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New Construction for Spherical Minimal Immersions

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Abstract. We describe a general method of manufacturing new minimal immersions between round spheres out of old ones. The resulting spherical minimal immersions are given analytically in terms of the harmonic projection operator and have higher source dimensions. Applied to classical examples, this gives an abundance of new minimal immersions of even-dimensional spheres.

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1. Introduction and Preliminaries

An isometric immersion $f: S^m_k \to S^n$, $m \geq 2$, of the m-sphere of constant curvature k into the unit n-sphere $S^n = S^n_1$ is said to be a spherical minimal immersion if f is minimal, that is the mean curvature of f vanishes. Each component f^j, $j = 0, \ldots, n$, of a spherical minimal immersion f is an eigenfunction of the Laplacian on S^m_k with eigenvalue m ([9]). Thus, $k = k_p = m/\lambda_p$ for some $p \geq 1$, where $\lambda_p = p(p+m-1)$ is the pth eigenvalue of the Laplacian acting on functions of S^m_1. This suggests scaling the metric on the domain to curvature 1 and considering a spherical minimal immersion $f: S^m \to S^n$, as a homothetic minimal immersion with homothety λ_p/m. In terms of the differential f_* of f, homothety amounts to the condition

$$\langle f_*(X), f_*(Y)\rangle = (\lambda_p/m)\langle X, Y\rangle \tag{1}$$

being satisfied for any vector fields X, Y on S^m.

An eigenfunction of the Laplacian on S^m corresponding to the eigenvalue λ_p (or clasically, a spherical harmonic of order p on S^m) is the restriction (to S^m) of a harmonic homogeneous polynomial (on $\mathbf{R}^{m+1}$) of degree p in the variables $x_0, \ldots, x_m \in \mathbf{R}$ ([13]). Thus, a spherical minimal immersion is a conformal immersion $f: S^m \to S^n$ that is the restriction of a harmonic homogeneous polynomial map $f: \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$; a map whose components f^j, $j = 0, \ldots, n$, are harmonic homogeneous polynomials of a fixed degree. (By homogeneity the conformality factor is constant and degree p corresponds to homothety λ_p/m.)

The oldest example of a spherical minimal immersion is the Veronese map $v: S^2 \to S^4$. It factors through the antipodal action and gives the well-known Veronese surface; the real projective plane imbedded minimally in S^4. A straightforward generalization (keeping $p = 2$) gives the Veronese maps $v_m: S^m \to S^{m(m+3)/2-1}$; $v_2 = v$. These all share the property that the components of v_m form an orthonormal basis in the space of quadratic spherical harmonics. For further generalization, let $\mathcal{H}_m^p$ denote the linear space of spherical harmonics of order p on S^m. We make $\mathcal{H}_m^p$ Euclidean by endowing it with the normalized L^2-scalar product

$$\langle h, h' \rangle = \frac{n(m,p)+1}{\mathrm{vol}(S^m)} \int_{S^m} h\, h'\, v_{S^m}, \quad h, h' \in \mathcal{H}_m^p,$$

where

$$n(m,p) + 1 = \dim \mathcal{H}_m^p = (m + 2p - 1)\frac{(m+p-2)!}{p!(m-1)!}. \tag{2}$$

We now take the elements of an orthonormal basis $\{f_{m,p}^j\}_{j=0}^{n(m,p)}$ in $\mathcal{H}_m^p$ as components of a map $f_{m,p}$ that, due to the large number of symmetries, turns out (see [14]) to be a spherical minimal immersion $f_{m,p}: S^m \to S^{n(m,p)}$ (with homothety λ_p/m); $f_{m,2} = v_m$. The map $f_{m,p}$ is called the standard minimal immersion; it is uniquely determined up to congruence on the range. For fixed p and m, $f_{m,p}$ has the largest range dimension among all full spherical minimal immersions. (Here fullness means that the image spans the range linearly, or equivalently, the components are linearly independent.) For p odd, $f_{m,p}$ is an imbedding. For p even, $f_{m,p}$ factors through the antipodal action on S^m and gives an imbedding of the real projective m-space into the sphere.

