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Constructions of Harmonic Polynomial Maps between Spheres

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Abstract. The complexity of λ_q-eigenmaps, i.e. homogeneous degree q harmonic polynomial maps $f\colon S^m \to S^n$, increases fast with the degree q and the source dimension m. Here we introduce a variety of methods of manufacturing new eigenmaps out of old ones. They include degree and source dimension raising operators. As a byproduct, we get estimates on the possible range dimensions of full eigenmaps and obtain a geometric insight of the harmonic product of λ_2-eigenmaps.

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1. Introduction and Preliminaries

It is well known that a spherical harmonic of order q, i.e. an eigenfunction of the Laplace–Beltrami operator Δ^{S^m} with eigenvalue S $\lambda_q = q(q+m-1)$, is the restriction (to S^m) of a homogeneous harmonic polynomial of degree q in $m+1$ variables. A map $f\colon S^m \to S_V$ into the unit sphere of a Euclidean vector space V is said to be a λ_q-*eigenmap* if all components of f are spherical harmonics of order q. A λ_q-eigenmap is a harmonic map with constant energy density $\lambda_q/2$ [1] and their 'classification' is a fundamental problem raised in [1]. Apart from the classical examples such as the Hopf map and the Veronese surfaces (or more generally, the standard minimal immersions), only a few explicit examples are known. The objective of this paper is to give various new constructions that give rise to a variety of new examples of eigenmaps between spheres. In Section 2 we define the degree raising operator that associates to a λ_q-eigenmap $f\colon S^m \to S^n$ a λ_{q+1}-eigenmap $f^+\colon S^m \to S^{(m+1)(n+1)-1}$. By iteration, this is then generalized to raising the degree by an arbitrary positive integer. In Section 3, we introduce the source dimension raising operator that associates to a λ_2-eigenmap $f\colon S^m \to S^n$ a λ_2-eigenmap $\tilde{f}\colon S^{m+1} \to S^{m+n+2}$. We obtain, as a corollary, ten new range dimensions for full λ_2-eigenmaps for $m \geq 5$. The main result here (for the lowest range dimension) asserts rigidity of the Hopf map among λ_2-eigenmaps $f\colon S^m \to$

S^2, $m \geq 2$. In Section 4 we define the harmonic product of two eigenmaps and show that this includes the degree raising operator discussed in Section 2. The main result of this section is a general existence theorem of the harmonic product for λ_2-eigenmaps. Finally, in Section 5, this result is used to give examples and nonexamples for the harmonic product.

2. Raising the Degree

Let H denote the harmonic projection operator [6]. H is the orthogonal projection from the vector space $\mathcal{P}^q$ of homogeneous polynomials in $m+1$ variables of degree q onto the linear subspace of harmonic polynomials of the same degree (cf. Vilenkin [6]).

Let $f\colon S^m \to S_V$ be a λ_q-eigenmap. We define

$$f^+\colon \mathbf{R}^{m+1} \to \mathbf{R}^{(m+1)(n+1)}$$

as follows. The components of f^+ are given in double indices $i = 0, \ldots, m$ and $j = 0, \ldots, n$ by

$$(f^+)_i^j = c_q^+ H(x_i f^j)$$

where

$$c_q^+ = \sqrt{\frac{2q+m-1}{q+m-1}}.$$

Our first result (proved in [5]) asserts that f^+ is a λ_{q+1}-eigenmap. Since the developments of the rest of this section depend on this claim, for completeness, we give here the details.

LEMMA 1. *f^+ maps the unit sphere to the unit sphere so that the restriction $f^+\colon S^m \to S^{(m+1)(n+1)-1}$ is a λ_{q+1}-eigenmap.*

Proof. By the harmonic projection formula [6] (or elementary computation), and harmonicity of the components f^j, we have

$$H(x_i f^j) = x_i f^j - \frac{\rho^2}{2q+m-1}\frac{\partial f^j}{\partial x_i},$$

where $\rho^2 = \Sigma_{i=0}^m x_i^2$. Homogeneity of f^j has two consequences. First, we have

$$\sum_{i=0}^m x_i \frac{\partial f^j}{\partial x_i} = q f^j.$$

Second, since f maps the unit sphere to the unit sphere, we have $\Sigma_{j=0}^n (f^j)^2 = \rho^{2q}$ as polynomials. Applying the Laplacian $\Delta = \Delta^{\mathbf{R}^{m+1}} = \Sigma_{i=0}^m \partial^2/\partial x_i^2$ to both sides and restricting to S^m, we obtain

$$\sum_{i=0}^m \sum_{j=0}^n \left(\frac{\partial f^j}{\partial x_i}\right)^2 = q(2q+m-1).$$

Thus

$$\begin{aligned}\sum_{i=0}^{m}\sum_{j=0}^{n} H(x_i f^j)^2 &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(x_i f^j - \frac{1}{2q+m-1}\frac{\partial f^j}{\partial x_i}\right)^2 \\ &= 1 - \frac{2}{2q+m-1}\sum_{j=0}^{n} f^j \sum_{i=0}^{m} x_i \frac{\partial f^j}{\partial x_i} + \\ &\quad + \frac{1}{(2q+m-1)^2}\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{\partial f^j}{\partial x_i}\right)^2 \\ &= 1 - \frac{2q}{2q+m-1} + \frac{q}{2q+m-1} \\ &= \frac{q+m-1}{2q+m-1}\end{aligned}$$

and the lemma follows.

A map between spheres is said to be *full* if the image is not contained in any proper great sphere. Restricting the range to the least great sphere it is contained in, a nonfull map can always be made full. Notice that f^+ is not full even if f is. In what follows we will denote a map and its full restriction by the same letter. Two maps $f_1\colon S^m \to S_{V_1}$ and $f_2\colon S^m \to S_{V_2}$ are said to be equivalent if there exists an isometry $A\colon V_1 \to V_2$ such that $f_2 = A \circ f_1$. After having f^+ made full it is clear that f_1 and f_2 equivalent implies that f_1^+ and f_2^+ are also equivalent. The converse, though much deeper, is also true and is proved in [5], namely, that the $+$ operation is injective on the set of equivalence classes of λ_q-eigenmaps.

We now generalize this to raising the degree by an arbitrary positive integer p by iteration. In what follows, we use standard multiindex notation, namely a multiindex $\alpha = (\alpha_0, \ldots, \alpha_m)$ always has nonnegative integer components, $|\alpha| = \Sigma_{i=0}^{m}\alpha_i$, $\alpha! = \prod_{i=0}^{m}\alpha_i$ and $x^\alpha = \prod_{i=0}^{m} x_i^{\alpha_i}$. Let $e^\alpha \in \mathcal{P}^p$, $|\alpha| = p$, be the homogeneous polynomial of degree p given by

$$e^\alpha(x) = \sqrt{\frac{p!}{\alpha!}}\, x^\alpha,$$

and define the scalar product in $\mathcal{P}^p$ such that $\{e^\alpha\}_{|\alpha|=p}$ is an orthonormal basis. Given a λ_q-eigenmap $f\colon S^m \to S^n$, we define

$$f^{+,p}\colon \mathbf{R}^{m+1} \to \mathcal{P}^p \otimes \mathbf{R}^{n+1}$$

by

$$(f^{+,p})^j_\alpha = c_{p,q} H(e^\alpha \cdot f^j), \quad |\alpha| = p,\ j = 0, \ldots, n,$$

where

$$c_{p,q} = \sqrt{\frac{(2q+m-1)(2q+m+1)\dots(2q+m+2p-3)}{(q+m-1)(q+m)\dots(q+m+p-2)}}.$$

