stands for positive semidefinite, '$\odot$' is the symmetric tensor product and the orthogonal complement is taken with respect to the standard scalar product $\langle C, C' \rangle = \text{ trace}\,(C \cdot C')$, $C, C' \in S^2(\mathcal{H}^p)$.

f_p is equivariant with respect to the homomorphism $\rho_p : SO(m+1) \to SO(\mathcal{H}^p)$ that is just the orthogonal $SO(m+1)$-module structure on $\mathcal{H}^p$ defined by $g \cdot h = h \circ g^{-1}$, $g \in SO(m+1)$ and $h \in \mathcal{H}^p$. Equivariance means that

$$f_p \circ g = \rho_p(g) \cdot f_p, \quad g \in SO(m+1).$$

$\mathcal{E}^p$ is a submodule of $S^2(\mathcal{H}^p)$, where the latter is endowed with the module structure induced from that of $\mathcal{H}^p$. Moreover, $\mathcal{L}^p \subset \mathcal{E}^p$ is an invariant subset. In fact, for a full eigenmap $f : S^m \to S_V$ with eigenvalue λ_p, we have

$$g \cdot \langle f \rangle = \langle f \circ g^{-1} \rangle, \quad g \in SO(m+1).$$

The work of DoCarmo-Wallach [3,12] gives the decomposition of $S^2(\mathcal{H}^p)\otimes_{\mathbf{R}}\mathbf{C}$ into irreducible components. (Since their proof contains an essential ingredient for our purposes here, we indicate the idea of the proof in Section 2.) We have, for $m \geq 3$:

$$S^2(\mathcal{H}^p) \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\triangle_0^p;\, u,v \text{ even}} V_m^{(u,v,0,\dots,0)}. \tag{1.2}$$

Here $\triangle_0^p \subset \mathbf{R}^2$ denotes the closed convex triangle with vertices $(0,0)$, (p,p) and $(2p,0)$ and $V_m^{(u_1,\dots,u_d)}$, $d = [|(m+1)/2|]$, stands for the complex irreducible $SO(m+1)$-module with highest weight vector $(u_1,\dots,u_d)$ whose components are with respect to the standard maximal torus in $SO(m+1)$. (Note that, for $m=3$, $V_m^{(u,v,0,\dots,0)}$ means $V_3^{(u,v)} \oplus V_3^{(u,-v)}$ unless $v=0$.) Moreover, $\mathcal{E}_p \otimes_{\mathbf{R}} \mathbf{C}$ is nontrivial iff $m \geq 3$ and $p \geq 2$ and, in this case, it consists of those components of the symmetric square that are not class 1 with respect to $(SO(m+1), SO(m))$. Hence the decomposition of $\mathcal{E}^p \otimes_{\mathbf{R}} \mathbf{C}$ is obtained by restricting the summation above to the subtriangle $\triangle_1^p \subset \triangle_0^p$ whose vertices are $(2,2)$, (p,p) and $(2p-2,2)$. Thus

$$\mathcal{E}^p \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\triangle_1^p;\, u,v \text{ even}} V_m^{(u,v,0,\dots,0)}. \tag{1.3}$$

Adding condition (1) to those defining $\mathcal{L}_p$, we obtain that the linear slice

$$\mathcal{M}^p = \mathcal{L}^p \cap \mathcal{F}^p,$$

where

$$\mathcal{F}^p = \operatorname{span}\{(f_p)_*(X)\check{}\odot (f_p)_*(Y)\check{}\,|\, X,Y \in T(S^m)\}^\perp$$

parametrizes the equivalence classes of full homothetic minimal immersions with homothety λ_p/m. Here $\check{}: T(V) \to V$ is the canonical map that translates tangent vectors to the origin. It follows that $\mathcal{M}^p$ is also a compact convex body. DoCarmo and Wallach [3,12] showed that $\mathcal{F}^p$ is nontrivial iff $m \geq 3$ and $p \geq 4$ and, in this case, we have

$$\mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C} \supset \sum_{(u,v)\in\triangle_2^p;\, u,v \text{ even}} V_m^{(u,v,0,\dots,0)}, \tag{1.4}$$

where $\triangle_2^p \subset \triangle_1^p$ is the subtriangle with vertices $(4,4)$, (p,p) and $(2p-4,4)$. They conjectured that the lower bound in (4) is actually sharp, i.e. that the modules

$$V_m^{(2\ell,2,0,\dots,0)}, \quad \ell = 1,\dots,p-1, \tag{1.5}$$

corresponding to the base of $\triangle_1^p$ are not components of $\mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C}$. In what follows we refer to this as the exact dimension conjecture (although it is actually about the space $\mathcal{F}^p$ itself). Note that $\dim V_m^{(u_1,\dots,u_d)}$ can be computed explicitly using the Weyl dimension formula.

The purpose of this paper is to show that the exact dimension conjecture is true:

Theorem 1. *For $m \geq 3$ and $p \geq 4$,*

$$V_m^{(2,2,0,\ldots,0)}, \ldots, V_m^{(2p-2,2,0,\ldots,0)}$$

are not components of $\mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C}$ so that we have

$$\mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\triangle_2^p; u,v \text{ even}} V_m^{(u,v,0,\ldots,0)}.$$

For $m = 3$ and $p = 4$ this was proved by Muto in [6] by explicit tensor computation. Our method is geometric; it uses eigenmaps and their effect on conformal fields and, in fact, it provides an analytic and geometric description of the eigenmaps parametrized by the components in (5). For the proof, we need three technical tools. First, in Section 2, we describe two operators on eigenmaps that raise and lower the degree. These have been studied in [8, 9] but our approach here concentrates on the connection between the degree raising-lowering operators and the DoCarmo-Wallach differential operator used to decompose the tensor product $\mathcal{H}^p \otimes \mathcal{H}^q$ leading to (2). The second tool, given in Section 3, is to study the effect of eigenmaps on conformal fields. This reduces the whole problem to finding, for fixed m and p, a single eigenmap $f : S^m \to S_V$ which satisfies a harmonicity property of a quadratic form in the derivatives of the components of f. As for the third tool, in Section 4 we show how the nonhomothetic property of an eigenmap can be carried over to eigenmaps of higher degree. Finally the examples needed to finish the proof of Theorem 1 are worked out in Section 5.

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2. Raising and lowering the degree. To decompose $S^2(\mathcal{H}^p)$ into irreducible components, DoCarmo and Wallach [3,12] first derived the recurrence formula

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{r=0}^{p} V_m^{(p+q-r,r,0,\ldots,0)} \oplus \left(\mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}\right), \quad p \geq q \geq 1, \ m \geq 3. \tag{2.6}$$

(In what follows, for notational simplicity, we denote $\mathcal{H}^p$ and its complexification by the same symbol. Since the representations we encounter here are absolutely irreducible, this will not lead to confusion.) The key role in the proof is played by the differential operator

$$D : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}$$

defined by

$$D(h \otimes h') = \sum_{i=0}^{m} \frac{\partial h}{\partial x_i} \otimes \frac{\partial h'}{\partial y_i}.$$

In fact, Young's theory applied to $\ker D$ gives the first summand on the right-hand-side of (6) and surjectivity of D is established by a careful induction argument with respect to m using the Branching Rule restricting representations from $SO(m+1)$ to the subgroup $SO(m)$.

