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Quadratic Eigenmaps between Spheres

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Abstract. A quadratic form $f: S^m \to S^n$ between spheres is separable if, up to isometries on the source and the range, the components of f are pure or mixed quadratic polynomials. The space parametrizing the separated quadratic eigenmaps f is shown here to fiber over a semi-algebraic set with each fiber a finite-dimensional compact convex body. For $m = 3$, this gives a new description of the parameter space of all quadratic eigenmaps $f: S^3 \to S^m$ as a fibration over an 'inflated tetrahedron' and generic hexagonal fibres.

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1. Introduction

A map $f: \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ is a *k-form* if the components of f are homogeneous polynomials of degree k. f is *spherical* if it maps the unit sphere to the unit sphere. In this case, we say that (the restriction) $f: S^m \to S^n$ is also a *k-form*. If, in addition, the components of $f: \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ are harmonic, or equivalently, the components of $f: S^m \to S^n \subset \mathbf{R}^{n+1}$ are spherical harmonics of order k, i.e. eigenfunctions of the spherical Laplacian on S^m with eigenvalue $\lambda_k = k(k+m-1)$, then we say that $f: S^m \to S^n$ is an *eigenmap* with eigenvalue λ_k. Eigenmaps are harmonic in the sense of Eells and Sampson [3], in fact, an eigenmap with eigenvalue λ_k is nothing but a harmonic map with constant energy density $\lambda_k/2$.

A k-form $f: S^m \to S^n$ is *full* if its image is not contained in any proper great sphere. Two k-forms $f_1, f_2: S^m \to S^n$ are *equivalent* if there exists an isometry $U \in \mathrm{O}(n+1)$ such that $f_2 = U \cdot f_1$.

For fixed m and k, the equivalence classes of full eigenmaps $f: S^m \to S^n$ with eigenvalue λ_k can be parametrized by a compact convex body $\mathcal{L}_m^k$ in a finite dimensional representation space of $\mathrm{O}(m+1)$ (cf. [10] or Section 2). $\mathcal{L}_m^k$ is called *the standard moduli space.*

Many examples of quadratic ($k = 2$) eigenmaps are known, in fact, in the first nonrigid range $m = 3$, a full classification, i.e. a geometric description of the 10-dimensional standard moduli space $\mathcal{L}_3^2$, is given in [9]. For fixed m, the complexity of $\mathcal{L}_m^k$ increases very fast with k. In [11], [12], however, we defined degree-raising and -lowering operators which associate to an eigenmap with eigenvalue λ_k, eigenmaps with eigenvalue $\lambda_{k\pm1}$. We proved that the degree-raising operator gives rise to an (equivariant) imbedding $\mathcal{L}_m^k \to \mathcal{L}_m^{k+1}$ whose cokernel is annihilated by the degree-lowering operator.

In view of this, the classification of all quadratic eigenmaps $f: S^m \to S^n$ can be singled out as an important part of the whole classification problem. This is still formidable as it contains, via the Hopf–Whitehead construction (cf. [2] or Section 6), the problem of classifying orthogonal multiplications posed and studied by Hurwitz and Radon nearly a hundred years ago (cf. [5], [6], [7]).

The purpose of this paper is to introduce a natural class of quadratic forms between spheres, called separable, which comprise most of the known examples and give a geometric description of their moduli space.

Let p be a quadratic polynomial in $x_0, \ldots, x_m$. We say that p is *pure* (resp. *mixed*) if $p(x) = \Sigma_{i=0}^m \alpha_i x_i^2$ (resp. $p(x) = \Sigma_{0 \leq i < j \leq m} \alpha_{ij} x_i x_j$). A quadratic form $f: S^m \to S^n$ is *separated* if f is equivalent to a quadratic form such that each component is pure or mixed. A quadratic form $f: S^m \to S^n$ is *separable* if there exists an isometry $a \in \mathrm{O}(m+1)$ such that $f \circ a$ is separated.

Using moduli space techniques, in Section 2 we prove the following:

THEOREM 1. *Every quadratic eigenmap $f: S^3 \to S^n$ is separable.*

In Section 3 we construct a space $\mathcal{S}_m$ (resp. $\mathcal{S}_m^0$) that parametrizes the equivalence classes of full separated forms $f: S^m \to S^n$ (resp. eigenmaps). $\mathcal{S}_m$, resp. $\mathcal{S}_m^0$, fibers over a semi-algebraic set Σ_m, resp. Σ_m^0, in $\mathbf{R}^{m(m+1)/2}$ and each fiber is a compact convex body in a finite-dimensional vector space. We determine dim $\mathcal{S}_m$ and dim $\mathcal{S}_m^0$, in particular, we have dim $\mathcal{S}_m^0 \simeq$ dim $\mathcal{L}_m^2$ as $m \to \infty$. In contrast to Theorem 1, however, we have:

THEOREM 2. *For $m \geq 4$, there exist nonseparable quadratic eigenmaps $f: S^m \to S^n$.*

In Section 4, we show that Σ_{m+1}^0 can be imbedded into Σ_m as a codimension 1 slice (i.e. the intersection of Σ_m with an affine hyperplane). As a byproduct, we obtain a source dimension-raising operator which associates to a quadratic eigenmap $f: S^m \to S^n$ a quadratic eigenmap $\tilde{f}: S^{m+1} \to S^{n+m+1}$. This gives rise to an imbedding between the respective standard moduli spaces.

These imbeddings are used in Section 5, for $m = 3$, to give a new and very explicit description of (separable and hence all) quadratic eigenmaps $f: S^3 \to S^n$.

In Section 6, we give a sharp lower bound for the range dimension of a separable eigenmap. As a corollary, we obtain that any quadratic eigenmap $f: S^4 \to S^4$ is nonseparable. Finally we show that the Hopf–Whitehead construction is just a special (rank 1) case of a more general construction of quadratic eigenmaps $f: S^m \to S^n$ and derive an explicit description of the rank 2 case.

2. Proof of Theorem 1

We begin by recalling some facts about the standard moduli spaces. Let $\mathcal{H}^k = \mathcal{H}^k_m$ denote the space of spherical harmonics of order k on S^m. Let $\{f^j_{\lambda_k}\}_{j=0}^{n(k)} \subset \mathcal{H}^k$ be an orthonormal basis with respect to the normalized L_2-scalar product

$$\langle h, h' \rangle = \frac{n(k)+1}{\mathrm{vol}(S^m)} \int_{S^m} hh' v_{S^m},$$

where v_{S^m} is the volume form on S^m, $\mathrm{vol}(S^m) = \int_{S^m} v_{S^m}$ is the volume of S^m and

$$n(k) + 1 = \dim\, \mathcal{H}^k = (m + 2k - 1)\frac{(m+k-2)!}{k!(m-1)!}. \tag{1}$$

We now define the standard minimal immersion $f_{\lambda_k}\colon S^m \to S^{n(k)}$ as a map with components $(f^0_{\lambda_k}, \ldots, f^{n(k)}_{\lambda_k})$ (cf. [1], [14]). f_{λ_k} is clearly full and different choices of the orthonormal basis give equivalent eigenmaps.

f_{λ_k} is universal in the sense that, for any eigenmap $f\colon S^m \to S^n$ with eigenvalue λ_k, there exists a linear map $A\colon \mathcal{H}^k \to \mathbf{R}^{n+1}$ such that $f = A \cdot f_{\lambda_k}$. Note that, A is surjective iff f is full.

