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Normed Bilinear Pairings for Semi-Euclidean Spaces near the Hurwitz-Radon Range

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Abstract

A codimension $c\,(\geq 0)$ orthogonal multiplication of type $(k,l;m;m+c)$ for semi-Euclidean spaces is a normed bilinear map $F:\mathbf{R}^{k,l}\times\mathbf{R}^m\to\mathbf{R}^{m+c}$, $k+l\leq m$, $k>0$, where the signatures of the semi-Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^{m+c}$ are suppressed. For the Hurwitz-Radon range, i.e. $c=0$, F gives rise to (and is determined by) a module over the Clifford algebra $C_{l,k-1}$ whose generators possess certain invariance properties with respect to the semi-Euclidean structure on the module. For $c=1$, we prove an Adem-type restriction-extension theorem to the effect that F (up to isometries on the source and the range) restricts to an orthogonal multiplication of type $(k,l;m;m)$ if m is even, and extends to an orthogonal multiplication of type $(k,l;m+1;m+1)$ if m is odd. The resulting types are in the Hurwitz-Radon range, thereby classified. The main results of the paper give a full description of codimension two full orthogonal multiplications F of type $(k,l;m;m+2)$. We show that, for m even, F extends to an orthogonal multiplication of type $(k,l;m+2;m+2)$. For m odd, we have $k+l=3$ and F restricts (again up to isometries on the source and the range) to an orthogonal multiplication of type $(k,l;m-1;m-1)$ which is a direct summand of F.
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1 Preliminaries and Statement of Results

In this article we study the existence of 'composition formulas' of the type

$$\begin{aligned}(x_1^2+\ldots+x_k^2-x_{k+1}^2-\ldots-x_{k+l}^2)(y_1^2+\ldots+y_p^2-y_{p+1}^2-\ldots-y_{p+q}^2)\\ = z_1^2+\ldots+z_r^2-z_{r+1}^2-\ldots-z_{r+s}^2,\end{aligned}\tag{1}$$

where $z_1,\ldots,z_{r+s}$ are homogeneous bilinear forms in the variables $x_1,\ldots,x_{k+l}$ and $y_1,\ldots,y_{p+q}$ which, in general, take their values in an arbitrary field $\mathbf{F}$ (of characteristic $\neq 2$). Here we (may and) will assume that $k+l\leq p+q\leq r+s$ and $k>0$. Though the 'positive definite case' (i.e. when $l=q=s=0$) has been posed by Hurwitz nearly a hundred years ago (cf. [7,8] and Radon [13]) and has an extensive literature (cf. the survey article [14] of Shapiro as well as his book [16] which are our general references here), our knowledge of the 'indefinite case' above is rather fragmentary.

The purpose of this paper is to establish existence and classification theorems for (1) over the reals assuming that the codimension $c = (r + s) - (p + q)$ is at most two. For the Hurwitz-Radon range ($c = 0$), full classification, in terms of Clifford modules with semi-Euclidean structures, is known although an explicit and thorough treatment is difficult to find (cf. Lawrynowicz and Rembieliński [10]).

Restricting ourselves to real variables, the problem is equivalent to finding all *orthogonal multiplications*, i.e. bilinear maps

$$F : \mathbf{R}^{k,l} \times \mathbf{R}^{p,q} \to \mathbf{R}^{r,s}$$

between semi-Euclidean spaces that satisfiy

$$|F(x,y)|^2 = |x|^2\,|y|^2, \qquad x \in \mathbf{R}^{k,l},\ y \in \mathbf{R}^{p,q}. \tag{2}$$

We express (2) by saying that F is *normed.* We say that F is *full* if the image of F spans the range $\mathbf{R}^{r,s}$. Here $\mathbf{R}^{a,b}$, $a, b \geq 0$, denotes $\mathbf{R}^{a+b}$ with the standard semi-Euclidean scalar product

$$\begin{aligned} \langle u, v\rangle &= u_1v_1 + \ldots u_av_a - u_{a+1}v_{a+1} - \ldots - u_{a+b}v_{a+b}, \\ & u = (u_1, \ldots, u_{a+b}),\ v = (v_1, \ldots, v_{a+b}). \end{aligned}$$

(For basic facts in semi-Euclidean geometry we refer to O'Neill [11].) Note that if the signature (a, b) of $\mathbf{R}^{a,b}$ is irrelevant, we simply write $\mathbf{R}^{a+b}$ and refer to it as a semi-Euclidean space. We also say that F above is an orthogonal multiplication of type $(k, l; p, q; r, s)$, or shortly, $(k, l; m; n)$, where $m = p + q$ and $n = r + s$.

Given two orthogonal multiplications $F_0 : \mathbf{R}^{k,l} \times \mathbf{R}^{m_0} \to \mathbf{R}^{n_0}$ and $F_1 : \mathbf{R}^{k,l} \times \mathbf{R}^{m_1} \to \mathbf{R}^{n_1}$, we define their direct sum as the orthogonal multiplication $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^n$, $m = m_0 + m_1$, $n = n_0 + n_1$, by $F(x, (y_0, y_1)) = (F_0(x, y_0), F_1(x, y_1))$, $x \in \mathbf{R}^{k,l}$, $y_0 \in \mathbf{R}^{m_0}$ and $y_1 \in \mathbf{R}^{m_1}$.

Let F be an orthogonal multiplication of type $(k, l; m; n)$ and F_0 of type $(k, l; m_0, n_0)$, where $m_0 \leq m$ and $n_0 \leq n$. We say that F restricts to F_0 (or that F_0 extends to F) if there exist isometric imbeddings $\phi : \mathbf{R}^{m_0} \to \mathbf{R}^m$ and $\psi : \mathbf{R}^{n_0} \to \mathbf{R}^n$ such that $F \circ (U \times \phi) = \psi \circ F_0$ for some isometry U on $\mathbf{R}^{k,l}$. If, in addition, the restriction $F|\mathbf{R}^{k,l} \times (\operatorname{im}\phi)^{\perp}$ maps into $(\operatorname{im}\psi)^{\perp}$ then we say that F_0 is a direct summand of F. In this case, up to isometries on the source and the range, F is the direct sum of two orthogonal multiplications of type $(k, l; m_0; n_0)$ and $(k, l; m - m_0; n - n_0)$.

Our first result describes the codimension one case in much the same way as the result of Adem [1,2] and Shapiro [15] for the definite case (over arbitrary fields).

Theorem 1 *Let $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^{m+1}$, $k + l \leq m$, $k > 0$, be an orthogonal multiplication between semi-Euclidean spaces. If m is even then F restricts to an orthogonal multiplication of type $(k, l; m; m)$. If m is odd then F extends to an orthogonal multiplication of type $(k, l; m + 1; m + 1)$.*

Our main result gives a complete description of all codimension two orthogonal multiplications in terms of those in the Hurwitz-Radon range as follows:

Theorem 2 *Let $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^{m+2}$, $k+l \le m$, $k > 0$, be a full orthogonal multiplication. If m is even then then F extends to an orthogonal multiplication of type $(k, l; m+2; m+2)$. If m is odd then $k + l = 3$ and F restricts to an orthogonal multiplication of type $(k, l; m-1; m-1)$ which is a direct summand of F.*

We finally recall the relevant and related results on the positive definite case $l = q = s = 0$. First, in 1940-41, Hopf and Stiefel (cf. [6] and [17]) proved that for the existence of a nonsingular bilinear pairing $\mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}^n$ it is necessary that $\binom{n}{j}$ is even for $n-k < j < m$. This, for $c = n - m = 2$, immediately gives the nonexistence of types $(k; m; m+2)$ for $m \equiv 1 \pmod 4$ and $k \ge 4$. Second, Berger and Friedland [3] and K.Y.Lam and Yiu [9] considered the problem of finding, for given m and n, the largest k such that a (not necessarily full) orthogonal multiplication of type $(k; m; n)$ exists. They found and listed the solutions for $c \le 5$ (with the exception of $c = 5$ with $m \equiv 27 \pmod{32}$). Note that maximality of k does not imply fullness of the corresponding orthogonal multiplication. Theorem 2 has been proved in [5] by the authors for the definite case and, for $c = 2$, the results described above follow from this. To reduce the length of the proof of Theorem 2 in the indefinite case we will heavily rely on what has already been proved in [5]. Although we will give generous outlines for the analogous arguments, we will concentrate on the main technical difficulty that one encounters in all semi-Euclidean spaces, i.e. the presence of degenerate, especially, null subspaces.

2 The Hurwitz-Radon Range; Clifford Modules

Let

$$F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^n, \qquad k + l \le m \le n,$$

be an orthogonal multiplication for semi-Euclidean spaces, where the signatures on $\mathbf{R}^m$ and $\mathbf{R}^n$ have not been explicitly indicated. Since F is normed, (by changing the signs of the scalar products, if necessary) we (may and) will assume that

$$k > 0.$$

Denote by $\{e_\alpha\}_{\alpha=1}^{k+l} \subset \mathbf{R}^{k,l}$ and $\{f_i\}_{i=1}^m \subset \mathbf{R}^m (= \mathbf{R}^{p,q})$ the standard orthonormal bases. We then have $|e_\alpha|^2 = \epsilon_\alpha$ with $\epsilon_\alpha = +1$ for $\alpha = 1, \ldots, k$ and $\epsilon_\alpha = -1$ for $\alpha = k+1, \ldots, k+l$ and $|f_i|^2 = \sigma_i$ (with $\sigma_i = +1$ for $i = 1, \ldots, p$ and $\sigma_i = -1$ for $i = p+1, \ldots, p+q = m$). For fixed α, F being normed implies that $\{u_i^\alpha = F(e_\alpha, f_i)\}_{i=1}^m$ is orthonormal in $\mathbf{R}^n$ with $|u_i^\alpha|^2 = \epsilon_\alpha \sigma_i$.

In particular, since $k > 0$, the signatures (p, q) of $\mathbf{R}^m$ and (r, s) of $\mathbf{R}^n$ satisfy

$$p \le r \quad \text{and} \quad q \le s,$$

and, if $l > 0$, then

$$|p - q| + |r - s| \le c,$$

where $c = n - m$ is the codimension of F. (These inequalities follow from the fact that if $x \in \mathbf{R}^{k,l}$ is spacelike (resp. timelike) then, for any $y \in \mathbf{R}^m$ non-null, the causal characters of y and $F(x, y)$ are the same (resp. opposite).)

For $\alpha, \beta = 1, \ldots, k + l$, let $P^{\beta\alpha} = (p_{ij}^{\beta\alpha})_{i,j=1}^m$ be the $m \times m$-matrix such that

$$u_i^\beta = \sum_{j=1}^m p_{ij}^{\beta\alpha} u_j^\alpha + n_i^{\beta\alpha}, \tag{3}$$

where $n_i^{\beta\alpha}$ is perpendicular to the vectors u_j^α, $j = 1, \ldots, m$, in $\mathbf{R}^n$. We call (3) 'the change of basis formula'. Polarization of (2) gives

$$\langle u_i^\alpha, u_j^\beta \rangle + \langle u_j^\alpha, u_i^\beta \rangle = 0, \qquad \alpha \neq \beta. \tag{4}$$

Substituting the change of basis formula into this, we obtain that $P^{\beta\alpha}$, *considered acting on the source* $\mathbf{R}^m$, is skew symmetric, i.e.

$$(P^{\beta\alpha})^\top = -P^{\beta\alpha}, \qquad \alpha \neq \beta, \tag{5}$$

where the transpose is taken with respect to the semi-Euclidean structure on $\mathbf{R}^m$. (In fact, using coordinates, we have $\sigma_i p_{ij}^{\beta\alpha} + \sigma_j p_{ji}^{\beta\alpha} = 0$ and skew symmetry follows.)

