Real Orthogonal Multiplications in Codimension Two

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Abstract

A codimension $c(\geq 0)$ orthogonal multiplication of type $(l, m, m+c)$ is a normed bilinear map $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+c}$, $l \leq m$. Between 1898-1923, Hurwitz and Radon gave a complete classification of all codimension zero orthogonal multiplications. In 1980, Adem showed that, for $c = 1$, F (up to isometries on the source and the range) extends to an orthogonal multiplication of type $(l, m+1, m+1)$ if m is odd, and restricts to an orthogonal multiplication of type (l, m, m) if m is even. The resulting types are covered by the Hurwitz-Radon classification. The main result of this paper is to show that, for $c = 2$ and m even, a full orthogonal multiplication F of type $(l, m, m+2)$ extends similarly to an orthogonal multiplication of type $(l, m+2, m+2)$. We also prove that, for $c = 2$ with m odd, the only possible dimensions are $l = 3$ and $m = 4r + 1$ for some $r \geq 1$ and (up to isometries again) F can be given explicitly in terms of (the restriction of) an orthogonal multiplication $\mathbf{R}^4 \times \mathbf{R}^{4r} \to \mathbf{R}^{4r}$ given by quaternionic vector space multiplication. These give a complete description of all orthogonal multiplications of codimension ≤ 2.

1 Introduction and Statement of Results

An *orthogonal multiplication of type* (l, m, n) is a bilinear map

$$F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n \qquad (1)$$

that satisfies

$$|F(x,y)|^2 = |x|^2\,|y|^2, \qquad x \in \mathbf{R}^l,\ y \in \mathbf{R}^m. \tag{2}$$

F is said to be *full* if the image of F spans $\mathbf{R}^n$.

It is a classical problem to give a complete list of all orthogonal multiplications of type (l, m, n). Note that the concept of orthogonal multiplication can be generalized over any field (of characteristic $\neq 2$). (Actually the very first classification theorem due to Hurwitz [11] was proved over $\mathbf{C}$.) Nevertheless, our main interest here is to classify real orthogonal multiplications so that we refer to the general algebraic results only briefly. For the relevant results as well as the many connections of orthogonal multiplications with other branches of mathematics we refer to the survey article of Shapiro [16] (cf. also the forthcoming book [18]).

For $m = n$, Hurwitz [11,12] and Radon [15] gave a complete solution to this problem that can be concisely stated as follows:

Decomposing m as

$$m = 2^{p+4q}(2r+1), \qquad 0 \leq p \leq 3,\ \ q, r \geq 0,$$

a full orthogonal multiplication $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^m$ exists iff

$$l \leq \rho(m), \tag{3}$$

where the Hurwitz-Radon function ρ is given by

$$\rho(m) = 2^p + 8q.$$

(Note that $p = 0, 1, 2, 3$ correspond to the real, complex, quaternion and Cayley multiplications.)

We briefly recall a proof (based on Clifford modules) since it will play an important part in the sequel. Let $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$ be an orthogonal multiplication and denote by $\{e_\alpha\}_{\alpha=1}^l \subset \mathbf{R}^l$ and $\{f_i\}_{i=1}^m \subset \mathbf{R}^m$ the standard orthonormal bases. For fixed $\alpha = 1, \ldots, l$, (2) implies that $\{u_i^\alpha = F(e_\alpha, f_i)\}_{i=1}^m$ is orthonormal in $\mathbf{R}^n$. For $m = n$, and this is what we assume now, it is then an orthonormal basis in $\mathbf{R}^m$. Hence, for $\alpha, \beta = 1, \ldots, l$, there exists an orthogonal $m \times m$-matrix $P^{\beta\alpha} = (p_{ij}^{\beta\alpha})_{ij=1}^m$ such that

$$u_i^\beta = \sum_{j=1}^m p_{ij}^{\beta\alpha} u_j^\alpha, \qquad i = 1 \ldots, m. \tag{4}$$

Moreover, polarization of (2) gives

$$\langle u_i^\alpha, u_j^\beta \rangle + \langle u_j^\alpha, u_i^\beta \rangle = 0, \qquad \alpha \neq \beta. \tag{5}$$

Substituting (4) into (5), we obtain that $P^{\beta\alpha}$ is skew symmetric, i.e.

$$(P^{\beta\alpha})^{\top} = -P^{\beta\alpha}, \qquad \alpha \neq \beta. \tag{6}$$

In particular, $P^{\beta\alpha}$, being orthogonal, is a complex structure on $\mathbf{R}^m$:

$$(P^{\beta\alpha})^2 = -I, \qquad \alpha \neq \beta. \tag{7}$$

We can further exploit (5) by substituting (4) into the first term of (5) and then switching α and β in (4) and substituting it into the second term in (5). We obtain

$$P^{\beta\alpha} = -P^{\alpha\beta}, \qquad \alpha \neq \beta. \tag{8}$$

Finally, iterating (4), we arrive at

$$P^{\gamma\alpha}P^{\beta\alpha} = -P^{\gamma\beta}, \qquad \alpha, \beta, \gamma \text{ distinct}.$$

Setting $U_\alpha = P^{\alpha l}$, $\alpha = 1, \ldots, l-1$, we obtain that $\{U_\alpha\}_{\alpha=1}^{l-1}$ is an anticommuting family of skew symmetric complex structures on $\mathbf{R}^m$. $\{U_\alpha\}_{\alpha=1}^{l-1}$ determines F up to normalization. More precisely, an orthogonal multiplication $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^m$ is said to be *normalized* if $F(e_l, X) = X$ for all $X \in \mathbf{R}^m$. Summarizing:

The set of normalized orthogonal multiplications $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^m$ is in bijective correspondence with the set of anticommuting skew symmetric complex structures $\{U_\alpha\}_{\alpha=1}^{l-1}$ and the correspondence is given by $U_\alpha X = F(e_\alpha, X)$.

Two orthogonal multiplications $F, F' : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$ are said to be *equivalent* if there exist orthogonal matrices $A \in O(l)$, $B \in O(m)$ and $C \in O(n)$ such that $F' = C \circ F \circ (A \times B)$. Clearly, for $m = n$, any orthogonal multiplication is equivalent to a normalized one.

An anticommuting family $\{U_\alpha\}_{\alpha=1}^{l-1}$ of skew symmetric complex structures on $\mathbf{R}^m$ gives rise to what is called a Clifford module structure on $\mathbf{R}^m$ over the Clifford algebra $C\ell_{l-1}$. Now, the classification of Clifford modules gives (3).

It is also a classical observation that a Clifford module structure on $\mathbf{R}^m$ over $C\ell_{l-1}$ gives rise to $l-1$ (pointwise) orthonormal vector fields $v_1, \ldots, v_{l-1}$ on the unit sphere S^{m-1}. A celebrated result of Adams [1,2] in 1962 states that the maximum number of (pointwise) linearly independent vector fields on spheres can be obtained through orthogonal multiplications, in particular, the only parallelizable spheres are S^1, S^3 and S^7.

In 1964, when the concept of harmonic maps between Riemannian manifolds has been introduced by Eells-Sampson [8], orthogonal multiplications gained considerable importance since they provide examples of harmonic polynomial maps between spheres. In fact, restriction of F in (1) to the product of the unit spheres is a harmonic map $S^{l-1} \times S^{m-1} \to S^{n-1}$. Also, for $l = m$, the Hopf-Whitehead construction associates to an orthogonal multiplication

$$F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$$

the map

$$f_F : S^{l+m-1} \to S^n$$

given by

$$f_F(x, y) = (|x|^2 - |y|^2, 2F(x, y)), \qquad x \in \mathbf{R}^l,\ y \in \mathbf{R}^m,\ |x|^2 + |y|^2 = 1.$$

The components of f_F are (restrictions of) quadratic polynomials. For $l = m$, they are also harmonic so that f_F is a quadratic harmonic eigenmap, or equivalently, a harmonic map of constant energy density in the sense of Eells-Sampson [8]. Again the complex, quaternion and Cayley multiplications give rise to the various classical Hopf maps between spheres. Note that the real tensor product $\otimes : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{lm}$ provides a further example (with maximal $n = lm$) to this situation.

In 1987, the second author showed in [20] that, for fixed l and m, the space of equivalence classes of full orthogonal multiplications of type (l, m, n) can be parametrized by the orbit space of an invariant compact convex body $\mathcal{L}$ in $so(k) \otimes so(l)$ under the action of $Ad \otimes Ad$. This however gives a little clue to the possible range dimensions $\max(l, m) \leq n \leq lm$. The range dimension stratifies $\mathcal{L}$, in particular, the minimum range dimension stratum (corresponding to the Hurwitz-Radon range) is a compact (algebraic) submanifold in the boundary of $\mathcal{L}$. This has been studied by Bier and Schwardman [7] and Shapiro [18] (Chapter 7). For $l = m = 2$, the determination of $\mathcal{L}$ is simple and the parameter space is a segment in $\mathbf{R}$ with interior points corresponding to $n = 4$ and the boundary vertices to $n = 2$. For $l = m = 3$, $\mathcal{L}$ has been determined by Parker [14] in 1983. She obtained that the possible range dimensions are $n = 4, 7, 8, 9$.

According to a recent rigidity result of Gauchman and Toth [10], a quadratic harmonic polynomial map $f : S^m \to S^2$ is the Hopf map (up to isometries of the source and the range), in particular $m = 3$. Moreover [20], there is no full quadratic harmonic map $f : S^3 \to S^3$, where fullness means that the image of f spans $\mathbf{R}^4$. Since these correspond to the lowest possible range dimensions, in view of the link between quadratic harmonic polynomial maps between spheres and orthogonal multiplications, it is natural to ask whether there exist orthogonal multiplications

$$F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$$

with small *codimension* $c = n - \max(l, m)$. To state the first result for codimension one, due to Adem [3,4], we first introduce the following definition:

Let $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$ and $F_0 : \mathbf{R}^{l_0} \times \mathbf{R}^{m_0} \to \mathbf{R}^n$ be orthogonal multiplications with $l_0 \leq l$ and $m_0 \leq m$. We say that F_0 is a *restriction* of F if there exist linear isometries $\phi : \mathbf{R}^{l_0} \to \mathbf{R}^l$ and $\psi : \mathbf{R}^{m_0} \to \mathbf{R}^m$ such that $F_0(X, Y) = F(\phi X, \psi Y)$.

Theorem 1 *Let $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+1}$ be a full orthogonal multiplication. Then m is odd, $l \leq \rho(m+1)$ and there exists an orthogonal multiplication $\bar{F} : \mathbf{R}^l \times \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$ such that F is equivalent to a restriction of $\bar{F}$.*

Note that Theorem 1 states, in particular, that, for m even, an orthogonal multiplication of type $(l, m, m+1)$ is not full, i.e. it restricts (on the range) to an orthogonal multiplication of type (l, m, m). Along with the extension for m odd, this is the original formulation of Adem's theorem.

Since the codimension two case will be modeled on the Adem's extension argument, we give a proof of Theorem 1 in Section 2. Note also that an instructive proof of the codimension one case (over arbitrary fields) was given by Shapiro in [17].

Theorem 2 *Let m be even and $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ a full orthogonal multiplication. Then $l \leq \rho(m+2)$ and there exists an orthogonal multiplication $\bar{F}$: $\mathbf{R}^l \times \mathbf{R}^{m+2} \to \mathbf{R}^{m+2}$ such that F is equivalent to a restriction of $\bar{F}$.*

The proof of Theorem 2 is the main (rather technical) result of the paper and is accomplished in Section 3.