THEOREM 1. *$f^{+,p}$ maps the unit sphere to the unit sphere so that it restricts to a λ_{p+q}-eigenmap $f^{+,p}\colon S^m \to S_{\mathcal{P}^p \otimes \mathbf{R}^{n+1}}$.*

Proof. We use induction with respect to p to show that

$$c_{p,q}^2 \sum_{|\alpha|=p} \sum_{j=0}^{n} H(e^\alpha f^j)^2 = \rho^{2(p+q)}. \tag{1}$$

For $p = 1$, this is the statement of the lemma above noting that $c_{1,q} = c_q^+$. Assuming that (1) is true for all λ_q-eigenmaps, we compute

$$\begin{aligned} c_{p+1,q}^2 \sum_{|\alpha|=p+1} \sum_{j=0}^{n} H(e^\alpha f^j)^2 &= c_{p+1,q}^2 \sum_{|\alpha|=p+1} \sum_{j=0}^{n} \times \\ &\quad \times \frac{(p+1)!}{\alpha_0! \dots \alpha_m!} H(x_0^{\alpha_0} \dots x_m^{\alpha_m} f^j)^2 \\ &= c_{p+1,q}^2 \sum_{i_0,\dots,i_p=0}^{m} \sum_{j=0}^{n} H(x_{i_0} \dots x_{i_m} f^j)^2 \\ &= c_{p+1,q}^2 \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{i_1,\dots,i_p=0}^{m} H(x_i H(x_{i_1} \dots x_{i_p} f^j))^2 \\ &= c_{p+1,q}^2 \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{|\alpha|=p} H(x_i H(e^\alpha f^j))^2. \end{aligned}$$

By the induction hypothesis, $c_{p,q} H(e^\alpha f^j)$ are components of a λ_{p+q}-eigenmap. Hence, by Lemma 1,

$$(c_{p+q}^+)^2 \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{|\alpha|=p} H(x_i c_{p,q} H(e^\alpha f^j))^2 = \rho^{2(p+q)}.$$

Combining this with the result above and noting that $c_{p+1,q} = c_{p,q} c_{p+q}^+$ the theorem follows.

3. Raising the Source Dimension

To manufacture new λ_q-eigenmaps out of old ones, the degree raising operation discussed in the previous section allows us to restrict ourselves to $q = 2$. As, for

$m = 3$, a complete classification of λ_2-eigenmaps is known (cf. [3]), we now describe an operation on λ_2-eigenmaps that raises the source dimension. In fact, given a λ_2-eigenmap $f\colon S^m \to S^n$, we define

$$\tilde{f}\colon \mathbf{R}^{m+2} \to \mathbf{R}^{n+m+3}$$

by

$$\tilde{f}(x) = \left(1 + \frac{1}{m(m+2)}\right)^{-1/2} \left(f(x), \sqrt{\frac{m+2}{m}}\left(x_{m+1}^2 - \frac{\rho^2}{m+2}\right),\right.$$
$$\left.\sqrt{2 + \frac{2}{m}}\, x_0 x_{m+1}, \ldots, \sqrt{2 + \frac{2}{m}}\, x_m x_{m+1}\right), \quad x = (x_0, \ldots, x_m).$$

PROPOSITION 1. *Given a λ_2-eigenmap $f\colon S^m \to S^n$, the induced map $\tilde{f}$ maps the unit sphere to the unit sphere so that it restricts to a λ_2-eigenmap $\tilde{f}\colon S^{m+1} \to S^{n+m+2}$. Moreover, f full implies that $\tilde{f}$ is full.*

Proof. Simple computation.

The maximum range dimension for a full λ_2-eigenmap $f\colon S^m \to S^n$ is

$$\frac{m(m+3)}{2} - 1$$

that is the multiplicity of the eigenvalue λ_2 minus one. For $m = 3$, the possible range dimensions of full λ_2-eigenmaps $f\colon S^3 \to S^n$ are $n = 2, 4, 5, 6, 7, 8$. Combining these with Proposition 1, we obtain the following:

COROLLARY 1. *For $m \geq 3$, there exist full λ_2-eigenmaps*

$$f\colon S^m \to S^{(m(m+3)/2)-r},$$

where

$$r = 1, 2, 3, 4, 5, 7.$$

For example, it follows that, for $m = 4$, full λ_2-eigenmaps $f\colon S^4 \to S^n$ exist for $n = 7, 9, 10, 11, 12, 13$. Moreover, $n = 4$ can be added to this list since the gradient of an isoparametric function gives a full λ_2-eigenmap. Note also that, by a result of R. Wood (cf. [7]), there is no full polynomial map $f\colon S^4 \to S^3$ so that $n = 3$ does not arise as a range dimension.

Remark. Precomposing these with the Hopf map $h\colon S^7 \to S^4$, we obtain λ_4-eigenmaps $f\colon S^7 \to S^n$, where $n = 4, 7, 9, 10, 11, 12, 13$.

Let $F\colon \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$ be an orthogonal multiplication, i.e. F is bilinear and

$|F(x, y)| = |x||y|$, $x, y \in \mathbf{R}^m$. The Hopf–Whitehead construction associates to F the λ_2-eigenmap

$$f_F: S^{2m-1} \to S^n$$

defined by

$$f_F(x, y) = (|x|^2 - |y|^2, 2F(x, y)).$$

Clearly, f_F is full iff F is onto. Note that, leaving harmonicity, a general result of R. Wood in [7] asserts that any quadratic polynomial map f between spheres is homotopic to an f_F associated to an orthogonal multiplication $F: \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$. Returning to the harmonic case ($l = m$) above, for $m = 2$, the possible range dimensions are $n = 2$ and 4 corresponding to the complex multiplication and the real tensor product (cf. [4] for further details). By [2], for $m = 3$, the possible range dimensions are $n = 4, 7, 8$ and 9 (the first corresponding to quaternionic multiplication). Combining these range dimensions with the ones obtained from Corollary 1 (for $m = 5$), it follows that there exist full λ_2-eigenmaps $f: S^5 \to S^n$ for $n = 4, 7, 8, 9, 13, 15, 16, 17, 18, 19$. For example, in contrast to the nonexistence of full λ_2-eigenmaps $f: S^4 \to S^3$, the lowest range dimension here gives a full λ_2-eigenmap $f: S^5 \to S^4$. In fact, restricting the quaternionic multiplication to a pair of three-dimensional subspaces gives a full orthogonal multiplication $F: \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^4$ and the Hopf–Whitehead construction provides a full λ_2-eigenmap $f_F: S^5 \to S^4$. Raising the source dimension as above, we obtain the following:

COROLLARY 2. *For $m \geq 5$, there exist full λ_2-eigenmaps*

$$f: S^m \to S^{(m(m+3)/2)-r},$$

where

$$r = 1, 2, 3, 4, 5, 7, 11, 12, 13, 16.$$

Finally, note that there is no orthogonal multiplication $F: \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ so that the Hopf–Whitehead construction does not give any λ_2-eigenmaps $f: S^5 \to S^3$.