We can further exploit (4) by substituting the change of basis formula into the first term of (4) and then repeating this in the second term with α and β switched. We obtain

$$\epsilon_\alpha P^{\alpha\beta} + \epsilon_\beta P^{\beta\alpha} = 0, \qquad \alpha \neq \beta. \tag{6}$$

From here on, for the rest of this section, we assume that we are in the Hurwitz-Radon range $m = n$ and give a brief account on the classification theory from the point of view that will be adopted for positive codimensions.

The change of basis formula reduces to

$$u_i^\beta = \sum_{j=1}^m p_{ij}^{\beta\alpha} u_j^\alpha.$$

Iterating this on two and three indices,we obtain

$$(P^{\beta\alpha})^2 = -\epsilon_\alpha \epsilon_\beta I, \qquad \alpha \neq \beta,$$

and

$$P^{\gamma\beta} = P^{\gamma\alpha} P^{\alpha\beta}, \qquad \alpha, \beta, \gamma \text{ distinct}.$$

Setting $U^\alpha = P^{1\alpha}$, $\alpha = 2, \ldots, k + l$, we obtain that $\{U^\alpha\}_{\alpha=2}^{k+l}$ is a family of skew symmetric transformations of $\mathbf{R}^m$ satisfying

$$\begin{aligned} (U^\alpha)^2 &= -\epsilon_\alpha I, & (7)\\ U^\alpha U^\beta &+ U^\beta U^\alpha = 0, \qquad \alpha \neq \beta. & (8) \end{aligned}$$

We now introduce the Clifford algebra $\mathcal{C}_{a,b}$, $a, b \geq 0$, associated to the semi-Euclidean vector space $\mathbf{R}^{a,b}$, as the algebra (over $\mathbf{R}$) given by the orthonormal basis $\{e_\alpha\}_{\alpha=1}^{a+b} \subset \mathbf{R}^{a,b}$ as generators and satisfying the relations $e_\alpha^2 = 1$, $\alpha = 1, \ldots, a$, and $e_\alpha^2 = -1$, $\alpha = a+1, \ldots, a+b$, and $e_\alpha e_\beta + e_\beta e_\alpha = 0$, $\alpha \neq \beta$. Let V be a $\mathcal{C}_{a,b}$-module and θ a (linear) 2-form on V. We say that θ is $(\pm)$invariant if, for $x, y \in V$, we have

$$\begin{aligned} \theta(e_\alpha x, e_\alpha y) &= \mp\theta(x, y), \quad \alpha = 1, \ldots, a, \\ \theta(e_\alpha x, e_\alpha y) &= \pm\theta(x, y), \quad \alpha = a+1, \ldots, a+b. \end{aligned}$$

In what follows, θ will be either semi-Euclidean (nondegenerate and symmetric) or symplectic (nondegenerate and skew).

Going back to orthogonal multiplications, we see that (7)-(8) translate into the single fact that $\mathbf{R}^m$ is a $\mathcal{C}_{l,k-1}$-module. Moreover, by skew symmetry and (7), we have

$$|U^\alpha y|^2 = \epsilon_\alpha |y|^2, \qquad y \in \mathbf{R}^m,$$

so that, $\mathbf{R}^m$ carries a (+)invariant semi-Euclidean structure. Finally, $\{U^\alpha\}_{\alpha=2}^{k+l}$ determines F up to an isometry on the source $\mathbf{R}^m$. More precisely, $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^m$, is said to be normalized if $F(e_1, x) = x$, $x \in \mathbf{R}^m$. Note that, since $k > 0$, any F can be normalized by precomposing it with the inverse of the isometry $x \to F(e_1, x)$. We summarize the above in the following:

Proposition 1 *The set of normalized orthogonal multiplications $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^m$, $k + l \leq m$, $k > 0$, is in one-to-one correspondence with the set of $\mathcal{C}_{l,k-1}$-modules that carry (+)invariant semi-Euclidean structure.*

It now remains to classify all (irreducible) $\mathcal{C}_{a,b}$-modules that carry (+)invariant semi-Euclidean structure.

We now recall the classification of Clifford algebras (cf. Porteous [12] or Benn and Tucker [4]). We denote by $F(d)$ the algebra of $d \times d$-matrices with entries in F, where $F = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$. Clearly, $\mathcal{C}_{1,0} \cong \mathbf{R} \oplus \mathbf{R}$, $\mathcal{C}_{1,1} \cong \mathbf{R}(2)$, $\mathcal{C}_{0,1} \cong \mathbf{C}$ and $\mathcal{C}_{0,2} \cong \mathbf{H}$ and, little computation shows that $\mathcal{C}_{0,3} \cong \mathbf{H} \oplus \mathbf{H}$ and $\mathcal{C}_{0,4} \cong \mathbf{H}(2)$. Now the isomorphisms

$$\begin{aligned} \mathcal{C}_{a+1,b} &\cong \mathcal{C}_{b+1,a} && (9) \\ \mathcal{C}_{a+1,b+1} &\cong \mathcal{C}_{a,b} \otimes \mathcal{C}_{1,1} && (10) \\ \mathcal{C}_{a,b+4} &\cong \mathcal{C}_{a,b} \otimes \mathcal{C}_{0,4} && (11) \end{aligned}$$

completely describe all Clifford algebras. (For an explicit tableau, cf. Porteous [12], p.250.) In particular, each $\mathcal{C}_{a,b}$ is isomorphic either to $F(d)$ or $F(d) \oplus F(d)$. The only irreducible module(s) over $F(d)$ $(\oplus F(d))$ being F^d with the usual matrix multiplication (precomposed with the projections), the only problem that remains is the existence of (+)invariant semi-Euclidean structure on these modules.

We can now apply a Hurwitz-type induction in the use of (10) to show that an irreducible $\mathcal{C}_{a,b}$-module carries a (+)invariant semi-Euclidean structure iff, for any $t \geq 0$, an irreducible

$C_{a+t,b+t}$-module carries a $(*)$invariant 2-form, where $*$ is the sign of $(-1)^t$, and it is semi-Euclidean if $t \equiv 0, 1 \pmod 4$ and symplectic if $t \equiv 2, 3 \pmod 4$. The existence of these forms are easily verified for low ranges of a and b. Using the isomorphisms (9)-(11) and periodicity (with period 8) in both coordinates a and b, we finally arrive at the following tableau:

7	∘	∘	∘	∘	∘	∘	∘	∘
6	∘	∘	∘	•	∘	•	•	•
5	∘	∘	∘	∘	∘	•	•	•
4	∘	•	∘	∘	∘	•	•	•
3	∘	∘	∘	∘	∘	∘	∘	∘
2	∘	•	•	•	∘	∘	∘	•
1	∘	•	•	•	∘	∘	∘	∘
0	∘	•	•	•	∘	•	∘	∘
	0	1	2	3	4	5	6	7

where ∘ and • at (a, b) means existence and nonexistence of a $(+)$invariant semi-Euclidean structure on an irreducible $C_{a,b}$-module. Notice that symmetry to the $y + 1 = x$ line is the consequence of (9). The table is extended periodically in both directions by period 8.

Remark. Irreducibility of the $C_{a,b}$-module is crucial for the (non)existence of a $(+)$invariant semi-Euclidean structure. In fact, it is easy to see that, given an irreducible $C_{a,b}$-module V, a $(+)$invariant semi-Euclidean structure always exists on the double of V.

Examples. The orthogonal multiplication $F : \mathbf{R}^{1,1} \times \mathbf{R}^{1,1} \to \mathbf{R}^{1,1}$, defined by $F((x, y), (u, v)) = (xu + yv, yu + xv)$ is reducible, i.e. it splits into the direct sum of two orthogonal multiplications. F corresponds to the (reducible) $C_{1,0}$-module $\mathbf{R}^2$ with the generator e acting as reflection in the line $u = v$. The situation is the same for the orthogonal multiplication $F : \mathbf{R}^{2,2} \times \mathbf{R}^{2,2} \to \mathbf{R}^{2,2}$, where, using complex coordinates, F is given by $F((z_1, z_2), (w_1, w_2)) = (z_1 w_1 + z_2 \bar{w}_2, z_1 w_2 + z_2 \bar{w}_1)$. Here the (reducible) $C_{2,1}$-module is $\mathbf{R}^2 \oplus \mathbf{R}^2$ with the generators e_1, e_2, e_3 acting as follows: e_1 (resp. e_2) switches the copies of $\mathbf{R}^2$ and acts as a reflection in the first axis (resp. in the $u = v$ line) on each copy; e_3 acts as the positive rotation by $\pi/2$ on each copy.

3 Proof of Theorem 1

Let $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^{m+1}$, $k + l \leq m$, $k > 0$, be a full orthogonal multiplication. We use the notations of the beginning of the previous section. For each $\alpha = 1, \ldots, k + l$, let $E^\alpha \in \mathbf{R}^{m+1}$ be the (uniquely determined) vector such that $\{u_1^\alpha, \ldots, u_m^\alpha, E^\alpha\}$ is an oriented orthonormal basis in $\mathbf{R}^{m+1}$ in the following sense: We rearrange $\{u_1^\alpha, \ldots, u_m^\alpha, E^\alpha\}$ in such a way that (i) The vectors with positive norms always precede those with negative norms; (ii) The lower index is always increasing unless it violates (i); (iii) E^α follows the vectors u_j^α unless it violates (i). With this, we assume that the rearranged system agrees with the standard orientation of $\mathbf{R}^{m+1}$. Since E^α is a unit vector, we have

$$|E^\alpha|^2 = \nu_\alpha = \pm 1.$$

Since $n_i^{\beta\alpha}$ in (3) is proportional to E^α, we rewrite the change of basis formula as

$$u_i^\beta = \sum_j p_{ij}^{\beta\alpha} u_j^\alpha + q_i^{\beta\alpha} E^\alpha, \tag{12}$$

where $q_i^{\beta\alpha} \in \mathbf{R}$.

We now recall that $P^{\beta\alpha}$ is, by (5), skew and satisfies (6). Moreover, substituting (12) into the orthonormality condition $\langle u_i^\beta, u_j^\beta \rangle = \epsilon_\beta \sigma_i \delta_{ij}$, we obtain

$$(P^{\beta\alpha})^2 = -\epsilon_\alpha \epsilon_\beta I + \epsilon_\alpha \nu_\alpha q^{\beta\alpha} (q^{\beta\alpha})^\top, \tag{13}$$

where $q^{\beta\alpha}$ stands for the vector with coordinates $q_i^{\beta\alpha}$ and the transpose is taken with respect to the semi-Euclidean structure in $\mathbf{R}^m$, i.e. $(q^{\beta\alpha})^\top$ is a row-vector with coordinates $\sigma_i q_i^{\beta\alpha}$, $i = 1, \ldots, m$.