To state the corresponding result for $c = 2$ and m odd, we first introduce the following example. Let $F_0 : \mathbf{R}^3 \times \mathbf{R}^{4r} \to \mathbf{R}^{4r}$, $r \geq 1$, be a full orthogonal multiplication that exists by the Hurwitz theorem since $\rho(4r) = 4$ (or, more explicitly, by a restriction of the quaternionic multiplication on a quaternionic vector space of dimension r). We now define $F_r : \mathbf{R}^3 \times \mathbf{R}^{4r+1} \to \mathbf{R}^{4r+3}$ by

$$F_r(e_\alpha, f_i) = \begin{cases} F_0(e_\alpha, f_i) & \text{if } i = 1, \ldots, 4r, \\ f_{4r+\alpha} & \text{if } i = 4r+1, \end{cases}$$

where $\{e_\alpha\} \subset \mathbf{R}^3$ and $\{f_i\} \subset \mathbf{R}^{4r+3}$ denote the standard orthonormal bases with canonical inclusion $\mathbf{R}^{4r+1} \subset \mathbf{R}^{4r+3}$ onto the span of the first $4r+1$ basis vectors. F_r is clearly a full orthogonal multiplication and is uniquely determined by the requirements that it is an extension of F_0 and $F_r(\cdot, f_{4r+1})$ is the canonical inclusion $\mathbf{R}^3 \to \mathbf{R}^{4r+3}$ onto the span of the last three basis vectors. We say that F_r *is induced by* F_0. We now state the following:

Theorem 3 *Let m be odd and $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ a full orthogonal multiplication. Then, $l = 3$ and $m = 4r+1$, for some $r \geq 1$, and F is equivalent to F_r (that is induced by an orthogonal multiplication $F_0 : \mathbf{R}^3 \times \mathbf{R}^{4r} \to \mathbf{R}^{4r}$).*

The (less technical) proof of Theorem 3 is contained in Section 4. Note that Theorem 3 can be thought of as a weak dual of Theorem 2 since F_r is inextendible to an orthogonal multiplication of type $(3, 4r+3, 4r+3)$ but actually restricts to an orthogonal multiplication of type $(3, 4r, 4r)$ covered by the Hurwitz-Radon classification.

Theorems 2-3 give a complete and explicit classification of all full codimension two orthogonal multiplications over $\mathbf{R}$. We now digress here and recall the known results for the codimension two case and relate them to Theorems 2-3.

First, in 1940-41, Hopf and Stiefel [9,19] showed that the existence of a nonsingular bilinear pairing $\mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+c}$ (such as an orthogonal multiplication) implies that $\binom{m+c}{k}$ is even for $m+c-l < k < m$. This, applied to $c = 2$ immediately gives the nonexistence of orthogonal multiplications of type $(l, m, m+2)$ for $m \equiv 1 \pmod 4$ and $l \geq 4$. This is a particular case of Theorem 3. (For further results using topological K-theory, cf. [16] along with the references there.) Over arbitrary fields, Adem [3] showed the nonexistence of orthogonal multiplications of type (4,5,7). This has been generalized by Yuzvinsky [21] to nonexistence of types $(4, m, m+2)$, provided again that $m \equiv 1 \pmod 4$. (For a simpler proof of Yuzvinsky's result, cf. [5].)

Second, Berger and Friedland [6] (over $\mathbf{R}$) and K.Y.Lam and Yiu [13] (over arbitrary fields) considered the (closely related) problem of finding, for given m and c, the largest l such that a (not necessarily full) orthogonal multiplication of type $(l, m, m+c)$ exists. They found a solution for $c \leq 5$ (with the exception of $c = 5$ with $m \equiv 27 \pmod{32}$). The special case $c = 2$ of their result follows from Theorems 2-3. Indeed, if an orthogonal multiplication of type $(l, m, m+2)$ exists then it is either full or it restricts (up to an isometry on the range) to a full orthogonal multiplication of type either $(l, m, m+1)$ or (l, m, m). Hence, by Theorems 1-2, the maximal value l for which the type $(l, m, m+2)$ exists is $\max\{\rho(m), \rho(m+1), \rho(m+2)\}$ if $m \not\equiv 1 \pmod 4$ and, by Theorem 3, it is 3 if $m \equiv 1 \pmod 4$. This is exactly what they obtained. As an example, let $m = 4$. The value of the maximal l is 4 and the corresponding orthogonal multiplication of type $(4, 4, 6)$ is clearly not full but (up to isometries) corresponds to quaternionic multiplication followed by the inclusion $\mathbf{R}^4 \subset \mathbf{R}^6$. For full orthogonal multiplications, the value of the maximal l is 2, so that we get the existence of the type $(2, 4, 6)$ that is not present in their list. For another example, let $m = 11$. The value of the maximal l is 4 which corresponds to the nonfull type $(4, 11, 13)$ that can be derived from quaternionic vector space multiplication. In contrast, full orthogonal multiplications of type $(l, 11, 13)$ do not exist. Finally, we note that, in general, the maximal values of l corresponding to not necessarily full and full orthogonal multiplications coincide iff $m \equiv 1, 2 \pmod 4$.

In Theorems 2-3, we can also specialize to $l = m$ and apply the result of Hurwitz-Radon to obtain the following:

Corollary 1 *Let $F : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^{m+1}$ be a full orthogonal multiplication. Then $m = 3$ or $m = 7$ and F is equivalent to a restriction of the quaternion or Cayley multiplications, respectively.*

Corollary 2 *Let $F : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ be a full orthogonal multiplication. Then $m = 2$ or $m = 6$ and F is equivalent to a restriction of the quaternion or Cayley multiplication, respectively.*

Remark. For $c \leq 2$, the constraint

$$2 \leq l \leq \rho(m + c) \tag{9}$$

is also sufficient for the existence of a full orthogonal multiplication $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+c}$. This can be seen as follows. By Hurwitz-Radon, we certainly have a full orthogonal multiplication $\bar{F} : \mathbf{R}^l \times \mathbf{R}^{m+c} \to \mathbf{R}^{m+c}$, in particular, $m + c$ is even. We obtain, by restriction, an orthogonal multiplication $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+c}$. Composing F with an isometry on the range, we may assume that the image of F is $\mathbf{R}^{m+c'} \subset \mathbf{R}^{m+c}$, $0 \leq c' \leq c$, with the canonical inclusion. Clearly, $c = c'$ iff F is full. In general, restricting to the image, we arrive at a full orthogonal multiplication $F' : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+c'}$. Now, for $c = 1$, we get $c' = 1$ since otherwise $c' = 0$ and this contradicts to m being odd. Finally, let $c = 2$, in particular, m is even. If $c' = 2$ then we are done. If $c' = 1$ then we apply Theorem 1 to $F' : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+1}$ and obtain a full extension $\mathbf{R}^l \times \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$ which is impossible since $m + 1$ is odd. Finally, let $c' = 0$, in particular, $l \leq \rho(m)$. This, combined with (9) and $c = 2$ gives $l = 2$ since, for $l \geq 3$, these inequalities translate into divisibility of m and $m + 2$ by 4 that is clearly impossible. It remains to show the existence of a full orthogonal multiplication $F : \mathbf{R}^2 \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ for m even. To prove this we restart with $\bar{F} : \mathbf{R}^2 \times \mathbf{R}^{m+2} \to \mathbf{R}^{m+2}$ which we assume to be normalized so that it is equivalent to a single skew symmetric complex structure U on $\mathbf{R}^{m+2}$ given by $U = \bar{F}(e_1, \cdot)$. Now let L be a codimension 2 linear subspace in $\mathbf{R}^{m+2}$ such that L and $U(L)$ span $\mathbf{R}^{m+2}$. The restriction $\bar{F}|\mathbf{R}^2 \times L \to \mathbf{R}^{m+2}$ is clearly full and (identifying L with a copy of $\mathbf{R}^m$) we are done.

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2 Codimension One Orthogonal Multiplications; Proof of Theorem 1

Let $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$ be a full orthogonal multiplication. We also let $\{e_\alpha\}_{\alpha=1}^l \subset \mathbf{R}^l$ and $\{f_i\}_{i=1}^m \subset \mathbf{R}^m$ denote the standard orthonormal bases and define

$$u_i^\alpha = F(e_\alpha, f_i) \in \mathbf{R}^n,$$

and agree that, unless stated otherwise, Greek indices α, β and γ run on $1, \ldots, l$ while Latin indices i, j and k take their values in $1, \ldots, m$. As in Section 1, (2) implies that, for fixed α, the vectors $u_1^\alpha, \ldots, u_m^\alpha$ are orthonormal in $\mathbf{R}^n$ and satisfy (5).

We now specialize to $n = m + 1$ for the rest of this section. For each α, let $E^\alpha \in \mathbf{R}^{m+1}$ be the (uniquely determined) vector such that $\{u_1^\alpha, \ldots, u_m^\alpha, E^\alpha\}$ is an oriented orthonormal basis in $\mathbf{R}^{m+1}$. In analogy with (4), we now write

$$u_i^\beta = \sum_j p_{ij}^{\beta\alpha} u_j^\alpha + q_i^{\beta\alpha} E^\alpha, \tag{10}$$

where $p_{ij}^{\beta\alpha}$, $q_i^\alpha \in \mathbf{R}$. The same argument as in Section 1 then leads to (6) and (8) which we state here as

$$(P^{\beta\alpha})^\top = -P^{\beta\alpha} = P^{\alpha\beta}, \tag{11}$$

where the matrix $P^{\beta\alpha}$ has entries $p_{ij}^{\beta\alpha}$.

Finally, substituting (10) into the orthonormality condition $\langle u_i^\beta, u_j^\beta \rangle = \delta_{ij}$, we obtain

$$(P^{\beta\alpha})^2 = -I + q^{\beta\alpha}(q^{\beta\alpha})^\top, \tag{12}$$

where $q^{\beta\alpha}$ stands for the vector with coordinates $q_i^{\beta\alpha}$ (cf. again (7)).

Lemma 1 *Let P be a skew symmetric $m \times m$-matrix and $q \in \mathbf{R}^m$ such that*

$$P^2 = -I + q \cdot q^\top. \tag{13}$$

Then q is nonzero iff m is odd. In this case q is a unit vector in the kernel of P.

Proof. Applying q to both sides of (13), we see that q is an eigenvector of P^2 with eigenvalue $|q|^2 - 1$. Restricting (13) to the orthogonal complement $q^\perp$ of q in $\mathbf{R}^m$, we obtain that $P^2|q^\perp = -I$, in particular, each (nonzero) vector in $q^\perp$ is an eigenvector of P^2 with eigenvalue -1. We now use the elementary but crucial fact that the nonzero eigenvalues of the square of a skew symmetric matrix have even multiplicities. Hence, if $q \neq 0$ then $|q|^2 - 1 \neq -1$ so that m is odd. Conversely, if m is odd then zero is an eigenvalue of P^2 so that $|q|^2 = 1$. Moreover, in either case, $Pq = 0$ and the lemma follows.

We now return to the original setting and observe that fullness of the orthogonal multiplication $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+1}$ implies that $q^{\beta\alpha} \neq 0$ for at least one choice of $\alpha \neq \beta$. By Lemma 1, m is odd. Again by (the converse statement in) Lemma 1, for all $\alpha \neq \beta$, we obtain

$$|q^{\beta\alpha}| = 1 \qquad \text{and} \qquad P^{\beta\alpha} q^{\beta\alpha} = 0. \tag{14}$$

Lemma 2 *We have*

$$q^{\beta\alpha} = -q^{\alpha\beta}, \qquad \alpha \neq \beta. \tag{15}$$

Proof. Consider the vector $\sum_i q_i^{\beta\alpha} u_i^\alpha \in \mathbf{R}^{m+1}$. By the first formula in (14), it is a unit vector and, by the second and (10), it is orthogonal to u_j^β for all j. We conclude that it is $\pm E^\beta$. We now claim that actually

$$E^\beta = -\sum_i q_i^{\beta\alpha} u_i^\alpha. \tag{16}$$

To prove this we work out the Gram determinant of the bases $\{u_1^\alpha, \ldots, u_m^\alpha, E^\alpha\}$ and $\{u_1^\beta, \ldots, u_m^\beta, -\sum_i q_i^{\beta,\alpha} u_i^\alpha\}$. Using (10), we arrive at

$$\begin{vmatrix} P^{\beta\alpha} & q^{\beta\alpha} \\ -(q^{\beta\alpha})^\top & 0 \end{vmatrix}$$

which is certainly positive since the corresponding matrix is skew. Thus (16) follows.