Associating to f the symmetric linear endomorphism

$$\langle f \rangle = A^{\top} A - I \in S^2(\mathcal{H}^k), \qquad (I = \text{identity})$$

establishes a parametrization of the space of equivalence classes of full eigenmaps $f\colon S^m \to S^n$ with eigenvalue λ_k by the compact convex body

$$\mathcal{L}^k_m = \{C \in \mathcal{E}^k_m | C + I \geq 0\}$$

in the linear subspace

$$\mathcal{E}^k_m = \mathrm{span}\{f_{\lambda_k}(x) \odot f_{\lambda_k}(x) | x \in S^m\}^{\perp} \subset S^2(\mathcal{H}^k).$$

Here '$\geq$' stands for positive semidefinite, '$\odot$' is the symmetric tensor product and the orthogonal complement is taken with respect to the standard scalar product $\langle C, C' \rangle = \mathrm{trace}((C')^{\top} \cdot C)$, $C, C' \in S^2(\mathcal{H}^k)$. $\mathcal{L}^k_m$ is said to be the *standard moduli space* of eigenmaps with eigenvalue λ_k. (For more details as well as for the general theory of moduli spaces, cf. [10].)

f_{λ_k} is equivariant with respect to the homomorphism $\rho_k\colon \mathrm{SO}(m+1) \to \mathrm{SO}(\mathcal{H}^k)$ that is just the orthogonal ($\mathrm{SO}(m+1)$-)module structure on $\mathcal{H}^k$ defined by $a \cdot h = h \circ a^{-1}$, $a \in \mathrm{SO}(m+1)$ and $h \in \mathcal{H}^k$. Equivariance is given explicitly by

$$f_{\lambda_k} \circ a = \rho_k(a) \cdot f_{\lambda_k}, \qquad a \in \mathrm{SO}(m+1).$$

$\mathcal{E}_m^k$ is a submodule of $S^2(\mathcal{H}^k)$, where the latter is endowed with the module structure induced from that of $\mathcal{H}^k$. Moreover, $\mathcal{L}_m^k \subset \mathcal{E}_m^k$ is an invariant subset. Explicitly, for a full eigenmap $f\colon S^m \to S^n$ with eigenvalue λ_k, we have

$$a \cdot \langle f \rangle = \langle f \circ a^{-1} \rangle, \qquad a \in \mathrm{SO}(m+1).$$

We now specialize to $k = 2$. For the standard minimal immersion $f_{\lambda_2}\colon S^m \to S^{(m(m+3)/2)-1}$, we take

$$f_{\lambda_2}(x) = \sqrt{\frac{m+1}{m}} \left(\left(x_i^2 - \frac{\rho^2}{m+1} \right)_{i=0,\dots,m}, (\sqrt{2} x_i x_j)_{0 \leq i < j \leq m} \right),$$

where $\rho^2 = x_0^2 + \cdots x_m^2$. Note that the first $m + 1$ components add up to zero so that the image is contained in the corresponding hypersphere of $S^{m(m+3)/2}$. In particular, f_{λ_2} is separated. Now let $f\colon S^m \to S^n$ be a full separated quadratic eigenmap. Up to equivalence, we may assume that f has only pure and mixed components with the former preceding the latter. Setting $f = A \cdot f_{\lambda_2}$, the matrix A consists of two rectangular blocks. Thus, we have

$$\langle f \rangle = A^\top A - I \in S^2(\mathbf{R}^m) \oplus S^2(\mathbf{R}^{m(m+1)/2}).$$

We obtain that the space of equivalence classes of full separated eigenmaps $f\colon S^m \to S^n$ can be parametrized by the compact convex slice

$$\mathcal{L}_m^2 \cap S^2(\mathbf{R}^m) \oplus S^2(\mathbf{R}^{m(m+1)/2}) \subset \mathcal{L}_m^2.$$

Its orbit space under $\mathrm{O}(m+1)$ in $\mathcal{L}_m^2$ parametrizes the space of equivalence classes of full separable eigenmaps.

We now set $m = 3$ and summarize the following facts on the standard moduli space $\mathcal{L}_3^2$. (The proofs are given in [9]; cf. also [10].) Using complex coordinates $(z, w) \in \mathbf{C}^2 = \mathbf{R}^4$, we first define the quadratic eigenmaps $H_\alpha\colon S^3 \to S^2$, $\alpha \in \mathbf{R}$, and $V\colon S^3 \to S^5$ as

$$H_\alpha(z, w) = (\mathrm{e}^{2i\alpha} z^2 + \bar{w}^2, 2\Im(\mathrm{e}^{i\alpha} z w)) \tag{2}$$

and

$$V(z, w) = (z^2, \sqrt{2}\, zw, w^2). \tag{3}$$

Then H_{α_1} and H_{α_2} are inequivalent iff $\alpha_1 \not\equiv \alpha_2 \pmod{\pi}$; $\{\langle H_\alpha \rangle | \alpha \in \mathbf{R}\}$ is the boundary of a (flat) 2-disk D on $\partial \mathcal{L}_3^2$ with center $\langle V \rangle$. The diagonal subgroup $\Gamma = \{\mathrm{diag}\,(\mathrm{e}^{i\theta}, \mathrm{e}^{i\theta}) | \theta \in \mathbf{R}\}$ rotates D by

$$\mathrm{diag}(\mathrm{e}^{i\theta}, \mathrm{e}^{i\theta}) \cdot \langle H_\alpha \rangle = \langle H_{a+2\theta} \rangle$$

and the antidiagonal subgroup $\Gamma' = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta}) | \theta \in \mathbf{R}\}$ leaves D pointwise fixed. Replacing w by its conjugate $\bar{w}$ in (2)–(3), we obtain the quadratic eigenmaps H'_α and V' and the disk D' which is rotated by Γ' and left fixed by Γ. Finally, the convex hull E of D and D' consists of segments with endpoints in D and D' and

$$\mathrm{SO}(4) \cdot E = \partial \mathcal{L}_3^2. \tag{4}$$

We now turn to the proof of Theorem 1. Let $f: S^3 \to S^n$ be a full quadratic eigenmap and set $C = \langle f \rangle \in \mathcal{L}_m^2$. Since the standard minimal immersion is separated and corresponds to the origin in $\mathcal{L}_3^2$, we may assume that $C \neq 0$. Consider the radial segment through C and let C' denote its intersection with $\partial \mathcal{L}_3^2$. By (4), performing an isometry on S^3 if necessary, we may assume that $C' \in E$. By the above, there exists a segment S in E with endpoints in D and D' such that $C' \in S$. We now use the subgroups Γ and Γ' to rotate D and D' such that the endpoints of S get to the radial segments connecting V and H_0 in D and V' and H'_0 in D'. These are separated so that the endpoints of the rotated S are also separated. Thus, by convexity, the rotated C' is also separated. The rotated C is on the radial segment between 0 and C' so it is also separated. Hence f is separable and the proof is complete.