Lemma 1 *For $\alpha \neq \beta$, we have*

(i) $P^{\beta\alpha} q^{\beta\alpha} = 0$;

(ii) $|q^{\beta\alpha}|^2 = \epsilon_\beta \nu_\alpha$, provided that $q^{\beta\alpha}$ is nonzero.

Proof. Assume that $q^{\beta\alpha}$ is nonzero. To simplify the notation, we suppress the upper double indices $\beta\alpha$ and put $\epsilon = \epsilon_\alpha \epsilon_\beta$ and $\nu = \epsilon_\alpha \nu_\alpha$. With these, (13) rewrites as

$$P^2 = -\epsilon I + \nu q q^\top. \tag{14}$$

We now introduce what will be called the 'P^3-trick' (to be used later in various instances). For $x \in \mathbf{R}^m$, we have

$$P^3 x = P^2(Px) = -\epsilon P x + \nu \langle q, Px \rangle q$$

and

$$P^3 x = P(P^2 x) = -\epsilon P x + \nu \langle q, x \rangle P q,$$

so that

$$\langle q, Px \rangle q = \langle q, x \rangle P q, \qquad x \in \mathbf{R}^m.$$

Taking scalar product of both sides by x and using skew symmetry of P, we obtain

$$\langle q, x \rangle \langle P q, x \rangle = 0.$$

Since $q \neq 0$, this certainly gives $Pq = 0$. Going back to (14) and using $P^2 q = 0$, we get $|q|^2 = \epsilon\nu$ and the proof is complete.

Lemma 2 *Let V be a semi-Euclidean vector space and P a skew endomorphism of V. If P is nonsingular then* $\dim V$ *is even.*

Proof. We have

$$\det P = \det P^\top = \det(-P) = (-1)^{\dim V} \det P.$$

Lemma 3 *m is odd provided there is at least one nonzero $q^{\beta\alpha}$.*

Proof. Take $V = (q^{\beta\alpha})^{\perp}$. By Lemma 1, V is nondegenerate and, by (13), $P^{\beta\alpha}|V$ satisfies the conditions of Lemma 2. Hence $\dim V = m - 1$ is even.

From now on, we assume that F is full. In terms of (12), this is equivalent to saying that at least one $q^{\beta\alpha}$ is nonzero; thus m is odd and, by Lemma 2, *all* $q^{\beta\alpha}$ are nonzero. Notice also that the first part of Theorem 1 is proved.

Lemma 4 *We have*

(i) $\epsilon_\alpha \nu_\alpha = \epsilon_\beta \nu_\beta$;

(ii) $E^\beta = -\epsilon_\alpha \nu_\alpha \sum_{i=1}^m \sigma_i q_i^{\beta\alpha} u_i^\alpha$.

Proof. Let $\hat{E}^{\beta\alpha}$ denote the right hand side in (ii). Using orthonormality of the vectors u_i^α, we compute

$$\begin{aligned} |\hat{E}^{\beta\alpha}|^2 &= \sum_{i=1}^m (q_i^{\beta\alpha})^2 |u_i^\alpha|^2 \\ &= \epsilon_\alpha |q^{\beta\alpha}|^2 \\ &= \epsilon_\alpha \epsilon_\beta \nu_\alpha (= \pm 1), \end{aligned}$$

where we used (ii) of Lemma 1. Next, changing the basis and using (i) of Lemma 1, we have

$$\begin{aligned} \langle \hat{E}^{\beta\alpha}, u_j^\beta \rangle &= -\epsilon_\alpha \nu_\alpha \sum_{i=1}^m \sigma_i q_i^{\beta\alpha} \langle u_i^\alpha, \sum_{k=1}^m p_{jk}^{\beta\alpha} u_k^\alpha + q_j^{\beta\alpha} E^\alpha \rangle \\ &= -\nu_\alpha \sum_{i=1}^m p_{ji}^{\beta\alpha} q_i^{\beta\alpha} = 0. \end{aligned}$$

Thus

$$\hat{E}^{\beta\alpha} = \pm E^\beta, \tag{15}$$

in particular, the norm square of both sides gives (i). It remains to show that we have positive sign in (15). To do this, we consider the orthonormal bases $\{u_1^\alpha, \ldots, u_m^\alpha, E^\alpha\}$ and $\{u_1^\beta, \ldots, u_m^\beta, \hat{E}^{\beta\alpha}\}$ and show that, with the order defined at the beginning of this section, they have the same orientation. This is done by case-by-case verification and, the proofs being very similar, we give the details only for two cases. Assume first that $\epsilon_\alpha = 1$, $\epsilon_\beta = 1$ and $\epsilon_\alpha \nu_\alpha = \epsilon_\beta \nu_\beta = 1$. Let (p, q) be the signature of the source $\mathbf{R}^m$ and write $P^{\beta\alpha}$ and $q^{\beta\alpha}$ in block form accordingly:

$$P^{\beta\alpha} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad q^{\beta\alpha} = \begin{bmatrix} r \\ s \end{bmatrix}.$$

Then the transfer matrix between the two (rearranged) bases is

$$\begin{bmatrix} A & r & B \\ -r^t & 0 & s^t \\ C & s & D \end{bmatrix},$$

where ^{t} means ordinary transpose. Now both

$$\begin{bmatrix} A & r \\ -r^t & 0 \end{bmatrix} \quad \text{and} \quad D$$

are skew and hence have positive determinants. It follows that the transfer matrix preserves space and time-orientation and we are done.

Second, assume that $\epsilon_\alpha = 1$, $\epsilon_\beta = -1$ and $\epsilon_\alpha \nu_\alpha = \epsilon_\beta \nu_\beta = 1$. In this case, we interchange the last $p+1$ rows and the first q rows in the transition matrix and then change the sign in the last q rows. We obtain a skew matrix. Its determinant is positive. The row operations amount to multiply this with $(-1)^{(p+1)q+q}$ to get the determinant of the transfer matrix. The sign is, however, positive since $m = p+q$ is odd and we are done.

Lemma 5 *For $\alpha \neq \beta$, we have*

$$q^{\alpha\beta} = -\epsilon_\alpha \epsilon_\beta q^{\beta\alpha}.$$

Proof. Changing the basis, and using Lemmas 1 and 4, we compute

$$\begin{aligned} \sum_{i=1}^m \sigma_i q_i^{\alpha\beta} u_i^\alpha &= \sum_{i,j=1}^m \sigma_i q_i^{\alpha\beta} p_{ij}^{\alpha\beta} u_j^\beta + |q^{\alpha\beta}|^2 E^\beta \\ &= -\sum_{i,j=1}^m \sigma_j p_{ji}^{\alpha\beta} q_i^{\alpha\beta} u_j^\beta + |q^{\alpha\beta}|^2 E^\beta \\ &= \epsilon_\alpha \nu_\beta E^\beta = -\epsilon_\alpha \epsilon_\beta \sum_{i=1}^m \sigma_i q_i^{\beta\alpha} u_i^\alpha \end{aligned}$$

and the lemma follows.

Lemma 6 *Let α, β and γ be pairwise distinct. We have*

(i) $P^{\gamma\alpha} P^{\alpha\beta} = P^{\gamma\beta} + \epsilon_\beta \nu_\beta q^{\gamma\alpha} (q^{\gamma\beta})^\top$;

(ii) $P^{\gamma\alpha} q^{\alpha\beta} = q^{\gamma\beta}$;

(iii) $\langle q^{\beta\alpha}, q^{\gamma\alpha} \rangle = 0$.

Proof. The proof is a simple computation and is completely analogous to the proof of formulas (17)-(19) in [5] and is therefore omitted.

Proof of Theorem 1. Let $\bar{U}^\alpha$, $\alpha = 2, \ldots, k+l$, be the linear transformation of $\mathbf{R}^{m+1}$ whose matrix, with respect to the orthonormal basis $\{u_1^1, \ldots, u_m^1, E^1\}$ is

$$\bar{U}^\alpha = \begin{bmatrix} P^{\alpha 1} & q^{\alpha 1} \\ -\nu_1 (q^{\alpha 1})^\top & 0 \end{bmatrix}.$$

The symmetry properties of P, the change of basis formula and Lemmas 1, 5 and 6 then translate into

$$\begin{aligned}
(\bar{U}^\alpha)^2 &= -\epsilon_\alpha I \\
\bar{U}^\alpha\bar{U}^\beta &+ \bar{U}^\beta\bar{U}^\alpha = 0, \qquad \alpha \neq \beta, \\
\langle \bar{U}^\alpha u, v\rangle &+ \langle u, \bar{U}^\alpha v\rangle = 0, \qquad u, v \in \mathbf{R}^{m+1}.
\end{aligned}$$

From now on, we think of the source $\mathbf{R}^m$ as being a linear subspace of $\mathbf{R}^{m+1}$ by the linear isometry $x \to F(e_1, x)$, in particular, we have $f_i = u_i^1$, $i = 1, \ldots, m$. We define the bilinear map

$$\bar{F} : \mathbf{R}^{k,l} \times \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$$

by

$$\bar{F}(x, y) = x_1 y + \sum_{\alpha=2}^{k+l} x_\alpha \bar{U}^\alpha y,$$

where $x = \sum_{\alpha=1}^{k+l} x_\alpha e_\alpha$. Using the properties of $\bar{U}^\alpha$ above, we compute

$$\begin{aligned}
|\bar{F}(x,y)|^2 &= x_1^2|y|^2 + \sum_{\alpha,\beta=2}^{k+l} x_\alpha x_\beta \langle \bar{U}^\alpha y, \bar{U}^\beta y\rangle \\
&= x_1^2|y|^2 - \sum_{\alpha,\beta=2}^{k+l} x_\alpha x_\beta \langle \bar{U}^\beta \bar{U}^\alpha y, y\rangle \\
&= x_1^2|y|^2 + \sum_{\alpha=2}^{k+l} \epsilon_\alpha x_\alpha^2 |y|^2 \\
&= |x|^2|y|^2
\end{aligned}$$

so that $\bar{F}$ is normed, i.e. an orthogonal multiplication. Finally, we have

$$\bar{F}(e_1, f_i) = f_i = F(e_1, f_i)$$

and, for $\alpha \geq 2$, the change of basis formula gives

$$\bar{F}(e_\alpha, f_i) = \bar{U}^\alpha u_i^1 = \sum_{j=1}^{m} p_{ij}^{\alpha 1} u_j^1 + q_i^{\alpha 1} E^1 = u_j^\alpha = F(e_\alpha, f_i)$$

so that $\bar{F}$ is an extension of F. The proof of Theorem 1 is complete.