On the other hand, by (10) and (14), we have

$$\begin{aligned}\sum_i q_i^{\alpha\beta} u_i^\alpha &= \sum_i q_i^{\alpha\beta}\Big(\sum_j p_{ij}^{\alpha\beta} u_j^\beta + q_i^{\alpha\beta} E^\beta\Big)\\ &= \sum_{i,j} p_{ij}^{\alpha\beta} q_i^{\alpha\beta} u_j^\beta + \sum_i (q_i^{\alpha\beta})^2 E^\beta = E^\beta.\end{aligned}$$

Combining this with (16), the lemma follows.

We now return to the main line and iterate (10) on (pairwise) distinct α, β and γ. Using (15) and (16), we have

$$\begin{aligned}u_i^\gamma &= \sum_k p_{ik}^{\gamma\alpha} u_k^\alpha + q_i^{\gamma\alpha} E^\alpha\\ &= \sum_k p_{ik}^{\gamma\alpha}\Big(\sum_j p_{kj}^{\alpha\beta} u_j^\beta + q_k^{\alpha\beta} E^\beta\Big) + q_i^{\gamma\alpha} \sum_j q_j^{\beta\alpha} u_j^\beta\\ &= \sum_j \Big(\sum_k p_{ik}^{\gamma\alpha} p_{kj}^{\alpha\beta} + q_i^{\gamma\alpha} q_j^{\beta\alpha}\Big) u_j^\beta + \sum_k p_{ik}^{\gamma\alpha} q_k^{\alpha\beta} E^\beta.\end{aligned}$$

In terms of matrices, this translates into

$$P^{\gamma\alpha} P^{\beta\alpha} = -P^{\gamma\beta} + q^{\gamma\alpha}(q^{\beta\alpha})^{\mathsf{T}} \tag{17}$$

and

$$P^{\gamma\alpha} q^{\alpha\beta} = q^{\gamma\beta}. \tag{18}$$

In particular, using (14), (18) and Lemma 2, we compute

$$\begin{aligned}\langle q^{\beta\alpha}, q^{\gamma\alpha}\rangle &= -\langle q^{\beta\alpha}, q^{\alpha\gamma}\rangle\\ &= -\langle q^{\beta\alpha}, P^{\alpha\beta} q^{\beta\gamma}\rangle\\ &= -\langle P^{\beta\alpha} q^{\beta\alpha}, q^{\beta\gamma}\rangle = 0.\end{aligned} \tag{19}$$

Finally, for $\alpha = 1, \ldots, l-1$, we define the skew symmetric $(m+1)\times(m+1)$-matrix $\bar{U}_\alpha$ as

$$\bar{U}_\alpha = \begin{bmatrix} -P^{\alpha l} & -q^{\alpha l}\\ (q^{\alpha l})^{\mathsf{T}} & 0\end{bmatrix}.$$

Direct calculation shows that (17)-(19) translate into the single fact that $\{\bar{U}_\alpha\}$ is an anticommuting family of skew symmetric complex structures on $\mathbf{R}^{m+1}$. To finish the proof of Theorem 1 we now consider the normalized orthogonal multiplication $\bar{F} : \mathbf{R}^l \times \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$ that corresponds to $\{\bar{U}_\alpha\}$. We claim that F is equivalent to the restriction $\bar{F}|\mathbf{R}^l \times \mathbf{R}^m$, with $\mathbf{R}^m \subset \mathbf{R}^{m+1}$ being the canonical inclusion (to the linear subspace $f_{m+1}^{\perp}$ spanned by $f_1, \ldots, f_m$). We first normalize F (by passing to an equivalent orthogonal multiplication) and retain all the previous notations for the normalized F. By (10), for $\alpha = 1, \ldots, l-1$, we have

$$\begin{aligned} F(e_\alpha, f_i) &= u_i^\alpha = \sum_j p_{ij}^{\alpha l} u_j^l + q_i^{\alpha l} E^l \\ &= \sum_j p_{ij}^{\alpha l} f_j + q_i^{\alpha l} f_{m+1} \\ &= \bar{F}(e_\alpha, f_i). \end{aligned}$$

Finally, because of the normalizations we made

$$F(e_l, f_i) = u_i^l = f_i = \bar{F}(e_l, f_i)$$

which completes the proof of Theorem 1.

We close this section by showing how Corollary 1 is derived from Theorem 1. Let $F : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^{m+1}$ be a full orthogonal multiplication. By Theorem 1, $m \leq \rho(m+1)$. Setting, as usual, $m+1 = 2^{p+4q}(2r+1)$, $0 \leq p \leq 3$, $q, r \geq 0$, this inequality translates into $2^{p+4q}(2r+1) \leq 2^p + 8q + 1$. It is now easy to see that this holds iff $p = 0, 1, 2, 3$ and $q, r = 0$. Since F is full, $p \neq 0, 1$. The remaining cases $p = 2$, 3 correspond to $m = 3, 7$.

First, let $m = 3$. By Theorem 1, $F : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^4$ is equivalent to a restriction of $\bar{F} : \mathbf{R}^3 \times \mathbf{R}^4 \to \mathbf{R}^4$. Again by Theorem 1 applied to $\bar{F}$ with the arguments switched (or by Hurwitz-Radon) $\bar{F}$ is further equivalent to a restriction of an orthogonal multiplication $\mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$ that is well-known to be equivalent to quaternionic multiplication. For $m = 7$, the argument is similar.

3 Orthogonal Multiplications $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ with m even; Proof of Theorem 2

We retain the notations introduced at the beginning of Section 2 and set $n = m+2$. For each α, we let E_1^α and E_2^α be orthonormal in $\mathbf{R}^{m+2}$ such that $\{u_1^\alpha, \ldots, u_m^\alpha, E_1^\alpha, E_2^\alpha\}$ is an oriented orthonormal basis in $\mathbf{R}^{m+2}$. In analogy with (10), we write

$$u_i^\beta = \sum_j p_{ij}^{\beta\alpha} u_j^\alpha + q_i^{\beta\alpha} E_1^\alpha + r_i^{\beta\alpha} E_2^\alpha, \qquad \alpha \neq \beta, \tag{20}$$

where $p_{ij}^{\beta\alpha}$, $q_i^{\beta\alpha}$, $r_i^{\beta\alpha} \in \mathbf{R}$. In what follows, due to its frequent use, instead of refering to this formula explicitly, in computations we will only indicate that we 'switch the bases'. As usual, we introduce the $m \times m$-matrix $P^{\beta\alpha}$ with entries $p_{ij}^{\beta\alpha}$ and the vectors $q^{\beta\alpha}$ and $r^{\beta\alpha}$ in $\mathbf{R}^m$ with coordinates $q_i^{\beta\alpha}$ and $r_i^{\beta\alpha}$, respectively. By the same proof as that of (11) and (12), for $\alpha \neq \beta$, we then obtain

$$(P^{\beta\alpha})^\top = -P^{\beta\alpha} = P^{\alpha\beta} \tag{21}$$

and

$$(P^{\beta\alpha})^2 = -I + q^{\beta\alpha}(q^{\beta\alpha})^\top + r^{\beta\alpha}(r^{\beta\alpha})^\top. \tag{22}$$

From now on we assume that m is even.

Lemma 3 *For $\alpha \neq \beta$, we have*

$$|q^{\beta\alpha}| = |r^{\beta\alpha}| \quad \text{and} \quad \langle q^{\beta\alpha}, r^{\beta\alpha}\rangle = 0.$$

Moreover, setting

$$\mu_{\beta\alpha} = |q^{\beta\alpha}| = |r^{\beta\alpha}|,$$

for $\mu_{\beta\alpha} \neq 0$, we have

$$\begin{aligned} P^{\beta\alpha} q^{\beta\alpha} &= \lambda_{\beta\alpha} r^{\beta\alpha}, \\ P^{\beta\alpha} r^{\beta\alpha} &= -\lambda_{\beta\alpha} q^{\beta\alpha}, \end{aligned} \tag{23}$$

where $\lambda_{\beta\alpha} \in \mathbf{R}$ such that

$$\lambda_{\beta\alpha}^2 = 1 - \mu_{\beta\alpha}^2. \tag{24}$$

Proof. In the following proof, for simplicity, we suppress $\beta\alpha$ in the upper and lower indices. Evaluating (22) on q and r, we obtain

$$\begin{aligned} P^2 q &= (|q|^2 - 1)q + \langle q, r\rangle r, \\ P^2 r &= \langle q, r\rangle q + (|r|^2 - 1)r. \end{aligned} \tag{25}$$

In particular, $V = \operatorname{span}\{q, r\}$ is invariant under P^2. So is $W = V^\perp$, in fact, by (22),

$$P^2|W = -I. \tag{26}$$

We first claim that $\dim V \neq 1$. Indeed, otherwise $P^2 = -I$ on $\mathbf{R}^m$ since m is even. In particular, $P^2 q = -q$ and $P^2 r = -r$. Comparing this with (25) we obtain $q = r = 0$ and this is a contradiction. Hence $\dim V = 0, 2$.

If $\dim V = 0$ then $q = r = 0$ and we are done. Hence we may assume that $\dim V = 2$, i.e. q and r are linearly independent. By (25), the characteristic equation for $P^2|V$ takes the form

$$t^2 - (|q|^2 + |r|^2 - 2)t + (|q|^2 - 1)(|r|^2 - 1) - \langle q, r\rangle^2 = 0. \tag{27}$$

Just as in the the proof of Lemma 1, here we use again the fact that the nonzero eigenvalues of P^2 have even multiplicities. Since $\dim W = m - 2$ is even, by (26), the only possibility is that $P^2|V$ has one eigenvalue of multiplicity two. Therefore the discriminant of (27) is zero and we obtain

$$(|q|^2 - |r|^2)^2 + 4\langle q, r\rangle^2 = 0$$

and the first statement of the lemma follows. We now turn to the proof of the second. By (25), V and W are eigenspaces of P^2 with eigenvalues $\mu^2 - 1 (\neq -1)$ and -1, respectively. Thus, V and W are also invariant under P. In terms of the orthogonal basis $\{q, r\} \subset V$, the skew symmetric P acts on V as

$$\begin{aligned} Pq &= \lambda r \\ Pr &= -\lambda q \end{aligned}$$

for some $\lambda \in \mathbf{R}$. Applying P to both sides of, say, the first of these equations and comparing it with (25) we arrive at $\lambda^2 = 1 - |q|^2$ and the proof is complete.

We now recall that in the codimension one case the crucial fact was the antisymmetry of $q^{\beta\alpha}$ given in Lemma 2. The proof depended essentially on formula (16). Though here the main technical difficulty is the nonuniqueness of E_1^α and E_2^α, for $\alpha \neq \beta$, it is nevertheless useful to introduce, for $\alpha \neq \beta$,

$$\begin{aligned} I_1^{\beta\alpha} &= \sum_i q_i^{\beta\alpha} u_i^\alpha - \lambda_{\beta\alpha} E_2^\alpha \\ I_2^{\beta\alpha} &= \sum_i r_i^{\beta\alpha} u_i^\alpha + \lambda_{\beta\alpha} E_1^\alpha. \end{aligned} \tag{28}$$

We now observe that the contents of Lemma 3 translate into the single fact that

$\{I_1^{\beta\alpha}, I_2^{\beta\alpha}\}$ *is an orthonormal basis in* $\operatorname{span}\{E_1^\beta, E_2^\beta\}$.