Setting $p = q$, we first describe the restriction $D|\mathcal{L}^p$ in terms of eigenmaps. Let H denote the harmonic projection operator [10]. H is the orthogonal projection from the vector space $\mathcal{P}^p$ of homogeneous polynomials in $m+1$ variables of degree p onto the linear subspace of harmonic polynomials.

Let $f : S^m \to S_V$ be a λ_p-eigenmap. We define the p-forms

$$f^{\pm} : \mathbf{R}^{m+1} \to V \otimes \mathcal{H}^1$$

by

$$(2.7) \qquad f^+ = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}} \sum_{i=0}^{m} H(x_i f) \otimes y_i \quad \text{and} \quad f^- = \sqrt{\frac{2}{\lambda_{2p}}} \sum_{i=0}^{m} \frac{\partial f}{\partial x_i} \otimes y_i.$$

The harmonic projection formula

$$(2.8) \qquad H(x_i f) = x_i f - \frac{\rho^2}{2p+m-1} \frac{\partial f}{\partial x_i}, \quad \rho^2 = |x|^2,$$

along with homogeneity of f easily implies that $f^{\pm}$ are spherical so that we obtain eigenmaps

$$f^{\pm} : S^m \to S_{V \otimes \mathcal{H}^1}$$

with eigenvalue $\lambda_{p\pm 1}$.

Theorem 2. *Let $f : S^m \to S_V$ be a full eigenmap with eigenvalue λ_p. Then we have*

$$D(\langle f \rangle) = \left(\frac{\lambda_{2p}}{2} \right) \langle f^- \rangle$$

and

$$D^{\top}(\langle f \rangle) = (p+1)^2 \left(\frac{\lambda_{2p}}{2\lambda_p} \right) \langle f^+ \rangle.$$

The proof will be accomplished is several steps. We first claim that $f_p^{\pm} \cong f_{p\pm 1}$. Indeed, since the Laplace operator commutes with the isometries on S^m it also commutes with the harmonic projection operator H. It follows that $f_p^{\pm} : S^m \to S_{\mathcal{H}^p \otimes \mathcal{H}^1}$ are equivariant with respect to the $SO(m+1)$-module structure on $\mathcal{H}^p \otimes \mathcal{H}^1$. This translates into $\langle f_p^{\pm} \rangle \in \mathcal{L}^{p\pm 1}$ being left fixed by $SO(m+1)$. Since $\mathcal{E}^{p\pm 1}$ have no trivial summands, $\langle f_p^{\pm} \rangle$ correspond to the origin and the equivalence follows.

To make this equivalence explicit, we introduce the $SO(m+1)$-module monomorphisms

$$\iota_\pm : \mathcal{H}^{p\pm 1} \to \mathcal{H}^p \otimes \mathcal{H}^1$$

by

$$\iota_-(h') = c_p^- \sum_{i=0}^m H(x_i h') \otimes y_i, \quad h' \in \mathcal{H}^{p-1},$$

and

$$\iota_+(h'') = c_p^+ \sum_{i=0}^m \frac{\partial h''}{\partial x_i} \otimes y_i, \quad h'' \in \mathcal{H}^{p+1},$$

where the requirement

(2.9) $$\iota_\pm^\top \iota_\pm = I$$

determines the value of the constants $c_p^\pm$. We now have

(2.10) $$\iota_\pm(f_{p\pm 1}(x)) = f_p^\pm(x), \quad x \in S^m.$$

Indeed, the linear span of the image of $f_p^\pm$ in $\mathcal{H}^p \otimes \mathcal{H}^1$ is a copy of $\mathcal{H}^{p\pm 1}$. By (6), $\mathcal{H}^p \otimes \mathcal{H}^1 = \mathcal{H}^{p-1} \oplus \mathcal{H}^{p+1} \oplus V^{(p,1,0,\dots,0)}$, in particular, the multiplicities $m[\mathcal{H}^{p\pm 1} : \mathcal{H}^p \otimes \mathcal{H}^1] = 1$ and (10) follows.

Let $f : S^m \to S_V$ be a full eigenmap with eigenvalue λ_p. Setting $f = A \circ f_p$, we have $f^\pm = (A \otimes I) f_p^\pm$. Thus, by (9), we have

$$\begin{aligned}\langle f^\pm \rangle &= \iota_\pm^\top (A^\top \otimes I)(A \otimes I)\iota_\pm - I \\ &= \iota_\pm^\top ((A^\top A - I) \otimes I)\iota_\pm \\ &= \iota_\pm^\top (\langle f \rangle \otimes I)\iota_\pm.\end{aligned}$$

In view of this we define

$$\Phi_p^\pm : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p\pm 1})$$

by

$$\Phi_p^\pm(C) = \iota_\pm^\top \circ (C \otimes I) \circ \iota_\pm.$$

Clearly, $\Phi_\pm$ are homomorphisms of $SO(m+1)$-modules. The previous computation amounts to

$$\Phi_p^\pm(\langle f \rangle) = \langle f^\pm \rangle,$$

in particular, $\Phi_p^\pm(\mathcal{L}^p) \subset \mathcal{L}^{p\pm 1}$.

To complete the proof of Theorem 2, we show that, on $S^2(\mathcal{H}^p)$,

(2.11) $$D = \left(\frac{\lambda_{2p}}{2}\right) \Phi_p^-$$

and

$$D^{\top} = (p+1)^2 \left(\frac{\lambda_{2p}}{2\lambda_p}\right) \Phi_p^{+}. \tag{2.12}$$

Combining these with the results of DoCarmo and Wallach, we obtain the following:

Theorem 3. *Φ_p^{+} is injective and restricts to an equivariant imbedding of $\mathcal{L}^p$ into $\mathcal{L}^{p+1}$. Φ_p^{-} is surjective with (complexified) kernel*

$$\sum_{r=0}^{[|p/2|]} V_m^{(2p-2r,2r,0,\ldots,0)}.$$

For the forthcoming computations we need the following identities:

$$\left\langle h, \frac{\partial h''}{\partial x_i}\right\rangle = \mu_p \langle H(x_i h), h''\rangle, \tag{2.13}$$

$$\iota_{-}^{\top}(h \otimes y_i) = \sqrt{\frac{2}{\lambda_{2p}}}\,\frac{\partial h}{\partial x_i}, \tag{2.14}$$

$$\iota_{+}^{\top}(h \otimes y_i) = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}}\, H(x_i h), \tag{2.15}$$

where $h \in \mathcal{H}^p$, $h'' \in \mathcal{H}^{p+1}$ and

$$\mu_p = (p+1)\frac{\lambda_{2p}}{2\lambda_p}.$$

(13) can be obtained by direct integration using the fact that spherical harmonics of different order are L^2-orthogonal. (14)-(15) are direct consequences of (13). Indeed, take the scalar product of both sides of (10) with $h \otimes y_i$ and transpose. (To indicate a different proof, first note that $h \mapsto H(x_i h)$, $h \in \mathcal{H}^p$, defines an $SO(m+1)$-module homomorphism of $\mathcal{H}^p \otimes \mathcal{H}^1$ onto $\mathcal{H}^{p+1}$. By (6), this homomorphism is a constant multiple of $\iota_{+}^{\top}$. The same applies to (14). Now (13) follows from (14) or (15).) Finally, note that (14)-(15) combined with (9) gives the value of $c_p^{\pm}$ as:

$$c_p^{+} = \frac{1}{p+1}\sqrt{\frac{2\lambda_p}{\lambda_{2p}}} \quad \text{and} \quad c_p^{-} = \frac{p}{\sqrt{\lambda_{2p}/2}}\,\frac{\lambda_{2(p-1)}}{2\lambda_{p-1}}. \tag{2.16}$$