4 Codimension Two Orthogonal Multiplications; General Setting

Let $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^{m+2}$, $k + l \leq m$, $k > 0$, be a full orthogonal multiplication. We retain the notations introduced at the beginning of Section 2. For each $\alpha = 1, \ldots, m$, we choose E_1^α and E_2^α in $\mathbf{R}^{m+2}$ such that $\{u_1^\alpha, \ldots, u_m^\alpha, E_1^\alpha, E_2^\alpha\}$ is an orthonormal basis of $\mathbf{R}^{m+2}$. We set the signs as

$$|E_1^\alpha|^2 = \nu'_\alpha \quad \text{and} \quad |E_2^\alpha|^2 = \nu''_\alpha.$$

For $\alpha \neq \beta$, the normal vectors $n_i^{\beta\alpha}$ in (3) are linear combinations of E_1^α and E_2^α. Thus, we can write the change of basis formula as

$$u_i^\beta = \sum_{j=1}^m p_{ij}^{\beta\alpha} u_j^\alpha + q_i^{\beta\alpha} E_1^\alpha + r_i^{\beta\alpha} E_2^\alpha, \qquad \alpha \neq \beta. \tag{16}$$

As usual, we denote by $q^{\beta\alpha}$ and $r^{\beta\alpha}$ the vectors in $\mathbf{R}^m$ whose components are $q_i^{\beta\alpha}$ and $r_i^{\beta\alpha}$, respectively.

For $\alpha \neq \beta$, let

$$V^{\beta\alpha} = \operatorname{span}\{q^{\beta\alpha}, r^{\beta\alpha}\}$$

and

$$W^{\beta\alpha} = (V^{\beta\alpha})^\perp.$$

Then $V^{\beta\alpha}$ and $W^{\beta\alpha}$ are linear subspaces of $\mathbf{R}^m$ of complementary dimension and they span $\mathbf{R}^m$ iff one (and hence both) are nondegenerate.

Lemma 7 *The normal vectors E_1^α and E_2^α can be chosen in such a way that, whenever* $\dim V^{\beta\alpha} = 1$, *we have*

$$q^{\beta\alpha} \neq \pm r^{\beta\alpha}.$$

Remark. We emphasize here that, at this point, we keep E_1^α and E_2^α arbitrary (subject only to orthogonality) provided that $\dim V^{\beta\alpha} \neq 1$.

Proof of Lemma 7. Let $\dim V^{\beta\alpha} = 1$ and assume that $q^{\beta\alpha} = \pm r^{\beta\alpha}$. We first consider the case $\nu_\alpha'\nu_\alpha'' = 1$. We change $\{E_1^\alpha, E_2^\alpha\}$ to $\{\tilde{E}_1^\alpha, \tilde{E}_2^\alpha\}$ such that

$$\begin{aligned} E_1^\alpha &= \cos t\, \tilde{E}_1^\alpha - \sin t\, \tilde{E}_2^\alpha \\ E_2^\alpha &= \sin t\, \tilde{E}_1^\alpha + \cos t\, \tilde{E}_2^\alpha. \end{aligned}$$

The change of basis formula then implies that

$$\begin{aligned} \tilde{q}^{\beta\alpha} &= \cos t\, q^{\beta\alpha} + \sin t\, r^{\beta\alpha} \\ \tilde{r}^{\beta\alpha} &= -\sin t\, q^{\beta\alpha} + \cos t\, r^{\beta\alpha}. \end{aligned}$$

Taking t (small) such that $\sin t \neq 0$ and $\cos t \neq 0$, we obtain $\tilde{q}^{\beta\alpha} \neq \tilde{r}^{\beta\alpha}$ (without affecting the other relations). If $\nu_\alpha'\nu_\alpha'' = -1$, instead of rotation, we use hyperbolic boost, i.e. the transformation

$$\begin{aligned} E_1^\alpha &= \cosh t \tilde{E}_1^\alpha + \sinh t \tilde{E}_2^\alpha, \\ E_2^\alpha &= \sinh t \tilde{E}_1^\alpha + \cosh t \tilde{E}_2^\alpha. \end{aligned}$$

Recall that $P^{\beta\alpha}$ possesses the symmetry properties (5)-(6). As in the earlier cases, using the change of basis formula (16) in the orthonormality of the vectors u_i^β, we arrive at

$$(P^{\beta\alpha})^2 = -\epsilon_\alpha\epsilon_\beta I + \epsilon_\alpha\nu_\alpha' q^{\beta\alpha}(q^{\beta\alpha})^\top + \epsilon_\alpha\nu_\alpha'' r^{\beta\alpha}(r^{\beta\alpha})^\top. \tag{17}$$

Fundamental Lemma *Let $F : \mathbf{R}^{k,l} \times \mathbf{R}^m \to \mathbf{R}^{m+2}$, $k + l \leq m$, $k > 0$, be a full orthogonal multiplication. Then, by a suitable change of basis in $\mathbf{R}^{k,l}$, we have the following three cases:*

(i) For all $\alpha \neq \beta$, $V^{\beta\alpha}$ is 2-dimensional and nondegenerate; m is even.

(ii) For all $\alpha \neq \beta$, $V^{\beta\alpha}$ is 2-dimensional and null; m is even.

(iii) For all $\alpha \neq \beta$, $V^{\beta\alpha}$ is 1-dimensional and nondegenerate; m is odd.

The proof of the Fundamental Lemma is the content of the next section. Then the proof of Theorem 2 will be accomplished in the following two sections corresponding to m even or odd. We finish this section by some preliminary considerations preparing the proof of the Fundamental Lemma.

Lemma 8 *Let* $\dim V^{\beta\alpha} = 1$. *We have*

(i) $P^{\beta\alpha} q^{\beta\alpha} = 0$;

(ii) Setting $r^{\beta\alpha} = c^{\beta\alpha} q^{\beta\alpha}$, $q^{\beta\alpha} \neq 0$, we have

$$(\nu_\alpha{}' + (c^{\beta\alpha})^2 \nu_\alpha{}'')|q^{\beta\alpha}|^2 = \epsilon_\beta.$$

$V^{\beta\alpha} = V^{\alpha\beta}$ and it is $P^{\beta\alpha}$-invariant and nondegenerate. Finally, m is odd.

Proof. We simplify the notations by suppressing the upper indices $\beta\alpha$ and setting $\nu' = \epsilon_\alpha \nu_\alpha{}'$ and $\nu'' = \epsilon_\alpha \nu_\alpha{}''$. Since q and r are linearly dependent, we may assume that $q \neq 0$ and $r = cq$. With this, (17) rewrites as

$$P^2 = -\epsilon I + (\nu' + c^2 \nu'') q q^{\mathsf{T}}.$$

By Lemma 7, $c \neq \pm 1$ so that $\nu' + c^2 \nu'' \neq 0$. We now use the P^3-trick as in the proof of Lemma 1 (with ν replaced by $\nu' + c^2\nu''$) to obtain (i)-(ii). Everything follows now (cf. Lemma 2) except that the upper indices in V can be switched. To do this, we return to indices and first note that, by the above, $W^{\beta\alpha}$ is the $-\epsilon_\alpha\epsilon_\beta$-eigenspace of $(P^{\beta\alpha})^2$. Since $(P^{\alpha\beta})^2 = (P^{\beta\alpha})^2$, (17) implies that $W^{\alpha\beta} \subset W^{\beta\alpha}$. Assume that $W^{\alpha\beta} \neq W^{\beta\alpha}$. Then $\dim V^{\alpha\beta} = 2$, i.e. $q^{\alpha\beta}$ and $r^{\alpha\beta}$ are linearly independent. Take $x \in W^{\beta\alpha} \setminus W^{\alpha\beta}$. Applying both sides of (17) (with α and β switched) to x and using that $q^{\alpha\beta}$ and $r^{\alpha\beta}$ are linearly independent, we obtain that $x \in (V^{\alpha\beta})^\perp = W^{\alpha\beta}$ which is a contradiction. Hence $W^{\beta\alpha} = W^{\alpha\beta}$ and, taking perpendicular complements, the statement follows.

Lemma 9 *Let* $\dim V^{\beta\alpha} = 2$. *Then $V^{\beta\alpha} = V^{\alpha\beta}$ and it is $P^{\beta\alpha}$-invariant. Moreover, we have*

$$\begin{aligned} P^{\beta\alpha} q^{\beta\alpha} &= \nu_\alpha{}' \lambda_{\beta\alpha} r^{\beta\alpha}, \\ P^{\beta\alpha} r^{\beta\alpha} &= -\nu_\alpha{}'' \lambda_{\beta\alpha} q^{\beta\alpha}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} (\lambda_{\beta\alpha})^2 &= \epsilon_\alpha \epsilon_\beta \nu_\alpha{}' \nu_\alpha{}'' - \epsilon_\alpha \nu_\alpha{}'' |q^{\beta\alpha}|^2 \\ &= \epsilon_\alpha \epsilon_\beta \nu_\alpha{}' \nu_\alpha{}'' - \epsilon_\alpha \nu_\alpha{}' |r^{\beta\alpha}|^2, \end{aligned}$$

in particular,

$$|r^{\beta\alpha}|^2 = \nu_\alpha{}'\nu_\alpha{}''|q^{\beta\alpha}|^2.$$

Furthermore, we have

$$\langle q^{\beta\alpha}, r^{\beta\alpha}\rangle = 0.$$

$V^{\beta\alpha}$ *is either nondegenerate or null and, in both cases, m is even.*

Proof. We again use the simplified notation. We first claim that W is the $-\epsilon$-eigenspace of P^2. Indeed, let x be a P^2-eigenvector with eigenvalue $-\epsilon$, i.e. $P^2x = -\epsilon x$. Applying x to both sides of (17), we then obtain $\langle q, x\rangle = \langle r, x\rangle = 0$ since q and r are linearly independent. Thus $x \in V^\perp = W$ and the claim follows.

Returning to indices for a moment, we see that $W^{\alpha\beta} \subset W^{\beta\alpha}$ since $(P^{\beta\alpha})^2 = (P^{\alpha\beta})^2$. But $W^{\beta\alpha}$ is of codimension two so that equality holds. Taking perpendicular complements, $V^{\beta\alpha} = V^{\alpha\beta}$ follows.

Returning to the index-free notation, we next claim that W is P-invariant. Indeed, given $x \in W$, we have $P^2(Px) = P(P^2x) = -\epsilon Px$ so that Px is a P^2-eigenvector with eigenvalue $-\epsilon$. By the above, $Px \in W$ and hence the claim. P is skew and so $V = W^\perp$ is also P-invariant.