Lemma 4 *For $\alpha \neq \beta$, we have*

$$\operatorname{span}\{q^{\beta\alpha}, r^{\beta\alpha}\} = \operatorname{span}\{q^{\alpha\beta}, r^{\alpha\beta}\},$$

and

$$\mu_{\beta\alpha} = \mu_{\alpha\beta}.$$

Proof. We claim that

$$q_i^{\beta\alpha} I_1^{\beta\alpha} + r_i^{\beta\alpha} I_2^{\beta\alpha} = q_i^{\alpha\beta} E_1^{\beta} + r_i^{\alpha\beta} E_2^{\beta}. \tag{29}$$

Switching the bases and using (21)-(24) and (28), we compute

$$\begin{aligned}
\sum_k p_{ik}^{\beta\alpha} u_k^{\beta} &= \sum_k p_{ik}^{\beta\alpha} \Big(\sum_j p_{kj}^{\beta\alpha} u_j^{\alpha} + q_k^{\beta\alpha} E_1^{\alpha} + r_k^{\beta\alpha} E_2^{\alpha} \Big) \\
&= \sum_j \Big(-\delta_{ij} + q_i^{\beta\alpha} q_j^{\beta\alpha} + r_i^{\beta\alpha} r_j^{\beta\alpha} \Big) u_j^{\alpha} + \lambda_{\beta\alpha} r_i^{\beta\alpha} E_1^{\alpha} - \lambda_{\beta\alpha} q_i^{\beta\alpha} E_2^{\alpha} \\
&= -u_i^{\alpha} + q_i^{\beta\alpha} I_1^{\beta\alpha} + r_i^{\beta\alpha} I_2^{\beta\alpha}.
\end{aligned}$$

Comparing this with (20) the claim follows.

Using the orthogonality relations between the various vectors involved, (29) implies

$$q^{\beta\alpha} = \langle E_1^{\beta}, I_1^{\beta\alpha} \rangle q^{\alpha\beta} + \langle E_2^{\beta}, I_1^{\beta\alpha} \rangle r^{\alpha\beta}$$

and

$$r^{\beta\alpha} = \langle E_1^{\beta}, I_2^{\beta\alpha} \rangle q^{\alpha\beta} + \langle E_2^{\beta}, I_2^{\beta\alpha} \rangle r^{\alpha\beta}$$

so that the first statement of the lemma follows.

Taking norm square of both sides of (29) and summing up with respect to i we obtain

$$|q^{\beta\alpha}|^2 + |r^{\beta\alpha}|^2 = |q^{\alpha\beta}|^2 + |r^{\alpha\beta}|^2$$

and the second also follows. The lemma is proved.

We now show that, without loss of generality, we may assume that either

$$0 < \mu_{\beta\alpha} < 1, \quad \text{for all } \alpha \neq \beta,$$

or

$$\mu_{\beta\alpha} = 1, \quad \text{for all } \alpha \neq \beta.$$

We do this by precomposing F with a suitable isometry on $\mathbf{R}^l$, or, what is the same, by changing $\{e_\alpha\}$ to a new orthonormal basis in $\mathbf{R}^l$ in finitely many steps each corresponding to a pair of indices in $1, \ldots, l$.

To describe one reduction step, let $\beta_0 \neq \gamma_0$ be fixed and define the new orthonormal basis $\{\tilde{e}_\alpha\}$ as obtained from $\{e_\alpha\}$ by rotation in the $\beta_0\gamma_0$-plane by angle $-\pi \leq t \leq \pi$ to be specified later. Equivalently, we set

$$\begin{aligned}
\tilde{e}_{\beta_0} &= \cos t \, e_{\beta_0} - \sin t \, e_{\gamma_0}, \\
\tilde{e}_{\gamma_0} &= \sin t \, e_{\beta_0} + \cos t \, e_{\gamma_0},
\end{aligned}$$

and $\tilde{e}_\alpha = e_\alpha$, for $\alpha \neq \beta_0, \gamma_0$. Then, with obvious notations, we have

$$\begin{aligned}\tilde{u}_i^{\beta_0} &= \cos t\, u_i^{\beta_0} - \sin t\, u_i^{\gamma_0},\\ \tilde{u}_i^{\gamma_0} &= \sin t\, u_i^{\beta_0} + \cos t\, u_i^{\gamma_0},\end{aligned}$$

and $\tilde{u}_i^\alpha = u_i^\alpha$, for $\alpha \neq \beta_0, \gamma_0$.

We now claim that, for $\alpha \neq \beta_0, \gamma_0$:

$$\begin{aligned}\tilde{q}^{\beta_0\alpha} &= \cos t\, q^{\beta_0\alpha} - \sin t\, q^{\gamma_0\alpha},\\ \tilde{r}^{\beta_0\alpha} &= \cos t\, r^{\beta_0\alpha} - \sin t\, r^{\gamma_0\alpha}.\end{aligned}$$

(Similar formulas can be obtained for $\tilde{q}^{\gamma_0\alpha}$ and $\tilde{r}^{\gamma_0\alpha}$.) Indeed, using the formula for changing the bases, we compute

$$\begin{aligned}\tilde{q}^{\beta_0\alpha}E_1^\alpha &+ \tilde{r}^{\beta_0\alpha}E_2^\alpha = \tilde{u}_i^{\beta_0} - \sum_j \langle \tilde{u}_i^{\beta_0}, u_j^\alpha\rangle u_j^\alpha\\ &= \cos t\,(u_i^{\beta_0} - \sum_j p_{ij}^{\beta_0\alpha}u_j^\alpha) - \sin t\,(u_i^{\gamma_0} - \sum_j p_{ij}^{\gamma_0\alpha}u_j^\alpha)\\ &= \cos t\,(q_i^{\beta_0\alpha}E_1^\alpha + r_i^{\beta_0\alpha}E_2^\alpha) - \sin t\,(q_i^{\gamma_0\alpha}E_1^\alpha + r_i^{\gamma_0\alpha}E_2^\alpha),\end{aligned}$$

and the claim follows. In particular, by continuity, for sufficiently small t, $\mu_{\beta\alpha} \neq 0, 1$ implies $\tilde{\mu}_{\beta\alpha} \neq 0, 1$.

Assume now that $\mu_{\beta\alpha} = 0$ for some $\alpha \neq \beta$. Since F is full, there exists γ such that $\mu_{\gamma\alpha} \neq 0$. Setting $\beta_0 = \beta$ and $\gamma_0 = \gamma$, we have

$$\tilde{\mu}_{\beta\alpha} = |\tilde{q}^{\beta\alpha}| = |\sin t\,|\,|q^{\gamma\alpha}| = |\sin t\,|\,\mu_{\gamma\alpha} \neq 0$$

for $t \notin \pi\mathbf{Z}$. Taking t sufficiently small, we see that the number of vanishing μ's decreased. Hence, after finitely many steps, retaining the earlier notations, we arrive at an orthogonal multiplication F for which $\mu_{\beta\alpha} \neq 0$ for all $\alpha \neq \beta$.

Assume now that $\mu_{\beta\alpha} \neq 1$ for some $\alpha \neq \beta$. Let $\gamma \neq \alpha, \beta$ and take $\beta_0 = \gamma$ and $\gamma_0 = \beta$ in the reduction step above. We obtain

$$\begin{aligned}\tilde{\mu}_{\gamma\alpha}^2 &= \frac{1}{2}(1 + \cos(2t))\mu_{\gamma\alpha}^2 + \frac{1}{2}(1 - \cos(2t))\mu_{\beta\alpha}^2\\ &- \sin(2t)\langle q^{\gamma\alpha}, q^{\beta\alpha}\rangle.\end{aligned}$$

If $\mu_{\gamma\alpha} = 1$ then the right hand side, as a function of t, is nonconstant. Hence we may pick t (sufficiently small) such that $\tilde{\mu}_{\gamma\alpha} \neq 1$. Retaining once again the original notations, we may therefore assume that $\mu_{\gamma\alpha} \neq 1$ for all $\gamma \neq \alpha$.

We now restart with $\mu_{\gamma\alpha} = \mu_{\alpha\gamma} \neq 0$. Repeating the procedure once again (with changed indices), we conclude that we may assume $\mu_{\delta\gamma} \neq 1$ for all $\gamma \neq \delta$.

Summarizing, from now on, we assume that

$$\mu_{\beta\alpha} \neq 0, \qquad \text{for all } \alpha \neq \beta,$$

and either

$$\lambda_{\beta\alpha} \neq 0, \qquad \text{for all } \alpha \neq \beta,$$

or

$$\lambda_{\beta\alpha} = 0, \qquad \text{for all } \alpha \neq \beta.$$

Lemma 5 *For $\alpha \neq \beta$, we have*

$$\lambda_{\beta\alpha} = -\lambda_{\alpha\beta}.$$

Proof. We first note that, according to (24) and Lemma 4, $\lambda^2_{\beta\alpha} = \lambda^2_{\alpha\beta}$ so that it is enough to prove that $\lambda_{\beta\alpha}\lambda_{\alpha\beta} < 0$. The first part of Lemma 4 states that $q^{\alpha\beta}$ and $r^{\alpha\beta}$ are expressible as a linear combinations of $q^{\beta\alpha}$ and $r^{\beta\alpha}$. We first make this more precise by showing the following

$$\begin{aligned} \lambda_{\alpha\beta} r^{\alpha\beta} &= \langle E_1^\beta, E_1^\alpha\rangle q^{\beta\alpha} + \langle E_1^\beta, E_2^\alpha\rangle r^{\beta\alpha} \\ -\lambda_{\alpha\beta} q^{\alpha\beta} &= \langle E_2^\beta, E_1^\alpha\rangle q^{\beta\alpha} + \langle E_2^\beta, E_2^\alpha\rangle r^{\beta\alpha}. \end{aligned} \tag{30}$$

We begin with the expansions

$$\begin{aligned} E_1^\alpha &= \sum_j q_j^{\beta\alpha} u_j^\beta + \langle E_1^\alpha, E_1^\beta\rangle E_1^\beta + \langle E_1^\alpha, E_2^\beta\rangle E_2^\beta, \\ E_2^\alpha &= \sum_j r_j^{\beta\alpha} u_j^\beta + \langle E_2^\alpha, E_1^\beta\rangle E_1^\beta + \langle E_2^\alpha, E_2^\beta\rangle E_2^\beta, \end{aligned} \tag{31}$$

that hold since, in the switching formula for the bases, the coefficient of E_1^α (resp. E_2^α) is $q_i^{\beta\alpha}$ (resp. $r_i^{\beta\alpha}$). Using these, we compute

$$\begin{aligned} u_i^\beta &= \sum_j p_{ij}^{\beta\alpha} u_j^\alpha + q_i^{\beta\alpha} E_1^\alpha + r_i^{\beta\alpha} E_2^\alpha \\ &= \sum_j p_{ij}^{\beta\alpha}\Big(\sum_k p_{jk}^{\alpha\beta} u_k^\beta + q_j^{\alpha\beta} E_1^\beta + r_j^{\alpha\beta} E_2^\beta\Big) \\ &+ q_i^{\beta\alpha}\Big(\sum_k q_k^{\beta\alpha} u_k^\beta + \langle E_1^\alpha, E_1^\beta\rangle E_1^\beta + \langle E_1^\alpha, E_2^\beta\rangle E_2^\beta\Big) \\ &+ r_i^{\beta\alpha}\Big(\sum_k r_k^{\beta\alpha} u_k^\beta + \langle E_2^\alpha, E_1^\beta\rangle E_1^\beta + \langle E_2^\alpha, E_2^\beta\rangle E_2^\beta\Big). \end{aligned}$$

Taking the E_1^β and E_2^β components, (30) follows.