As for the minimum range dimension, we have the following rigidity result:

THEOREM 2. *Let $f: S^m \to S^2$ be a full λ_2-eigenmap. Then $m = 3$ and, up to isometries on the source and the range, f is the Hopf map.*

Remarks. 1. Hence the only unsettled range dimensions of a full λ_2-eigenmap $f: S^4 \to S^n$ are $n = 5$, 6 and 8.

2. For any degree q, we can define

$$N(q) = \min\{n \mid \text{there exists a full } \lambda_q\text{-eigenmap } f: S^m \to S^n \text{ for some } m \geq 2\}$$

Clearly, $N(q) \leq 2q$, for any q (because of the Veronese map $f\colon S^2 \to S^{2q}$) but, for q even, $N(q) \leq q$ as composition of the Hopf map and the Veronese map shows.

PROOF OF THEOREM 2. Let $f\colon S^m \to S^n$ be a λ_2-eigenmap. Using coordinates, we write

$$f(x) = \sum_{i=0}^{m} a_i x_i^2 + \sum_{0 \leq i < j \leq m} a_{ij} x_i x_j,$$

where $a_i,\ a_{ij} \in \mathbf{R}^{n+1}$, $i = 0, \ldots, m$ and $0 \leq i < j \leq m$. To simplify the notation, we set $a_{ij} = a_{ji}$, so that a_{ij} is defined for all distinct indices $0 \leq i,\ j \leq m$. Harmonicity is equivalent to

$$\sum_{i=0}^{m} a_i = 0. \tag{2}$$

Writing out the condition that f maps the unit sphere into the unit sphere, we obtain the following:

$$| a_i |= 1, \tag{3}$$

$$\langle a_i,\ a_{ij} \rangle = 0, \quad i,\ j \text{ distinct}, \tag{4}$$

$$| a_{ij} |^2 + 2\langle a_i,\ a_j \rangle = 2, \quad i,\ j \text{ distinct}, \tag{5}$$

$$\langle a_i,\ a_{jk} \rangle + \langle a_{ij},\ a_{ik} \rangle = 0, \quad i,\ j,\ k \text{ distinct}, \tag{6}$$

$$\langle a_{ij},\ a_{kl} \rangle + \langle a_{ik},\ a_{jl} \rangle + \langle a_{il},\ a_{jk} \rangle = 0, \quad i,\ j,\ k,\ l \text{ distinct}. \tag{7}$$

We say that a system of vectors $\{a_i,\ a_{ij}\} \subset \mathbf{R}^{n+1}$ is *feasible* if it satisfies (2)–(7). A feasible system is *generic* if $a_i \neq \pm a_j$ for all $i,\ j$ distinct. We first show that by precomposing f with a suitable isometry on the source, the associated feasible system of vectors can be made generic (provided that $m \geq 2$).

Let $0 \leq r < s \leq m$ and consider the rotation in the $x_r x_s$-plane by angle ϕ. Denoting the new coordinates by the superscript ϕ, we obtain

$$x_r^\phi = x_r \cos\phi + x_s \sin\phi,$$

$$x_s^\phi = -x_r \sin\phi + x_s \cos\phi,$$

and the rest of the coordinates are unchanged. For the new system of vectors $\{a_i^\phi\}$ we have

$$a_r^\phi = a_r \cos^2\phi + a_s \sin^2\phi - a_{rs} \sin\phi \cos\phi,$$

$$a_s^\phi = a_r \sin^2\phi + a_s \cos^2\phi + a_{rs} \sin\phi \cos\phi,$$

with the rest of the vectors unchanged. Clearly, $a_r + a_s = a_r + a_s$. Since the a_i's are unit vectors, we have the following three cases:

Case 1: a_r and a_s are linearly independent. By (3)–(5), a_r, a_s and a_{rs} span a three-dimensional subspace in which we have rotation around the axis $a_r + a_s$ by angle 2ϕ.

Case 2: $a_r + a_s$. Everything stays fixed.

Case 3: $a_r = -a_s$. By (5), $| a_{rs} | = 2$ and so a_r and $a_{rs}/2$ is an orthonormal basis in the plane they span. The opposite pair of vectors a_r^ϕ and a_s^ϕ is obtained from a_r and a_s by rotation with angle 2ϕ in this plane.

We now prove the claim about making a system of feasible vectors generic. Given a feasible system of vectors, assume that $a_i = a_j$, for some $i \neq j$. By (2), there exists a_k that is different from these. We can now rotate a_i and a_k corresponding to Case 1 or Case 3 to get three vectors such that each two are linearly independent. In this way we can decrease the number of identical vectors without increasing the number of opposite pairs. Since the number of vectors is finite we arrive at a feasible system of vectors in which the vectors a_i are all distinct. Assume now that $a_i = -a_j$. Since $m \geq 2$ there exists a_k that is linearly independent from these. We now rotate a_i and a_k as in Case 1 to get all three linearly independent without creating new identical pairs. After finitely many steps, we arrive at a generic system.

We now assume that $n = 2$ and denote the vector cross product in $\mathbf{R}^3$ by $\times$. By (3)–(5), we have

$$a_{ij} = \epsilon_{ij} \sqrt{\frac{2}{1 + \langle a_i, a_j \rangle}} \, a_i \times a_j, \quad i \neq j, \tag{8}$$

where $\epsilon_{ij} = \pm 1$ is antisymmetric in ij. We now claim that any three distinct vectors a_i, a_j and a_k span $\mathbf{R}^3$. This is clear, since otherwise a_{ij}, a_{jk} and a_{ki} were parallel, contradicting (5) and (6).

We claim that, for any distinct indices i, j and k, we have

$$\langle a_i, a_j \rangle + \langle a_j, a_k \rangle + \langle a_k, a_i \rangle = -1. \tag{9}$$

Letting

$$\mu_{ij} = \langle a_i, a_j \rangle,$$

we substitute (8) into (6) and obtain

$$\epsilon_{jk} \sqrt{\frac{2}{1 + \mu_{jk}}} \langle a_i, a_j \times a_k \rangle + \\ + \epsilon_{ij}\epsilon_{ik} \frac{2}{\sqrt{(1 + \mu_{ij})(1 + \mu_{ik})}} \langle a_i \times a_j, a_i \times a_k \rangle = 0.$$