Turning to the proof of (11), we let $C \in S^2(\mathcal{H}^p)$ and compute

$$D(C) = D\Big(\sum_{j,\ell=0}^{n(p)} c_{j\ell} f_p^j \otimes f_p^\ell\Big)$$

$$= \sum_{i=0}^{m} \sum_{j,\ell=0}^{n(p)} c_{j\ell} \frac{\partial f_p^j}{\partial x_i} \otimes \frac{\partial f_p^\ell}{\partial y_i}$$

$$= \sum_{i=0}^{m} \sum_{j,\ell=0}^{n(p)} \sum_{r,s=0}^{n(p-1)} c_{j\ell} \left\langle \frac{\partial f_p^j}{\partial x_i}, f_{p-1}^r \right\rangle \left\langle \frac{\partial f_p^\ell}{\partial y_i}, f_{p-1}^s \right\rangle f_{p-1}^r \otimes f_{p-1}^s$$

$$= \mu_{p-1}^2 \sum_{i=0}^{m} \sum_{r,s=0}^{n(p-1)} \langle C(H(x_i f_{p-1}^r)), H(y_i f_{p-1}^s)\rangle f_{p-1}^r \otimes f_{p-1}^s$$

$$= \mu_{p-1} \sum_{i=0}^{m} \sum_{r,s=0}^{n(p-1)} \left\langle \frac{\partial}{\partial x_i} C(H(x_i f_{p-1}^r)), f_{p-1}^s \right\rangle f_{p-1}^r \otimes f_{p-1}^s.$$

Thus

$$D(C)(h') = \mu_{p-1} \sum_{i=0}^{m} \frac{\partial}{\partial x_i} C(H(x_i h')), \quad h' \in \mathcal{H}^{p-1}. \tag{2.17}$$

On the other hand,

$$\begin{aligned}\Phi_p^-(C)(h') &= \iota_-^\top (C \otimes I)\iota_-(h') \\ &= (p\lambda_{2(p-1)}/(\lambda_{2p}\lambda_{p-1})) \sum_{i=0}^{m} \frac{\partial}{\partial x_i} C(H(x_i h')).\end{aligned}$$

Comparing this with (17), we find that (11) follows.

Finally, we prove (12) by showing that, up to a constant multiple, Φ^+ is the transpose of Φ^-. We compute, for $C \in S^2(\mathcal{H}^p)$ and $C' \in S^2(\mathcal{H}^{p-1})$:

$$\langle \Phi_p^-(C), C' \rangle = \sum_{l=0}^{n(p-1)} \langle (\iota_-^\top (C \otimes I)\iota_-) f_{p-1}^\ell, C' f_{p-1}^\ell \rangle$$

$$= \left(\frac{p\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}} \right) \sum_{i=0}^{m} \sum_{l=0}^{n(p-1)} \left\langle \frac{\partial}{\partial x_i} (C(H(x_i f_{p-1}^\ell)), C' f_{p-1}^\ell \right\rangle$$

$$= \left(\frac{p\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}} \right) \sum_{i=0}^{m} \sum_{j=0}^{n(p)} \sum_{l=0}^{n(p-1)} \langle H(x_i f_{p-1}^\ell), f_p^j \rangle \left\langle \frac{\partial}{\partial x_i} (C f_p^j), C' f_{p-1}^\ell \right\rangle$$

$$= \left(\frac{2}{\lambda_{2p}} \right) \sum_{i=0}^{m} \sum_{j=0}^{n(p)} \sum_{l=0}^{n(p-1)} \left\langle f_{p-1}^\ell, \frac{\partial f_p^j}{\partial x_i} \right\rangle \left\langle \frac{\partial}{\partial x_i} (C f_p^j), C' f_{p-1}^\ell \right\rangle$$

$$= \left(\frac{p^2}{\lambda_{2p}}\right)\left(\frac{\lambda_{2(p-1)}}{\lambda_{p-1}}\right)\sum_{i=0}^{m}\sum_{j=0}^{n(p)}\langle \iota_+(Cf_p^j), (C'\otimes I)\iota_+ f_p^j\rangle$$

$$= \left(\frac{p^2}{\lambda_{2p}}\right)\left(\frac{\lambda_{2(p-1)}}{\lambda_{p-1}}\right)\langle C, \Phi_{p-1}^+(C')\rangle.$$

Thus, (12) follows.

3. Conformal fields and eigenmaps. Let $f : S^m \to S_V$ be a full eigenmap with eigenvalue λ_p. We define the symmetric 2-tensor $\Psi(f)$ on S^m by

$$\Psi(f)(X,Y) = \langle f_*(X), f_*(Y)\rangle - \left(\frac{\lambda_p}{m}\right)\langle X, Y\rangle, \tag{3.18}$$

where X and Y are vector fields on S^m. By definition, $\Psi(f) = 0$ iff f is homothetic. In what follows, unless stated otherwise, we will always consider f a harmonic p-form $f : \mathbf{R}^{m+1} \to V$. Then (18) defines $\Psi(f)$ as a symmetric 2-tensor on $\mathbf{R}^{m+1}$.

We now restrict $\Psi(f)$ to conformal fields on S^m. Given $a \in \mathbf{R}^{m+1}$, the *conformal field* X^a is a vector field on S^m defined by

$$(X_x^a)\check{} = a - \langle a, x\rangle x, \quad x \in S^m.$$

Setting $\Psi(f)(a,b) = \Psi(f)(X^a, X^b)$, we obtain that $\Psi(f)(a,b) = 0$ for all $a, b \in \mathbf{R}^{m+1}$, iff f is homothetic. This is because pointwise the conformal fields span each tangent space.