We now derive two groups of formulas from (30). First, using the orthogonality of $q^{\beta\alpha}$ and $r^{\beta\alpha}$, we obtain

$$\begin{aligned}
\lambda_{\alpha\beta}\langle q^{\alpha\beta}, q^{\beta\alpha}\rangle &= -\mu_{\alpha\beta}^2\langle E_1^\alpha, E_2^\beta\rangle \\
\lambda_{\alpha\beta}\langle q^{\alpha\beta}, r^{\beta\alpha}\rangle &= -\mu_{\alpha\beta}^2\langle E_2^\alpha, E_2^\beta\rangle \\
\lambda_{\alpha\beta}\langle r^{\alpha\beta}, q^{\beta\alpha}\rangle &= \mu_{\alpha\beta}^2\langle E_1^\alpha, E_1^\beta\rangle \\
\lambda_{\alpha\beta}\langle r^{\alpha\beta}, r^{\beta\alpha}\rangle &= \mu_{\alpha\beta}\langle E_2^\alpha, E_1^\beta\rangle.
\end{aligned}$$

Eliminating $\mu_{\alpha\beta}^2 = \mu_{\beta\alpha}^2$, we obtain

$$\begin{aligned}
\lambda_{\alpha\beta}\langle q^{\alpha\beta}, q^{\beta\alpha}\rangle &+ \lambda_{\beta\alpha}\langle r^{\alpha\beta}, r^{\beta\alpha}\rangle = 0 \\
\lambda_{\alpha\beta}\langle q^{\alpha\beta}, r^{\beta\alpha}\rangle &- \lambda_{\beta\alpha}\langle r^{\alpha\beta}, q^{\beta\alpha}\rangle = 0 \\
\lambda_{\alpha\beta}\langle E_1^\alpha, E_1^\beta\rangle &+ \lambda_{\beta\alpha}\langle E_2^\alpha, E_2^\beta\rangle = 0 \\
\lambda_{\alpha\beta}\langle E_1^\alpha, E_2^\beta\rangle &- \lambda_{\beta\alpha}\langle E_2^\alpha, E_1^\beta\rangle = 0.
\end{aligned} \tag{32}$$

Second, by working out the entries of the Gram matrix of both sides of the equations in (30), we obtain

$$\begin{aligned}
\langle E_1^\alpha, E_1^\beta\rangle^2 &+ \langle E_2^\alpha, E_1^\beta\rangle^2 = \lambda_{\alpha\beta}^2 \\
\langle E_1^\alpha, E_2^\beta\rangle^2 &+ \langle E_2^\alpha, E_2^\beta\rangle^2 = \lambda_{\alpha\beta}^2 \\
\langle E_1^\alpha, E_1^\beta\rangle\langle E_1^\alpha, E_2^\beta\rangle &+ \langle E_2^\alpha, E_1^\beta\rangle\langle E_2^\alpha, E_2^\beta\rangle = 0.
\end{aligned} \tag{33}$$

From now on we assume that $\lambda_{\alpha\beta} \neq 0$ since otherwise the lemma clearly holds. The Gram determinant D of the bases $\{u_i^\alpha, E_1^\alpha, E_2^\alpha\}$ and $\{u_i^\beta, E_1^\beta, E_2^\beta\}$ is positive (since the bases have the same orientation) and, using (20), it turns out to be

$$D = \begin{vmatrix} P^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ (q^{\beta\alpha})^\top & \langle E_1^\alpha, E_1^\beta\rangle & \langle E_1^\alpha, E_2^\beta\rangle \\ (r^{\beta\alpha})^\top & \langle E_2^\alpha, E_1^\beta\rangle & \langle E_2^\alpha, E_2^\beta\rangle \end{vmatrix}.$$

We now perform row operations on D corresponding the elementary matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & -\langle E_1^\alpha, E_2^\beta\rangle & -\langle E_2^\alpha, E_2^\beta\rangle \\ 0 & \langle E_1^\alpha, E_1^\beta\rangle & \langle E_2^\alpha, E_1^\beta\rangle \end{bmatrix}$$

and the effect is, in view of (30) and (32), the following

$$\begin{vmatrix} -\langle E_1^\alpha, E_2^\beta\rangle & -\langle E_2^\alpha, E_2^\beta\rangle \\ \langle E_1^\alpha, E_1^\beta\rangle & \langle E_2^\alpha, E_1^\beta\rangle \end{vmatrix} \cdot D = \lambda_{\alpha\beta}^2 \begin{vmatrix} P^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ -(q^{\alpha\beta})^\top & 0 & \lambda_{\alpha\beta} \\ -(r^{\alpha\beta})^\top & -\lambda_{\alpha\beta} & 0 \end{vmatrix}.$$

The determinant on the right hand side is nonnegative since its matrix is skew symmetric. Thus

$$\langle E_1^\alpha, E_1^\beta\rangle\langle E_2^\alpha, E_2^\beta\rangle - \langle E_1^\alpha, E_2^\beta\rangle\langle E_2^\alpha, E_1^\beta\rangle \geq 0.$$

Combining this with the last two equations of (32), we obtain

$$-\frac{\lambda_{\alpha\beta}}{\lambda_{\beta\alpha}}\left(\langle E_1^\alpha, E_1^\beta\rangle^2 + \langle E_1^\alpha, E_2^\beta\rangle^2\right) \geq 0.$$

The proof is complete.

Lemma 6 *If $\alpha \neq \beta$ then there exists an angle $\theta^{\alpha\beta}$, $-\pi \leq \theta^{\alpha\beta} \leq \pi$, such that we have the following:*

(i)

$$\begin{aligned} q^{\beta\alpha} &= \cos\theta^{\alpha\beta} q^{\alpha\beta} - \sin\theta^{\alpha\beta} r^{\alpha\beta}, \\ r^{\beta\alpha} &= \sin\theta^{\alpha\beta} q^{\alpha\beta} + \cos\theta^{\alpha\beta} r^{\alpha\beta}, \end{aligned}$$

(ii)

$$\theta^{\alpha\beta} = -\theta^{\beta\alpha},$$

(iii)

$$\begin{aligned} I_1^{\beta\alpha} &= \cos\theta^{\alpha\beta} E_1^\beta - \sin\theta^{\alpha\beta} E_2^\beta, \\ I_2^{\beta\alpha} &= \sin\theta^{\alpha\beta} E_1^\beta + \cos\theta^{\alpha\beta} E_2^\beta, \end{aligned}$$

(iv)

$$\begin{aligned} E_1^\beta &= \sum_j q_j^{\alpha\beta} u_j^\alpha - \lambda_{\alpha\beta}(\sin\theta^{\alpha\beta} E_1^\alpha - \cos\theta^{\alpha\beta} E_2^\alpha), \\ E_2^\beta &= \sum_j r_j^{\alpha\beta} u_j^\alpha - \lambda_{\alpha\beta}(\cos\theta^{\alpha\beta} E_1^\alpha + \sin\theta^{\alpha\beta} E_2^\alpha). \end{aligned}$$

Remark. We will refer to these as the 'angular switching formulas'.

Proof. By Lemmas 3 and 4, the orthogonal pairs $\{q^{\beta\alpha}, r^{\beta\alpha}\}$ and $\{q^{\alpha\beta}, r^{\alpha\beta}\}$ span the same subspace so that we have

$$\begin{aligned} q^{\beta\alpha} &= \cos\theta^{\alpha\beta} q^{\alpha\beta} - \sin\theta^{\alpha\beta} r^{\alpha\beta}, & (34) \\ r^{\beta\alpha} &= \pm(\sin\theta^{\alpha\beta} q^{\alpha\beta} + \cos\theta^{\alpha\beta} r^{\alpha\beta}), & (35) \end{aligned}$$

for some angle $-\pi \leq \theta^{\alpha\beta} \leq \pi$. In particular,

$$\begin{aligned} \langle q^{\alpha\beta}, q^{\beta\alpha}\rangle &= \mu_{\alpha\beta}^2 \cos\theta^{\alpha\beta}, \\ \langle r^{\alpha\beta}, q^{\beta\alpha}\rangle &= -\mu_{\alpha\beta}^2 \sin\theta^{\alpha\beta}. \end{aligned} \tag{36}$$

Case I: $\lambda_{\alpha\beta} \neq 0$. By Lemma 5, the first two equations in (32) reduce to

$$\begin{aligned} \langle q^{\alpha\beta}, q^{\beta\alpha}\rangle &= \langle r^{\alpha\beta}, r^{\beta\alpha}\rangle, \\ \langle q^{\alpha\beta}, r^{\beta\alpha}\rangle &= -\langle r^{\alpha\beta}, q^{\beta\alpha}\rangle. \end{aligned} \tag{37}$$

Combining these, we have

$$\begin{aligned} r^{\beta\alpha} &= \mu_{\alpha\beta}^{-2}\langle r^{\beta\alpha}, q^{\alpha\beta}\rangle q^{\alpha\beta} + \mu_{\alpha\beta}^{-2}\langle r^{\beta\alpha}, r^{\alpha\beta}\rangle r^{\alpha\beta} \\ &= \sin\theta^{\alpha\beta} q^{\alpha\beta} + \cos\theta^{\alpha\beta} r^{\alpha\beta}, \end{aligned}$$

so that (35) has positive sign and the first angular switching formulas follow.

Inverting (i) we also obtain (ii).

Next, we use the crucial relations (29) in the proof of Lemma 4, to obtain

$$\begin{aligned} \mu_{\alpha\beta}^2 I_1^{\beta\alpha} &= \langle q^{\alpha\beta}, q^{\beta\alpha}\rangle E_1^\beta + \langle r^{\alpha\beta}, q^{\beta\alpha}\rangle E_2^\beta, \\ \mu_{\alpha\beta}^2 I_2^{\beta\alpha} &= \langle q^{\alpha\beta}, r^{\beta\alpha}\rangle E_1^\beta + \langle r^{\alpha\beta}, r^{\beta\alpha}\rangle E_2^\beta. \end{aligned}$$

Now the angular switching formulas (iii) for the normal vectors follow from (36)-(37).

Finally, we turn to (iv). We first compare (30) and (33) to obtain

$$\begin{aligned} \langle E_1^\alpha, E_1^\beta\rangle &= \langle E_2^\alpha, E_2^\beta\rangle = -\lambda_{\alpha\beta}\sin\theta^{\alpha\beta}, \\ \langle E_1^\alpha, E_2^\beta\rangle &= -\langle E_2^\alpha, E_1^\beta\rangle = -\lambda_{\alpha\beta}\cos\theta^{\alpha\beta}. \end{aligned}$$

Substituting these into (31), we obtain (iv).

Case II: $\lambda_{\alpha\beta} = 0$. If positive sign holds in (35) then we proceed exactly the same way as in Case I above. To complete the proof we finally show that in (35) negative sign is impossible. According to the proof of Lemma 5, the Gram determinant D of the bases $\{u_i^\alpha, E_1^\alpha, E_2^\alpha\}$ and $\{u_i^\beta, E_1^\beta, E_2^\beta\}$ reduces to

$$D = \begin{vmatrix} P^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ (q^{\beta\alpha})^\top & 0 & 0 \\ (r^{\beta\alpha})^{top} & 0 & 0 \end{vmatrix},$$

since, by (30), we have

$$\langle E_1^\alpha, E_1^\beta \rangle = \langle E_1^\alpha, E_2^\beta \rangle = \langle E_2^\alpha, E_1^\beta \rangle = \langle E_2^\alpha, E_2^\beta \rangle = 0.$$

By (34) and (35) with negative sign, D rewrites as

$$\begin{vmatrix} P^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ \cos\theta^{\alpha\beta}(q^{\alpha\beta})^\top - \sin\theta^{\alpha\beta}(r^{\alpha\beta})^\top & 0 & 0 \\ -\sin\theta^{\alpha\beta}(q^{\alpha\beta})^\top - \cos\theta^{\alpha\beta}(r^{\alpha\beta})^\top & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -\cos\theta^{\alpha\beta} & \sin\theta^{\alpha\beta} \\ \sin\theta^{\alpha\beta} & \cos\theta^{\alpha\beta} \end{vmatrix} \begin{vmatrix} P^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ -(q^{\alpha\beta})^\top & 0 & 0 \\ -(r^{\alpha\beta})^\top & 0 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} P^{\alpha\beta} & q^{\alpha\beta} & r^{\alpha\beta} \\ -(q^{\alpha\beta})^\top & 0 & 0 \\ -(r^{\alpha\beta})^\top & 0 & 0 \end{vmatrix} \leq 0$$

since the matrix of the last determinant is skew. This is a contradiction and the lemma follows.