On the other hand, we have

$$\begin{aligned}\langle a_i \times a_j,\, a_i \times a_k\rangle &= \langle a_j,\, a_k\rangle - \langle a_i,\, a_k\rangle\langle a_i,\, a_j\rangle \\ &= \mu_{jk} - \mu_{ij}\mu_{ik}\end{aligned}$$

so that

$$\langle a_i,\, a_j \times a_k\rangle = \epsilon_{ij}\epsilon_{jk}\epsilon_{ki}\sqrt{2}\sqrt{\frac{1+\mu_{jk}}{(1+\mu_{ij})(1+\mu_{ik})}}\,(\mu_{jk} - \mu_{ij}\mu_{ik}), \tag{10}$$

where we used antisymmetry of the ϵ's. The left-hand side is the (signed) volume of the parallelepiped spanned by a_i, a_j and a_k. We now take a cyclic permutation of i, j and k and note that $\epsilon_{ij}\epsilon_{jk}\epsilon_{ki}$ does not change. We obtain

$$(1+\mu_{jk})(\mu_{jk} - \mu_{ij}\mu_{ki}) = (1+\mu_{ki})(\mu_{ki} - \mu_{ij}\mu_{jk}),$$

or equivalently

$$(1+\mu_{ij}+\mu_{jk})\mu_{jk} = (1+\mu_{ij}+\mu_{ki})\mu_{ki}. \tag{11}$$

Taking a cyclic permutation of i, j and k and subtracting it from (11), we arrive at

$$(1+\mu_{ij}+\mu_{jk}+\mu_{ki})(\mu_{ij} - \mu_{ki}) = 0.$$

Thus either (9) holds or $\mu_{ij} = \mu_{ik}$. Taking a cyclic permutation again and noting that (9) remains invariant, it remains to study the case when

$$\langle a_i,\, a_j\rangle = \langle a_j,\, a_k\rangle = \langle a_k,\, a_i\rangle. \tag{12}$$

To finish the proof of the claim, we now show that this implies $\langle a_i,\, a_j\rangle = -1/3$. For this, we compute the volume of the parallelepiped spanned by a_i, a_j and a_k. We obtain that the volume is $(1-\mu)^2(1+2\mu)$, where μ is the common value of (12). Substituting this into (10), we arrive at $1+3\mu = 0$ and (9) follows.

Finally, we add a new vector a_l to a_i, a_j and a_k. (Note that a_l exists since $m \geq 3$.) By the above, (9) is valid with i, j or k replaced by l. Using these four equations, we obtain that

$$\langle a_i + a_j + a_k + a_l,\, a_i + a_j\rangle = 0.$$

Similarly

$$\langle a_i + a_j + a_k + a_l,\, a_k + a_l\rangle = 0.$$

Adding these, we obtain

$$a_i + a_j + a_k + a_l = 0.$$

Thus, a_l is uniquely determined by a_i, a_j and a_k and using that the system is generic we get $m = 4$. By [3], any λ_2-eigenmap $f\colon S^3 \to S^2$ is, up to isometries on the source and the range, the Hopf map and the proof is complete.

Remark. Given an orthogonal multiplication F: $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$, setting $x = (x_1, \ldots, x_m)$ and $y = (x_{m+1}, \ldots, x_{2m})$, we can write

$$F(x, y) = \sum_{i=0}^{m} \sum_{j=m+1}^{2m} a_{ij} x_i x_j.$$

The vectors $a_{ij} \in \mathbf{R}^n$ satisfy the relations

$$\begin{aligned} |\, a_{ij} \,| &= 2, \\ \langle a_{ij}, a_{ik} \rangle &= 0, \quad i, j, k \text{ distinct}, \end{aligned}$$

and (7), where we assume that $a_{ij} = a_{ji}$ and that a_{ij} is zero if $1 \leq i, j \leq m$ or $m + 1 \leq i, j \leq 2m$. Hence f_F corresponds to the feasible system of vectors $\{a_i, a_{ij}\} \subset \mathbf{R}^{n+1}$, where $a_1 = \cdots = a_m = -a_{m+1} = \cdots = -a_{2m}$ being orthogonal to all a_{ij}'s.

4. The Harmonic Product

We first reformulate Theorem 1 in a more convenient setting. Let

$$\bar{f}_{\lambda_p}: \mathbf{R}^{m+1} \to \mathcal{P}^p$$

be the harmonic polynomial map given by

$$\bar{f}_{\lambda_p}(x) = c_{p,0} \sum_{|\alpha|=p} H(e^\alpha)(x) e^\alpha,$$

where the constant is given before Theorem 1. The proof of Theorem 1 shows that $\bar{f}_{\lambda_p}$ maps the unit sphere into the unit sphere so that it restricts to a λ_p-eigenmap $\bar{f}_{\lambda_p}$: $S^m \to S_{\mathcal{P}^p}$.

Let $\mathcal{H}^p$ denote the space of spherical harmonics of order p on S^m. We endow $\mathcal{H}^p$ with the normalized scalar product

$$\langle g, g' \rangle = \frac{n(\lambda_p) + 1}{\mathrm{vol}(S^m)} \int_{S^m} g g' v_{S^m},$$

where

$$n(\lambda_p) + 1 = \dim \mathcal{H}^p$$

and v_{S^m} is the volume form on S^m with total volume $\mathrm{vol}(S^m)$. Given an orthonormal basis $\{f^j_{\lambda_p}\} \subset \mathcal{H}^p$, we define the standard minimal immersion [4]

$$f_{\lambda_p}: S^m \to S^{n(\lambda_p)}$$

by

$$f_{\lambda_p}(x) = \sum_{j=0}^{n(\lambda_p)} f^j_{\lambda_p}(x) f^j_{\lambda_p}, \quad x \in S^m.$$

THEOREM 3. *$\bar{f}_{\lambda_p}$, when made full, is equivalent to the standard minimal immersion f_{λ_p}.*

Proof. This is a simple consequence of the fact that an equivariant eigenmap is standard. Equivariance of $\bar{f}_{\lambda_p}$ can be seen easily since the harmonic projection operator commutes with the actions of SO$(m+1)$ on $\mathcal{P}^p$ and on $\mathcal{H}^p$ given by precomposition with the inverse. This latter is because H is built from the powers of the Laplace–Beltrami operator.

We now introduce a law of composition for eigenmaps. Let $f\colon S^m \to S_V$ be a λ_p-eigenmap and $g\colon S^m \to S_W$ a λ_q-eigenmap. Their tensor product $f \otimes g\colon S^m \times S^m \to S_{V\otimes W}$ is a λ_{p+q}-eigenmap. (For the general properties of the tensor product of eigenmaps, cf. [4].) Restricting to the diagonal $S^m \subset S^m \times S^m$ is, in general, not an eigenmap and so, instead, we think of $f \otimes g$ as a harmonic polynomial map of $\mathbf{R}^{m+1} \times \mathbf{R}^{m+1}$ to $V \otimes W$ and restrict to the diagonal $\mathbf{R}^{m+1} \subset \mathbf{R}^{m+1} \times \mathbf{R}^{m+1}$ to get a polynomial map $(f \otimes g)\,|_{\mathbf{R}^{m+1}}\colon \mathbf{R}^{m+1} \to V \otimes W$. Finally we take the harmonic part of each component with respect to an(y) orthonormal basis to get a harmonic polynomial map $H((f \otimes g)\,|_{\mathbf{R}^{m+1}})\colon \mathbf{R}^{m+1} \to V \otimes W$. If it maps the unit sphere into the unit sphere (up to a constant multiple) then its restriction is called the *harmonic product* of f and g denoted by $f \lozenge g\colon S^m \to S_{V\otimes W}$. We now show that Theorem 1 can be reinterpreted in terms of the harmonic product as follows:

THEOREM 4. *For any λ_q-eigenmap $f\colon S^m \to S^n$ the harmonic product $f_{\lambda_p} \lozenge f$ exists.*

For the proof, we need the following:

LEMMA 2. *For $g \in \mathcal{H}^{p+1}$ and $g' \in \mathcal{H}^p$, we have*

$$\left\langle \frac{\partial g}{\partial x_i}, g' \right\rangle = \mu_p \langle g, H(x_i g')\rangle_{p+1},$$

where

$$\mu_p = (p+1)\frac{2p+m-1}{p+m-1}.$$

Proof. Simple computation (cf. also [5]).