In 1967 Calabi [1] proved that any full spherical minimal immersion $f: S^2 \to S^n$ is standard. (This amounts to showing that $n = n(2,p) = 2p$ for some p and orthonormality of the components of f.) Using higher fundamental forms, in 1971 DoCarmo and Wallach [3] derived a general rigidity result which implies that, for $p \leq 3$, any full spherical minimal immersion $f: S^m \to S^n$ with homothety λ_p/m is standard. Moreover, their main result asserts that, for $m \geq 3$ and $p \geq 4$, the space of full spherical minimal immersions $f: S^m \to S^n$ with homothety λ_p/m (mod out by congruences on the ranges) is a compact convex body $\mathcal{M}_m^p$ in a nontrivial linear subspace $\mathcal{F}_m^p$ of the symmetric square $S^2(\mathcal{H}_m^p)$. In fact, using representation theory of the orthogonal group, they gave a (positive) lower bound for $\dim(\mathcal{M}_m^p) = \dim(\mathcal{F}_m^p)$ when $m \geq 3$ and $p \geq 4$. Whether this lower bound was exact remained unsolved until 1994 when, using a completely different (analytical) approach, the author [11] showed that the lower bound is the actual dimension. The lowest nonrigid range $m = 3$ and $p = 4$ corresponds to the 18-dimensional parameter space $\mathcal{M}_3^4$. (This particular dimension was calculated by Muto [8] using explicit tensor calculus.) In 1996, Ziller and the author [12] proved that $\mathcal{M}_3^4$ is the convex hull of two linear slices corresponding to $\mathrm{SU}(2)$- and $\mathrm{SU}(2)'$-equivariant spherical minimal immersions $f: S^3 \to S_V$, where $\mathrm{SO}(4) = \mathrm{SU}(2) \cdot \mathrm{SU}(2)'$ is the standard local product structure.

Although a lot of effort has been made to obtain examples of spherical minimal immersions, or equivalently, minimal isometric imbeddings of spherical space forms into spheres, apart from the standard minimal immersions, examples are known only for specific domain dimensions (and specific equivariance properties). After an initial example of Mashimo [6], [7] and some more by Wang and Ziller [15] using 'equivariant constructions', a groundbreaking work of DeTurck and Ziller [2] resulted in a host of new spherical minimal immersions. The list is extensive and shows that every homogeneous spherical space form admits a minimal isometric immersion into a sphere. All these examples possess large groups of symmetries that act transitively on the domain. In particular, and this is relevant for this paper, all domains are odd dimensional. Using an entirely different approach, Escher [4] subsequently derived a necessary condition for the existence of a minimal imbedding of (nonhomogeneous) three-dimensional spherical space forms. In particular, the lens space $L(5, 2)$ cannot be imbedded into any sphere using a spherical minimal immersion with homothety $\lambda_p/3$ if $p < 28$.

Apart from the standard minimal immersions, no examples of spherical minimal immersions are known from even-dimensional domains. The objective of this paper is to describe a very general method that associates to a set of spherical minimal immersions from S^m a spherical minimal immersion from S^{m+1}. More precisely, we have:

THEOREM 1. *Let $m \geq 3$ and $p \geq 4$. Given full spherical minimal immersions $f_q: S^m \to S^{n_q}$ with homothety λ_q/m, $q = 1, \ldots, p$, there exists a full spherical minimal immersion $\tilde{f}: S^{m+1} \to S^N$, where $N = \sum_{q=1}^{p}(n_q + 1)$.*

Remark 1. By rigidity, $n_q = n(m, q)$, for $q = 1, 2, 3$.

Remark 2. A special case of the construction has been used in [5] to study higher fundamental forms of spherical minimal immersions.

Remark 3. It will be clear from the construction that if there is a common (finite) symmetry group $G \subset \mathrm{SO}(m + 1)$ of all f_q, that is, $f_q \circ a = f_q$ for all $a \in G$ and $q = 1, \ldots, p$, then, under the standard inclusion $\mathrm{SO}(m + 1) \subset \mathrm{SO}(m + 2)$, G is also a symmetry group of $\tilde{f}$.