Up to this point we discussed relations between two orthonormal sets of vectors $\{u_i^\alpha\}$ and $\{u_i^\beta\}$ so that most of the formulas involved only two indices α and β. We now bring in a third set of orthonormal vectors, say $\{u_i^\gamma\}$, and obtain equations for three indices.

Lemma 7 *Let α, β and γ be distinct. We have*

(i)

$$P^{\gamma\alpha}P^{\beta\alpha} = -P^{\gamma\beta} + q^{\gamma\alpha}(q^{\beta\alpha})^\top + r^{\gamma\alpha}(r^{\beta\alpha})^\top,$$

(ii)

$$\begin{aligned} P^{\gamma\alpha}q^{\alpha\beta} &= q^{\gamma\beta} + \lambda_{\beta\alpha}(-\sin\theta^{\alpha\beta}q^{\gamma\alpha} + \cos\theta^{\alpha\beta}r^{\gamma\alpha}), \\ P^{\gamma\alpha}r^{\alpha\beta} &= r^{\gamma\beta} - \lambda_{\beta\alpha}(\cos\theta^{\alpha\beta}q^{\gamma\alpha} + \sin\theta^{\beta\alpha}r^{\gamma\alpha}), \end{aligned}$$

(iii)

$$\begin{aligned} P^{\gamma\alpha}q^{\beta\alpha} &= \lambda_{\beta\alpha}r^{\gamma\alpha} + \cos\theta^{\alpha\beta}q^{\gamma\beta} - \sin\theta^{\alpha\beta}r^{\gamma\beta}, \\ P^{\gamma\alpha}r^{\beta\alpha} &= -\lambda_{\beta\alpha}q^{\gamma\alpha} + \sin\theta^{\alpha\beta}q^{\gamma\beta} + \cos\theta^{\alpha\beta}r^{\gamma\beta}. \end{aligned}$$

(iv)

$$\begin{aligned} \langle q^{\alpha\gamma}, q^{\alpha\beta}\rangle &= \langle r^{\alpha\gamma}, r^{\alpha\beta}\rangle \\ &= -\lambda_{\beta\gamma}\sin\theta^{\beta\gamma} + \lambda_{\beta\alpha}\lambda_{\gamma\alpha}\cos(\theta^{\alpha\beta} + \theta^{\gamma\alpha}), \\ \langle q^{\alpha\gamma}, r^{\alpha\beta}\rangle &= -\langle q^{\alpha\beta}, r^{\alpha\gamma}\rangle \\ &= \lambda_{\beta\gamma}\cos\theta^{\beta\gamma} - \lambda_{\beta\alpha}\lambda_{\alpha\gamma}\sin(\theta^{\alpha\beta} + \theta^{\gamma\alpha}), \end{aligned}$$

(v)

$$\begin{aligned} \langle q^{\gamma\alpha}, q^{\beta\alpha}\rangle &= \langle r^{\gamma\alpha}, r^{\beta\alpha}, r^{\beta\alpha}\rangle = \lambda_{\beta\alpha}\lambda_{\alpha\gamma} - \lambda_{\beta\gamma}\sin(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta\gamma\alpha), \\ \langle q^{\gamma\alpha}, r^{\beta\alpha}\rangle &= -\langle q^{\beta\alpha}, r^{\gamma\alpha}\rangle = \lambda_{\beta\gamma}\cos(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}), \end{aligned}$$

(vi)

$$\begin{aligned} \langle q^{\gamma\alpha}, q^{\alpha\beta}\rangle &= \langle r^{\gamma\alpha}, r^{\alpha\beta}\rangle \\ &= -\lambda_{\beta\gamma}\sin(\theta^{\beta\gamma} + \theta^{\gamma\alpha}) + \lambda_{\beta\alpha}\lambda_{\alpha\gamma}\cos\theta^{\alpha\beta}, \\ \langle q^{\gamma\alpha}, r^{\alpha\beta}\rangle &= -\langle r^{\gamma\alpha}, q^{\alpha\beta}\rangle \\ &= \lambda_{\beta\gamma}\cos(\theta^{\beta\gamma} + \theta^{\gamma\alpha}) - \lambda_{\beta\alpha}\lambda_{\alpha\gamma}\sin\theta^{\alpha\beta}. \end{aligned}$$

Remark. We will refer to (i)-(vi) as the three-index formulas.

Proof. As in the codimension one case, we switch the bases twice and use the angular switching formulas for the normal vectors to obtain

$$\begin{aligned} u_i^\gamma &= \sum_j p_{ij}^{\gamma\alpha}u_j^\alpha + q_i^{\gamma\alpha}E_1^\alpha + r_i^{\gamma\alpha}E_2^\alpha \\ &= \sum_j p_{ij}^{\gamma\alpha}\Big(\sum_k p_{jk}^{\alpha\beta}u_k^\beta + q_j^{\alpha\beta}E_1^\beta + r_j^{\alpha\beta}E_2^\beta\Big) \end{aligned}$$

$$
\begin{aligned}
&+ \quad q_i^{\gamma\alpha}\Big(\sum_j q_j^{\beta\alpha}u_j^\beta - \lambda_{\beta\alpha}(\sin\theta^{\beta\alpha}E_1^\beta - \cos\theta^{\beta\alpha}E_2^\beta)\Big)\\
&+ \quad r_i^{\gamma\alpha}\Big(\sum_j r_j^{\beta\alpha}u_j^\beta - \lambda_{\beta\alpha}(\cos\theta^{\beta\alpha}E_1^\beta + \sin\theta^{\beta\alpha}E_2^\beta)\Big).
\end{aligned}
$$

Comparing this with the expansion

$$u_i^\gamma = \sum_j p_{ij}^{\gamma\beta}u_j^\beta + q_i^{\gamma\beta}E_1^\beta + r_i^{\gamma\beta}E_2^\beta,$$

we arrive at (i) and (ii). Using the switching formula in (ii) we easily get (iii). To get further, we now work out both sides of

$$(P^{\beta\gamma}P^{\gamma\alpha})\,q^{\alpha\beta} = P^{\beta\gamma}(P^{\gamma\alpha}q^{\alpha\beta}). \tag{38}$$

For the left hand side, by (i) and (ii), we obtain

$$
\begin{aligned}
(P^{\beta\gamma}P^{\gamma\alpha})q^{\alpha\beta} &= (P^{\beta\alpha} - q^{\beta\gamma}(q^{\alpha\gamma})^\top - r^{\beta\gamma}(r^{\alpha\gamma})^\top)q^{\alpha\beta}\\
&= -\lambda_{\alpha\beta}r^{\alpha\beta} - \langle q^{\alpha\gamma}, q^{\alpha\beta}\rangle q^{\beta\gamma} - \langle r^{\alpha\gamma}, q^{\alpha\beta}\rangle r^{\beta\gamma}.
\end{aligned}
$$

On the other hand, using (ii) repeatedly, we compute

$$
\begin{aligned}
P^{\beta\gamma}(P^{\gamma\alpha}q^{\alpha\beta}) &= P^{\beta\gamma}(q^{\gamma\beta} + \lambda_{\beta\alpha}(\sin\theta^{\beta\alpha}q^{\gamma\alpha} + \cos\theta^{\beta\alpha}r^{\gamma\alpha}))\\
&= -\lambda_{\gamma\beta}r^{\gamma\beta} + \lambda_{\beta\alpha}\sin\theta^{\beta\alpha}(q^{\beta\alpha} + \lambda_{\alpha\gamma}(\sin\theta^{\alpha\gamma}q^{\beta\gamma} + \cos\theta^{\alpha\gamma}r^{\beta\gamma}))\\
&+ \lambda_{\beta\alpha}\cos\theta^{\beta\alpha}(r^{\beta\alpha} + \lambda_{\alpha\gamma}(-\cos\theta^{\alpha\gamma}q^{\beta\gamma} + \sin\theta^{\alpha\gamma}r^{\beta\gamma}))\\
&= -\lambda_{\gamma\beta}(\sin\theta^{\beta\gamma}q^{\beta\gamma} + \cos\theta^{\beta\gamma}r^{\beta\gamma}) + \lambda_{\beta\alpha}r^{\alpha\beta}\\
&- \lambda_{\beta\alpha}\lambda_{\alpha\gamma}\cos(\theta^{\alpha\beta} + \theta^{\gamma\alpha})q^{\beta\gamma} - \lambda_{\beta\alpha}\lambda_{\alpha\gamma}\sin(\theta^{\alpha\beta} + \theta^{\gamma\alpha})r^{\beta\gamma}.
\end{aligned}
$$

Substituting these back to (38) and comparing coefficients we arrive at two of the equations in (iv). The rest of (iv) can be obtained in a similar way. By making use of the switching formula as above, we can recover (v) and (vi) from (iv). The lemma follows.

Lemma 8 *If α, β and γ are distinct and $\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) \neq 0$ then*

$$\lambda_{\alpha\beta}\lambda_{\beta\gamma}\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) - \lambda_{\gamma\alpha}\sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) = 0. \tag{39}$$

Proof. Here we use skew symmetry of $P^{\gamma\alpha}$ and work out the left hand side of the equality

$$\langle P^{\gamma\alpha}q^{\alpha\beta}, q^{\beta\alpha}\rangle + \langle q^{\alpha\beta}, P^{\gamma\alpha}q^{\beta\alpha}\rangle = 0.$$

By (ii) and (iii) of Lemma 7, we have

$$\begin{aligned}\langle q^{\gamma\beta} + \lambda_{\beta\alpha}(-\sin\theta^{\alpha\beta} q^{\gamma\alpha} + \cos\theta^{\alpha\beta} r^{\gamma\alpha}), q^{\beta\alpha}\rangle & \\ +\langle q^{\alpha\beta}, \lambda_{\beta\alpha} r^{\gamma\alpha} + \cos\theta^{\alpha\beta} q^{\gamma\beta} - \sin\theta^{\alpha\beta} r^{\gamma\beta}\rangle &= 0\end{aligned}$$

so that we arrive at

$$\cos\theta^{\alpha\beta}(\lambda_{\gamma\alpha}\sin(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) + \lambda_{\beta\alpha}\lambda_{\beta\gamma}(1 + \cos(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}))) = 0.$$

Using the assumption, we obtain (39) multiplied by $\cos\theta^{\alpha\beta}$. Similarly, using skew symmetry of $P^{\gamma\alpha}$ on $q^{\alpha\beta}$ and $r^{\beta\alpha}$, we arrive at (39) multiplied by $\sin\theta^{\alpha\beta}$ and this completes the proof.

