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Mappings of moduli spaces for harmonic eigenmaps and minimal immersions between spheres

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1. Introduction and preliminaries.

Let $\mathcal{H}^p=\mathcal{H}^p_{S^m}$ denote the vector space of spherical harmonics of order $p\geqq 1$ on the Euclidean m-sphere S^m, $m\geqq 2$. We think of a spherical harmonic as a degree p homogeneous harmonic polynomial in $m+1$ variables or as an eigenfunction of the Laplace-Beltrami operator Δ^{S^m} with eigenvalue $\lambda_p=p(p+m-1)$ (obtained from the polynomial by restriction to S^m). A map $f: S^m\to S_V$ into the unit sphere of a Euclidean vector space V is said to be a *λ_p-eigenmap* if all components of f belong to $\mathcal{H}^p$, i.e., for $\mu\in V^*$, we have $\mu\circ f\in\mathcal{H}^p$. (Note that a λ_p-eigenmap is nothing but a harmonic map with energy density $\lambda_p/2$ [2].) $f: S^m\to S_V$ is *full* if the image of f in V spans V. In general, restricting to *span im* $f\cap S_V$, f gives rise to a full λ_p-eigenmap that we will denote by the same symbol. Two λ_p-eigenmaps $f_1: S^m\to S_{V_1}$ and $f_2: S^m\to S_{V_2}$ are *equivalent* if there exists an isometry $U: V_1\to V_2$ such that $f_2=U\circ f_1$.

The universal example of a λ_p-eigenmap is given by the standard minimal immersion $f_{\lambda_p}: S^m\to S_{\mathcal{H}^p}$ defined by

$$f_{\lambda_p}(x)=\sum_{j=0}^{n(\lambda_p)} f^j_{\lambda_p}(x)f^j_{\lambda_p},$$

where $\{f^j_{\lambda_p}\}_{j=0}^{n(\lambda_p)}\subset\mathcal{H}^p$ is an orthonormal basis with respect to the normalized L_2-scalar product

$$\langle h, h'\rangle_p=\frac{n(\lambda_p)+1}{vol(S^m)}\int_{S^m} hh'v_{S^m}. \tag{1}$$

Here v_{S^m} is the volume form on S^m, $vol(S^m)=\int_{S^m}v_{S^m}$ is the volume of S^m and

$$n(\lambda_p)+1=\dim\mathcal{H}^p=(2p+m-1)\frac{(p+m-1)!}{(p+1)!(m-1)!}. \tag{2}$$

f_{λ_p} is clearly full and does not depend on the orthonormal basis.

f_{λ_p} is universal in the sense that, for any λ_p-eigenmap $f: S^m\to S_V$, there exists a linear map $A:\mathcal{H}^p\to V$ such that $f=A\circ f_{\lambda_p}$. Clearly, A is surjective iff f is full.

Associating to f the symmetric linear endomorphism

$$\langle f\rangle_{\lambda_p}=A^{\mathsf{T}}A-I\in S^2(\mathcal{H}^p), \qquad (I=\text{identity})$$

establishes a parametrization of the space of equivalence classes of full λ_p-eigenmaps $f: S^m\to S_V$ by the compact convex body

$$\mathcal{L}_{\lambda_p}=\{C\in\mathcal{E}_{\lambda_p}\mid C+I\geqq 0\} \tag{3}$$

in the linear subspace

$$\mathcal{E}_{\lambda_p}=\operatorname{span}\{f_{\lambda_p}(x)\odot f_{\lambda_p}(x)\mid x\in S^m\}^{\perp}\subset S^2(\mathcal{H}^p). \tag{4}$$

Here '$\geqq$' stands for positive semidefinite, '$\odot$' for the symmetric tensor product and the orthogonal complement is taken with respect to the standard scalar product

$$\langle C, C'\rangle=\sum_{j=0}^{n(\lambda_p)}\langle Cf_{\lambda_p}^j, C'f_{\lambda_p}^j\rangle_p, \qquad C, C'\in S^2(\mathcal{H}^p). \tag{5}$$

$\mathcal{L}_{\lambda_p}$ is said to be the (standard) *moduli space* of λ_p-eigenmaps. (For more details as well as for the general theory of moduli spaces, cf. [**5**].) The classification of λ_p-eigenmaps raised in [**2**] as a fundamental problem in harmonic map theory is thereby equivalent to describing $\mathcal{L}_{\lambda_p}$.

f_{λ_p} is equivariant with respect to the homomorphism $\rho_{\lambda_p}: SO(m+1)\to SO(\mathcal{H}^p)$ that is just the orthogonal ($SO(m+1)$-)module structure on $\mathcal{H}^p$ defined by $a\cdot h=h\circ a^{-1}$, $a\in SO(m+1)$ and $h\in\mathcal{H}^p$. Equivariance is given explicitly by

$$f_{\lambda_p}\circ a=\rho_{\lambda_p}(a)\circ f_{\lambda_p}, \qquad a\in SO(m+1). \tag{6}$$