3. Moduli Spaces of Separable Quadratic Forms

Let $f: S^m \to S^n$ be a quadratic form. Using coordinates, f can be written as

$$f(x) = \sum_{i=0}^{m} a_i x_i^2 + \sum_{0 \le i < j \le m} a_{ij} x_i x_j,$$

where $a_i, a_{ij} \in \mathbf{R}^{n+1}$, $i = 0, \ldots, m$ and $0 \le i < j \le m$. To simplify the notation, we set $a_{ij} = a_{ji}$, so that a_{ij} is defined for all distinct indices $0 \le i, j \le m$. The condition that f is spherical is equivalent to the following:

$$|a_i| = 1, \tag{5}$$

$$\langle a_i, a_{ij} \rangle = 0, \quad i, j \text{ distinct} \tag{6}$$

$$|a_{ij}|^2 + 2\langle a_i, a_j \rangle = 2, \quad i, j \text{ distinct} \tag{7}$$

$$\langle a_i, a_{jk} \rangle + \langle a_{ij}, a_{ik} \rangle = 0, \quad i, j, k \text{ distinct} \tag{8}$$

$$\langle a_{ij}, a_{kl} \rangle + \langle a_{ik}, a_{jl} \rangle + \langle a_{il}, a_{jk} \rangle = 0, \quad i, j, k, l \text{ distinct.} \tag{9}$$

We say that a system of vectors $\{a_i, a_{ij}\} \subset \mathbf{R}^{n+1}$ is *feasible* if it satisfies (5)–(9). Note that f is harmonic, i.e. an eigenmap, iff

$$\sum_{i=0}^{m} a_i = 0. \tag{10}$$

We introduce the rank of the quadratic form $f\colon S^m \to S^n$ as

$$\operatorname{rank} f = \dim \operatorname{span} \{a_i | i = 0, \dots, m\}. \tag{11}$$

As an example, we see that the standard minimal immersion $f_{\lambda_2}\colon S^m \to S^{(m(m+3)/2)-1}$ can be characterized by saying that it is separated, $\langle a_i, a_j \rangle = -1/m$, $i \neq j$, and $\{a_{ij}\}$ is orthogonal with $|a_{ij}|^2 = 2(m+1)/m$. Clearly, $\operatorname{rank} f_{\lambda_k} = m$.

Given a quadratic form $f\colon S^m \to S^n$ with associated feasible system of vectors $\{a_i, a_{ij}\}$, for $U \in \mathrm{O}(n+1)$, the equivalent quadratic form $U \cdot f$ has $\{Ua_i, Ua_{ij}\}$ as the associated feasible system of vectors. Hence, f is separated iff

$$\langle a_i, a_{jk} \rangle = 0, \quad \text{for all } i, j, k = 0, \dots, m, \ j \neq k.$$

In this case, we will always take f (in its equivalence class) such that $\{a_i\} \subset \mathbf{R}^p$, $p = \operatorname{rank} f$, where $\mathbf{R}^p \subset \mathbf{R}^{n+1}$ is the linear subspace spanned by the first p coordinates, and $\{a_{ij}\} \subset \mathbf{R}^{q+1}$, $p + q = n$, where $\mathbf{R}^{q+1} = (\mathbf{R}^p)^\perp$.

Given a full separated form $f\colon S^m \to S^n$ with associated feasible system of vectors $\{a_i, a_{ij}\}$, we let $F\colon \mathbf{R}^{m+1} \to \mathbf{R}^{q+1}$ denote the full quadratic map defined by

$$F(x) = \sum_{0 \le i < j \le m} a_{ij} x_i x_j.$$

The orthogonality relations (7)–(9) translate into

$$|F(x)|^2 = \sum_{0 \le i < j \le m} \mu_{ij} x_i^2 x_j^2, \tag{12}$$

where $\mu_{ij} = |a_{ij}|^2$. We call $\mu = (\mu_{ij})_{0 \le i < j \le m} \in \mathbf{R}^{m(m+1)/2}$ the *signature* of F. We will also think of the signature μ as a symmetric matrix in $S^2(\mathbf{R}^{m+1})$ with zero diagonal entries, i.e. we put $\mu_{ij} = \mu_{ji}$, $i \neq j$, and $\mu_{ii} = 0$.

F determines f up to equivalence. In fact, μ determines the Gram matrix $G(\mu)$ of the system of vectors $\{a_i\}$ via (7) since

$$\langle a_i, a_j \rangle = 1 - \frac{|a_{ij}|^2}{2} = 1 - \frac{\mu_{ij}}{2}. \tag{13}$$

We have $\operatorname{rank} G(\mu) = \operatorname{rank} f = p$ and $G(\mu)$ determines $\{a_i\}$ within $\operatorname{span} \{a_i\} = \mathbf{R}^p$ up to isometry. By birth, $G(\mu)$ is positive semidefinite.

Conversely, let $F\colon \mathbf{R}^{m+1} \to \mathbf{R}^{q+1}$ be a quadratic map satisfying (12), and assume that

$$G(\mu) = \left(1 - \frac{\mu_{ij}}{2}\right)_{i,j=0}^{m} \ge 0, \qquad \mu_{ij} = \mu_{ji}, \quad \mu_{ii} = 0.$$

Then, up to isometry, there exists $\{a_i\} \subset \mathbf{R}^p$, $p = \operatorname{rank} G(\mu)$ satisfying (13). Putting these vectors together in $\mathbf{R}^p \oplus \mathbf{R}^{q+1}$, we arrive at a separated form $f\colon S^m \to S^n$, $n = p + q$. Finally, f is an eigenmap iff the entries of each row in $G(\mu)$ add up to zero, i.e. iff

$$\sum_{i=0}^{m} \mu_{ij} = 2(m+1) \quad \text{for all } j = 0, \ldots, m.$$

The discussion above warrant to introduce the *signature space*

$$\Sigma_m = \Bigg\{ \mu = (\mu_{ij}) \in \mathbf{R}^{m(m+1)/2} | G(\mu) = \left(1 - \frac{\mu_{ij}}{2}\right)^m_{i,j=0} \geq 0,\ \mu_{ij} = \mu_{ji},\ \mu_{ii} = 0 \Bigg\}.$$

By the above, to each full separated form $f\colon S^m \to S^n$ there corresponds a full quadratic map $F\colon \mathbf{R}^{m+1} \to \mathbf{R}^{q+1}$ of signature $\mu \in \Sigma_m$ and this correspondence is one-to-one.