Lemma 9 *If α, β and γ are distinct and $\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) \neq 0$ then*

$$\lambda_{\alpha\beta}^2 + \lambda_{\beta\gamma}^2 + \lambda_{\gamma\alpha}^2 - 1 - 4\lambda_{\beta\gamma}^2\lambda_{\alpha\beta}^2\cos^2\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) = 0. \tag{40}$$

Proof. We first symmetrize (i) of Lemma 7:

$$\begin{aligned}P^{\gamma\alpha}P^{\beta\alpha} &+ P^{\beta\alpha}P^{\gamma\alpha} \\ &= q^{\gamma\alpha}(q^{\beta\alpha})^{\top} + q^{\beta\alpha}(q^{\gamma\alpha})^{\top} + r^{\gamma\alpha}(r^{\beta\gamma})^{\top} + r^{\beta\alpha}(r^{\gamma\alpha})^{\top}.\end{aligned} \tag{41}$$

We now apply $q^{\beta\alpha}$ to both sides of this equation. Using Lemmas 3 and 5 along with the three-index formulas in Lemma 7, a tedious but elementary computation leads to

$$\begin{aligned}\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})q^{\gamma\alpha} &- \sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})r^{\gamma\alpha} \\ -\lambda_{\beta\gamma}\sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})q^{\beta\alpha} &- \lambda_{\beta\gamma}\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})r^{\beta\alpha} \\ +\lambda_{\alpha\beta}\sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} - \theta^{\gamma\alpha})q^{\beta\gamma} &+ \lambda_{\alpha\beta}\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} - \theta^{\gamma\alpha})r^{\beta\gamma} = 0.\end{aligned}$$

In a similar vein, applying both sides of (41) to $r^{\beta\alpha}$, we arrive at

$$\begin{aligned}-\sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})q^{\gamma\alpha} &- \cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})r^{\gamma\alpha} \\ -\lambda_{\beta\gamma}\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})q^{\beta\alpha} &+ \lambda_{\beta\gamma}\sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha})r^{\beta\alpha} \\ +\lambda_{\alpha\beta}\cos\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} - \theta^{\gamma\alpha})q^{\beta\gamma} &- \lambda_{\alpha\beta}\sin\frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} - \theta^{\gamma\alpha})r^{\beta\gamma} = 0.\end{aligned}$$

Eliminating $r^{\gamma\alpha}$ from the last two equations, we obtain

$$q^{\gamma\alpha} = \lambda_{\beta\gamma}r^{\beta\alpha} + \lambda_{\alpha\beta}(\sin\theta^{\gamma\alpha}q^{\beta\gamma} - \cos\theta^{\gamma\alpha}r^{\beta\gamma}).$$

Taking norm square of both sides and using (iv) of Lemma 7, (40) follows.

Lemma 10 *If α, β and γ are distinct then either*

$$\cos \frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) = 0$$

or

$$\lambda_{\alpha\beta} = \lambda_{\beta\gamma} = \lambda_{\gamma\alpha} = 0.$$

Proof. We assume that both are false, in particular, by the second, all λ's are nonzero (cf. the discussion before Lemma 5). By cyclic permuation of the indices in (40) of Lemma 9, we obtain

$$\lambda^2_{\alpha\beta} = \lambda^2_{\beta\gamma} = \lambda^2_{\gamma\alpha} = \lambda, \text{ (say)}$$

and so (40) implies

$$\cot^2 \frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) = \frac{3\lambda - 1}{4\lambda^2 - 3\lambda + 1}.$$

On the other hand, (39) of Lemma 8 reduces to

$$\cot^2 \frac{1}{2}(\theta^{\alpha\beta} + \theta^{\beta\gamma} + \theta^{\gamma\alpha}) = \frac{1}{\lambda}.$$

Comparing the two values of the cotangent we obtain $\lambda = 1$, i.e. $\mu_{\alpha\beta} = \mu_{\beta\gamma} = \mu_{\gamma\alpha} = 0$ which is a contradiction. The lemma follows.

Until now the normal vectors E_1^α and E_2^α were subject to the only condition that the orthonormal basis $\{u_i^\alpha, E_1^\alpha, E_2^\alpha\}$ is oriented. We now specify a unique choice for these vectors. First, for $\alpha = l$, we take E_1^l and E_2^l arbitrary (but fixed). Then, for each $\beta \neq l$, we define

$$\begin{aligned} E_1^\beta &= -I_1^{\beta l} = -\sum_j q_j^{\beta l} u_j^l + \lambda_{\beta l} E_2^l, \\ E_2^\beta &= -I_2^{\beta l} = -\sum_j r_j^{\beta l} u_j^l - \lambda_{\beta l} E_1^l. \end{aligned}$$

The angular switching formula (iii) of Lemma 6 has two implications. First, $\{u_i^\beta, E_1^\beta, E_2^\beta\}$ has the same orientation as $\{u_i^l, E_1^l, E_2^l\}$ so that the definition above makes sense. Second, we actually have

$$\cos\theta^{\beta l} = -1 \quad \text{and} \quad \sin\theta^{\beta l} = 0, \quad \beta \neq l.$$

so that

$$\theta^{\beta l} = \pm\pi, \qquad \beta \neq l. \tag{42}$$

It now follows from the angular switching formula (ii) of Lemma 6 that

$$q^{\beta l} = -q^{l\beta} \quad \text{and} \quad r^{\beta l} = -r^{l\beta}, \quad \beta \neq l.$$

This is a special case of the following crucial lemma:

Lemma 11 *For $\beta \neq \gamma$, we have*

$$q^{\beta\gamma} = -q^{\gamma\beta} \quad \text{and} \quad r^{\beta\gamma} = -r^{\gamma\beta}, \tag{43}$$

or equivalently,

$$\theta^{\beta\gamma} = \pm\pi. \tag{44}$$

Proof. The second statement is clearly equivalent to the first by the angular switching formula (ii) of Lemma 6. By (42), we may assume that $\beta, \gamma \neq l$. By Lemma 10, we have to consider only two cases.

Case I. $\cos\frac{1}{2}(\theta^{l\beta} + \theta^{\beta\gamma} + \theta^{\gamma l}) = 0$. Here (44) clearly follows from (42).

Case II. $\lambda_{\beta\alpha} = 0$ for all $\alpha \neq \beta$. In this case (ii) of Lemma 7 reduces greatly, in particular, it gives $P^{\gamma l}q^{l\beta} = q^{\gamma\beta}$ and this is what allows to pass information from $q^{l\beta}$ to $q^{\gamma\beta}$. More precisely, using (i) of Lemma 7 and (42), we compute

$$\begin{aligned}
\langle q^{\gamma\beta}, q^{\beta\gamma}\rangle &= \langle P^{\gamma l}q^{l\beta}, P^{\beta l}q^{l\gamma}\rangle = \langle q^{l\beta}, P^{\gamma l}P^{l\beta}q^{l\gamma}\rangle \\
&= \langle q^{l\beta}, (P^{\gamma\beta} - q^{\gamma l}(q^{\beta l})^{\mathsf{T}} - r^{\gamma l}(r^{\beta l})^{\mathsf{T}})\rangle \\
&= \langle q^{l\beta}, P^{\gamma\beta}q^{l\gamma}\rangle - \langle q^{\beta l}, q^{l\gamma}\rangle^2 - \langle q^{\beta l}, r^{l\gamma}\rangle\langle r^{\beta l}, q^{l\gamma}\rangle.
\end{aligned}$$

By (ii) of Lemma 7, the first term equals $\langle q^{l\beta}, q^{\beta l}\rangle = -|q^{l\beta}|^2 = -\mu_{l\beta}^2 = -1$ and, by (vi), the second and the third vanish. Thus the first part of (43) follows. The proof of the second part is similar.

We finally turn to the proof of Theorem 2. For $\alpha = 1, \ldots, l-1$, we define the skew symmetric $(m+2) \times (m+2)$-matrix $\bar{U}_\alpha$ as

$$\bar{U}_\alpha = \begin{bmatrix} -P^{\alpha l} & -q^{\alpha l} & -r^{\alpha l} \\ (q^{\alpha l})^{\mathsf{T}} & 0 & -\lambda_{\alpha l} \\ (r^{\alpha l})^{\mathsf{T}} & \lambda_{\alpha l} & 0 \end{bmatrix}.$$

Calculation shows that (22)-(24) translate into

$$\bar{U}_\alpha^2 = -I,$$

while the three-index formulas (i) and (iii) of Lemma 7 along with Lemma 11 imply

$$\bar{U}_\alpha \bar{U}_\beta + \bar{U}_\beta \bar{U}_\alpha = 0, \qquad \alpha \neq \beta.$$

We now consider the normalized orthogonal multiplication $\bar{F} : \mathbf{R}^l \times \mathbf{R}^{m+2} \to \mathbf{R}^{m+2}$ that corresponds to the family $\{\bar{U}_\alpha\}$ of skew symmetric anticommuting complex structures on $\mathbf{R}^{m+2}$. It remains to prove that F is equivalent to the restriction $\bar{F}|\mathbf{R}^l \times \mathbf{R}^m$, with $\mathbf{R}^m \subset \mathbf{R}^{m+2}$ being the canonical inclusion (to $\{f^{m+1}, f^{m+2}\}^\perp$ spanned by $f_1, \ldots, f_m$). We may assume that F is normalized. (Note that normalization does not change the coefficients in (20).) The proof (of Theorem 2 and Corollary 2) is now an exact analogue of that of the codimension one case.

4 Orthogonal multiplications $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^{m+2}$ with m odd; Proof of Theorem 3

We now set $n = m + 2$ with m odd and retain the notations introduced at the beginning of Section 3. In particular, we have formula (20) for switching the bases along with (21) and (22).

Lemma 12 *For $\alpha \neq \beta$, the kernel of $P^{\beta\alpha}$ is one dimensional. Moreover, there exists a unit vector $E^{\beta\alpha}$ in the orthogonal complement of $\{u_1^\alpha, \ldots, u_m^\alpha\} \subset \mathbf{R}^{m+2}$ such that*

$$u_i^\beta = \sum_j p_{ij}^{\beta\alpha} u_j^\alpha + s_i^{\beta\alpha} E^{\beta\alpha}, \tag{45}$$

where $s^{\beta\alpha} \in \mathbf{R}^m$ is a unit vector in the kernel of $P^{\beta\alpha}$ that satisfies

$$(P^{\beta\alpha})^2 = -I + s^{\beta\alpha}(s^{\beta\alpha})^\top. \tag{46}$$

Proof. We first observe that (25)-(26) in the proof of Lemma 3 are valid since, in deriving them, we have not used the parity of m. Suppressing the indices $\beta\alpha$ again, next we note that $\dim \ker P \geq 1$ since P is a skew symmetric $m \times m$-matrix and m is odd. Using once again the fact that the nonzero eigenvalues of P^2 have even multiplicities, (26) implies that $V = \ker P$ is one dimensional. Setting $q = \xi \cdot t$ and $r = \eta \cdot t$, where $t \in V$ is a unit vector, we define

$$s = \sqrt{\xi^2 + \eta^2} \cdot t.$$

Replacing q and r in (22) with s, (46) follows. We are now in the position to apply Lemma 1 and conclude that s is a unit vector in the kernel of P, in particular, $\xi^2 + \eta^2 = 1$. Returning to the indices, we now define

$$E^{\beta\alpha} = \xi_{\beta\alpha} E_1^\alpha + \eta_{\beta\alpha} E_2^\alpha. \tag{47}$$

Clearly, $\{u_1^\alpha, \ldots, u_m^\alpha, E^{\beta\alpha}\}$ is orthonormal in $\mathbf{R}^{m+2}$ and (20) for the change of bases reduces to (45). The lemma follows.

Lemma 13 *For $\alpha \neq \beta$, we have*

$$s^{\beta\alpha} = \pm s^{\alpha\beta}.$$

Proof. We first note that, formula (45) for switching the bases implies

$$E^{\beta\alpha} = \sum_i s_i^{\beta\alpha} u_i^\beta \tag{48}$$

since $s^{\beta\alpha}$ is a unit vector. Using this and switching the bases twice, we compute

$$\begin{aligned} u_i^\beta &= \sum_k p_{ik}^{\beta\alpha} u_k^\alpha + s_i^{\beta\alpha} E^{\beta\alpha} \\ &= \sum_k p_{ik}^{\beta\alpha} (\sum_j p_{kj}^{\alpha\beta} u_j^\beta + s_k^{\alpha\beta} E^{\alpha\beta}) \\ &+ s_i^{\beta\alpha} \sum_j s_j^{\beta\alpha} u_j^\beta. \end{aligned}$$

Taking the normal ($E^{\alpha\beta}$-)components of both sides, it follows that $s^{\alpha\beta}$ is in the kernel of $P^{\beta\alpha}$. The kernel is, however, one dimensional so that the statement follows from Lemma 12.