PROOF OF THEOREM 4. Choosing an orthonormal basis in $\mathcal{H}^p$ for the components of the standard minimal immersion $f_{\lambda_p}\colon S^m \to S^{n(\lambda_p)}$ and using Lemma 2, we compute

$$\sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n} H(f^j_{\lambda_p} f^l) = \frac{1}{p!} \sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n} H\left(\sum_{i_1,\ldots,i_p=0}^{m} \frac{\partial^p f^j_{\lambda_p}}{\partial x_{i_1} \ldots \partial x_{i_p}} x_{i_1} \ldots x_{i_p} f^l \right) \times$$
$$\times H(f^j_{\lambda_p} f^l)$$
$$= \frac{1}{p!} \sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} H(x^\alpha f^l) \times$$
$$\times H\left(\frac{\partial^p f^j_{\lambda_p}}{\partial x_0^{\alpha_0} \ldots \partial x_m^{\alpha_m}} f^j_{\lambda_p} f^l \right)$$
$$= \frac{\mu_0 \ldots \mu_{p-1}}{p!} \sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} \times$$
$$\times H(x^\alpha f^l) H(\langle f^j_{\lambda_p}, H(x^\alpha) \rangle f^j_{\lambda_p} f^l)$$
$$= c^2_{p,0} \sum_{l=0}^{n} \sum_{|\alpha|=p} \frac{p!}{\alpha!} H(x^\alpha f^l)^2$$
$$= c^2_{p,0} \sum_{l=0}^{n} \sum_{|\alpha|=p} H(e^\alpha f^l)^2.$$

The proof of Theorem 1, however, shows that this is equal to

$$\frac{c^2_{p,0}}{c^2_{p,q}} \rho^{2(p+q)}$$

which completes the proof.

The harmonic product is clearly commutative and associative, i.e. with obvious notations, we have

$$f \lozenge (g \lozenge h) = (f \lozenge g) \lozenge h,$$

provided that $f \lozenge g$, $g \lozenge h$ and either side exist. In particular, by Theorem 4,

$$(f \lozenge g)^+ = f^+ \lozenge g = f \lozenge g^+,$$

where the equalities mean equivalence. Therefore to study the existence of the harmonic product we may assume that the factors are λ_2-eigenmaps.

The rest of this section is devoted to the proof of a useful criterion for the existence of the harmonic product of λ_2-eigenmaps. Let $f\colon S^m \to S^u$ and $g\colon S^m \to$

S^v be λ_2-eigenmaps with components $(f^j)_{j=0}^u$ and $(g^l)_{l=0}^v$. In what follows, the indices i, k, r, s will run on $0, \ldots, m$ while j (resp. l) will take values on $0, \ldots, u$ (resp. $0, \ldots, v$). We now set

$$f_{ik} = \sum_{j=0}^{u} f^j \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} \quad \text{and} \quad g_{ik} = \sum_{l=0}^{v} g^l \frac{\partial^2 g^l}{\partial x_i\, \partial x_k}.$$

(Note that the partial derivatives are actually constants.)

THEOREM 5. *Given λ_2-eigenmaps $f\colon S^m \to S^u$ and $g\colon S^m \to S^v$, the harmonic product $f \Diamond g$ exists iff*

$$\sum_{i,k=0}^{m} f_{ik} g_{ik} = c \cdot \rho^4, \quad c = \text{constant}. \tag{13}$$

Proof. The harmonic projection of a degree 4 polynomial $\phi \in \mathcal{P}^4$ is given by

$$H(\phi) = \phi - \frac{\rho^2}{2(m+5)} \Delta\phi + \frac{\rho^4}{8(m+3)(m+5)} \Delta^2\phi$$

(cf. Vilenkin [6]). In particular, we have

$$\begin{aligned} H(f^j g^l) &= f^j g^l - \frac{\rho^2}{2(m+5)} \sum_i \frac{\partial f^j}{\partial x_i} \frac{\partial g^l}{\partial x_i} + \\ &\quad + \frac{\rho^4}{2(m+3)(m+5)} \sum_{i,k} \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} \frac{\partial^2 g^l}{\partial x_i\, \partial x_k}. \end{aligned}$$

Up to a constant multiple, this is the jl-component of the harmonic product. We claim that the norm square of the corresponding vector is

$$\begin{aligned} \sum_{j,l} H(f^j g^l)^2 &= \left(\frac{m+1}{m+5} + c(f, g)\right) \rho^8 + \frac{\rho^4}{2(m+3)(m+5)^2} \times \\ &\quad \times \left(4(m+4) \sum_{i,k} f_{ik} g_{ik} - \rho^2 \Delta \left(\sum_{i,k} f_{ik} g_{ik}\right)\right) \end{aligned} \tag{14}$$

where the constant $c(f, g)$ is given by

$$c(f, g) = \frac{1}{4(m+3)^2(m+5)^2} \sum_{j,l} \left(\sum_{i,k} \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} \frac{\partial^2 g^l}{\partial x_i\, \partial x_k}\right)^2. \tag{15}$$

To show this, we first decompose

$$\sum_{j,l} H(f^j g^l)^2 = A_1 + A_2 + A_3 + B_{12} + B_{13} + B_{23},$$

where

$$
\begin{aligned}
A_1 &= \sum_{j,l} (f^j g^l)^2, \\
A_2 &= \frac{\rho^4}{(m+5)^2} \sum_{j,l} \left(\sum_i \frac{\partial f^j}{\partial x_i} \frac{g^l}{\partial x_i} \right)^2, \\
A_3 &= \frac{\rho^8}{4(m+3)^2(m+5)^2} \sum_{j,l} \left(\sum_{i,k} \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} \frac{\partial^2 g^l}{\partial x_i\, \partial x_k} \right)^2, \\
B_{12} &= -\frac{2\rho^2}{m+5} \sum_{i,j,l} f^j g^l \frac{\partial f^j}{\partial x_i} \frac{\partial g^l}{\partial x_i}, \\
B_{13} &= \frac{\rho^4}{(m+3)(m+5)} \sum_{i,j,k,l} f^j g^l \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} \frac{\partial^2 g^l}{\partial x_i\, \partial x_k}, \\
B_{23} &= -\frac{\rho^6}{(m+3)(m+5)} \sum_{i,j,l,r,s} \frac{\partial f^j}{\partial x_i} \frac{\partial g^l}{\partial x_i} \frac{\partial^2 f^j}{\partial x_r\, \partial x_s} \frac{\partial^2 g^l}{\partial x_r\, \partial x_s}.
\end{aligned}
$$

We now simplify each term in a tedious but straightforward manner. It is clear that $A_1 = \rho^8$ since f and g are both λ_2-eigenmaps between spheres. As for A_2, we compute

$$
\begin{aligned}
\sum_{j,l} \left(\sum_i \frac{\partial f^j}{\partial x_i} \frac{\partial g^l}{\partial x_i} \right)^2 &= \sum_{i,k} \left(\sum_j \frac{\partial f^j}{\partial x_i} \frac{\partial f^j}{\partial x_k} \right) \left(\sum_l \frac{\partial g^l}{\partial x_i} \frac{\partial g^l}{\partial x_k} \right) \\
&= \sum_{i,k} \left(\frac{1}{2} \frac{\partial^2 \rho^4}{\partial x_i\, \partial x_k} - f_{ik} \right) \left(\frac{1}{2} \frac{\partial \rho^4}{\partial x_i\, \partial x_k} - g_{ik} \right).
\end{aligned}
$$