For eigenmaps, we have to restrict ourselves to the slice

$$\Sigma_m^0 = \left\{ \mu \in \Sigma_m \,\middle|\, \sum_{i=0}^{m} \mu_{ij} = 2(m+1) \text{ for all } j = 0, \ldots, m \right\}$$

of the signature space Σ_m.

Let $\mu \in \Sigma_m$. We define $F_\mu\colon \mathbf{R}^m \to \mathbf{R}^{q(\mu)+1}$, $q(\mu) + 1 = \sharp\{\mu_{ij} \neq 0 | 0 \leq i < j \leq m\}$, by

$$F_\mu(x) = (\sqrt{\mu_{ij}}\, x_i x_j)_{0 \leq i < j \leq m},$$

where we discard the zero components on the right-hand side. F_μ is a full quadratic map with signature μ. Moreover, F_μ is universal in the sense that, for any full quadratic map $F\colon \mathbf{R}^{m+1} \to \mathbf{R}^{q+1}$ with signature μ, we have $F = A \cdot F_\mu$, where A is a (uniquely determined) $(m+1) \times (q(\mu)+1)$-matrix. As usual, we set

$$\langle F \rangle = A^\top A - I \in S^2(\mathbf{R}^{q(\mu)+1}),$$

and

$$\mathcal{L}_\mu = \{C \in \mathcal{E}_\mu | C + I \geq 0\},$$

where

$$\mathcal{E}_\mu = \operatorname{span}\{F_\mu(x) \odot F_\mu(x) | x \in \mathbf{R}^{m+1}\}^\perp \subset S^2(\mathbf{R}^{q(\mu)+1}).$$

By the usual DoCarmo and Wallach argument (cf. [1], [14]), we find that the correspondence $F \to \langle F \rangle$ parametrizes the equivalence classes of full quadratic maps $F: \mathbf{R}^{m+1} \to \mathbf{R}^{q+1}$ with signature μ by the compact convex body $\mathcal{L}_\mu$ of $\mathcal{E}_\mu$.

We conclude that

$$\mathcal{S}_m = \bigcup_{\mu \in \Sigma_m} \mathcal{L}_\mu \quad \left(\text{resp. } \mathcal{S}_m^0 = \bigcup_{\mu \in \Sigma_m^0} \mathcal{L}_\mu\right)$$

parametrizes the equivalence classes of full separated forms $f: S^m \to S^n$ (resp. full separated eigenmaps $f: S^m \to S^n$).

Given $\mu \in \Sigma_m$, we now describe $\mathcal{L}_\mu$. Assume first that $\mu_{ij} \neq 0$, for all $i \neq j$. Writing out the condition $C \in \mathcal{E}_\mu$ in coordinates, we obtain

$$\sum_{i,j,k,l} \sqrt{\mu_{ij}}\sqrt{\mu_{kl}}\, c_{ij,kl} x_i x_j x_k x_l = 0$$

so that if $i,\ j,\ k,\ l$ are not distinct then $c_{ij,kl}$ vanishes and if $i,\ j,\ k,\ l$ are all distinct then

$$\sqrt{\mu_{ij}}\sqrt{\mu_{kl}} c_{ij,kl} + \sqrt{\mu_{ik}}\sqrt{\mu_{jl}} c_{ik,jl} + \sqrt{\mu_{il}}\sqrt{\mu_{jk}} c_{il,jk} = 0. \tag{14}$$

In particular, if $\mu_{ij} \neq 0$, for $i \neq j$, we have

$$\dim \mathcal{L}_\mu = 2\binom{m+1}{4}.$$

If $\mu_{ij} = 0$ for some $i \neq j$ then $c_{ij,kl}$ (and, by symmetry, $c_{kl,ij}$) do not exist for all $k,\ l$ and they are missing from these relations (14). Thus, in general,

$$\dim\ \mathcal{L}_\mu \leq 2\binom{m+1}{4}.$$

(Alternatively, all the previous relations remain in effect if we assume that, whenever $\mu_{ij} = 0$ then $c_{ij,kl} = c_{kl,ij} = 0$, for all $k,\ l$.)

(14) actually gives somewhat more about the geometry of $\mathcal{L}_\mu$. For simplicity, assume that all μ's occurring in (14) are nonzero. Let $P_{ijkl} \subset S^2(\mathbf{R}^{m(m+1)/2})$ be the 3-dimensional linear subspace given by setting all c's other than $c_{ij,kl}$, $c_{ik,jl}$ and $c_{il,jk}$ equal to zero. In P_{ijkl}, the set of points $(c_{ij,kl},\ c_{ik,jl},\ c_{il,jk})$ for which $C + I \geq 0$ holds is the cube $[-1,\ 1]^3$. To get $P_{ijkl} \cap \mathcal{L}_\mu$, by (14), this cube has to be intersected with the plane through the origin with normal vector $(\sqrt{\mu_{ij}\mu_{kl}},\ \sqrt{\mu_{ik}\mu_{jl}},\ \sqrt{\mu_{il}\mu_{jk}})$. The intersection is a hexagon or a quadrangle. We will exploit this argument in a more concrete setting for $m = 3$ in Section 5.

By definition, we have $\dim \Sigma_m = m(m+1)/2$ and $\dim \Sigma_m^0 = (m-2)(m+1)/2$ so that the dimension of the space of equivalence classes of full separable eigenmaps is at most $2\binom{m+1}{4} + m^2 - 1$. On the other hand, we have

$$\begin{aligned}\dim \mathcal{L}_m^2 &= 1/2(n(2)+1)(n(2)+2) - (n(0)+1) \\ &\quad -(n(2)+1) - (n(4)+1),\end{aligned}$$

where $n(k)$ is given in (1) (cf. [10, p. 91]). Comparing these dimensions, we arrive at

$$\dim \mathcal{L}_m^2 - \dim \mathcal{S}_m^0 \geq \frac{(m-3)m(m+1)}{2}$$

and Theorem 2 follows. Note also that both dimensions on the left-hand side are degree 4 polynomials in m with equal leading coefficients so that the asymptotic formula of Section 1 is valid.