We now let α, β and γ distinct. Switching bases twice and, using (48), we obtain

$$\begin{aligned} u_i^\gamma &= \sum_k p_{ik}^{\gamma\alpha} u_k^\alpha + s_i^{\gamma\alpha} E^{\gamma\alpha} \\ &= \sum_k p_{ik}^{\gamma\alpha} (\sum_j p_{kj}^{\alpha\beta} u_j^\beta + s_k^{\alpha\beta} E^{\alpha\beta}) \\ &+ s_i^{\gamma\alpha} \sum_j s_j^{\gamma\alpha} u_j^\gamma. \end{aligned}$$

Taking the u_j^β-components of both sides, we arrive at

$$P^{\gamma\beta} = P^{\gamma\alpha} P^{\alpha\beta} + s^{\gamma\alpha} (s^{\gamma\alpha})^\top P^{\gamma\beta} \tag{49}$$

and this is the analogue of (17). We now apply both sides of (49) to $s^{\gamma\beta}$ and conclude that

$$P^{\gamma\alpha}P^{\alpha\beta}s^{\gamma\beta} = 0.$$

The kernel of $P^{\gamma\alpha}$ is, however, spanned by $s^{\gamma\alpha}$ so that we obtain

$$P^{\alpha\beta}s^{\gamma\beta} = \sigma \cdot s^{\gamma\alpha}, \quad \sigma \in \mathbf{R}.$$

To determine σ, using (46), we compute

$$\begin{aligned} \sigma^2 &= |P^{\alpha\beta}s^{\gamma\beta}|^2 \\ &= -\langle s^{\gamma\beta}, (P^{\alpha\beta})^2 s^{\gamma\beta}\rangle \\ &= 1 - \langle s^{\alpha\beta}, s^{\gamma\beta}\rangle^2. \end{aligned}$$

Summarizing, we have

$$P^{\alpha\beta}s^{\gamma\beta} = \pm\sqrt{1 - \langle s^{\alpha\beta}, s^{\gamma\beta}\rangle^2} \cdot s^{\gamma\alpha}. \tag{50}$$

We now use the fact that $P^{\alpha\beta}$ is skew along with Lemma 13 and obtain

$$\begin{aligned} (1 - \langle s^{\alpha\beta}, s^{\beta\gamma}\rangle^2)\langle s^{\beta\gamma}, s^{\gamma\alpha}\rangle &= 0, \\ (1 - \langle s^{\beta\gamma}, s^{\gamma\alpha}\rangle^2)\langle s^{\gamma\alpha}, s^{\alpha\beta}\rangle &= 0, \\ (1 - \langle s^{\gamma\alpha}, s^{\alpha\beta}\rangle^2)\langle s^{\alpha\beta}, s^{\beta\gamma}\rangle &= 0, \end{aligned}$$

where the last two equations are obtained by cyclic permutations. All vectors in this system are of unit length so that we conclude:

For α, β and γ distinct, the set $\{s^{\alpha\beta}, s^{\beta\gamma}, s^{\gamma\alpha}\}$ either consists of parallel vectors or is orthonormal.

Now let α, β, γ and δ be four distinct indices. Applying the conclusion above to each triple, the possible configurations A,B,C,D and E can be put into the following table:

	A	B	C	D	E
α, β, γ	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\perp$
α, β, δ	$\parallel$	$\parallel$	$\parallel$	$\perp$	$\perp$
α, γ, δ	$\parallel$	$\parallel$	$\perp$	$\perp$	$\perp$
β, γ, δ	$\parallel$	$\perp$	$\perp$	$\perp$	$\perp$

Here, e.g. $\perp$ in the first (α, β, γ-)row and last (E-)column means that $s^{\alpha\beta}$, $s^{\beta\gamma}$ and $s^{\gamma\alpha}$ are orthonormal.

We now claim that Cases B,C and D are impossible.

Cases B-C. We have $s^{\alpha\beta} \| s^{\beta\gamma} \perp s^{\beta\delta} \| s^{\alpha\beta}$ and this is a contradiction.

Case D. By (50), we have

$$\begin{aligned} P^{\delta\beta} s^{\gamma\beta} &= \pm\sqrt{1 - \langle s^{\delta\beta}, s^{\gamma\beta}\rangle^2} \cdot s^{\gamma\delta} = \pm s^{\gamma\delta} \\ P^{\delta\beta} s^{\alpha\beta} &= \pm\sqrt{1 - \langle s^{\delta\beta}, s^{\alpha\beta}\rangle^2} \cdot s^{\alpha\delta} = \pm s^{\alpha\delta}. \end{aligned}$$

On the other hand, $s^{\gamma\beta} \| s^{\alpha\beta}$ while $s^{\gamma\delta} \perp s^{\alpha\delta}$ which is a contradiction.

We are left with Cases A and E. In fact, it readily follows from these that the *whole* set $\{s^{\beta\alpha} \mid \alpha \neq \beta\}$ either consists of parallel vectors or is orthonormal.

(Extended) Case A. All $s^{\beta\alpha}$, $\alpha \neq \beta$ are parallel. By (45), $E^{\beta\alpha}$ switches sign with $s^{\beta\alpha}$ so that we may assume that

$$s^{\beta\alpha} = s, \qquad \alpha \neq \beta,$$

where $s \in \mathbf{R}^m$ is a unit vector. In (48), we can write $E^\beta = E^{\beta\alpha}$ since it no longer depends on the second index. Again by (48), the set $\{E^\beta\}$ is orthonormal since

$$E^\beta = \sum_i s_i u_i^\beta = \sum_i s_i F(e_\beta, f_i) = F(e_\beta, s). \tag{51}$$

On the other hand, by (47), all $E^\beta = E^{\beta\alpha}$, $\beta \neq \alpha$, are contained in a 2-dimensional linear subspace so that $l \leq 3$. Note that $l = 1$ cannot happen because F is full. For $l = 2$, we have span im $F =$ span $\{u_i^1, u_i^2, E^1, E^2\}$ = span $\{u_i^1, E^2\}$, where we used (45) and (48). The latter is of dimension $\leq m+1$ and this again contradicts to fullness. Thus $l = 3$.

Setting $W = s^\perp \subset \mathbf{R}^m$, we now consider the restriction

$$F_0 = F | \mathbf{R}^3 \times W : \mathbf{R}^3 \times W \to \mathbf{R}^{m+2}.$$

By the very definition of the normal vector, for $\alpha \neq \beta$, we have

$$\langle F(e_\alpha, f_i), E^\beta \rangle = 0,$$

in particular,

$$\langle F(e_\alpha, X), E^\beta \rangle = 0, \qquad X \in W.$$

On the other hand, by (51), we have

$$\langle F(e_\beta, X), E^\beta \rangle = \langle F(e_\beta, X), F(e_\beta, s) \rangle = \langle X, s \rangle$$

and this vanishes iff $X \in W$. Hence the image of F_0 is contained in $Z = \{E^1, E^2, E^3\}^\perp \subset \mathbf{R}^{m+2}$. The latter is of dimension $m-1$ so that

$$F_0 : \mathbf{R}^3 \times W \to Z$$

is onto and $\dim W = \dim Z$ and Hurwitz-Radon classification applies to F_0. In particular, $m = 4r+1$ for some $r \geq 1$. Looking back to the way F_0 is derived from F, we see that F is equivalent to F_r induced by F_0 as defined in the introduction. Theorem 3 follows in this case.

(Extended) Case E. $\{s^{\beta\alpha} \mid \alpha \neq \beta\}$ is orthonormal. To complete the proof of Theorem 3, we show that this is impossible. Let α, β and γ be distinct and $a, b, c, d \in \mathbf{R}$ arbitrary. Using orthonormality, Lemma 12 and (50) (with $\sigma = \pm 1$), we compute

$$\begin{aligned}
(a^2+b^2)(c^2+d^2) &= |F(ae_\beta + be_\gamma, cs^{\beta\alpha} + ds^{\gamma\alpha}|^2 \\
&= |\sum_i (cs_i^{\beta\alpha} + ds_i^{\gamma\alpha})(au_i^\beta + bu_i^\gamma)|^2 \\
&= |\sum_{ij} (cs_i^{\beta\alpha} + ds_i^{\gamma\alpha})(ap_{ij}^{\beta\alpha} + bp_{ij}^{\gamma\alpha}) \\
&+ acE^{\beta\alpha} + bdE^{\gamma\alpha}|^2 \\
&= |(\pm ad \pm bc) \sum_j s_j^{\beta\gamma} u_j^\alpha + acE^{\beta\alpha} + bdE^{\gamma\alpha}|^2 \\
&= a^2d^2 + b^2c^2 \pm 2abcd + a^2c^2 + b^2d^2 + 2abcd\langle E^{\beta\alpha}, E^{\gamma\alpha}\rangle.
\end{aligned}$$

Thus, we have

$$\langle E^{\beta\alpha}, E^{\gamma\alpha}\rangle = \pm 1,$$

or equivalently,

$$E^{\beta\alpha} \| E^{\gamma\alpha}.$$

The formula for the change of bases then implies that the image of F is contained in the linear span of $\{u_i^\alpha, E^{\beta\alpha}\}$ *for fixed* $\alpha \neq \beta$. The dimension of this span is $m+1$ and this contradicts to fullness of F. The proof of Theorem 3 is complete.

Finally, the proof of Corollary 2 goes along the same lines as that of Corollary 1. (Note that, by Theorem 3, m is even.)

References

1. J.F.Adams, *Vector-fields on spheres*, Bull.Am.Math.Soc. **68** (1962)39-41.

2. J.F.Adams, *Vector-fields on spheres*, Ann. Math. **75** (1962)603-632.

3. J.Adem, *On the Hurwitz problem over an arbitrary field I*, Bol.Soc.Mat.Mexicana 25(1980)29-51.

4. J.Adem, *On the Hurwitz problem over an arbitrary field II*, Bol.Soc.Mat.Mexicana 26(1981)29-41.

5. J.Adem, *On admissible triples over an arbitrary field*, Bull.Soc.Math. de Belgique, 38(1986)33-35.

6. M.A.Berger and S.Friedland, *The generalized Radon-Hurwitz numbers*, Compositio Math. 59(1986)113-146.

7. T.Bier and U.Schwardman, *Räume normierte Bilinearformen und Cliffordstrukturen*, preprint, 1982.

8. J.Eells and J.H.Sampson, *Harmonic mappings of Riemannian manifolds*, Amer.J. Math. **86** (1964)109-160.

9. H.Hopf, *Ein topologischer Beitrag zur reellen Algebra*, Comment.Math.Helv. 13(1940/41)219-239.

10. H.Gauchman and G.Toth, *Constructions of harmonic polynomial maps between spheres* (in preparation).

11. A.Hurwitz, *Über die Komposition der quadratischen Formen von beliebig vielen Variabeln*, Nach.v. der Ges. der Wiss., Göttingen, Math. Phys. Kl. (1898)309-316. Reprinted in Math. Werke II, 565-571.

12. A.Hurwitz, *Über der Komposition der quadratischen Formen*, Math.Ann. **88** (1923)1-25.

13. K.Y.Lam and P.Yiu, *Sums of squares formulae near the Hurwitz-Radon range*, Contemporary Mathematics, 58, Part II (1987)51-56.

14. M.Parker, *Orthogonal multiplications in small dimensions*, Bull.London Math. Soc. **15** (1983)368-372.

15. J.Radon, *Lineare scharen orthogonale Matrizen*, Abh.Math.Sem. Univ.Hamburg 1(1922)1-14.

16. D.Shapiro, *Products of sums of squares*, Expo.Math. 2 (1984)235-261.

17. D.Shapiro, *On the Hurwitz problem over an arbitrary field*, Bol.Soc.Mat. Mexicana 29, No.1(1984)1-4.

18. ~~D.Shapiro, *Compositions of quadratic forms*~~ (in preparation)

19. E.Stiefel, *Über Richtungsfelder in den projektiven Räumen und einen Satz aus der reellen Algebra*, Comment.Math.Helv. 13(1941)201-218.

20. G.Toth, *Harmonic maps and minimal immersions through representation theory*, Academic Press, Boston, (1990).

21. S.Yuzvinsky, *On the Hopf condition over an arbitrary field*, Bol.Soc.Mat. Mexicana 28(1983)1-7.