According to a result of [12], the possible range dimensions of full quartic ($p = 4$) spherical minimal immersions $f: S^3 \to S^n$ are

$$n = 9, 14, 15, 18, 19, 20, 22, 23, 24.$$

Using Theorem 1, we obtain the following sequence of range dimensions for full quartic minimal immersions $\tilde{f}: S^4 \to S^N$:

$$N = 39, 44, 45, 48, 49, 50, 52, 53, 54.$$

Precomposition of spherical harmonics and minimal immersions with isometries on S^m gives rise to compatible orthogonal $\mathrm{SO}(m+1)$-module structures on $\mathcal{H}_m^p$, $S^2(\mathcal{H}_m^p)$ and $\mathcal{F}_m^p$ with $\mathcal{M}_m^p \subset \mathcal{F}_m^p$ being $\mathrm{SO}(m+1)$-invariant.

THEOREM 2. *Let $m \geq 3$ and $p \geq 4$. There exists a linear imbedding*

$$\Phi\colon \mathcal{F}_m^4 \times \cdots \times \mathcal{F}_m^p \to \mathcal{F}_{m+1}^p$$

that is equivariant with respect to the standard inclusion $\mathrm{SO}(m+1) \subset \mathrm{SO}(m+2)$. Φ *restricts to an equivariant imbedding*

$$\phi\colon \mathcal{M}_m^4 \times \cdots \times \mathcal{M}_m^p \to \mathcal{M}_{m+1}^p$$

onto a linear slice of $\mathcal{M}_{m+1}^p$.

2. Spherical Harmonics

Since construction of the new spherical minimal immersions will use the harmonic projection operator, in this section we summarize some well-known facts from the theory of spherical harmonics ([13]). Let $\mathcal{P}_{m+1}^p$ denote the space of homogeneous polynomials of degree p in the variables $x_0, \ldots, x_m$. Viewing a spherical harmonic as a (harmonic) polynomial gives $\mathcal{H}_m^p \subset \mathcal{P}_{m+1}^p$. Letting $r^2 = x_0^2 + \cdots + x_m^2 \in \mathcal{P}_{m+1}^2$, with respect to the L^2-scalar product on $\mathcal{P}_{m+1}^p$, the orthogonal complement of $\mathcal{H}_m^p$ in $\mathcal{P}_{m+1}^p$ is $r^2 \cdot \mathcal{P}_{m+1}^{p-2}$. The harmonic projection operator is the orthogonal projection $H : \mathcal{P}^p \to \mathcal{H}^p$ (with kernel $r^2\mathcal{P}_{m+1}^{p-2}$). H can be expressed in terms of powers of the Laplacian:

$$H(g) = g + \sum_{i=1}^{[|p/2|]} \frac{(-1)^i \triangle^i g}{2^i i!(2p+m-3)\ldots(2p+m-2i-1)} r^{2i},$$
$$g \in \mathcal{P}_{m+1}^p. \tag{3}$$

We now add the variable x_{m+1} (replace m by $m+1$) and consider $h \in \mathcal{H}_{m+1}^p$. Letting $x = (x_0, \ldots, x_m)$, we have ([13])

$$h(x, x_{m+1}) = \sum_{q=0}^{p} x_{m+1}^{p-q} h_q(x) + (r^2 + x_{m+1}^2) g(x, x_{m+1}),$$

where $h_q \in \mathcal{H}_m^q$, $q = 0, \ldots, p$, are uniquely determined by h. Taking harmonic projection of both sides, we obtain the orthogonal decomposition

$$h(x, x_{m+1}) = \sum_{q=0}^{p} H(x_{m+1}^{p-q} h_q(x)), \quad h_q \in \mathcal{H}_m^q,\ q = 0, \ldots, m. \tag{4}$$

In terms of the spaces of spherical harmonics, this translates into

$$\mathcal{H}^p_{m+1}|_{\mathrm{SO}(m+1)} = \sum_{q=0}^{p} \mathcal{H}^q_m, \tag{5}$$

where $\mathcal{H}^q_m$ means the image of $x^{p-q}_{m+1}\mathcal{H}^q_m$ under the harmonic projection H. This is legitimate since (being a homomorphism of SO$(m+1)$-modules) H is injective on these subspaces.