We now use the P^3-trick again. Given $x \in \mathbf{R}^m$, by (17), we have

$$P(P^2x) = -\epsilon Px + \nu'\langle q, x\rangle Pq + \nu''\langle r, x\rangle Pr$$

and

$$P^2(Px) = -\epsilon Px + \nu'\langle q, Px\rangle q + \nu''\langle r, Px\rangle r.$$

Combining these, we arrive at

$$\nu'\langle q, x\rangle Pq + \nu''\langle r, x\rangle Pr + \nu'\langle Pq, x\rangle q + \nu''\langle Pr, x\rangle r = 0. \tag{19}$$

V is P-invariant so, for a moment, we can set

$$\begin{aligned} Pq &= aq + b'r \\ Pr &= b''q + dr. \end{aligned}$$

Substituting these back to (19) and using linear independence of q and r, we obtain

$$\begin{aligned} 2\nu'a\langle q, x\rangle + (\nu'b' + \nu''b'')\langle r, x\rangle &= 0 \\ (\nu'b' + \nu''b'')\langle q, x\rangle + 2\nu''d\langle r, x\rangle &= 0. \end{aligned}$$

These are valid for all $x \in \mathbf{R}^m$ and so we finally arrive at

$$a = d = 0$$
$$\nu'b' + \nu''b'' = 0,$$

and (18) follows. In particular

$$P^2q = -\nu'\nu''\lambda^2 q.$$

On the other hand

$$P^2q = -\epsilon q + \nu'|q|^2 q + \nu''\langle q, r\rangle r.$$

Comparing, we obtain orthogonality of q and r and the first equation for λ^2. The second can be obtained by considering P^2r. To prove the last two statements, assume that V is degenerate, i.e. $|q|^2|r|^2 = \langle q, r\rangle^2 = 0$. By the above, this, however, means that $|q|^2 = |r|^2 = 0$ so that V is null.

Finally, we show that m is even. This is clear if V is nondegenerate since W is then nondegenerate and codimension two and Lemma 2 applies. Assume now that V is degenerate, hence null, i.e. $V \subset W$. V is P-invariant and $P|W$ generates a finite cyclic group so that there exists a P-invariant linear subspace $V' \subset W$ complementary to V in W. V' is nondegenerate. Indeed, if $v' \in V'$ is perpendicular to V' then it is perpendicular to all W since $v' \in W = V^\perp$ and V with V' span W. Thus, $v' \in V$ and so $v' = 0$. Moreover, V' is codimension two in W and so it is codimension four in $\mathbf{R}^m$. Lemma 2 applies to V' yielding that m is even. The proof is complete.

Lemma 10 *If m is even then $\nu_\alpha'\nu_\alpha''$ does not depend on α.*

Proof. The space $\mathbf{R}^{m+2}$ has signature (r, s) so its determinant (discriminant) is $(-1)^s$. Using the orthonormal basis $\{u_1^\alpha, \ldots, u_m^\alpha, E_1^\alpha, E_2^\alpha\}$ we get another expression for the determinant: $(-1)^s = (\epsilon_\alpha)^m \sigma_1 \ldots \sigma_m \nu_\alpha' \nu_\alpha''$. If m is even it follows that $\nu_\alpha'\nu_\alpha''$ does not depend on α.

5 Proof of the Fundamental Lemma

For any $\gamma \neq \delta$, we set

$$\Delta^{\delta\gamma} = |q^{\delta\gamma}|^2|r^{\delta\gamma}|^2 - \langle q^{\delta\gamma}, r^{\delta\gamma}\rangle^2.$$

Step 1. Assume that for a pair of indices $\alpha \neq \beta$, we have $\Delta^{\beta\alpha} \neq 0$. We show that, after a suitable change of the basis in $\mathbf{R}^{k,l}$, we can get $\Delta^{\delta\gamma} \neq 0$, for all $\gamma \neq \delta$.

Fix an index $\gamma \neq \alpha$, β. Change the orthonormal basis $\{e_\alpha\}_{\alpha=1}^{k+l}$ in $\mathbf{R}^{k,l}$ to another orthonormal basis $\{\tilde{e}_\alpha\}_{\alpha=1}^{k+l}$ as follows:

If $\epsilon_\beta\epsilon_\gamma = 1$ then let

$$\begin{aligned} \tilde{e}_\gamma &= \cos t\, e_\gamma - \sin t\, e_\beta, \\ \tilde{e}_\beta &= \sin t\, e_\gamma + \cos t\, e_\beta, \\ \tilde{e}_\delta &= e_\delta, \text{ if } \delta \neq \beta, \gamma. \end{aligned}$$

If $\epsilon_\beta\epsilon_\gamma = -1$ then we apply a hyperbolic boost by replacing the trigonometric functions above by their hyperbolic analogues.

From now on we give details only for the case $\epsilon_\beta\epsilon_\gamma = 1$, the treatment of the opposite sign being very similar. Then, we have

$$\tilde{u}_i^\gamma = \cos t\, u_i^\gamma - \sin t\, u_i^\beta$$

and using the change of basis formula for both terms on the right hand side (going down to u_j^α), with obvious notations, we arrive at

$$\begin{aligned}\tilde{q}^{\gamma\alpha} &= \cos t\, q^{\gamma\alpha} - \sin t\, q^{\beta\alpha},\\ \tilde{r}^{\gamma\alpha} &= \cos t\, r^{\gamma\alpha} - \sin t\, r^{\beta\alpha}.\end{aligned} \tag{20}$$

Assume now that $\Delta^{\gamma\alpha} = 0$. Using the transformation rules above, an easy computation gives

$$\tilde{\Delta}^{\gamma\alpha} = -1/2\,\Delta^{\beta\alpha}\cos(2t) + A + B\sin(2t) + C\cos(4t) + D\sin(4t),$$

where A, B, C and D do not depend on t. Since $\Delta^{\beta\alpha} \neq 0$, $\tilde{\Delta}^{\gamma\alpha}$, as a function of t, is nonconstant. Thus, by a small change of t, we get $\tilde{\Delta}^{\gamma\alpha} \neq 0$. Repeating this process finitely many times, and retaining the earlier notation, we can now assume that $\Delta^{\gamma\alpha} \neq 0$ for all $\gamma\,(\neq \alpha)$. By Lemma 9, $V^{\gamma\alpha} = V^{\alpha\gamma}$ so that $\Delta^{\alpha\gamma} \neq 0$ also follows. Changing the basis in $\mathbf{R}^{k,l}$ once again, we obtain $\Delta^{\delta\gamma} \neq 0$, for all $\gamma \neq \delta$.

This however means that, for all $\gamma \neq \delta$, $V^{\delta\gamma}$ is 2-dimensional and nondegenerate and we arrive at Case (i) of the Fundamental Lemma. Note that, by Lemma 9, m is even.

Step 2. We can now assume that

(1) For all $\gamma \neq \delta$, either $V^{\delta\gamma}$ is 2-dimensional and null or $\dim V^{\delta\gamma} \leq 1$;

(2) Property (1) is preserved under any change of the basis in $\mathbf{R}^{k,l}$. (If property (1) is not preserved under any change of the basis in $\mathbf{R}^{k,l}$, we will return to the situation of Step 1 after a suitable change of the basis.

Assume that for a pair of indices $\alpha \neq \beta$, $\dim V^{\beta\alpha} = 2$. We show that, after a suitable change of the basis in $\mathbf{R}^{k,l}$, we have $\dim V^{\delta\gamma} = 2$, for all $\gamma \neq \delta$. As in Step 1, we fix $\gamma \neq \alpha, \beta$ and apply the same transformations.

Assume that $\dim V^{\gamma\alpha} \leq 1$ along with $\dim \tilde{V}^{\gamma\alpha} \leq 1$. Setting $q^{\gamma\alpha} = \lambda\xi$ and $r^{\gamma\alpha} = \mu\xi$, $\lambda, \mu \in \mathbf{R}$ and $0 \neq \xi \in \mathbf{R}^m$, we claim that $\xi \in \operatorname{span}\{q^{\beta\alpha}, r^{\beta\alpha}\}$. Indeed, in the opposite case, for $\sin t \neq 0$, the linear span of ξ, $-\sin t\, q^{\beta\alpha}$ and $-\sin t\, r^{\beta\alpha}$ is 3-dimensional. Hence the linear span of ξ, $\tilde{q}^{\gamma\alpha}$ and $\tilde{r}^{\gamma\alpha}$ is also 3-dimensional and this is a contradiction since $\tilde{q}^{\gamma\alpha}$ and $\tilde{r}^{\gamma\alpha}$ are linearly dependent. Thus, we can write

$$\xi = aq^{\beta\alpha} + br^{\beta\alpha}$$

so that

$$\begin{aligned}\tilde{q}^{\gamma\alpha} &= (\lambda a\cos t - \sin t)q^{\beta\alpha} + \lambda b\cos t\, r^{\beta\alpha},\\ \tilde{r}^{\gamma\alpha} &= \mu a\cos t\, q^{\beta\alpha} + (\mu b\cos t - \sin t)r^{\beta\alpha}.\end{aligned}$$

By linear dependence, the determinant of the coefficients, i.e.

$$\sin t\cos t\,(\tan t - (\lambda a + \mu b))$$

must vanish. Hence, if we take $0 < t < \pi/2$ with $\tan t \neq \lambda a + \mu b$ then $\tilde{q}^{\gamma\alpha}$ and $\tilde{r}^{\gamma\alpha}$ become linearly independent. Repeating this process finitely many times, and retaining the earlier notations, we arrive at a system, where $\dim V^{\gamma\alpha} = 2$ for all $\gamma\,(\neq \alpha)$. By Lemma 9, $V^{\gamma\alpha} = V^{\alpha\gamma}$ so that $\dim V^{\alpha\gamma} = 2$. Now the process applied again gives $\dim V^{\delta\gamma} = 2$, for all $\gamma \neq \delta$. Since we are in Step 2, all these spaces are null and we land in Case (ii) of the Fundamental Lemma. Note that m is even by Lemma 9.

Step 3. We can now assume that

(1) For all $\gamma \neq \delta$, we have $\dim V^{\delta\gamma} \leq 1$;

(2) Property (1) is preserved under any change of the basis in $\mathbf{R}^{k,l}$. (Otherwise we return to Case 1 or 2.)

Assume that for a pair of indices $\alpha \neq \beta$, we have $\dim V^{\beta\alpha} = 1$. We may assume that $q^{\beta\alpha} \neq 0$. Changing the basis in $\mathbf{R}^{k,l}$ in exactly the same way as above, we obtain $q^{\gamma\alpha} \neq 0$, in particular, $\dim V^{\gamma\alpha} = 1$, for all $\gamma\,(\neq \alpha)$. By Lemma 8, $V^{\gamma\alpha} = V^{\alpha\gamma}$ and we can finish the process as above. Again by Lemma 8, we arrive at Case (iii) of the Fundamental Lemma and m is odd.

Step 4. We are left with the case when $\dim V^{\delta\gamma} = 0$ for all $\gamma \neq \delta$. This however contradicts to fullness of F as can be readily seen from the change of basis formula.

6 Proof of Theorem 2

If m is odd we are in Case (iii) of the Fundamental Lemma. The proof of Theorem 2 in this case is completely analogous to the proof of Theorem 3 in [5] and is therefore omitted.

In what follows we give a unified treatment of Cases (i)-(ii) of the Fundamental Lemma. Throughout, we assume that m is even and, for $\alpha \neq \beta$, $V^{\beta\alpha}$ is 2-dimensional and nondegenerate or null. We use the notations introduced in Section 4 including (16)-(17) and Lemma 9. We set

$$\begin{aligned} \mu_{\beta\alpha} &= |q^{\beta\alpha}|^2 = \nu_\alpha{}'\nu_\alpha{}''|r^{\beta\alpha}|^2 \\ &= \epsilon_\beta\nu_\alpha{}' - \epsilon_\alpha\nu_\alpha{}''\lambda^2_{\beta\alpha}. \end{aligned} \tag{21}$$

Note that, by Lemma 10, $\nu_\alpha{}'\nu_\alpha{}''$ is independent of α.