$\mathcal{E}_{\lambda_p}$ is a submodule of $S^2(\mathcal{H}^p)$, where the latter is endowed with the module structure induced from that of $\mathcal{H}^p$. Moreover, $\mathcal{L}_{\lambda_p}\subset\mathcal{E}_{\lambda_p}$ is an invariant subset. Explicitly, for a full λ_p-eigenmap $f: S^m\to S_V$, we have

$$a\cdot\langle f\rangle_{\lambda_p}=\langle f\circ a^{-1}\rangle_{\lambda_p}, \qquad a\in SO(m+1).$$

The work of DoCarmo-Wallach [**1**] [**8**] gives the decomposition of $S^2(\mathcal{H}^p\otimes_{\mathbf{R}}\mathbf{C})$ into irreducible components. We have, for $m\geqq 3$:

$$S^2(\mathcal{H}^p\otimes_{\mathbf{R}}\mathbf{C})\cong\sum_{(a,b)\in\bar{\Delta}^p;\ a,b\ \text{even}} V_{m+1}^{(a,b,0,\cdots,0)}. \tag{7}$$

Here $\bar{\Delta}^p\subset\mathbf{R}^2$ denotes the closed convex triangle with vertices $(0, 0)$, $(2p, 0)$ and (p, p) and $V_{m+1}^{(a_1,\cdots,a_l)}$, $l=[\,|(m+1)/2|\,]$, stands for the complex irreducible $SO(m+1)$-module with highest weight vector $(a_1, \cdots, a_l)$ whose components are with respect to the standard maximal torus in $SO(m+1)$. (Note that, for $m=3$, $V_{m+1}^{(a,b,0,\cdots,0)}$ means $V_4^{(a,b)}\oplus V_4^{(a,-b)}$.) Moreover, $\mathcal{E}_{\lambda_p}\otimes_{\mathbf{R}}\mathbf{C}$ is nontrivial iff $m\geqq 3$ and $p\geqq 2$ and, in this case, it consists of those components of the symmetric square that are not class 1 with respect to $(SO(m+1), SO(m))$. Hence the decomposition of $\mathcal{E}_{\lambda_p}\otimes_{\boldsymbol{R}}\boldsymbol{C}$ is obtained by restricting the summations above to the subtriangle $\Delta^p\subset\bar{\Delta}^p$ whose vertices are $(2, 2)$, $(2p-2, 2)$ and (p, p).

Since $\Delta^p\subset\Delta^{p+1}$ we may think of $\mathcal{E}_{\lambda_p}\otimes_{\boldsymbol{R}}\boldsymbol{C}$ as a submodule of $\mathcal{E}_{\lambda_{p+1}}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. In view of this, it is natural to ask whether $\mathcal{L}_{\lambda_p}$ can be equivariantly imbedded into $\mathcal{L}_{\lambda_{p+1}}$. One objective of this paper is to give an affirmative answer to this question. The importance of this imbedding is twofold. First, as explained in [5], the complexity of (the boundary of) $\mathcal{L}_{\lambda_p}$ increases rapidly with p so that knowing the moduli space for low values of p (e.g., for $m=3$ and $p=2$, cf. [4]) we gain an insight as to what happens in the moduli space for higher values of p. Second, our explicit construction of the equivariant imbedding that we are about to describe gives a whole new series of concrete examples for λ_p-eigenmaps for higher values of p.

A homothetic immersion $f: S^m\to S_V$ is minimal iff it is a (harmonic) eigenmap [2]. For minimal λ_p-eigenmaps the homothety constant is λ_p/m. Thus adding the condition

$$|f_*(X)|^2=\frac{\lambda_p}{m}|X|^2, \tag{8}$$

for any vector field X on S^m, to those of $\mathcal{L}_{\lambda_p}$ defines the subset

$$\mathcal{M}_{\lambda_p}\subset\mathcal{L}_{\lambda_p}$$

that parametrizes the equivalence classes of full minimal λ_p-eigenmaps. More precisely, we have [5]:

$$\mathcal{M}_{\lambda_p}=\mathcal{L}_{\lambda_p}\cap\mathcal{F}_{\lambda_p},$$

where

$$\mathcal{F}_{\lambda_p}=\operatorname{span}\{f_*(X)^\vee\odot f_*(X)^\vee\,|\,X\in T(S^m)\}^\perp$$