In (4), all higher order Laplacians of $x^{p-q}_{m+1}h_q(x)$ can be worked out explicitly so that, using (3), we obtain

$$H(x^{p-q}_{m+1}h_q(x)) = K_{m,p,q}(r^2, x_{m+1})h_q(x), \tag{6}$$

where $K_{m,p,q}$ is a polynomial in r^2 and x_{m+1} and is independent of h_q. More explicitly, we have

$$K_{m,p,q}(x, x_{m+1}) \tag{7}$$

$$= \frac{(p-q)!\Gamma(m/2+q)}{2^{p-q}\Gamma(m/2+p)} \times$$

$$\times\, (r^2 + x^2_{m+1})^{(p-q)/2} G^{m/2+q}_{p-q}\left(\frac{x_{m+1}}{(r^2+x^2_{m+1})^{1/2}}\right), \tag{8}$$

where Γ is the Gamma function and G^a_d is the Gegenbauer polynomial:

$$G^a_d(t) \;=\; \frac{2^d\Gamma(a+d)}{d!\Gamma(a)}\left[t^d - \frac{d(d-1)}{2^2(a+d-1)}t^{d-2} + \right.$$

$$\left. +\; \frac{d(d-1)(d-2)(d-3)}{2^4\cdot 1\cdot 2(a+d-1)(a+d-2)}t^{d-4} - \cdots\right].$$

For fixed a, the normalized Gegenbauer polynomials

$$2^{a-1}\Gamma(d)\left(\frac{2(a+d)d!}{\pi\Gamma(2a+d)}\right)^2 G^a_d(t), \quad d = 0, 1, \ldots,$$

form an orthonormal system on the interval $[-1, 1]$ with respect to the weight $(1-t^2)^{a-1/2}$.

3. Decomposition of the Standard Minimal Immersion

Each component of the standard minimal immersion $f_{m+1,p}\colon S^{m+1} \to S^{n(m+1,p)}$ is in $\mathcal{H}^p_{m+1}$ which splits according to (5).

PROPOSITION. *The standard minimal immersion* $f_{m+1,p}\colon S^{m+1} \to S^{n(m+1,p)}$ *can be written as*

$$f_{m+1,p}(x, x_{m+1}) = (c_{m,p,q}H(x^{p-q}_{m+1}f_{m,q}(x)))_{0\le q\le p} \quad (f_{m,0} = 1), \tag{9}$$

where

$$c_{m,p,q} = \binom{p}{q}(m+2q-1)\frac{(m+q-2)!}{(m+p-1)!}\frac{[m(m+2)\dots(m+2p-2)]^2}{m(m+1)\dots(m+p+q-1)}. \quad (10)$$

Remark. By definition, $f_{m,0} = 1$.

Proof. For each q, we fix an orthonormal basis $\{f_{m,q}^{j_q}\}_{j_q=0}^{n(m,q)} \subset \mathcal{H}_m^q$ and this defines $f_{m,q}$. The proposition will now follow if we can show that the spherical harmonics in (8) are orthonormal. This amounts to working out the integral

$$\int_{S^{m+1}} H(x_{m+1}^{p-q} f_{m,q}^{j_q}(x)) H(x_{m+1}^{p-q'} f_{m,q'}^{j'_{q'}}(x)) v_{S^{m+1}}.$$

By (6)–(7), in terms of the Gegenbauer polynomials, up to constant multiple, the integrand rewrites as

$$\int_{S^{m+1}} f_{m,q}^{j_q}(x) f^{j'_{q'}}(x) G_{p-q}^{m/2+q}(x_{m+1}) G_{p-q'}^{m/2+q'}(x_{m+1}) v_{S^{m+1}}.$$