We now make use of the fact that

$$
\frac{\partial^2 \rho^4}{\partial x_i\, \partial x_k} = 4\delta_{ik}\rho^2 + 8x_i x_k
$$

along with (harmonicity and) homogeneity of the components of f and g:

$$
\sum_{i,k} x_i x_k \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} = 2f^j \quad \text{and} \quad \sum_{i,k} x_i x_k \frac{\partial^2 g^l}{\partial x_i\, \partial x_k} = 2g^l
$$

to arrive at

$$
A_2 = \frac{4\rho^8}{m+5} + \frac{\rho^4}{(m+5)^2} \sum_{i,k} f_{ik} g_{ik}. \tag{16}
$$

We now notice that A_3 is a constant multiple of ρ^8; in fact

$$A_3 = c(f,\, g)\rho^8, \tag{17}$$

where $c(f,\, g)$ is given in (14). Turning to the mixed terms, we compute

$$\begin{aligned} B_{12} &= -\frac{2\rho^2}{m+5}\sum_i\left(\sum_j f^j\frac{\partial f^j}{\partial x_i}\right)\left(\sum_l g^l\frac{\partial g^l}{\partial x_i}\right)\\ &= -\frac{\rho^2}{2(m+5)}\sum_i\left(\frac{\partial\rho^4}{\partial x_i}\right)^2\\ &= -\frac{8\rho^8}{m+5}. \end{aligned} \tag{18}$$

By the very definition of f_{ik} and g_{ik}, we have

$$B_{13} = \frac{\rho^4}{(m+3)(m+5)}\sum_{i,k} f_{ik}g_{ik}. \tag{19}$$

Finally, we have

$$B_{23} = -\frac{\rho^6}{2(m+3)(m+5)^2}\,\Delta\left(\sum_{i,k} f_{ik}g_{ik}\right) \tag{20}$$

and this can be seen by working out the right-hand side (and using harmonicity of f^j and g^l once again). Putting (16)–(20) together, (14) follows.

We now turn to the proof of the theorem. First, assuming (13), we have

$$\Delta\left(\sum_{i,k} f_{ik}g_{ik}\right) = 4c(m+3)\rho^2$$

so that

$$\sum_{j,l} H(f^jg^l)^2 = \left(\frac{m+1}{m+5} + \frac{2c}{(m+3)(m+5)^2} + c(f,\, g)\right)\rho^8. \tag{21}$$

Conversely, assume that the harmonic product exists, i.e. $\Sigma_{j,l}H(f^jg^l)^2$ is a constant multiple of ρ^8. By (14), this means that $\mu = \Sigma_{i,k}f_{ik}g_{ik}$ satisfies the equation

$$4(m+4)\mu - \rho^2\,\Delta\mu = c'\rho^4, \quad c' = \text{constant}. \tag{22}$$

Taking Δ of both sides and using homogeneity of $\Delta\mu$, we obtain that

$$\Delta\mu = c''\rho^2,$$

where

$$c'' = \frac{\Delta^2\mu + 4(m+3)c'}{m+3}.$$

Finally, substituting this back to (22), we get

$$\mu = \frac{c' + c''}{4(m+4)}\rho^4$$

and the proof is complete.

The normalizing constant for the harmonic product can be determined explicitly in terms of c in (13) by the following:

COROLLARY 3. *Assume that* (13) *holds. Then*

$$\sum_{j,l} H(f^j g^l)^2 = \left(\frac{m+1}{m+5} + \frac{c}{2(m+3)(m+5)}\right)\rho^8. \tag{23}$$

Proof. Taking Δ^2 of both sides of (13), we obtain

$$\sum_{i,k,r,s} \frac{\partial^2 f_{ik}}{\partial x_r\,\partial x_s}\,\frac{\partial^2 g_{ik}}{\partial x_r\,\partial x_s} = 2c(m+1)(m+3).$$

Using this, we compute

$$\begin{aligned}\sum_{j,l}\left(\sum_{i,k}\frac{\partial^2 f^j}{\partial x_i\,\partial x_k}\,\frac{\partial^2 g^l}{\partial x_i\,\partial x_k}\right)^2 &= \sum_{i,j,k,l,r,s}\frac{\partial^2 f^j}{\partial x_i\,\partial x_k}\,\frac{\partial^2 g^l}{\partial x_i\,\partial x_k}\,\frac{\partial^2 f^j}{\partial x_r\,\partial x_s}\,\frac{\partial^2 g^l}{\partial x_r\,\partial x_s}\\ &= \sum_{i,k,r,s}\frac{\partial^2 f_{ik}}{\partial x_r\,\partial x_s}\,\frac{\partial^2 g_{ik}}{\partial x_r\,\partial x_s}\\ &= 2c(m+1)(m+3).\end{aligned}$$

Comparing this with the formula for A_3, we obtain

$$c(f,\, g) = \frac{(m+1)c}{2(m+3)(m+5)^2}.$$

Substituting this into (21) we arrive at (23).

We now specialize to the case when $f = g$ that is we turn to the existence of the harmonic square $f \diamond f$. Let $f\colon S^m \to S^n$ be a full λ_2-eigenmap and assume that the harmonic square $f \diamond f$ exists. By Theorem 5, we then have $\Sigma_{i,k} f_{ik}^2 = c\rho^2$ so that (by restriction)

$$\hat{f}\colon S^m \to S^{m(m+2)}$$

defined by

$$\hat{f} = \frac{1}{\sqrt{c}} (f_{ik})_{i,k=0}^{m}$$

is clearly a λ_2-eigenmap.

THEOREM 6. *$\hat{f}$, when made full, is equivalent to f.*

Before the proof we need the following:

LEMMA 3. $\operatorname{span}\{f_{ik}\}_{i,k=0}^{m} = \operatorname{span}\{f^j\}_{j=0}^{n}$.

Proof. Since the f_{ik} are linear combinations of f^j, we have $\operatorname{span}\{f_{ik}\} \subset \operatorname{span}\{f^j\}$. Assuming that the inclusion is proper, let α: $\operatorname{span}\{f^j\} \to \mathbf{R}$ be a nonzero linear functional that vanishes on $\operatorname{span}\{f_{ik}\}$. Setting $c_j = \alpha(f^j)$, we have

$$0 = \alpha(f_{ik}) = \alpha\left(\sum_j f^j \frac{\partial^2 f^j}{\partial x_i\, \partial x_k}\right) = \sum_j c_j \frac{\partial^2 f^j}{\partial x_i\, \partial x_k}.$$

Hence

$$\sum_{i,j,k} c_j \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} x_i x_k = 2 \sum_j c_j f^j = 0$$

that is a contradiction since f is full and not all the c_j are zero.