We now extend the conformal field X^a to $\mathbf{R}^{m+1}$ by

$$(X_x^a)\check{} = a - \frac{\langle a, x\rangle}{\rho^2}x, \quad x \in \mathbf{R}^{m+1},\ \rho^2 = |x|^2 \tag{3.19}$$

and, on $\mathbf{R}^{m+1}$, define

$$\Psi(f)(a,b) = \langle f_*(X^a), f_*(X^b)\rangle - \left(\frac{\lambda_p}{m}\right)\langle X^a, X^b\rangle \rho^{2(p-1)}.$$

Lemma 1. *Given $a, b \in \mathbf{R}^{m+1}$, $\Psi(f)(a,b)$ is a homogeneous polynomial of degree $2p-2$.*

Proof. Let ∂_a, $a \in \mathbf{R}^{m+1}$, denote the directional derivative at a. We claim that

$$\Psi(f)(a,b) = -\langle f, \partial_a\partial_b f\rangle + p(p-1) \times \left(\left(1+\frac{1}{m}\right)\langle a, x\rangle\langle b, x\rangle - \left(\frac{1}{m}\right)\langle a, b\rangle\rho^2\right)\rho^{2(p-2)}. \tag{3.20}$$

Lemma 1 clearly follows from this. This formula will be a useful computational tool in the sequel; note for example that the coefficient of $\rho^{2(p-2)}$ is harmonic. Turning to the proof of (20), we first note that

$$f_*(X^a)\check{} = \partial_a f - p\frac{\langle a, x\rangle}{\rho^2} f.$$

and

$$\begin{aligned}\langle f, \partial_a f\rangle &= \frac{1}{2}\,\partial_a |f|^2 \\ &= \frac{1}{2}\,\partial_a \rho^{2p} = p\langle a, x\rangle \rho^{2(p-1)}.\end{aligned}$$

Using these, straightforward computation gives

$$\begin{aligned}\Psi(f)(a,b) = \langle \partial_a f, \partial_b f\rangle &+ \left(\frac{\lambda_p}{m} - p^2\right)\langle a, x\rangle\langle b, x\rangle \rho^{2(p-2)} \\ &- \left(\frac{\lambda_p}{m}\right)\langle a, b\rangle \rho^{2(p-1)}.\end{aligned} \tag{3.21}$$

Finally, we have

$$\langle \partial_a f, \partial_b f\rangle = -\langle f, \partial_a\partial_b f\rangle + \partial_a\langle f, \partial_b f\rangle.$$

We now work out the second term on the right-hand-side to arrive at (20). $\square$

Lemma 1 can be reformulated by saying that, for any full eigenmap $f : S^m \to S_V$ with eigenvalue λ_p, $\Psi(f)$ defines a symmetric bilinear map

$$\Psi(f) : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathcal{P}^{2p-2},$$

where, as usual, we identify $\mathbf{R}^{m+1}$ with $\mathcal{H}^1$. We now notice that $\Psi(f)$ depends only on the equivalence class of f. Indeed, setting $f = A \circ f_p$ and using that f_p is homothetic, i.e. it satisfies (1), we compute

$$\begin{aligned}\Psi(f)(X,Y) &= \langle A(f_p)_*(X)\check{}, A(f_p)_*(Y)\check{}\rangle \\ &\quad - \langle (f_p)_*(X)\check{}, (f_p)_*(Y)\check{}\rangle \\ &= \langle (A^\top A - I)(f_p)_*(X)\check{}, (f_p)_*(Y)\check{}\rangle \\ &= \langle \langle f\rangle (f_p)_*(X)\check{}, (f_p)_*(Y)\check{}\rangle.\end{aligned}$$

In view of this, for $C \in \mathcal{E}^p$, we define

$$\Psi(C) : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathcal{P}^{2p-2}$$

by

$$\Psi_p(C)(a,b) = \langle C(f_p)_*(X^a)\check{}, (f_p)_*(X^b)\check{}\rangle, \quad a, b \in \mathcal{H}^1.$$

Lemma 2. *We have*

$$\Psi(C)(a,b) = \langle \partial_a C f_p, \partial_b f_p\rangle. \tag{3.22}$$

Proof. This follows from (21) by taking the difference $\Psi(f)(a,b)-\Psi(f_p)(a,b)$, where the second term is zero because f_p is homothetic. □

Lemma 3. *We have*

$$\triangle^{p-1}\Psi(C)(a,b) = 0.$$

Proof. The components of f are harmonic homogeneous polynomials of degree p. Thus

$$\begin{aligned}\triangle^{p-1}\Psi(C)(a,b) &= 2^{p-1} \sum_{i_1,\dots,i_{p-1}=0}^{m} \left\langle C\partial_a \frac{\partial^{p-1} f}{\partial x_{i_1}\dots\partial x_{i_{p-1}}}, \partial_b \frac{\partial^{p-1} f}{\partial x_{i_1}\dots\partial x_{i_{p-1}}}\right\rangle \\ &= 2^{p-2}\partial_a\partial_b \sum_{i_1,\dots,i_{p-1}=0}^{m} \left\langle C \frac{\partial^{p-1} f}{\partial x_{i_1}\dots\partial x_{i_{p-1}}}, \frac{\partial^{p-1} f}{\partial x_{i_1}\dots\partial x_{i_{p-1}}}\right\rangle \\ &= \frac{1}{2}\partial_a\partial_b\triangle^{p-1}\langle Cf, f\rangle = 0.\end{aligned}$$

□

Lemma 4. $\Psi(C)$ *is traceless.*

Proof. Setting $a = b = e_i$ in (20) and summing up with respect to $i = 0,\dots,m$, the statement follows from the harmonicity of f. □

We now introduce the notation

$$\mathcal{P}_0^{2q} = \{\psi \in \mathcal{P}^{2q} \mid \triangle^q \psi = 0\}.$$

Lemma 3 says that, for $C \in \mathcal{E}^p$, $\Psi(C)$ defines a linear map

$$\Psi(C) : S^2(\mathcal{H}^1) \to \mathcal{P}_0^{2p-2}.$$

Moreover, the trivial summand in the decomposition

$$S^2(\mathcal{H}^1) = \mathcal{H}^0 \oplus \mathcal{H}^2$$

corresponds to the trace so, using Lemma 4, we finally arrive at the linear map

$$\Psi(C) : \mathcal{H}^2 \to \mathcal{P}_0^{2p-2}.$$

Equivalently, we will think of $\Psi(C)$ as an element of $\mathcal{P}_0^{2p-2} \otimes \mathcal{H}^2$.