We introduce the transformation $\gamma\colon [0,\pi] \times S^m \to S^{m+1}$ defined by

$$\gamma(\theta, x) = (\sin\theta \cdot x, \cos\theta \cdot |x|), \quad x \in S^m,\ \theta \in [0,\pi].$$

This maps the cylinder $[0,\pi] \times S^m$ to S^{m+1} with 0 and π corresponding to the North and South poles in S^{m+1} relative to the equator S^m. The determinant of the Jacobian of γ being $(-1)^m|x| \sin^m\theta$, the integral transforms into

$$\int_0^\pi G_{p-q}^{m/2+q}(\cos\theta) G_{p-q'}^{m/2+q'}(\cos\theta) \sin^{m+q+q'}\theta \, \mathrm{d}\theta \int_{S^m} f_{m,q}^{j_q}(x) f_{m,q'}^{j'_{q'}}(x) v_{S^m}$$

and orthogonality follows from orthogonality of the Gegenbauer polynomials. It remains to work out the normalization constants $c_{m,p,q}$ in (9). To simplify the matters, for fixed $q = 0, \dots, p$, we replace the component $f_{m,q}^j$ by an arbitrary spherical harmonic $h_q \in \mathcal{H}_m^q$ of unit length

$$|h_q|^2 = \frac{n(m,q)+1}{\mathrm{vol}(S^m)} \int_{S^m} h_q(x)^2 v_{S^m} = 1.$$

Using the calculus above, we compute

$$\begin{aligned} &\int_{S^{m+1}} H(x_{m+1}^{p-q} h_q(x))^2 v_{S^{m+1}} \\ &\quad = \left(\frac{(p-q)!\Gamma(m/2+q)}{2^{p-q}\Gamma(m/2+p)}\right)^2 \times \\ &\qquad \times \int_{S^{m+1}} G_{p-q}^{m/2+q}(x_{m+1})^2 h_q(x)^2 v_{S^{m+1}} \end{aligned}$$

$$= \left(\frac{(p-q)!\Gamma(m/2+q)}{2^{p-q}\Gamma(m/2+p)} \right)^2 \times$$

$$\times \int_0^\theta G_{p-q}^{m/2+q}(\cos\theta)^2 \sin^{m+2q}\theta \,\mathrm{d}\theta \int_{S^m} h_q(x)^2 v_{S^m}$$

$$= \frac{\pi(p-q)!\Gamma(m+p+q)}{2^{2p+m-1}(m/2+p)\Gamma(m/2+p)^2} \int_{S^m} h_q(x)^2 v_{S^m}.$$

Taking into account the various normalizing constants, we arrive at

$$\begin{aligned} |H(x_{m+1}^{p-q} h_q(x))|^2 &= \frac{n(m+1,p)+1}{\mathrm{vol}\,(S^{m+1})} \int_{S^{m+1}} H(x_{m+1}^{p-q} h_q(x))^2 v_{S^{m+1}} \\ &= \frac{\pi(p-q)!\Gamma(m+p+q)}{2^{2p+m-1}(m/2+p)\Gamma(m/2+p)^2} \times \\ &\quad \times \frac{n(m+1,p)+1}{n(m,q)+1} \frac{\mathrm{vol}\,(S^m)}{\mathrm{vol}\,(S^{m+1})}. \end{aligned}$$

The value of the ratio of the spherical volumes is

$$\frac{\mathrm{vol}(S^m)}{\mathrm{vol}(S^{m+1})} = \frac{2^{m-2} m \Gamma(m/2)^2}{\pi \Gamma(m)}.$$

(Note that this also follows from our computations for $p = q = 0$.) Using this and the dimension formula (2), the value of $c_{m,p,q}$ in (9) now follows.