PROOF OF THEOREM 5. Let

$$A\colon \mathbf{R}^{n+1} \to \mathbf{R}^{(m+1)^2}$$

be the linear map defined by

$$A(y) = \frac{1}{\sqrt{c}} \left(\sum_j \frac{\partial^2 f^j}{\partial x_i\, \partial x_k} y^j\right)_{i,k=0}^{m}, \quad y = (y^j)_{j=0}^{n} \in \mathbf{R}^{n+1}.$$

Then, we have $\hat{f} = A \circ f$. By Lemma 3, the image V of A is of dimension $n+1$. Hence A is a linear isomorphism between $\mathbf{R}^{n+1}$ and V and, by assumption on the existence of the harmonic square, it maps the unit sphere into the unit sphere. Thus, A is an isometry and the proof is complete.

5. Existence and Nonexistence of the Harmonic Product

5.1. THE GENERALIZED HOPF MAP

Using complex coordinates $z_0, \ldots, z_m \in \mathbf{C}^{m+1}$, the generalized Hopf map

$$h\colon S^{2m+1} \to S^{(3m(m+1)/2)-1}$$

is defined by

$$h(z_0, \ldots, z_m) = \left(\frac{1}{\sqrt{m}} (| z_i |^2 - | z_k |^2), \sqrt{\frac{2(m+1)}{m}} \, \Re(z_i \bar{z}_k), \right.$$

$$\left. \sqrt{\frac{2(m+1)}{m}} \, \Im(z_i \bar{z}_k) \right)_{0 \le i < k \le m} .$$

h is clearly a λ_2-eigenmap. For future reference, we note that, setting $z_i = x_i + \sqrt{-1}\, x_{\bar{i}}$, $i = 0, \ldots, m$ and $\bar{i} = m+1, \ldots, 2m+1$, we have

$$h_{ii} = h_{\bar{i}\bar{i}} = \frac{2(m+1)}{m} (x_i^2 + x_{\bar{i}}^2) - \frac{2}{m+3} \rho^2, \tag{24}$$

$$h_{ik} = h_{\bar{i}\bar{k}} = \frac{2(m+1)}{m} (x_i x_k + x_{\bar{i}} x_{\bar{k}}), \quad i \neq k, \tag{25}$$

$$h_{i\bar{k}} = -h_{\bar{i}k} = \frac{2(m+1)}{m} (x_i x_{\bar{k}} - x_k x_{\bar{i}}). \tag{26}$$

Let $f\colon S^{2m+1} \to S^n$ be a λ_2-eigenmap. We now recall that a real spherical harmonic ϕ of order 2 on S^{2m+1} (such as f^j), written as a harmonic polynomial in the complex variables z_i and $\bar{z}_i$ decomposes as

$$\phi = \phi_{||} + \phi_{\times},$$

where the 'pure part' $\phi_{||}$ is the sum of those monomials in ϕ that are of degree 2 in z_i or degree 2 in $\bar{z}_i$ and the 'mixed part' $\phi_{\times}$ contains those monomials in ϕ that are of degree 1 in z_i and degree 1 in $\bar{z}_i$. (Note that $\phi_{||}$ (resp. $\phi_{\times}$) are often refered as the $(2,0) + (0,2)$-part (resp. the $(1,1)$-part) of ϕ.) Clearly, $\phi_{||}$ and $\phi_{\times}$ are both harmonic.

We now view f as a vector-valued function whose components are spherical harmonics of order 2 and decompose accordingly:

$$f = f_{||} + f_{\times}$$

into pure and mixed parts, where $f_{||}$, $f_{\times}\colon S^{2m+1} \to \mathbf{R}^{n+1}$ are vector-valued functions. It is natural to ask when will these map into the unit sphere, i.e. when will the pure and mixed parts of a λ_2-eigenmap be again λ_2-eigenmaps.

THEOREM 7. *The harmonic product $h \lozenge f$ of the generalized Hopf map $h\colon S^{2m+1} \to S^{(3m(m+1)/2)-1}$ and a λ_2-eigenmap $f\colon S^{2m+1} \to S^n$ exists iff both the pure and mixed parts of f are λ_2-eigenmaps.*

Proof. We use criterion (13) in Theorem 5 for the existence of $h \lozenge f$. By (24)–(26), we have

$$\sum_{a,b=0}^{2m+1} h_{ab} f_{ab} = \sum_i \left(\frac{2(m+1)}{m} (x_i^2 + x_{\bar{i}}^2) - \frac{2}{m+3} \rho^2 \right) (f_{ii} + f_{\bar{i}\bar{i}}) +$$

$$+ \sum_{i \neq k} \left(\frac{2(m+1)}{m} (x_i x_k + x_{\bar{i}} x_{\bar{k}})(f_{ik} + f_{\bar{i}\bar{k}}) + \right.$$

$$+ \frac{2(m+1)}{m} (x_i x_{\bar{k}} - x_k x_{\bar{i}}) f_{i\bar{k}} +$$

$$\left. + \frac{2(m+1)}{m} (x_k x_{\bar{i}} - x_i x_{\bar{k}}) f_{\bar{i}k} \right)$$

$$= \frac{2(m+1)}{m} \sum_{a,b=0}^{2m+1} f_{ab} x_a x_b + \frac{2(m+1)}{m} \sum_{a,b=0}^{2m+1} f_{a,b} y_a y_b,$$

where we set $y_i = -x_{\bar{i}}$, $y_{\bar{i}} = x_i$ (and the ρ^2 term cancels because of harmonicity). For the two terms on the right-hand side, with obvious notations, we have

$$\sum_{a,b} f_{ab} x_a x_b = \sum_{a,b,j} f^j \frac{\partial^2 f^j}{\partial x_a \, \partial x_b} x_a x_b = 2 \sum_j (f^j)^2 = 2\rho^2,$$

$$\sum_{a,b} f_{ab} y_a y_b = \sum_{a,b,j} f^j \frac{\partial^2 f^j}{\partial y_a \, \partial y_b} = 2 \sum_j f^j(x) f^j(y).$$

Putting these together, we obtain

$$\sum_{a,b=0}^{2m+1} h_{ab} f_{ab} = \frac{4(m+1)}{m} \rho^4 + \frac{4(m+1)}{m} \sum_j f^j(x) f^j(y)$$

$$= \frac{4(m+1)}{m} \rho^4 + \frac{4(m+1)}{m} \sum_j f^j(z) f^j(\sqrt{-1}\, z),$$

where we used that $y_i + \sqrt{-1}\, y_{\bar{i}} = -x_{\bar{i}} + \sqrt{-1}\, x_i = \sqrt{-1}\, z_i$ and $f^j(z)$ means f^j written in the z-variables, etc.