4. Hierarchy between the Signature Spaces

We first define an imbedding $\phi\colon \Sigma_{m+1}^0 \to \Sigma_m$ with image

$$\left\{ \mu \in \Sigma_m \;\middle|\; \sum_{0 \leq i < j \leq m} \mu_{ij} = m(m+2) \right\} \tag{15}$$

a codimension 1 slice of Σ_m. Given $\bar{\mu} \in \Sigma_{m+1}^0$, we define $\mu \in \mathbf{R}^{m(m+1)/2}$ as $\bar{\mu} \in \mathbf{R}^{(m+1)(m+2)/2}$ with the components $\bar{\mu}_{i,m+1}$, $i = 0, \ldots, m$ deleted. If $\bar{\mu}$ is considered as a matrix, μ corresponds to $\bar{\mu}$ with the last row and column deleted. In each row of the Gram-matrix $G(\bar{\mu})$ the entries add up to zero, so that we have

$$\bar{\mu}_{i,m+1} = 2(m+2) - \sum_{j=0}^{m} \bar{\mu}_{ij}. \tag{16}$$

In particular, the determinant of $G(\bar{\mu})$ vanishes. Hence, $G(\bar{\mu}) \geq 0$ iff $G(\mu) \geq 0$. Thus, $\mu \in \Sigma_m$. Moreover, μ completely determines $\bar{\mu}$ by (16), since the bar on the right-hand side can be deleted. Finally, in the last row of $G(\bar{\mu})$ the entries add up to zero and since these are also the entries of the last column, using (16), we arrive at

$$\sum_{0 \leq i < j \leq m} \mu_{ij} = m(m+2). \tag{17}$$

It follows that, $\phi\colon \bar{\mu} \mapsto \mu$ is a bijection between Σ_{m+1}^0 and the slice in (15).

Let $\tilde{\mu} \in \Sigma^0_{m+1}$ with $\mu \in \Sigma_m$ satisfying (17). Given a quadratic map $\tilde{F}$: $\mathbf{R}^{m+2} \to \mathbf{R}^{n+1}$ with signature $\tilde{\mu}$, restriction to $x_{m+1} = 0$ is a quadratic map F: $\mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ with signature μ. Conversely, given a quadratic map F: $\mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ with signature μ, we define $\tilde{F}$: $\mathbf{R}^{m+2} \to \mathbf{R}^{m+n+2}$ by

$$\tilde{F}(x) = \left(F(x),\ \left(\sqrt{\tilde{\mu}_{i,m+1}}\, x_i x_{m+1}\right)_{i=0,\dots,m}\right), \tag{18}$$

where $\tilde{\mu}_{i,m+1}$ is given in (16). Clearly, $\tilde{F}$ is a quadratic map with signature $\tilde{\mu}$.

We now use this to define a map ψ: $\Sigma^0_m \to \Sigma^0_{m+1}$. Let $\mu \in \Sigma^0_m$. Since the entries in each row of $G(\mu)$ add up to zero, we have $\Sigma^m_{j=0}\mu_{ij} = 2(m+1)$, for all $i = 0, \dots, m$. Summing up with respect to i, we get

$$\sum_{0 \le i < j \le m} \mu_{ij} = (m+1)^2.$$

The normalized signature

$$\frac{m(m+2)}{(m+1)^2}\mu \tag{19}$$

belongs to Σ_m since the coefficient is in (0, 1) and Σ_m is convex. Moreover, the way we normalized, it satisfies (17). Thus, there is a unique $\tilde{\mu} \in \Sigma^0_{m+1}$ whose ϕ-image is (19). Now, associating $\tilde{\mu}$ to μ defines the map ψ: $\Sigma^0_m \to \Sigma^0_{m+1}$.

Let f: $S^m \to S^n$ be a full separated form and F: $\mathbf{R}^{m+1} \to \mathbf{R}^{q+1}$ the associated quadratic map with signature $\mu \in \Sigma^0_m$. Then

$$\frac{\sqrt{m(m+2)}}{m+1}F$$

has signature (19). Using the extension above, we arrive at

$$\tilde{F}(x) = \left(\frac{\sqrt{m(m+2)}}{m+1}F(x),\ \left(\sqrt{2\frac{m+2}{m+1}}x_i x_{m+1}\right)_{i=0,\dots,m}\right),$$

where $\tilde{F}$: $\mathbf{R}^{m+2} \to \mathbf{R}^{n+m+2}$ is a quadratic map with signature $\tilde{\mu}$. To $\tilde{F}$ there corresponds a separated eigenmap $\tilde{f}$: $S^{m+1} \to S^{n+m+1}$ determined up to equivalence. In coordinates, we have

$$\tilde{F}(x) = \left(\frac{\sqrt{m(m+2)}}{m+1}f(x),\ \frac{m+2}{m+1}\left(x^2_{m+1} - \frac{\rho^2}{m+2}\right),\right.$$
$$\left.\left(\sqrt{2\frac{m+2}{m+1}}x_i x_{m+1}\right)_{i=0,\dots,m}\right).$$

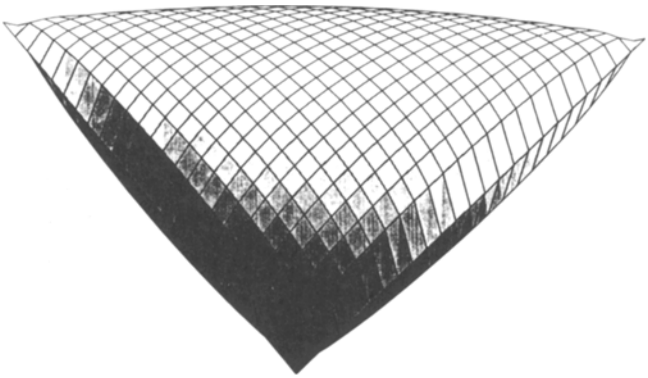

Fig. 1.

In fact, this formula now works without the assumption of f being separated. Associating $\tilde{f}$ to f is the source dimension-raising operator that has been found in [4]. The corresponding map between the standard moduli spaces is injective.

5. The Parameter Space for Low Source Dimensions

We first let $m = 1$. The signature space $\Sigma_1 = [0, 4] \subset \mathbf{R}$ with $\Sigma_1^0 = \{4\}$. Setting $\mu = \mu_{12} = 4 \sin^2(\alpha/2)$, $0 \leq \alpha \leq \pi$, we obtain the quadratic forms $f_\mu\colon S^1 \to S^2$, where

$$f_\mu(x, y) = \left(\cos\left(\frac{\alpha}{2}\right)(x^2 + y^2),\ \sin\left(\frac{\alpha}{2}\right)(x^2 - y^2),\ 2\sin\left(\frac{\alpha}{2}\right)xy\right).$$

Clearly, f_μ wraps S^1 twice around the intersection circle of the 2-sphere and the cone with x-axis as the axis of symmetry and opening half-angle $\alpha/2$ at the origin, the vertex. $\mu = 4$ corresponds to the complex multiplication $z \to z^2$ restricted to S^1. Finally, for each $\mu \in \Sigma_1$, $\mathcal{L}_\mu$ consists of a single point.