4. Proof of Theorem 1

For $q = 1, \ldots, p$, let $f_q : S^m \to S^{n_q}$, be a full spherical minimal immersion with homothety λ_q/m. In analogy with (8), we define $\tilde{f} : \mathbf{R}^{m+2} \to \mathbf{R}^N$, $N + 1 = \sum_{q=1}^p (n_q + 1)$ by

$$\tilde{f}(x, x_{m+1}) = \left(c_{m,p,q} H(x_{m+1}^{p-q} f_q(x)) \right)_{0 \le q \le p-1}, \quad (f_0 = 1). \tag{11}$$

We first show that $\tilde{f}$ maps S^{m+1} to S^N by comparing it with the standard minimal immersion $f_{m+1,p}$. Using the decomposition (8) and (6)–(7), we have

$$\begin{aligned} &|\tilde{f}|^2 - |f_{m+1,p}|^2 \\ &= \sum_{q=0}^p c_{m,p,q}^2 (|H(x_{m+1}^{p-q} f_q(x))|^2 - |H(x_{m+1}^{p-q} f_{m,q}(x))|^2) \\ &= \sum_{q=0}^p c_{m,p,q}^2 K_{m,p,q}(r^2, x_{m+1})^2 (|f_q(x)|^2 - |f_{m,q}(x)|^2) = 0. \end{aligned}$$

Thus $\tilde{f}$ maps spheres to spheres. Since it is a harmonic polynomial map, to prove Theorem 1, it remains to verify that the restriction $\tilde{f}\colon S^{m+1} \to S^N$ is homothetic with homothety $\lambda_p/(m+1)$. ($\tilde{f}$ is clearly full since its components are linearly independent.)

We now recall ([11]) a useful criterion of homothety. Let $f : S^m \to S^n$ be (the restriction of) a (harmonic) homogeneous polynomial map of degree p. We introduce the symmetric 2-tensor $\Psi(f)$ on S^m by

$$\Psi(f)(X,Y) = \langle f_*(X), f_*(Y)\rangle - (\lambda_p/m)\langle X, Y\rangle,$$

where X and Y are vector fields on S^m, and note that, by (1), homothety of f is equivalent to the vanishing of $\Psi(f)$. As f, $\Psi(f)$ extends to $\mathbf{R}^{m+1}$:

$$\Psi(f)(X,Y) = \langle f_*(X), f_*(Y)\rangle - (\lambda_p/m)\langle X, Y\rangle r^{2(p-1)},$$

where X and Y are vector fields on $\mathbf{R}^{m+1}$ and the correction factor $r^{2(p-1)}$ is for (future) homogeneity of $\Psi(f)$. We now evaluate $\Psi(f)$ on a pair of conformal fields. A conformal field X^a, $a \in \mathbf{R}^{m+1}$, is, on S^m, given by

$$X^a_x = a - \langle a, x\rangle x, \quad x \in S^m,$$

and, on $\mathbf{R}^{m+1}$, by

$$X^a_x = a - \frac{\langle a, x\rangle}{r^2} x, \quad x \in \mathbf{R}^{m+1}.$$

Given $a, b \in \mathbf{R}^{m+1}$, $\Psi(f)(a,b) = \Psi(f)(X^a, X^b)$ is a homogeneous polynomial of degree $2(p-1)$ ([11]). More explicitly, we have the following useful computational formula:

$$\begin{aligned} \Psi(f)(a,b) \;=\; & -\langle \partial_a f, \partial_b f\rangle \\ & + ((\lambda_p/m) - p^2)\langle a, x\rangle\langle b, x\rangle r^{2(p-2)} \\ & - (\lambda_p/m)\langle a, b\rangle r^{2(p-1)}. \end{aligned} \tag{12}$$

Here ∂_a and ∂_b denote directional derivatives in the direction of a and b. Since the conformal fields span each tangent space in S^m, homothety of f is equivalent to the vanishing of $\Psi(f)(a,b)$ for all $a, b \in \mathbf{R}^{m+1}$.