Lemma 5. *For $g \in SO(m+1)$, we have*

$$\Psi(g \cdot C)(g{\cdot}a,\ g{\cdot}b) = \Psi(C)(a,b) \circ g^{-1}.$$

Proof. This is again a straightforward computation in the use of equivariance of f_p. Since $\mathcal{L}^p$ spans $\mathcal{E}^p$, we can also take $C = \langle f \rangle$ with $f : S^m \to S_V$ a full eigenmap with eigenvalue λ_p. Then the claim reduces to

$$\Psi(f \circ g^{-1})(g{\cdot}a,\ g{\cdot}b) = \Psi(f)(a,b) \circ g^{-1}. \tag{3.23}$$

We now use

$$\begin{aligned} g_*^{-1}(X^{ga})\check{}_x &= g^{-1}(ga - \langle ga, x\rangle x) \\ &= a - \langle a, g^{-1}x\rangle g^{-1}x = (X^a)\check{}_{g^{-1}x}, \quad x \in S^m. \end{aligned}$$

and arrive at (23)

By Lemma 5, we obtain that the correspondence $C \mapsto \Psi(C)$ defines a homomorphism

$$\Psi : \mathcal{E}^p \to \mathcal{P}_0^{2p-2} \otimes \mathcal{H}^2$$

between $SO(m+1)$-modules. We now decompose

$$\mathcal{P}_0^{2p-2} = \sum_{\ell=1}^{p-1} \mathcal{H}^{2\ell} \rho^{2(p-\ell-1)} \tag{3.24}$$

into irreducible $SO(m+1)$-modules. By (6), for each $\ell = 1, \ldots, p-1$, we have

$$\mathcal{H}^{2\ell} \otimes \mathcal{H}^2 = \sum_{(u,v)\in\triangle_0^{(2\ell,2)};\, u,v \text{ integer}} V_m^{(u,v,0,\ldots,0)}, \tag{3.25}$$

where $\triangle^{(2\ell,2)}$ has vertices $(2\ell-2,\ 0)$, $(2\ell, 2)$ and $(2\ell+2,\ 0)$. The only common term in (3) and (25) is $V_m^{(2\ell,2,0,\ldots,0)}$. On the other hand

$$\ker \Psi = \mathcal{F}^p$$

so that the lower estimate (4) of DoCarmo and Wallach follows immediately. Moreover, for $\ell = 1, \ldots, p-1$, $V_m^{(2\ell,2,0,\ldots,0)} \not\subset \mathcal{F}^p$ iff $\Psi | V_m^{(2\ell,2,0,\ldots,0)} \neq 0$ iff Ψ is injective on $V_m^{(2\ell,2,0,\ldots,0)}$.

Remark 1. DoCarmo and Wallach used Frobenius Reciprocity to prove that a lower bound for $\mathcal{F}^p$ is given by the sum of those irreducible components of $\mathcal{E}^p$ which, when restricted to $SO(m)(\subset SO(m+1))$, contain no copies of $\mathcal{H}^0$ and $\mathcal{H}^2$. Applying the Branching Rule to the components of $\mathcal{E}^p$ they arrived at (4). The method above gives an alternative way to prove (4) without the use of induced representations. Note also that again by the work of DoCarmo-Wallach, (3) can be obtained without the use of Frobenius Reciprocity. Another proof of (3) will be indicated at the end of Section 4.

Remark 2. By Theorem 1, the image of $\Psi : \mathcal{E}^p \to \mathcal{P}_0^{2p-2} \otimes \mathcal{H}^2$ is

$$\sum_{\ell=1}^{p-1} V_m^{(2\ell,2,0,\dots,0)}.$$

For fixed $\ell = 1, \dots, p-1$, consider $\mathcal{H}^{2\ell} \otimes \mathcal{H}^2$ as an $SO(m+1)$-submodule of the Weyl's space $\otimes^{2\ell+2}\mathbf{C}$ of tensors of rank $2\ell+2$ (cf. [3, 12]). The kernel of D in $\mathcal{H}^{2\ell} \otimes \mathcal{H}^2$ is the $SO(m+1)$-submodule of tensors whose contraction with respect to any two arguments is zero. Finally, $V_m^{(2\ell,2,0,\dots,0)} \subset \ker D$ corresponds to the Young tableau $\Sigma_{(2\ell,2)}$ with two rows of row lengths $2l$ and 2 (cf. again [3,12]). Theorem 1 implies that given a traceless symmetric bilinear map $\Psi : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathcal{H}^{2\ell}$ sufficiently close to zero such that Ψ possesses the symmetries prescribed by the Young symmetrizer corresponding to $\Sigma_{(2\ell,2)}$, there exists a full eigenmap $f : S^m \to S^n$ (actually, $n = n(p)$) with eigenvalue λ_p such that $\Psi = \Psi(f)$. Moreover f can be made unique by requiring $\langle f \rangle \in V_m^{(2\ell,2,0,\dots,0)}$.

Let $C \in \mathcal{E}^p$ and decompose

$$C = \sum_{(u,v)\in\triangle_1^p;\, u,v \text{ even}} C^{u,v}$$

as in (3). Then

$$\Psi(C) = \sum_{\ell=1}^{p-1} \Psi(C^{2\ell,2})$$

and, for $a, b \in \mathcal{H}^1$, $\Psi(C^{2\ell,2})(a,b)$ is the harmonic homogeneous polynomial of degree 2ℓ multiplied by $\rho^{2(p-\ell-1)}$ as in (24). Summarizing, we arrive at the following:

Theorem 4. *Let $C \in \mathcal{E}^p$ and write*

$$\Psi(C)(a,b) = \sum_{\ell=1}^{p-1} h_\ell^{a,b} \rho^{2(p-\ell-1)}. \tag{3.26}$$

If, for some $a, b \in \mathcal{H}^1$, we have $h_\ell^{a,b} \neq 0$ then

$$V_m^{(2\ell,2,0,\dots,0)} \not\subset \mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C}.$$

Theorem 4 thus reduces the exact dimension conjecture to finding, for each $m \geq 3$ and $p \geq 4$, an eigenmap $f : S^m \to S_V$ with eigenvalue λ_p such that, for $C = \langle f \rangle$, the harmonic coefficients in (26) are nonzero.

4. Conformal fields and raising and lowering the degree. We write $\Psi_p = \Psi : \mathcal{E}^p \to \mathcal{P}^{2p-2} \otimes \mathcal{H}^2$ to indicate the dependence of Ψ on p.

Theorem 5. *For $C \in \mathcal{E}^p$, we have*

$$\Psi_{p+1}(\Phi_p^+(C))(a,b) = \frac{\lambda_4/4}{\lambda_p/p}\Psi_p(C)(a,b)\rho^2 + \frac{p^2}{\lambda_p\lambda_{2p}}\triangle(\Psi_p(C)(a,b))\rho^4 \tag{4.27}$$

and

$$\triangle(\Psi_p(C)(a,b)) = \lambda_{2p}\Psi_{p-1}(\Phi_p^-(C)(a,b)). \tag{4.28}$$

Corollary 1. *Let $1 \leq \ell \leq p-1$. Then $V_m^{(2\ell,2,0,\ldots,0)} \not\subset \mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C}$ iff $V_m^{(2\ell,2,0,\ldots,0)} \not\subset \mathcal{F}^q \otimes_{\mathbf{R}} \mathbf{C}$ for (some or) all $q \geq p$.*

Proof of Corollary 1. Without loss of generality, we set $q = p+1$. Assume $V_m^{(2\ell,2,0,\ldots,0)} \subset \ker \Psi_{p+1}$. By (28), we have $V_m^{(2\ell,2,0,\ldots,0)} \subset \ker(\Psi_p \circ \Phi_{p+1}^-)$ so that $V_m^{(2\ell,2,0,\ldots,0)} \subset \ker(\Psi_p \circ \Phi_{p+1}^- \circ \Phi_p^+)$. On the other hand, by Theorem 3, $\Phi_{p+1}^- \circ \Phi_p^+$ is an isomorphism on $V_m^{(2\ell,2,0,\ldots,0)}$ for $0 \leq \ell \leq p-1$ so that $V_m^{(2\ell,2,0,\ldots,0)} \subset \ker \Psi_p$. The proof of the converse is analogous (in the use of (27)). □

A general rigidity theorem of DoCarmo and Wallach asserts that any homothetic minimal immersion $f : S^m \to S_V$ with homothety λ_p/m is equivalent to the standard minimal immersion if $p \leq 3$. This means that, for $p \leq 3$, $\mathcal{F}^p$ is trivial. Corollary 1 then gives:

Corollary 2. *For $m \geq 3$ and $p \geq 3$,*

$$V_m^{(2,2,0,\ldots,0)} \quad \text{and} \quad V_m^{(4,2,0,\ldots,0)}$$

are not components of $\mathcal{F}^p$.