Recall that in the codimension one case (Section 3) a crucial fact was (anti)symmetry of $q^{\beta\alpha}$ in the upper indices as proved in Lemma 5. The proof depended on (ii) of Lemma 4 which, in turn, was due to the unicity of E^β. Here the main technical difficulty is the nonuniqueness of E_1^β and E_2^β.

Lemma 11 *We have*

$$\lambda^2_{\alpha\beta} = \lambda^2_{\beta\alpha},$$

in particular

$$\mu_{\alpha\beta} = \epsilon_\alpha\epsilon_\beta\nu_\alpha{}'\nu_\beta{}'\mu_{\beta\alpha}.$$

Proof. By Lemma 9, we have $(P^{\beta\alpha})^2|V^{\beta\alpha} = -\nu_\alpha{}'\nu_\alpha{}''\lambda^2_{\beta\alpha}I$ since this holds for $q^{\beta\alpha}$ and $r^{\beta\alpha}$ and they span $V^{\beta\alpha}$. Switching α and β, again by Lemma 9, the left hand side remains unchanged so that we obtain $\nu_\alpha'\nu_\alpha''\lambda_{\beta\alpha}^2 = \nu_\beta'\nu_\beta''\lambda_{\alpha\beta}^2$ and Lemma 10 gives the first statement. The second follows from this and (21).

Lemma 12 *Without loss of generality, we may assume that either $\lambda_{\delta\gamma} \neq 0$, for all $\gamma \neq \delta$, or $\lambda_{\delta\gamma} = 0$, for all $\gamma \neq \delta$.*

Notation. In what follows, we write $\lambda \neq 0$ (resp. $\lambda = 0$) to indicate that $\lambda_{\delta\gamma} \neq 0$ (resp. $\lambda_{\delta\gamma} = 0$) for all $\gamma \neq \delta$.

Proof of Lemma 12. Assume that, for a pair of indices $\alpha \neq \beta$, we have $\lambda_{\beta\alpha} \neq 0$. By (21), $\mu_{\beta\alpha} \neq \epsilon_\beta\nu_\alpha'$. We show that, after a suitable change of the basis in $\mathbf{R}^{k,l}$, we can get $\lambda_{\delta\gamma} \neq 0$, for all $\gamma \neq \delta$.

Fix an index $\gamma \neq \alpha$, β and assume that $\epsilon_\beta\epsilon_\gamma = 1$ (the other case can be treated similarly). We now apply the rotation of Step 1 in the proof of the Fundamental Lemma. Taking norm square of both sides of (20), we obtain

$$\tilde{\mu}_{\gamma\alpha} = 1/2(\mu_{\gamma\alpha} - \mu_{\beta\alpha})\cos(2t) - \langle q^{\gamma\alpha}, q^{\beta\alpha}\rangle \sin(2t) + 1/2(\mu_{\gamma\alpha} + \mu_{\beta\alpha}).$$

If $\lambda_{\gamma\alpha} = 0$ then $\mu_{\gamma\alpha} = \epsilon_\gamma\nu_\alpha' = \epsilon_\beta\nu_\alpha' \neq \mu_{\beta\alpha}$ so that $\tilde{\mu}_{\gamma\alpha}$, as a function of t, is noncontant. Thus, by a small change of t, we get $\lambda_{\gamma\alpha} \neq 0$. Repeating this finitely many times, and retaining the earlier notation, we can assume that $\lambda_{\gamma\alpha} \neq 0$, for all $\gamma\,(\neq \alpha)$. By Lemma 11, we also have $\lambda_{\alpha\gamma} \neq 0$. Changing the basis again, we obtain $\lambda_{\delta\gamma} \neq 0$, for all $\gamma \neq \delta$.

We now impose an orientability constraint on the choice of E_1^α and E_2^α. We first denote by $\mathcal{B}^\alpha$ the orthonormal basis $\{u_1^\alpha, \ldots, u_m^\alpha, E_1^\alpha, E_2^\alpha\}$ in $\mathbf{R}^{m+2}$, where the vectors appear in this order. Next, we rearrange this basis in such a way that (i) The vectors with positive norms always procede those with negative norms; (ii) The lower index is always increasing unless it violates (i); (iii) E_1^α and E_2^α always follows u_j^α unless it violates (i). We denote the rearranged basis by $\mathcal{B}_\pm^\alpha$ and we choose E_1^α and E_2^α such that the orientation of $\mathcal{B}_\pm^\alpha$ agrees with the standard orientaion of $\mathbf{R}^{m+2}$. Let $T^{\alpha\beta}$ denote the transfer matrix from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. In block form

$$T^{\alpha\beta} = \begin{bmatrix} p^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ \epsilon_\beta\nu_\alpha'(q^{\beta\alpha})^\top & \nu_\beta'\langle E_1^\alpha, E_1^\beta\rangle & \nu_\beta''\langle E_1^\alpha, E_2^\beta\rangle \\ \epsilon_\beta\nu_\alpha''(r^{\beta\alpha})^\top & \nu_\beta'\langle E_2^\alpha, E_1^\beta\rangle & \nu_\beta''\langle E_2^\alpha, E_2^\beta\rangle \end{bmatrix} \tag{22}$$

Lemma 13 *We have*

$$\det(T^{\alpha\beta}) = \nu_\alpha'\nu_\beta'(\epsilon_\alpha\epsilon_\beta)^{m/2}.$$

Proof. We write $\mathcal{O}(\mathcal{B}^\alpha) = \pm 1$ if $\mathcal{B}^\alpha$ is ($\pm$)-oriented in $\mathbf{R}^{m+2}$. Since $T^{\alpha\beta}$ is the transition matrix from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$, we have $\det(T^{\alpha\beta}) = \mathcal{O}(\mathcal{B}^\alpha)\mathcal{O}(\mathcal{B}^\beta)$. Now the proof can be settled by case-by-case verification, where the cases correspond to the various signs $\epsilon_\alpha, \epsilon_\beta, \nu_\alpha', \nu_\alpha''$ and ν_β' (cf. Lemma 10). For example, consider the case when $\epsilon_\alpha = \epsilon_\beta = -1$, $\nu_\alpha' = -\nu_\alpha'' = -\nu_\beta' = \nu_\beta'' = 1$. Looking at $\mathcal{B}^\alpha$ and $\mathcal{B}_\pm^\alpha$, we see that we need $pq + p$ transpositions of (adjacent) vectors to obtain one from the other. Hence $\mathcal{O}(\mathcal{B}^\alpha) = (-1)^{pq+p}$. In a similar vein, $\mathcal{O}(\mathcal{B}^\beta) = (-1)^{pq+p+1}$. Combining these, we arrive at $\det(T^{\alpha\beta}) = -1 = \nu_\alpha'\nu_\beta'(\epsilon_\alpha\epsilon_\beta)^{m/2}$.

Lemma 14 *We have*

$$\lambda_{\alpha\beta} = (-1)^{q+1}(\epsilon_\alpha\epsilon_\beta)^{m/2}\nu_\alpha{}'\nu_\beta{}''\lambda_{\beta\alpha}.$$

Proof. By Lemma 11, we may assume that $\lambda \neq 0$. The proof of the following formulas is similar to that of formulas (30) in [5] (and is omitted):

$$\begin{aligned} -\epsilon_\alpha\epsilon_\beta\lambda_{\alpha\beta}q^{\alpha\beta} &= \langle E_1^\alpha, E_2^\beta\rangle q^{\beta\alpha} + \langle E_2^\alpha, E_2^\beta\rangle r^{\beta\alpha}, \\ -\epsilon_\alpha\epsilon_\beta\lambda_{\alpha\beta}r^{\alpha\beta} &= \langle E_1^\alpha, E_1^\beta\rangle q^{\beta\alpha} + \langle E_2^\alpha, E_1^\beta\rangle r^{\beta\alpha}. \end{aligned} \tag{23}$$

Iterating (23) on $\alpha\beta$ and $\beta\alpha$, we obtain

$$\langle E_1^\alpha, E_2^\beta\rangle\langle E_2^\alpha, E_1^\beta\rangle - \langle E_1^\alpha, E_1^\beta\rangle\langle E_2^\alpha, E_2^\beta\rangle = \lambda_{\alpha\beta}\lambda_{\beta\alpha}. \tag{24}$$

Let $S^{\alpha\beta}$ denote the $(m+2)\times(m+2)$-matrix

$$S^{\alpha\beta} = \operatorname{diag}[\sigma_1, \ldots, \sigma_m] \oplus \begin{bmatrix} -\nu_\beta{}'\langle E_1^\alpha, E_2^\beta\rangle & -\nu_\beta{}''\langle E_2^\alpha, E_2^\beta\rangle \\ \nu_\beta{}'\langle E_1^\alpha, E_1^\beta\rangle & \nu_\beta{}''\langle E_2^\alpha, E_1^\beta\rangle \end{bmatrix}.$$

The product matrix $S^{\alpha\beta}T^{\alpha\beta}$, after dividing the last two rows by $\epsilon_\alpha\nu_\alpha{}'\nu_\beta{}'\lambda_{\alpha\beta}$, becomes skew, in particular, the determinant of $S^{\alpha\beta}T^{\alpha\beta}$ is nonnegative. By Lemma 13, we obtain

$$(\epsilon_\alpha\epsilon_\beta)^{m/2}\nu_\alpha{}'\nu_\beta{}' \det(S^{\alpha\beta}) \geq 0. \tag{25}$$

By (24), $\det(S^{\alpha\beta}) = (-1)^{q+1}\nu_\beta{}'\nu_\beta{}''\lambda_{\alpha\beta}\lambda_{\beta\alpha}$. This, along with (25) gives

$$(-1)^{q+1}(\epsilon_\alpha\epsilon_\beta)^{m/2}\nu_\alpha{}'\nu_\beta{}''\lambda_{\alpha\beta}\lambda_{\beta\alpha} \geq 0.$$

Dividing by $\lambda_{\alpha\beta}^2 = \lambda_{\beta\alpha}^2$ the lemma follows.