Next, let $m = 2$. Setting $\mu = (a, b, c)$, we have

$$\Sigma_2 = \{(a, b, c) \in [0, 4]^3 | a^2 + b^2 + c^2 - 2(ab + bc + ca) + abc \leq 0\}.$$

The tetrahedron spanned by the vectors (4, 4, 0), (4, 0, 4) and (0, 4, 4) is contained in Σ_2; in fact, its edges are the intersections of Σ_2 with the faces of the cube $[0, 4]^3$. $\Sigma_2^0 = \{(3, 3, 3)\}$ corresponds to the standard minimal immersion $f_{\lambda_2}\colon S^2 \to S^4$. (The signature space Σ_2 looks like an 'inflated tetrahedron', cf. the top view computer image in Figure 1.)

Finally, let $m = 3$. We will only work out Σ_3^0. The imbedding $\phi\colon \Sigma_3^0 \to \Sigma_2$, defined in Section 4, associates to (a, b, c, c, b, a) with $a + b + c = 8$ the point (a, b, c). Hence, identifying Σ_3^0 with its image, we see that Σ_3^0 is the triangle Δ with vertices (4, 4, 0), (4, 0, 4) and (0, 4, 4). The center (8/3, 8/3, 8/3) of Δ corresponds to the standard minimal immersion $f_{\lambda_2}\colon S^3 \to S^8$. Let $\mu = (a,\ b,\ c,\ c,\ b,\ a) \in \Sigma_3^0$ and determine $\mathcal{L}_\mu$. Using coordinates $x = (x,\ y,\ u,\ v) \in \mathbf{R}^4$, we have

$$F_\mu(x) = (\sqrt{a}\, xy,\ \sqrt{b}\, xu,\ \sqrt{c}\, xv,\ \sqrt{c}\, yu,\ \sqrt{b}\, yv,\ \sqrt{a}\, uv).$$

Assume first that μ has no zero components, i.e. the corresponding point in Δ is not a vertex. Evaluating

$$\langle C,\ F_\mu(x) \odot F_\mu(x)\rangle = 0, \tag{20}$$

for $C \in S^2(\mathbf{R}^6)$, we find that C has antidiagonal entries $\alpha,\ \beta,\ \gamma,\ \gamma,\ \beta$ and α with $a\alpha + b\beta + c\gamma = 0$ and all other entries are zero. $C + I \geq 0$ translates into $(\alpha,\ \beta,\ \gamma) \in [-1,\ 1]^3 \subset \mathbf{R}^3$. Thus $\mathcal{L}_\mu$ can be visualized as the intersection of the cube $[-1,\ 1]^3$ with the plane through the origin with normal vector $(a,\ b,\ c) \in \Delta$. If $(a,\ b,\ c)$ is in the interior of the triangle Δ then the intersection is a hexagon. If $(a,\ b,\ c)$ is on an edge of Δ but is not a vertex then the intersection is a quadrangle. Finally, if $(a,\ b,\ c)$ is one of the vertices of Δ then, evaluating (20), we find that C has antidiagonal entries $\alpha,\ -\alpha,\ -\alpha,\ \alpha$ and all other entries are zero. Thus $\mathcal{L}_\mu = [-1,\ 1]$.

We can also determine the corresponding eigenmaps, or what is the same, the quadratic maps $F\colon \mathbf{R}^4 \to \mathbf{R}^{q+1}$ via $F = A \cdot F_\mu$, where $A = \sqrt{C+I}$. (Note that, F, obtained this way is, in general, not full.) Using that, for $|x| \leq 1$

$$\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}^{1/2} = \begin{bmatrix} \sqrt{1+x} + \sqrt{1-x} & \sqrt{1+x} - \sqrt{1-x} \\ \sqrt{1+x} - \sqrt{1-x} & \sqrt{1+x} + \sqrt{1-x} \end{bmatrix}$$

and taking an appropriate representative in the equivalence class of F, we arrive at

$$\begin{aligned} F(x) = \Bigg(&\sqrt{\frac{a(1+\alpha)}{2}}(xy+uv),\ \sqrt{\frac{a(1-\alpha)}{2}}(xy-uv), \\ &\sqrt{\frac{b(1+\beta)}{2}}(xu+yv),\ \sqrt{\frac{b(1-\beta)}{2}}(xu-yv), \\ &\sqrt{\frac{c(1+\gamma)}{2}}(xv+yu),\ \sqrt{\frac{c(1-\gamma)}{2}}(xv-yu) \Bigg). \end{aligned}$$

The corresponding full separated eigenmaps $f\colon S^3 \to S^n$ can be written down explicitly. For brevity, we mention here only the range dimensions n. The interior of the hexagon corresponds to eigenmaps with $n = 8$, the edges (without the vertices) to $n = 7$ and the vertices to $n = 6$. The interior of the quadrangle corresponds to

$n = 7$, the edges to $n = 6$ and two pairs of opposite vertices to $n = 5$ and $n = 4$. Finally, the interior of the segment $[-1, 1]$ belongs to the range dimension $n = 4$ and the vertices to $n = 2$. Finally, note that, by Theorem 1, precomposing these by isometries on S^3 we obtain all full quadratic eigenmaps $f\colon S^3 \to S^n$. (Note also that, by fullness, $n = 3$ does not occur as a range dimension.)

Remark. Using the notations of the proof of Theorem 1, we see that $\mathcal{S}_3^0$ can be identified with the 4-dimensional slice cut out from $\mathcal{L}_3^2$ by the linear subspace spanned by the four (linearly independent) vectors $\langle H_0 \rangle$, $\langle H_0' \rangle$, $\langle V \rangle$ and $\langle V' \rangle$. Using the geometric description of $\mathcal{L}_3^2$, it is easy to see how the fiber structure fits in the slice. For example, the opposite of $\langle V \rangle$ in $\partial \mathcal{L}_3^2$ is $\langle H \rangle$ (cf. [10, p. 97]), where $H\colon S^3 \to S^2$ is the Hopf map. Similarly, the opposite of $\langle V' \rangle$ is the 'conjugate' $\langle H' \rangle$ and the three segments connecting $\langle H_0 \rangle$, $\langle H_{\pi/2} \rangle$ and $\langle H \rangle$ with their conjugates correspond to the three fibers $\mathcal{L}_\mu$ at the three vertices μ of the triangle Δ. The interior of the segment with endpoints $\langle H_0 \rangle$ and $\langle H_{\pi/2}' \rangle$ corresponds to the trace of a vertex of the quadrangle $\mathcal{L}_\mu$ as μ slides along an edge of Δ.