Remark. The exact dimension problem is equivalent to $\Psi_p | V_m^{(2p-2,2,0,\ldots,0)} \neq 0$, for all $p \geq 4$, since we can then use induction with respect to p.

Proof of Theorem 5. We work out only (27) since the proof of (28) is entirely analogous and technically much simpler. Using (22), we have

$$\Psi_{p+1}(\Phi_p^+(C)(a,b)) = \langle (C \otimes I)((f_p^+)_* X^a)\check{}, ((f_p^+)_* X^b)\check{}\rangle. \tag{4.29}$$

By homogeneity, we have

$$((f_p^+)_* X_x^a)\check{} = X_x^a(f_p^+) = \partial_a(f_p^+) - (p+1)\left(\frac{1}{\rho^2}\right)\langle a, x\rangle f_p^+.$$

Substituting this back to (29) and using

$$\langle (C \otimes I) f_p^+, f_p^+ \rangle = \langle \Phi_p^+(C) f_{p+1}, f_{p+1} \rangle = 0$$

we arrive at

$$\begin{aligned}\Psi_{p+1}(\Phi_p^+(C)(a,b)) &= \langle (C \otimes I)\partial_a f_p^+, \partial_b f_p^+ \rangle \\ &= \left(\frac{\lambda_{2p}}{2\lambda_p}\right) \sum_{i=0}^m \langle \partial_a H(x_i(Cf_p)), \partial_b H(x_i f_p) \rangle.\end{aligned}$$

Differentiating the harmonic projection formula (8), for $h \in \mathcal{H}^p$, we have

$$\begin{aligned}\partial_a H(x_i h) = a_i h + x_i \partial_a h - \left(\frac{4p}{\lambda_{2p}}\right) \langle a, x \rangle \frac{\partial h}{\partial x_i} \\ - \left(\frac{2p}{\lambda_{2p}}\right) \rho^2 \frac{\partial(\partial_a h)}{\partial x_i}.\end{aligned} \tag{4.30}$$

For $h = Cf_p$, we write the four terms on the right-hand-side as $A_1 + A_2 + A_3 + A_4$. Replacing a with b, for $h = f_p$, we write this sum as $B_1 + B_2 + B_3 + B_4$. It remains to compute $\sum_{r,s=1}^4 \langle A_r, B_s \rangle$, where summation with respect to $i = 0, \ldots, m$ has been suppressed. This we do term by term as follows:

$$\langle A_1, B_1 \rangle = \langle a, b \rangle \langle Cf_p, f_p \rangle = 0;$$

$$\begin{aligned}\langle A_1, B_2 \rangle &= \langle a, x \rangle \langle Cf_p, \partial_b f_p \rangle \\ &= \frac{1}{2} \langle a, x \rangle \langle Cf_p, f_p \rangle = 0;\end{aligned}$$

$$\langle A_1, B_3 \rangle = -\left(\frac{4p}{\lambda_{2p}}\right) \langle b, x \rangle \langle Cf_p, \partial_a f_p \rangle = 0;$$

$$\begin{aligned}\langle A_1, B_4 \rangle &= -\left(\frac{2p}{\lambda_{2p}}\right) \langle f_p, \partial_a \partial_b f_p \rangle \rho^2 \\ &= \left(\frac{2p}{\lambda_{2p}}\right) \langle \partial_a Cf_p, \partial_b f_p \rangle \rho^2 \\ &= \left(\frac{2p}{\lambda_{2p}}\right) \Psi_p(C)(a,b) \rho^2;\end{aligned}$$

$$\langle A_2, B_2 \rangle = \Psi_p(C)(a,b)\rho^2;$$

$$\langle A_2, B_3 \rangle = -\left(\frac{4p^2}{\lambda_{2p}}\right) \langle b, x \rangle \langle \partial_a Cf_p, f_p \rangle = 0;$$

$$\langle A_2, B_4 \rangle = -\left(\frac{2p(p-1)}{\lambda_{2p}}\right) \langle \partial_a Cf_p, \partial_b f_p \rangle \rho^2$$

$$= -\left(\frac{2p(p-1)}{\lambda_{2p}}\right)\Psi_p(C)(a,b)\rho^2;$$

$$\langle A_3, B_3\rangle = \left(\frac{16p^2}{\lambda_{2p}^2}\right)\langle a,x\rangle\langle b,x\rangle\sum_{i=0}^m\left\langle\frac{\partial(Cf_p)}{\partial x_i}, \frac{\partial f_p}{\partial x_i}\right\rangle$$

$$= \left(\frac{8p^2}{\lambda_{2p}^2}\right)\langle a,x\rangle\langle b,x\rangle\triangle\langle Cf_p, f_p\rangle = 0;$$

$$\langle A_3, B_4\rangle = \left(\frac{8}{\lambda_{2p}^2}\right)\langle a,x\rangle\sum_{i=0}^m\left\langle\frac{\partial(Cf_p)}{\partial x_i}, \frac{\partial(\partial_b f_p)}{\partial x_i}\right\rangle\rho^2 = 0;$$

$$\langle A_4, B_4\rangle = \left(\frac{4p^2}{\lambda_{2p}^2}\right)\sum_{i=0}^m\left\langle\frac{\partial(\partial_a f_p)}{\partial x_i}, \frac{\partial(\partial_b f_p)}{\partial x_i}\right\rangle\rho^4$$

$$= \left(\frac{2p^2}{\lambda_{2p}^2}\right)\triangle(\Psi_p(C)(a,b))\rho^4.$$

Putting these together, (27) follows.