We introduce, for $\alpha \neq \beta$, an auxiliary set of vectors by

$$\begin{aligned} I_1^{\beta\alpha} &= \sum_{j=1}^m \sigma_j q_j^{\beta\alpha} u_j^\alpha - \epsilon_\alpha\nu_\alpha{}'\nu_\alpha{}''\lambda_{\beta\alpha}E_2^\alpha, \\ I_2^{\beta\alpha} &= \sum_{j=1}^m \sigma_j r_j^{\beta\alpha} u_j^\alpha + \epsilon_\alpha\nu_\alpha{}'\nu_\alpha{}''\lambda_{\beta\alpha}E_1^\alpha. \end{aligned} \tag{26}$$

These vectors are analogous to the vectors given by formula (28) in [5]. In the same way as in [5] the following lemma can be proved:

Lemma 15

(i) For $\alpha \neq \beta$, $I_1^{\beta\alpha}$ and $I_2^{\beta\alpha}$ form an orthonormal basis in $\operatorname{span}\{E_1^\beta, E_2^\beta\}$ *and*

$$|I_1^{\beta\alpha}|^2 = \epsilon_\alpha\epsilon_\beta\nu_\alpha{}' \text{ and } |I_2^{\beta\alpha}|^2 = \epsilon_\alpha\epsilon_\beta\nu_\alpha{}''.$$

(ii) We have

$$q_i{}^{\alpha\beta}E_1^\beta + r_i{}^{\alpha\beta}E_2^\beta = \epsilon_\beta\nu_\alpha{}'q_i{}^{\beta\alpha}I_1{}^{\beta\alpha} + \epsilon_\beta\nu_\alpha{}''r_i{}^{\beta\alpha}I_2{}^{\beta\alpha}.$$

From now on we split the proof into two cases corresponding to the cases (i) and (ii) of the Fundamental Lemma.

Case 1. *$V^{\alpha\beta}$ is nondegenerate.*

The proof in this case is similar to the proof of Theorem 2 in [5]. We will sketch the argument omitting most of the details. First we obtain the 'angular switching formulas' and the 'three-index formulas' similar to the ones in Lemmas 6 and 7 in [5]. (If $V^{\alpha\beta}$ is indefinite, hyperbolic functions must be used in place of trigonometric functions.) Then we specify a choice of vectors E_1^α and E_2^α. We take E_1^1 and E_2^1 arbitrary and, for $\alpha \neq 1$, we set $E_1^\alpha = \epsilon_\alpha \nu_1' I_1^{\alpha 1}$, $E_2^\alpha = (\epsilon_\alpha)^{m/2}\nu_1' I_2^{\alpha 1}$ if $\nu_\alpha'\nu_\alpha'' = 1$ and $E_1^\alpha = \epsilon_\alpha I_1^{\alpha 1}$, $E_2^\alpha = -\epsilon_\alpha{}^{m/2} I_2^{\alpha 1}$ if $\nu_\alpha'\nu_\alpha'' = -1$. Then, as in [5], we obtain the crucial equalities

$$\begin{aligned} q^{\beta\alpha} &= -q^{\alpha\beta}, \\ \epsilon_\alpha{}^{m/2} r^{\beta\alpha} &= -\epsilon_\beta{}^{m/2+1} r^{\alpha\beta}. \end{aligned} \tag{27}$$

Finally, we finish Case 1 by showing that the conclusion of Theorem 2 holds in this case. As usual, we think of $\mathbf{R}^m$ as being a linear subspace of $\mathbf{R}^{m+2}$ via the linear isometry $x \to F(e_1, x)$, so that we have $f_i = u_i^1$, $i = 1, \ldots, m$. Let $\bar{U}^\alpha$, $\alpha = 2, \ldots, k+l$, be the linear transformation of $\mathbf{R}^{m+2}$ whose matrix, with respect to the orthonormal basis $\{u_1^1, \ldots, u_m^1, E_1^1, E_2^1\}$ is

$$\bar{U}^\alpha = \begin{bmatrix} p^{\alpha 1} & q^{\alpha 1} & r^{\alpha 1} \\ -\nu_1'(q^{\alpha 1})^\top & 0 & \nu_1''\lambda_{\alpha 1} \\ -\nu_1''(r^{\alpha 1})^\top & -\nu_1'\lambda_{\alpha 1} & 0 \end{bmatrix}.$$

Direct computation, in the use of the angular switching and three index formulas and (27) shows that

$$\begin{gathered} (\bar{U}^\alpha)^2 = -\epsilon_\alpha I, \\ \bar{U}^\alpha \bar{U}^\beta + \bar{U}^\beta \bar{U}^\alpha = 0. \end{gathered}$$

We also have

$$\langle \bar{U}^\alpha u, v\rangle + \langle u, \bar{U}^\alpha v\rangle = 0, \quad u, v \in \mathbf{R}^{m+2},$$

as can be checked by case-by-case verification. Now the proof is identical with that of the Theorem 1 in Section 3.

Case 2. *$V^{\beta\alpha}$ is null for all $\alpha \neq \beta$.*

By (21), $\lambda_{\beta\alpha}^2 = \epsilon_\alpha\epsilon_\beta\nu_\alpha'\nu_\alpha''$ so that

$$\lambda_{\beta\alpha} = \pm 1 \text{ and } \epsilon_\alpha\epsilon_\beta = \nu_\alpha'\nu_\alpha''.$$

Since, by Lemma 10, $\nu_\alpha'\nu_\alpha''$ is independent of α, we have to distinguish two subcases as follows:

Subcase 1. $k = l = 1$ and $\nu_\alpha' = -\nu_\alpha''$, for $\alpha = 1, 2$.

Subcase 2. $l = 0$ and $\nu_\alpha{}' = \nu_\alpha{}''$, for $\alpha = 1, \ldots, k$.

We consider first Subcase 1 and show that $F : \mathbf{R}^{1,1} \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ is the restriction of an orthogonal multiplication $\bar{F} : \mathbf{R}^{1,1} \times \mathbf{R}^{m+2} \to \mathbf{R}^{m+2}$ in the Hurwitz-Radon range.

Setting $P = P^{21}$, $q = q^{21}$, $r = r^{21}$ and $V = V^{21}$, by Lemma 9, we have

$$\begin{aligned} Pq &= \lambda r, \\ Pr &= \lambda q, \end{aligned}$$

where $\lambda = \pm 1$. Since V is null, we have

$$|q|^2 = |r|^2 = \langle q, r \rangle = 0$$

and

$$P^2 = I + \nu' q q^\top + \nu'' r r^\top,$$

where $\nu' = \nu_1'$ and $\nu'' = \nu_1''$.

We now think of the source as being a linear subspace of $\mathbf{R}^{m+2}$ by the linear isometry $x \to F(e_1, x)$, or equivalently, we set $f_i = u_i^1$, $i = 1, \ldots, m$. Let $\bar{U}$ be the linear transformation of $\mathbf{R}^{m+2}$ whose matrix, with respect to the basis $\{u_1^1, \ldots, u_m^1, E_1^1, E_2^1\}$, is

$$\bar{U} = \begin{bmatrix} P & q & r \\ -\nu' q^\top & 0 & -\lambda \\ -\nu'' r^\top & -\lambda & 0 \end{bmatrix}.$$

A direct calculation, in the use of the computational rules above, shows that

$$\bar{U}^2 = I$$

and

$$\langle \bar{U} u, v \rangle + \langle u, \bar{U} v \rangle = 0, \qquad u, v \in \mathbf{R}^{m+2}.$$

We now define the bilinear map $\bar{F} : \mathbf{R}^{1,1} \times \mathbf{R}^{m+2} \to \mathbf{R}^{m+2}$ by

$$\bar{F}(x, y) = x_1 y + x_2 \bar{U} y,$$

where $x = x_1 e_1 + x_2 e_2$. The rest follows in the same way as the proof of the codimension one case.

We now turn to Subcase 2. We have $\epsilon_\alpha = 1$ and $\nu_\alpha{}' = \nu_\alpha{}''$, for all $\alpha = 1, \ldots, k$. We may assume that actually $\nu_\alpha{}' = \nu_\alpha{}'' = 1$ for all $\alpha = 1, \ldots, k$ (since otherwise we change the signs of the semi-Euclidean structures on $\mathbf{R}^m$ and $\mathbf{R}^{m+2}$). Thus, we have

$$\epsilon_\alpha = \nu_\alpha{}' = \nu_\alpha{}'' = 1 \quad \text{and} \quad \lambda_{\alpha\beta} = \pm 1.$$

By Lemma 14,

$$\lambda_{\alpha\beta} = (-1)^{q+1} \lambda_{\beta\alpha}. \tag{28}$$

Our main purpose is to show that $k \leq 2$.

Lemma 16 *Without loss of generality, we may assume that, for α, β and γ distinct:*

$$\begin{aligned}\langle q^{\gamma\alpha}, q^{\beta\alpha}\rangle &= 0,\\ \langle r^{\gamma\alpha}, r^{\beta\alpha}\rangle &= 0,\\ \langle q^{\gamma\alpha}, r^{\beta\alpha}\rangle &+ \langle q^{\beta\alpha}, r^{\gamma\alpha}\rangle = 0.\end{aligned}$$

Proof. We change the basis of $\mathbf{R}^k$ as in Step 1 of the proof of the Fundamental Lemma in Section 5. Using (20), we obtain

$$\begin{aligned}|\tilde{q}^{\gamma\alpha}|^2 &= -\sin(2t)\langle q^{\gamma\alpha}, q^{\beta\alpha}\rangle,\\ |\tilde{r}^{\gamma\alpha}|^2 &= -\sin(2t)\langle r^{\gamma\alpha}, r^{\beta\alpha}\rangle,\\ \langle \tilde{q}^{\gamma\alpha}, \tilde{r}^{\gamma\alpha}\rangle &= -1/2\sin(2t)(\langle q^{\gamma\alpha}, r^{\beta\alpha}\rangle + \langle q^{\beta\alpha}, r^{\gamma\alpha}\rangle).\end{aligned}$$

If any of the claimed equations fails to be satisfied then we choose a small t and land in Case (i) of the Fundamental Lemma according to its proof.

We now prove the angular switching formulas:

Lemma 17

(i) There exists $\theta^{\alpha\beta} \in [-\pi, \pi]$ such that

$$\begin{aligned}q^{\beta\alpha} &= \cos\theta^{\alpha\beta} q^{\alpha\beta} - \sin\theta^{\alpha\beta} r^{\alpha\beta},\\ r^{\beta\alpha} &= (-1)^q(\sin\theta^{\alpha\beta} q^{\alpha\beta} + \cos\theta^{\alpha\beta} r^{\alpha\beta}).\end{aligned}$$

(ii)

$$\theta^{\beta\alpha} = (-1)^{q+1}\theta^{\alpha\beta}.$$

Proof. The following formulas are analogous to (32) and (33) in [5] and can be proved in the same way as in [5]:

$$\begin{aligned}\langle E_1^\alpha, E_1^\beta\rangle &= (-1)^q\langle E_2^\alpha, E_2^\beta\rangle,\\ \langle E_1^\alpha, E_2^\beta\rangle &= (-1)^{q+1}\langle E_2^\alpha, E_1^\beta\rangle,\\ \langle E_2^\alpha, E_1^\beta\rangle^2 &+ \langle E_2^\alpha, E_2^\beta\rangle^2 = 1.\end{aligned} \tag{29}$$

Using the last equation in (29), we introduce the angle $\theta^{\alpha\beta} \in [-\pi, \pi]$ by

$$\begin{aligned}\langle E_2^\alpha, E_1^\beta\rangle &= -\lambda_{\beta\alpha}\cos\theta^{\alpha\beta},\\ \langle E_2^\alpha, E_2^\beta\rangle &= \lambda_{\beta\alpha}\sin\theta^{\alpha\beta}.\end{aligned} \tag{30}$$

Then, by (29),

$$\begin{aligned}\langle E_1^\alpha, E_2^\beta\rangle &= (-1)^q\lambda_{\beta\alpha}\cos\theta^{\alpha\beta},\\ \langle E_1^\alpha, E_1^\beta\rangle &= (-1)^q\lambda_{\beta\alpha}\sin\theta^{\alpha\beta},\end{aligned} \tag{31}$$

Substituting (30) and (31) into (23), we obtain (i). Inverting (i) we arrive at (ii).