6. Orthogonal Multiplications and Quadratic Forms of Low Rank

An orthogonal multiplication of type (p, q, n) is a bilinear map $F\colon \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$, which satisfies

$$|F(x, y)|^2 = |x|^2 |y|^2, \qquad x \in \mathbf{R}^p, \quad y \in \mathbf{R}^q. \tag{21}$$

Orthogonal multiplications (or more generally, nonsingular bilinear pairings, cf. [8]) of type (p, n, n) exist iff $p \leq \rho(n)$, where the Hurwitz–Radon function ρ is defined by setting $\rho(n) = 8a + 2^b$ with $n = 2^{4a+b}c$, $0 \leq b \leq 3$ and c is odd.

The Hopf–Whitehead construction associates to an orthogonal multiplication F of type (p, p, n) the quadratic eigenmap $f_F\colon S^{2p-1} \to S^n$ by

$$f_F(x, y) = (|x|^2 - |y|^2, 2F(x, y)).$$

(For more details, cf. [2] and [10].)

For the next theorem, recall that the rank of a separated form $f\colon S^m \to S^n$ is the dimension of the linear span of the vectors a_i, $i = 0, \ldots, m$.

PROPOSITION 1. *Up to equivalence, the rank-1 separated eigenmaps are obtained from an orthogonal multiplication by the Hopf–Whitehead construction.*

Proof. Consider the set of feasible vectors $\{a_i, a_{ij}\}$ associated to a full separated eigenmap $f\colon S^m \to S^n$. Rank-1 signifies that all a_i, $i = 0, \ldots, m$, are collinear. Since their sum is zero, $m + 1$ is even, say $m + 1 = 2p$, and we may assume that $a_0 = \cdots = a_{p-1} = -a_p = \cdots = -a_{2p-1}$. Since they are unit vectors, for $i \neq j$, $\mu_{ij} = 2(1 - \langle a_i, a_j \rangle)$ is 4 if $|i - j| \leq p$ and zero otherwise. Thus, (12) reduces to (21) (with a factor of 4) and the proposition follows.

THEOREM 3. *Let $f\colon S^m \to S^n$ be a separated eigenmap of rank p. Then, we have*

$$p\left(1+\frac{m+1}{p+1}\right) \leq n+1. \tag{22}$$

Moreover, if equality holds then $p+1|m+1$ and $p|n+1$ and

$$\frac{m+1}{p+1} \leq \rho\left(p\frac{m+1}{p+1}\right).$$

(In particular, if p is not a multiple of 4 then $(m+1)/(p+1) = 1,\ 2,\ 4,\ 8$.)

COROLLARY. *The gradient of an isoparametric function on S^4 is a nonseparable quadratic eigenmap $f\colon S^4 \to S^4$.*

PROOF OF THE COROLLARY. Assume that $f\colon S^4 \to S^4$ is separable. Precomposing f by an isometry, we may assume that f is separated. By the inequality of Theorem 3, $p = 1$. Now Proposition 1 contradicts to the fact that the source dimension is even.

PROOF OF THEOREM 3. We first note that, by the orthogonality relations (5) and (7), $a_{ij} = 0$ iff $a_i = a_j$. Hence, the relation $\sim$ on $\{0,\ldots,\ m\}$ defined by $i \sim j$ if $a_{ij} = 0$ is an equivalence. Let $C_1,\ldots,\ C_s$ denote the equivalence classes and set $z_l = \sharp C_l,\ l = 1,\ldots,\ s$. By the definition of $\sim$, there are exactly s distinct vectors in $\{a_0,\ldots,\ a_m\}$ and, by (10), they are linearly dependent. Hence

$$p+1 \leq s. \tag{23}$$

On the other hand, $\Sigma_{l=1}^{s} z_l = m+1$ so that

$$sz_1 = s\ \min\{z_l | l = 1,\ldots,\ s\} \leq m+1, \tag{24}$$

where we assumed that the minimum is attained at z_1. Let $i_l \in C_l,\ l = 1,\ldots,\ s$, and consider the system of vectors $\{a_{i_l j}\}_{j=0}^{m} \subset \mathbf{R}^{q+1}$. By (8), this system is orthogonal so that its nonzero vectors, of which we have $m+1-z_l$ in number, form a linearly independent system. Thus

$$m+1-z_l \leq q+1. \tag{25}$$

Combining this, for $l = 1$, with (24), we obtain

$$(m+1)\left(1-\frac{1}{s}\right) \leq q+1.$$

Adding p to both sides and using (23) and $p+q = n$, (22) follows.

Assume now that equality holds in (22). Equivalently

$$p\frac{m+1}{p+1} = q+1 \tag{26}$$

so that $p+1$ divides $m+1$ and p divides $q+1$ and hence $n+1$. Combining (25) and (26), we obtain

$$\frac{m+1}{p+1} \le z_l, \quad l = 1, \ldots, s.$$

Summing up with respect to l, we get $s \le p+1$ so that, by (23), $s = p+1$. Retracing the steps $z_1 = \cdots = z_s = (m+1)/(p+1)$ follows. Thus, in $\{a_i\} \subset \mathbf{R}^p$, we have exactly $p+1$ distinct vectors (whose sum is zero by harmonicity). Setting $r = (m+1)/(p+1)$, by (7), F can be thought of as a map

$$F\colon \mathbf{R}^r \times \cdots \times \mathbf{R}^r \to \mathbf{R}^{pr}$$

with the domain consisting of $p+1$ components of $\mathbf{R}^r$ such that

$$|F(x_0, \ldots, x_p)|^2 = \sum_{0 \le i < j \le p} \mu_{ij} |x_i|^2 |x_j|^2, \qquad x_i \in \mathbf{R}^r, \quad i = 0, \ldots, p.$$

Differentiating both sides at $(0, x_1, \ldots, x_p)$ in the direction x_0 twice, we obtain

$$|F'(x_0, x_1, \ldots, x_p)|^2 = |x_0|^2 \left(\sum_{j=1}^{p} \mu_{0j} |x_j|^2 \right),$$

where

$$F'(x_0, x_1, \ldots, x_p) = \left.\frac{\partial}{\partial t}\right|_{t=0} F(tx_0, x_1, \ldots, x_p).$$

This means that

$$F'\colon \mathbf{R}^r \times \mathbf{R}^{pr} \to \mathbf{R}^{pr}$$

is a nonsingular bilinear pairing. By classical result, $r \le \rho(pr)$ (cf. [8]).