We are now ready to show that $\tilde{f}$ is homothetic. As before we compare $\tilde{f}$ with $f_{m+1,p}$. Noticing that the last two terms in (11) do not depend on f, for $a, b \in \mathbf{R}^{m+2}$, we compute

$$\begin{aligned} \Psi(\tilde{f})(a,b) \;-\; & \Psi(f_{m+1,p})(a,b) \\ = \; & -\langle \partial_a \tilde{f}, \partial_b \tilde{f}\rangle + \langle \partial_a f_{m+1,p}, \partial_b f_{m+1,p}\rangle \\ = \; & -\sum_{q=0}^{p} c^2_{m,p,q}\big(\langle \partial_a H(x^{p-q}_{m+1} f_q(x)), \partial_b H(x^{p-q}_{m+1} f_q(x))\rangle \\ & - \langle \partial_a H(x^{p-q}_{m+1} f_{m,q}(x)), \partial_b H(x^{p-q}_{m+1} f_{m,q}(x))\rangle\big). \end{aligned}$$

By (6), taking the directional derivative of each term gives two terms, e.g. we have

$$\begin{aligned}\partial_a H(x_{m+1}^{p-q} f_q(x)) &= \partial_a (K_{m,p,q}(r^2, x_{m+1}) f_q(x)) \\ &= \partial_a (K_{m,p,q}(r^2, x_{m+1})) f_q(x) + \\ &\quad + K_{m,p,q}(r^2, x_{m+1}) \partial_a f_q(x)\end{aligned}$$

so that

$$\langle \partial_a H(x_{m+1}^{p-q} f_q(x), \partial_b H(x_{m+1}^{p-q} f_q(x)\rangle$$

splits into four terms. The term involving

$$\partial_a K_{m,p,q} \partial_b K_{m,p,q}$$

cancels with the analogous term for the standard minimal immersions. The 'mixed terms' are independent of f_q since e.g.

$$\langle \partial_a f_q, f_q \rangle = \tfrac{1}{2} \partial_a |f_q|^2 = \tfrac{1}{2} \partial_a r^{2q}.$$

Thus they also cancel. Finally, the derivative terms, e.g.

$$\langle \partial f_q, \partial_b f_q \rangle$$

cancel with the analogous terms for the strandard minimal immersion since f_q is homothetic. Theorem 1 follows.

5. Proof of Theorem 2

Let $f\colon S^m \to S^n$ be a full spherical minimal immersion with homothety λ_p/m. Since the components of the standard minimal immersion $f_{m,p}$ span $\mathcal{H}_m^p$, there is a unique $(n+1) \times (n(m,p)+1)$-matrix (of maximal rank) such that $f = A \cdot f_{m,p}$. Associating to f the symmetric matrix $\langle f \rangle = A^\top A - I \in S^2(\mathbf{R}^{n(m,p)+1}) = S^2(\mathcal{H}_m^p)$ gives rise to the DoCarmo–Wallach parametrization of all full spherical minimal immersions $f\colon S^m \to S^n$ with homothety λ_p/m by the compact convex body $\mathcal{M}_m^p \subset \mathcal{F}_m^p$. The linear subspace $\mathcal{F}_m^p \subset S^2(\mathcal{H}_m^p)$ is obtained by translating the condition of homothety of a spherical minimal immersion in terms of the point in $S^2(\mathcal{H}_m^p)$ it is represented with. (Note that $\langle f \rangle$ determines f only up to congruence on the range.)

Given a full spherical minimal immersion $f_q\colon S^m \to S^{n_q}$ with homothety λ_q/m, $q = 1, \ldots, p$, we have $f_q = A_q \cdot f_{m,q}$ for an $(n_q+1) \times (n(m,q)+1)$-matrix A_q so that the corresponding parameter point in $\mathcal{M}_m^q$ is $\langle f_q \rangle = A_q^\top A_q - I$ in $S^2(\mathcal{H}_m^q)$. By the definition of $\tilde{f}$, it is clear that, under the isomorphism (5), the diagonal matrix

$$\langle f_4 \rangle \times \cdots \times \langle f_p \rangle$$

corresponds to $\langle \tilde{f} \rangle$. (Note that $\langle f_q \rangle$ is zero for $q = 1, 2, 3$.) This shows that Φ is obtained from the natural inclusion

$$S^2(\mathcal{H}^4_m) \times \cdots \times S^2(\mathcal{H}^p_m) \to S^2(\mathcal{H}^p_{m+1})$$

via (5). Theorem 2 follows.

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