Remark. The idea in Section 3 can be used to prove (3). Indeed, for a full harmonic p-form $f : \mathbf{R}^{m+1} \to V$, we define $\Psi^0(f) = |f|^2 - \rho^{2p} \in \mathcal{P}^{2p}$. Clearly, $\Psi^0(f)$ depends only on the equivalence class of f. Setting $f = A \circ f_p$, we obtain $\Psi^0(f) = \langle Cf_p, f_p\rangle$, where $C = A^\top A - I \in S^2(\mathcal{H}^p)$. Adopting this as the definition of Ψ^0 on $S^2(\mathcal{H}^p)$, we obtain a homomorphism $\Psi^0 : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p}$ of $SO(m+1)$-modules with $\ker\Psi^0 = \mathcal{E}^p$. Once we prove that Ψ^0 is onto, (3) will follow, since $\mathcal{P}^{2p} = \sum_{\ell=0}^p \mathcal{H}^{2\ell}\rho^{2(p-\ell)}$. We now take $f : \mathbf{R}^{m+1} \to \mathbf{R}$ given by $f(x) = H(x_0^p)$. Computation in the use of the harmonic projection formula shows that $\Psi^0(f) = H(x_0^p)^2 - \rho^{2p}$ has nonzero component in $\mathcal{H}^{2p}$. Finally, we use induction with respect to p along with the analogue of Theorem 5.

5. Examples. Case I. $m = 2m_0+1$ *is odd.* The advantage here is that we can use complex terminology. All eigenmaps will be of the form $f : S^{2m_0+1} \to S^{2n_0+1}$ and we assume that f is the restriction of a spherical harmonic p-form $f : \mathbf{C}^{m_0+1} \to \mathbf{C}^{n_0+1}$, $p \geq 4$, with components f^j, $j = 0, \ldots, n_0$, where f^j is a complex polynomial in the variables $z_0, \bar{z}_0, \ldots, z_{m_0}, \bar{z}_{m_0}$ (of (combined) degree p).

We first derive an expression for $\Psi(f)(a,b)$. For our purposes, it will be sufficient to locate the components of $\Psi(f)(a,b)$ in $\mathcal{H}^{2p-2}$ and $\mathcal{H}^{2p-4}\rho^2$. Hence, in the computations below we will use congruences mod ρ^4. Finally, we need only to consider $a = e_0 = (1, 0, \ldots, 0)$ and $b = e_1 = (i, 0, \ldots, 0)$ in $\mathbf{C}^{m_0+1}$. Setting

$z_j = x_j + iy_j$, $j = 0, \ldots, m_0$, we have

$$\partial_{e_0}\partial_{e_1} = \frac{\partial^2}{\partial x_0 \partial y_0} = i\left(\frac{\partial^2}{\partial z_0^2} - \frac{\partial^2}{\partial \bar{z}_0^2}\right).$$

We now compute

$$\begin{aligned}
\Psi(f)(e_0, e_1) &\equiv \sum_{j=0}^{n_0} \Im\left(\bar{f}^j\left(\frac{\partial^2 f^j}{\partial z_0^2} - \frac{\partial^2 f^j}{\partial \bar{z}_0^2}\right)\right) \\
&\equiv \sum_{j=0}^{n_0} \Im\left(\frac{\partial^2 f^j}{\partial z_0^2}\bar{f}^j + f^j \frac{\partial^2 \bar{f}^j}{\partial z_0^2}\right) \\
&\equiv \sum_{j=0}^{n_0} \Im\left(\frac{\partial^2}{\partial z_0^2}|f|^2 - 2\frac{\partial f^j}{\partial z_0}\frac{\partial \bar{f}^j}{\partial z_0}\right) \\
&\equiv -2\sum_{j=0}^{n_0} \Im\left(\frac{\partial f^j}{\partial z_0}\frac{\partial \bar{f}^j}{\partial z_0}\right) (\bmod \rho^4). \qquad (5.31)
\end{aligned}$$

The last congruence is because

$$\frac{\partial^2 |f|^2}{\partial z_0^2} = \frac{\partial^2 \rho^{2p}}{\partial z_0^2} = p(p-1)\rho^{2(p-2)}\bar{z}_0^2$$

and this is a multiple of ρ^4 for $p \geq 4$. Note that the main advantage of (31) is that the holomorphic and antiholomorphic components of f cancel.

Theorem 6. *Given $m = 2m_0 + 1$ odd and $p = 2q$ even, $q \geq 2$, there exists a full eigenmap $f : S^{2m_0+1} \to S^{2N-3}$, $N = \binom{m+p}{p}$, with eigenvalue λ_p such that, in the decomposition*

$$\Psi(f)(e_0, e_1) = \sum_{\ell=1}^{p-1} h_l \rho^{2(p-\ell-1)}$$

we have

$$h_{p-1} \neq 0 \quad \text{and} \quad h_{p-2} \neq 0.$$

Applying Theorem 4, we obtain

$$V_{2m_0+1}^{(2p-2,2,0,\ldots,0)}, \quad V_{2m_0+1}^{(2p-4,2,0,\ldots,0)} \not\subset \mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C}$$

and so, induction with respect to q in the use of Corollary 1 gives Theorem 1 for m odd.

Proof of Theorem 6. We start with the complex Veronese map $F_p : S^{2m_0+1} \to S^{2N-1}$, given by

$$F_p(x) = \left(\sqrt{\frac{p!}{i_0! \ldots i_m!}} z_0^{i_0} \ldots z_m^{i_m}\right)_{i_0+\ldots+i_m=p;\, i_0,\ldots,i_m \geq 0}.$$

By (31), $\Psi(F_p)(e_0, e_1) \equiv 0 \,(\bmod \rho^4)$, so we need to modify F_p. This we will do by replacing three components of F_p by two spherical harmonics of order p. From now on we assume that $p = 2q$ is even. The components to be deleted are

$$(32) \qquad \sqrt{\frac{(2q)!}{(q-1)!(q+1)!}} z_0^{q-1} z_1^{q+1}, \quad \sqrt{\frac{(2q)!}{(q-1)!(q+1)!}} z_0^{q+1} z_1^{q-1}, \quad \text{and} \quad \frac{\sqrt{(2q)!}}{q!} z_0^q z_1^q$$

and the components to be added are

$$(5.33) \quad \sqrt{\frac{(2q)!}{(q-1)!(q+1)!}} (|z_0|^2 - |z_1|^2) z_0^{q-1} z_1^{q-1} \quad \text{and} \quad \sqrt{\frac{3q+1}{q+1}} \frac{\sqrt{(2q)!}}{q!} z_0^q z_1^q.$$

Since the sum of squares of the absolute values of the terms in (32) is the same as in (33), we obtain a full eigenmap $f : S^{2m_0+1} \to S^{2N-3}$ with eigenvalue λ_{2q}. It remains to determine $\Psi(f)(e_0, e_1)$ modulo ρ^4. Since f has only one nonholomorphic component, the right-hand-side of (31) reduces to a single term. Differentiating, we obtain

$$(5.34) \quad \Psi(f)(e_0, e_1) \equiv \frac{2(2q)!}{(q-1)!(q+1)!} \Im(q\psi_{q-1,q-1} - (q-1)\psi_{q-2,q}) (\bmod \rho^4),$$

where

$$\psi_{k,\ell}(z) = z_0^2 |z_0|^{2k} |z_1|^{2\ell}, \quad k, \ell \geq 0.$$

Using the complex form of the Laplacian

$$\triangle \psi_{k,\ell} = 4(k(k+2)\psi_{k-1,\ell} + \ell^2 \psi_{k,\ell-1}),$$

where we agree that $\psi_{k,\ell}$ with a negative subscript is zero. $\psi_{k,\ell}$ has degree $2(k+\ell+1)$. The harmonic projection formula then gives

$$\begin{aligned} \psi_{k,\ell} &\equiv H(\psi_{k,\ell}) + \frac{\rho^2}{4(2(k+\ell+1)+m_0-1)} H(\triangle \psi_{k,\ell}) \\ &\equiv H(\psi_{k,\ell}) + \frac{\rho^2}{2(k+\ell)+m_0+1} (k(k+2)H(\psi_{k-1,\ell}) + \ell^2 H(\psi_{k,\ell-1})) \,(\bmod \rho^4). \end{aligned}$$