We now apply Theorem 5 to conclude that $h \lozenge f$ exists iff

$$\sum_j f^j(z) f^j(\sqrt{-1}\, z) = \omega \cdot \rho^4, \quad \omega = \text{constant}. \tag{27}$$

Given a spherical harmonic ϕ of order 2 on S^{2m+1}, we have $\phi_\times(\sqrt{-1}\, z) = \phi_\times(z)$ and $\phi_{||}(\sqrt{-1}\, z) = -\phi_{||}(z)$ so that we have $\phi(z)\phi(\sqrt{-1}\, z) = (\phi_\times(z) + \phi_{||}(z))(\phi_\times(z) - \phi_{||}(z))$. Applying this to the left-hand side of (27), we obtain

$$\sum_j f^j(z) f^j(\sqrt{-1}\, z) = \sum_j (f^j_\times(z))^2 - \sum_j (f^j_{||}(z))^2. \tag{28}$$

On the other hand, we have

$$\begin{aligned}\sum_j (f^j_\times(z))^2 + \sum_j (f^j_\|(z))^2 &= \tfrac{1}{2}\sum_j (f^j_\times(z) + f^j_\|(z))^2 + \\ &\quad + \tfrac{1}{2}\sum_j (f^j_\times(z) - f^j_\|(z))^2 \\ &= \tfrac{1}{2}\sum_j (f^j(z))^2 + \tfrac{1}{2}\sum_j (f^j(\sqrt{-1}\, z))^2 \\ &= \tfrac{1}{2}\sum_j (f^j(x))^2 + \tfrac{1}{2}\sum_j (f^j(y))^2 \\ &= \tfrac{1}{2}\rho^4 + \tfrac{1}{2}\rho^4 = \rho^4.\end{aligned}$$

Combining this with (27)–(28), the theorem follows.

5.2. THE GENERALIZED COMPLEX VERONESE MAP

Using the same notations as above, the generalized complex Veronese map

$$v\colon S^{2m+1} \to S^{(m+1)(m+2)-1}$$

is defined by

$$v(z_0, \ldots, z_m) = (\Re(z_i^2),\ \Im(z_i^2),\ \sqrt{2}\,\Re(z_r z_s),\ \sqrt{2}\,\Im(z_r z_s))_{0\le i\le m; 0\le r<s\le m}.$$

It is clear that v is a λ_2-eigenmap. The following result is the exact analogue of Theorem 7 and hence the proof is omitted.

THEOREM 8. *The harmonic product $v \lozenge f$ of the complex Veronese map $v\colon S^{2m+1} \to S^{(m+1)(m+2)-1}$ and a λ_2-eigenmap $f\colon S^{2m+1} \to S^n$ exists iff both the pure and mixed parts of f are λ_2-eigenmaps.*

In particular, since h is mixed and v is pure, we obtain that $h \lozenge h$, $h \lozenge v$ and $v \lozenge v$ exist. Note that, for $m = 1$ the λ_2-eigenmaps of S^3 have been classified (cf. [3]) and from that further examples can be easily obtained.

5.3. THE σ-EXTENSION OF A λ_2-EIGENMAP

To obtain examples for nonexistence of the harmonic product we first generalize the method of raising the source dimension of Section 3.

Let $f\colon S^m \to S^n$ be a λ_2-eigenmap and define

$$\tilde{f}\colon \mathbf{R}^{m+2} \to \mathbf{R}^{N+1}$$

by

$$\tilde{f}(x) = \left(Af(x),\ B\left(x_{m+1}^2 - \frac{\rho^2}{m+2}\right),\ Cx_i x_{m+1},\right.$$

$$\left. D\left(x_i^2 - \frac{\rho^2}{m+2}\right),\ Dx_r x_s \right)_{0\leq i,r,s\leq m;r\neq s},$$

where $\rho^2 = x_0^2 + \cdots + x_{m+1}^2$ and $A,\ B,\ C,\ D$ are constants with $A \neq 0$. Simple computation shows that $\tilde{f}$ maps the unit sphere into the unit sphere iff there exists an angle σ satisfying

$$\cos^2\sigma > \frac{m+1}{m+2} \tag{29}$$

such that

$$A = \frac{(m+2)\sqrt{m}}{m+1}\sqrt{\cos^2\sigma - \frac{m+1}{m+2}},$$

$$B = \frac{m+2}{m+1}\cos\sigma,$$

$$C = \sqrt{\frac{2(m+2)}{m+1}},$$

$$D = \frac{m+2}{\sqrt{m+1}}\sin\sigma.$$

Given σ satisfying (29), the restriction $f^\sigma = \tilde{f}\colon S^{m+1} \to S^N$ defined above is called the σ-extension of f. (Note that $N = n + 2m + 3 + m(m+1)/2$.) Clearly, $\sigma = 0$ reduces to the case treated in Section 3.

THEOREM 9. *Let $f\colon S^m \to S^u$ and $g\colon S^m \to S^v$ be λ_2-eigenmaps and σ and θ such that*

$$\cos^2\sigma,\ \cos^2\theta > \frac{m+1}{m+2}.$$

Assume that $f \diamond g$ exists. Then the harmonic product $f^\sigma \diamond g^\theta$ of the extensions does not exist.

Proof. Straightforward computation yields

$$f_{ii}^{\sigma} = A^2 f_{ii} + 2D^2 x_i^2 - \frac{2A^2}{m} x_{m+1}^2 = \left(\frac{2A^2}{m(m+2)} - \frac{2D^2}{m+2}\right)\rho^2,$$

$$f_{ik}^{\sigma} = A^2 f_{ik} + 2D^2 x_i x_k, \quad i \neq k,$$

$$f_{m+1,m+1}^{\sigma} = 2B^2 x_{m+1}^2 - \frac{2B^2}{m+2}\rho^2,$$

$$f_{i,m+1}^{\sigma} = C^2 x_i x_{m+1}$$

and similarly for g. In what follows we use (13) of Theorem 5 to conclude the nonexistence of $f^{\sigma} \lozenge g^{\theta}$. By assumption, we certainly have (13). By elementary (but long) calculation, we obtain

$$\sum_{a,b=0}^{m+1} f_{ik}^{\sigma} g_{ik}^{\theta} = K\rho^4 + Lx_{m+1}^4 + Mx_{m+1}^2\rho^2,$$

where

$$L = A_f^2 A_g^2 \left(c + \frac{4(m+1)}{m}\right) + \frac{4(m+1)}{m}(A_f^2 D_g^2 + A_g^2 D_f^2) + $$
$$+ 4D_f^2 D_g^2 + 4B_f^2 B_g^2 - \frac{8(m+2)^2}{(m+1)^2},$$

$$M = -\left(2c + \frac{8(m+1)}{m^2(m+2)}\right) A_f^2 A_g^2 - \frac{8(m+1)^2}{m(m+2)}(A_f^2 D_g^2 + A_g^2 D_f^2) -$$
$$- \frac{8}{m+2} B_f^2 B_g^2 - \frac{8(m+1)}{m+2} D_f^2 D_g^2 + \frac{8(m+2)^2}{(m+1)^2},$$

where the constants corresponding to f and g are indicated by subscripts. (The corresponding expression for K is irrelevant.) Assume now that $f^{\sigma} \lozenge g^{\theta}$ exists. Then, by Theorem 5, $L = M = 0$. In particular, we have

$$0 = 2L + M = (mD_f^2 + (m+1)A_f^2)(mD_g^2 + (m+1)A_g^2) +$$
$$+ m^2(m+1)B_f^2 B_g^2 - \frac{m^2(m+2)^3}{(m+1)^2}. \tag{30}$$

We now use the actual values of $A,\ B,\ C,\ D$ to obtain $mD^2 + (m+1)A^2 = m(m+2)/(m+1)$. Substituting this into (30), we get $B_f^2 B_g^2 = (m+2)^2/(m+1)^2$ or, equivalently, $\cos^2\sigma \cos^2\theta = (m+1)^2/(m+2)^2$ which clearly contradicts the assumptions.

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