Remark. The inequality is sharp for the Hopf maps. Note also that for $F\colon \mathbf{R}^4 \to \mathbf{R}^4$ defined by

$$F(x, y, u, v) = \left(\frac{1}{\sqrt{2}}(xy - uv), \frac{1}{\sqrt{2}}(xu - yv), xv + yu \right)$$

gives a rank 3 quadratic form $f: S^3 \to S^5$ for which equality holds in (22). (The assumption that f is an eigenmap was used only to derive (23) and can be replaced by saying that the distinct vectors in $\{a_i\}$ form a linearly dependent set.)

Now let $f: S^m \to S^n$ be a rank-2 full separated form with associated feasible system $\{a_i, a_{ij}\}$. Up to equivalence, we may assume that the vectors a_i are contained in the plane $\mathbf{R}^2$ spanned by the first two coordinate axes in $\mathbf{R}^{n+1}$ and that

$$a_0 = (1, 0), \qquad a_i = (\cos\,\theta_i, \sin\,\theta_i), \quad i = 1, \ldots, m.$$

We find that, up to equivalence, $f = f_\theta$, where

$$f_\theta(x) = \left(x_0^2 + \sum_{i=1}^m \cos\,\theta_i x_i^2, \sum_{i=1}^m \sin\,\theta_i x_i^2, 2F_\theta(x) \right),$$

where $F_\theta: \mathbf{R}^{m+1} \to \mathbf{R}^{n-1}$ is a full quadratic map with signature

$$\mu = \left(\sin^2 \left(\frac{\theta_i - \theta_j}{2} \right) \right)_{0 \le i < j \le m}.$$

(Here, we set $\theta_0 = 0$.)

Finally, f_θ is an eigenmap iff

$$1 + \sum_{i=1}^m \cos\,\theta_i = 0 \quad \text{and} \quad \sum_{i=1}^m \sin\,\theta_i = 0.$$

We now work out a concrete example (which is a rank-2 analogue of the Hopf–Whitehead construction). Let $m + 1 = 3p$ and assume that $a_0 = \cdots = a_{p-1} = (0, 1)$, $a_p = \cdots = a_{2p-1} = (-1/2, \sqrt{3}/2)$ and $a_{2p} = \cdots = a_{3p-1} = (-1/2, -\sqrt{3}/2)$. By (12), up to a contant factor, the quadratic map $F: \mathbf{R}^p \times \mathbf{R}^p \times \mathbf{R}^p \to \mathbf{R}^n$ satisfies

$$|F(x, y, z)|^2 = |x|^2|y|^2 + |y|^2|z|^2 + |z|^2|x|^2, \quad x, y, z \in \mathbf{R}^p \tag{27}$$

and the associated separated eigenmap $f: S^{3p-1} \to S^{n+1}$ has the form

$$f(x, y, z) = \left(|x|^2 - \frac{1}{2}(|y|^2 + |z|^2), \frac{\sqrt{3}}{2}(|y|^2 - |z|^2), \sqrt{3}\, F(x, y, z) \right).$$

Given two orthogonal multiplications $G_1, G_2: \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$, we say that (G_1, G_2) has a *common extension* if there exists an orthogonal multiplication $G: \mathbf{R}^m \times \mathbf{R}^{2m} \to \mathbf{R}^n$ such that (with obvious notations) we have

$$G(x, y, 0) = G_1(y, x) \quad \text{and} \quad G(x, 0, z) = G_2(x, z),$$
$$x, y, z \in \mathbf{R}^m, \quad (y, z) \in \mathbf{R}^{2m}.$$

(Note the switched argument in the first relation.) A necessary and sufficient condition for the existence of G is that

$$\langle G_1(y, x), G_2(x, z)\rangle = 0, \quad x, y, z \in \mathbf{R}^m.$$

Indeed, if G exists, we have

$$\begin{aligned}|G(x, y, z)|^2 &= |G_1(y, x) + G_2(x, z)|^2 \\ &= |x|^2(|y|^2 + |z|^2) + \langle G_1(y, x), G_2(x, z)\rangle.\end{aligned}$$

On the other hand, $|G(x, y, z)|^2 = |x|^2(|y|^2 + |z|^2)$ and (27) follows. For the converse, we set

$$G(x, y, z) = G_1(y, x) + G_2(x, z).$$

Given three orthogonal multiplications G_1, G_2, G_3: $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$, we say that (G_1, G_2, G_3) has a *cyclic extension* if each pair (G_1, G_2), (G_2, G_3) and (G_3, G_1) has a common extension.

PROPOSITION 2. *There is a one-to-one correspondence between the set of quadratic maps* F: $\mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$ *satisfying (27) and the set of triples* (G_1, G_2, G_3) *of orthogonal multiplications which have cyclic extensions. The correspondence is given by setting* $G_1(y, z) = F(0, y, z)$, $G_2(z, x) = F(x, 0, z)$ *and* $G_3(x, y) = F(x, y, 0)$.

Proof. We use the idea in the proof of Theorem 3. Taking $\partial^2/\partial t^2|_{t=0}$ of both sides of the equation

$$|F(tx, y, z)|^2 = t^2|x|^2(|y|^2 + |z|^2) + |y|^2|z|^2,$$

we obtain

$$|F'(x, y, z)|^2 = |x|^2(|y|^2 + |z|^2), \tag{28}$$

where $F'(x, y, z) = \partial/\partial t F(tx, y, z)|_{t=0}$. (28) says that F' is an orthogonal multiplication as a quadratic map F': $\mathbf{R}^m \times \mathbf{R}^{2m} \to \mathbf{R}^n$, in particular, it is bilinear. We therefore have

$$F'(x, y, z) = F'(x, y, 0) + F'(x, 0, z).$$

On the other hand, $F'(x, y, 0) = \partial/\partial t\, F(tx, 0, z)|_{t=0} = F(x, 0, z)$ since, by (27), the latter is also an orthogonal multiplication and hence bilinear. We obtain that

$$F'(x, y, z) = F(x, y, 0) + F(x, 0, z),$$

where we used the same argument for the second term. Combining this with the Taylor formula:

$$\begin{aligned} F(x, y, z) &= F(0, y, z) + F'(x, y, z) \\ &= F(0, y, z) + F(x, 0, z) + F(x, y, 0). \end{aligned}$$

Taking the norm square of both sides we obtain

$$\begin{aligned} &\langle F(0, y, z), F(x, 0, z)\rangle + \langle F(x, 0, z), F(x, y, 0)\rangle \\ &\quad + \langle F(x, y, 0), F(0, y, z)\rangle = 0. \end{aligned}$$

The terms on the right-hand side vanish separately since they correspond to polynomials with different homogeneity. We now define (G_1, G_2, G_3) as in the proposition above and conclude that this triple has a cyclic extension. The converse is clear.

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