Substituting this back to (34), we arrive at

$$\begin{aligned} \frac{(q+1)!(q-1)!}{2(2q)!} \Psi(f)(e_0, e_1) \equiv{}& qH(\Im\psi_{q-1,q-1}) - (q-1)H(\Im\psi_{q-2,q}) \\ &+ \frac{q(q-1)}{4(q-1)+m_0+1} \Big(H(\Im\psi_{q-2,q-1}) \\ &+ (q-1)H(\Im\psi_{q-1,q-2}) - (q-2)H(\psi_{q-3,q}) \Big) \\ &(\bmod \rho^4). \end{aligned}$$

To complete the proof, we need to show that

$$qH(\Im\psi_{q-1,q-1}) - (q-1)H(\Im\psi_{q-2,q}) \neq 0, \tag{5.35}$$

and

$$H(\Im\psi_{q-2,q-1}) + (q-1)H(\Im\psi_{q-1,q-2}) - (q-2)H(\Im\psi_{q-3,q}) \neq 0. \tag{5.36}$$

We prove (35); the verification of (36) is analogous. Assuming the contrary of (35) means that there exists a polynomial φ such that

$$q\Im\psi_{q-1,q-1} - (q-1)\Im\psi_{q-2,q} = \rho^2\varphi,$$

or in coordinates

$$\Im(z_0^2)(q|z_0|^2 - (q-1)|z_1|^2)|z_0|^{2(q-2)}|z_1|^{2(q-1)} = (|z_0|^2 + \ldots + |z_{m_0}|^2)\varphi.$$

Clearly $m_0 = 1$. Dividing by the irreducible factors (over $\mathbf{R}$), this reduces to

$$q|z_0|^2 - (q-1)|z_1^2 = c(|z_0|^2 + |z_1|^2)$$

where $c \in \mathbf{C}$. This is impossible so that Theorem 6 follows. □

Case II. $m = 2(m_0+1)$ *is even.* Although the following argument works in both cases, it gives an example only implicitly. For this reason, we saw no harm splitting the treatment into two cases. Moreover, to construct the example here, we use some of the computations of Case I. First we note that the components of the eigenmaps we consider here are complex valued spherical harmonics (of real or complex variables). To imitate Case I, we single out the first four real coordinates x_0, x_1, x_2, x_3 and rewrite them in terms of $z_0 = x_0 + ix_1$ and $z_1 = x_2 + ix_3$ and their conjugates.

Lemma 6. *For each $m = 2(m_0+1)$ and $p = 2q$ even, there exists a full eigenmap $F : S^{2(m_0+1)} \to S^n$ with eigenvalue λ_p which contains (a constant multiple of)*

$$z_0^{q-1}z_1^{q+1} \quad \text{and} \quad z_0^{q+1}z_1^{q-1}. \tag{5.37}$$

Proof. We use induction with respect to q. For $q = 1$, we define $F : S^{2(m_0+1)} \to S^n$ by

$$F(z,t) = \Bigg(z_0^2, \ldots, z_{m_0}^2, (\sqrt{2}z_iz_j)_{0\le i<j\le m_0}, \sqrt{2 + \frac{2}{m+1}}\, tz_0, \ldots, \sqrt{2 + \frac{2}{m+1}}\, tz_{m_0}, t^2 - \frac{(|z_0|^2 + \ldots + |z_{m_0}|^2)}{m_0+1} \Bigg),$$

where $z = (z_0, \ldots, z_{m_0}) \in \mathbf{C}^{m_0+1}$ and $t \in \mathbf{R}$. For the general induction step, assume that for fixed q an eigenmap F with two of its coordinates as in (37) exists. We now raise the degree twice and consider $(F^+)^+$. For $r, s = 0, 1, 2, 3$, (up to a constant multiple) it certainly contains

$$H(x_r H(x_s z_0^{q\pm1} z_1^{q\mp1})) = H(x_r x_s z_0^{q\pm1} z_1^{q\mp1}).$$

Now, in general, if ψ' and ψ'' are components of an eigenmap, then replacing them by $(1/\sqrt{2})(\psi' + \psi'')$ and $(1/\sqrt{2})(\psi' - \psi'')$ gives a new eigenmap. Thus, modifying $(F^+)^+$, we arrive at an eigenmap wich contains

$$H(\Re(z_0 z_1) z_0^{q\pm1} z_1^{q\mp1}) \quad \text{and} \quad H(\Im(z_0 z_1) z_0^{q\pm1} z_1^{q\mp1}).$$

Again, in general, if ψ' and ψ'' are components of an eigenmap, then replacing them by $(1/\sqrt{2})(\psi' + i\psi'')$ and $(1/\sqrt{2})(\psi' - i\psi'')$ gives a new eigenmap. Applying this to the situation above we arrive at the eigenmap claimed in the lemma. (Note that the harmonic projection operator can now be omitted since the corresponding polynomials are holomorphic).

We now restart with $F : S^{2(m_0+1)} \to S^n$ as in the lemma and replace the two components in (37) by

$$(|z_0|^2 - |z_1|^2) z_0^{q-1} z_1^{q-1} \quad \text{and} \quad z_0^q z_1^q.$$

with suitable constant multiples. We denote by f the eigenmap thus obtained. (Note that the coefficients of (37) in F are equal.) We are now in the situation of Case I to apply (31) to the difference

$$\Psi(f)(e_0, e_1) - \Psi(F)(e_0, e_1).$$

We obtain that (again up to a constant multiple) this difference has nonvanishing harmonic coefficients h_{p-1} and h_{p-2}. Now the argument used in Case I applies since either f or F has the required nonvanishing property. Theorem 1 follows.

Remark. The role of the spherical harmonic

$$(|z_0|^2 - |z_1|^2) z_0^{q-1} z_1^{q-1}$$

is crucial. We realized this (after many searches among the classical eigenmaps) when we worked out the components of the quartic eigenmap $f : S^7 \to S^7$ obtained by lifting the Hopf map $h : S^3 \to S^2$ to a quadratic eigenmap $\tilde{h} : S^4 \to S^7$ and precomposing it with the quaternionic Hopf map.

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Department of Mathematics
Rutgers University
Camden, New Jersey 08102

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