In the following lemmas the three-index formulas are analogous to those in Lemma 5 in [5] and they can be proved in the same way.

Lemma 18 *For α, β and γ distinct, we have*

(i)

$$P^{\gamma\alpha}P^{\beta\alpha} = -P^{\gamma\beta} + q^{\gamma\alpha}(q^{\beta\alpha})^\top + r^{\gamma\alpha}(r^{\beta\alpha})^\top;$$

(ii)

$$\begin{aligned}P^{\gamma\alpha}q^{\beta\alpha} &= (-1)^{q+1}\lambda_{\alpha\beta}r^{\gamma\alpha} + \cos\theta^{\alpha\beta}q^{\gamma\beta} - \sin\theta^{\alpha\beta}r^{\gamma\beta},\\ P^{\gamma\alpha}r^{\beta\alpha} &= (-1)^q(\lambda_{\alpha\beta}q^{\gamma\alpha} + \sin\theta^{\alpha\beta}q^{\gamma\beta} + \cos\theta^{\alpha\beta}r^{\gamma\beta});\end{aligned}$$

(iii)

$$\begin{aligned}\langle q^{\gamma\alpha}, q^{\beta\alpha}\rangle &= (-1)^q\lambda_{\alpha\beta}\lambda_{\gamma\alpha} - (-1)^q\lambda_{\gamma\beta}\sin(\theta^{\alpha\gamma} + \theta^{\beta\alpha} - \theta^{\beta\gamma}),\\ \langle q^{\gamma\alpha}, r^{\beta\alpha}\rangle &= (-1)^{q+1}\lambda_{\gamma\beta}\cos(\theta^{\alpha\gamma} + \theta^{\beta\alpha} - \theta^{\beta\gamma}),\\ \langle r^{\gamma\alpha}, r^{\beta\alpha}\rangle &= (-1)^q\lambda_{\alpha\beta}\lambda_{\gamma\alpha} - \lambda_{\gamma\beta}\sin(\theta^{\alpha\gamma} + \theta^{\beta\alpha} + \theta^{\beta\gamma}).\end{aligned}$$

Lemma 19 *We have*

(i) q is even;

(ii)

$$\begin{aligned}q^{\beta\alpha} &= \cos\theta^{\alpha\beta}q^{\alpha\beta} - \sin\theta^{\alpha\beta}r^{\alpha\beta}\\ r^{\beta\alpha} &= \sin\theta^{\alpha\beta}q^{\alpha\beta} + \cos\theta^{\alpha\beta}r^{\alpha\beta}.\end{aligned}$$

(iii)

$$\theta^{\beta\alpha} = -\theta^{\alpha\beta}.$$

(iv)

$$\lambda_{\beta\alpha} = -\lambda_{\alpha\beta};$$

(v)

$$\begin{aligned}\sin(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) &= \lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha},\\ \cos(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) &= 1.\end{aligned}$$

(vi)

$$\begin{aligned} I_1{}^{\beta\alpha} &= \cos\theta^{\alpha\beta} E_1^{\beta} - \sin\theta^{\alpha\beta} E_2^{\beta}, \\ I_2{}^{\beta\alpha} &= \sin\theta^{\alpha\beta} E_1^{\beta} + \cos\theta^{\alpha\beta} E_2^{\beta}. \end{aligned}$$

Proof. By the orthogonality relations in Lemma 16, the first and third equations in (iii) of Lemma 18 reduce to

$$\sin(\theta^{\alpha\gamma} + \theta^{\beta\alpha} - \theta^{\beta\gamma}) = \lambda_{\alpha\beta}\lambda_{\gamma\alpha}\lambda_{\gamma\beta} = (-1)^q \lambda_{\alpha\beta}\lambda_{\gamma\alpha}\lambda_{\gamma\beta} \tag{32}$$

and (i) follows. Now (i) and (ii) in Lemma 17 and (28) imply (ii), (iii) and (iv), respectively, and (32) reduces to (v). Substituting (ii) into the formula in Lemma 15 (ii), we get (vi).

We now specify the vectors E_1^α and E_2^α. Let E_1^1 and E_2^1 arbitrary (subject to the orientability constraint). For $\alpha \neq 1$, we set

$$E_1^\alpha = I_1^{\alpha 1} \text{ and } E_2^\alpha = I_2^{\alpha 1}.$$

We note that the orientability constraint is preserved.

For the rest of this section the indices α, β, γ and δ take their values in $2, \ldots, k$, and, for simplicity, we set $\lambda_\alpha = \lambda_{\alpha 1}$, $P^\alpha = \lambda_\alpha P^{\alpha 1}$, $q^\alpha = \lambda_\alpha q^{\alpha 1}$, $r^\alpha = \lambda_\alpha r^{\alpha 1}$ and $\theta^\alpha = \theta^{\alpha 1}$.

By the switching formulas in Lemma 19 (vi)

$$\begin{aligned} \theta^\alpha &= 0, \\ \lambda_\alpha q^{1\alpha} &= q^\alpha, \\ \lambda_\alpha r^{1\alpha} &= r^\alpha, \\ \sin\theta^{\alpha\beta} &= -\lambda_\alpha\lambda_\beta\lambda_{\alpha\beta}, \\ \cos\theta^{\alpha\beta} &= 0, \end{aligned}$$

where the last two equations are obtained by taking $\gamma = 1$ in (vi) of Lemma 19. Now we will use the following equation:

$$q^{\gamma\alpha} = \lambda_{\beta\gamma} + \lambda_{\alpha\beta}(\sin\theta^{\gamma\alpha} q^{\beta\gamma} - \cos\theta^{\gamma\alpha} r^{\beta\gamma}). \tag{33}$$

This equation is contained in the proof of Lemma 9 in [5]. In our situation the proof is completely analogous. (Note that the conditions of Lemma 9 in [5] are automatically satisfied.) Setting $\beta = 1$ and then $\gamma = 1$ in (33), we obtain

$$\begin{aligned} q^{\alpha\beta} &= -\lambda_\alpha\lambda_\beta r^\beta - \lambda_{\alpha\beta} q^\alpha, \\ r^{\alpha\beta} &= \lambda_\alpha\lambda_\beta q^\beta - \lambda_{\alpha\beta} r^\alpha. \end{aligned}$$

We now assume that $k \geq 3$ and get a contradiction. This means that there exist $\alpha \neq \beta$ with

$$\lambda_{\alpha\beta} = \lambda_\alpha\lambda_\beta$$

(cf. (ii) of Lemma 32). We can now rewrite the three index formulas in terms of single index vectors as follows:

$$\begin{aligned} (P^\alpha)^2 &= -I + q^\alpha(q^\alpha)^\top + r^\alpha(r^\alpha)^\top, \\ (P^\beta)^2 &= -I + q^\beta(q^\beta)^\top + r^\beta(r^\beta)^\top; \end{aligned} \tag{34}$$

$$P^\alpha P^\beta + P^\beta P^\alpha = q^\alpha(q^\beta)^\top + q^\beta(q^\alpha)^\top + r^\alpha(r^\beta)^\top + r^\beta(r^\alpha)^\top; \tag{35}$$

$$\begin{aligned} P^\alpha q^\alpha &= r^\alpha, \\ P^\alpha r^\alpha &= -q^\alpha, \\ P^\alpha q^\beta &= -q^\alpha + r^\alpha - r^\beta, \\ P^\alpha r^\beta &= -q^\alpha + q^\beta - r^\alpha; \end{aligned} \tag{36}$$

and

$$\begin{aligned} P^\beta q^\alpha &= q^\beta - r^\alpha + r^\beta, \\ P^\beta r^\alpha &= q^\alpha - q^\beta + r^\beta, \\ P^\beta q^\beta &= r^\beta, \\ P^\beta r^\beta &= -q^\beta; \end{aligned} \tag{37}$$

and all scalar products of q^α, q^β, r^α, r^β vanish.

Lemma 20 *We have*

$$q^\alpha(q^\alpha)^\top + r^\alpha(r^\alpha)^\top = q^\beta(q^\beta)^\top + r^\beta(r^\beta)^\top.$$

Proof. We use (34)-(37) in the identity

$$-(P^\beta)^2 P^\alpha + P^\beta(P^\alpha P^\beta + P^\beta P^\alpha) = -P^\alpha(P^\beta)^2 + (P^\alpha P^\beta + P^\beta P^\alpha)P^\beta \tag{38}$$

and obtain an equation for q^α, q^β, r^α, r^β. Switching α and β in (38) and comparing the two equations thus obtained the lemma follows.

Lemma 21 *q^α, q^β, r^α and r^β are linearly independent.*

Proof. Assume that

$$aq^\alpha + bq^\beta + cr^\alpha + dr^\beta = 0$$

with $a^2 + c^2, b^2 + d^2 > 0$. Applying P^α to both sides and eliminating r^β from the two equations, we get

$$q^\beta = Aq^\alpha + Br^\alpha.$$

We now apply P^α and P^β to this and obtain

$$Bq^\alpha - Ar^\alpha + (A-B)q^\beta + (A+B+1)r^\beta = 0$$

and

$$r^\beta = (B-1)q^\alpha - (A-1)r^\alpha.$$

Putting all these together, we get

$$(A^2+B^2-A+B-1)q^\alpha - (A^2+B^2+A-B-1)r^\alpha = 0$$

so that

$$A = B = \pm\sqrt{2}/2.$$

Hence

$$\begin{aligned} q^\beta &= \pm\sqrt{2}/2(q^\alpha + r^\alpha), \\ r^\beta &= (\pm\sqrt{2}/2 - 1)(q^\alpha - r^\alpha). \end{aligned}$$

Finally, taking P^β of both sides of these equations we get a contradiction to the linear independence of q^α and r^α.

Finally, let $x \notin W^{\alpha\beta}$. Applying Lemma 20, we get

$$\langle q^\alpha, x\rangle q^\alpha + \langle r^\alpha, x\rangle r^\alpha - \langle q^\beta, x\rangle q^\beta - \langle r^\beta, x\rangle r^\beta = 0.$$

By Lemma 21, all scalar products vanish. On the other hand, by the very definition of $W^{\alpha\beta}$ this means that $x \in W^{\alpha\beta}$. This is a contradiction and so $k \le 2$ follows.

By fullness of F, actually $k = 2$. To prove that F extends to an orthogonal multiplication $\bar{F} : \mathbf{R}^2 \times \mathbf{R}^{m+2} \to \mathbf{R}^{m+2}$ we use the same argument as in (the end of) subcase 1 with

$$\bar{U} = \begin{bmatrix} P & q & r \\ -q^\top & 0 & \lambda \\ -r^\top & -\lambda & 0 \end{bmatrix},$$

where $q = q^1$, $r = r^1$ and $\lambda = \lambda_1$.

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