is a submodule of $\mathcal{E}_{\lambda_p}\subset S^2(\mathcal{H}^p)$. Here $^\vee: T(\boldsymbol{R}^{m+1})\to\boldsymbol{R}^{m+1}$ is the canonical map that translates tangent vectors to the origin. It follows that the moduli space $\mathcal{M}_{\lambda_p}$ for minimal immersions is also a compact convex body. DoCarmo and Wallach [1][8] showed that $\mathcal{F}_{\lambda_p}$ is nontrivial iff $m\geqq 3$ and $p\geqq 4$ and, in this case, we have

$$\mathcal{F}_{\lambda_p}\otimes_{\boldsymbol{R}}\boldsymbol{C}\supset\sum_{(a,b)\in\Delta_0^p;\,a,b\text{ even}}V_{m+1}^{(a,b,0,\cdots,0)}, \tag{9}$$

where $\Delta_0^p\subset\Delta^p$ is the subtriangle with vertices $(4, 4)$, $(2p-4, 4)$ and (p, p). They conjectured that the lower bound in (9) is actually sharp, i.e. that the modules

$$V_{m+1}^{(2l,2,0,\cdots,0)},\qquad l=1,\cdots,p-1, \tag{10}$$

corresponding to the base of Δ^p are not components of $\mathcal{F}_{\lambda_p}\otimes_{\boldsymbol{R}}\boldsymbol{C}$.

In the second part of the paper we show that the construction for eigenmaps preserves minimality and thereby gives an equivariant imbedding of $\mathcal{M}_{\lambda_p}$ into $\mathcal{M}_{\lambda_{p+1}}$. Using the adjoint (of the module extension of this imbedding), we prove that, for $m=3$ and for any $p\geqq 4$, the first three in the sequence in (10)

$$V_4^{(2,\pm 2)},\quad V_4^{(4,\pm 2)} \quad\text{and}\quad V_4^{(6,\pm 2)}$$

are not components of $\mathcal{F}_{\lambda_p}\otimes_{\boldsymbol{R}}\boldsymbol{C}$ which is a step towards the positive resolution of the conjecture.

2. Raising and lowering the degree.

Let H denote the harmonic projection operator [7]. H is the orthogonal projection from the vector space of homogeneous polynomials in $m+1$ variables of a given degree onto the linear subspace of harmonic polynomials.

Let $f: S^m \to S_V$ be a λ_p-eigenmap. We define

$$f^{\pm}: \boldsymbol{R}^{m+1} \longrightarrow \boldsymbol{R}^{(m+1)(n+1)}$$

as follows. The components of $f^{\pm}$ are given in double indices $i=0, \cdots, m$ and $j=0, \cdots, n$ by

$$(f^+)_i^j = c_p^+ H(x_i f^j) \quad\text{and}\quad (f^-)_i^j = c_p^- \frac{\partial f^j}{\partial x_i}, \tag{11}$$

where

$$c_p^+ = \sqrt{\frac{2p+m-1}{p+m-1}} \quad\text{and}\quad c_p^- = \frac{1}{\sqrt{p(2p+m-1)}}\,. \tag{12}$$

PROPOSITION 1. *We have*

$$f^{\pm}(S^m) \subset S^{(m+1)(n+1)-1}$$

so that the restrictions $f^{\pm}: S^m \to S^{(m+1)(n+1)-1}$ *are* $\lambda_{p\pm 1}$*-eigenmaps.*

PROOF. By the harmonic projection formula [7] (or elementary computation), we have

$$H(x_i f^j) = x_i f^j - \frac{\rho^2}{2p+m-1}\frac{\partial f^j}{\partial x_i},$$

where $\rho^2 = \sum_{i=0}^m x_i^2$. Homogeneity of the components f^j has two consequences. First, we have

$$\sum_{i=0}^m x_i \frac{\partial f^j}{\partial x_i} = p f^j\,.$$

Second, since $f(S^m) \subset S^n$, we have $\sum_{j=0}^n (f^j)^2 = \rho^{2p}$ as polynomials. Applying the Laplacian $\Delta = \Delta^{R^{m+1}} = \sum_{i=0}^m \partial^2/\partial x_i^2$ to both sides and restricting to S^m, we obtain

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{\partial f^j}{\partial x_i}\right)^2 = p(2p+m-1) \tag{13}$$

and the statement for f^- follows. Using these, we compute (on S^m):

$$\sum_{i=0}^{m}\sum_{j=0}^{n} H(x_i f^j)^2 = \sum_{i=0}^{m}\sum_{j=0}^{n}\left(x_i f^j - \frac{1}{2p+m-1}\frac{\partial f^j}{\partial x_i}\right)^2$$

$$=1-\frac{2}{2p+m-1}\sum_{j=0}^{n} f^j \sum_{i=0}^{m} x_i \frac{\partial f^j}{\partial x_i} + \frac{1}{(2p+m-1)^2}\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{\partial f^j}{\partial x_i}\right)^2$$

$$=1-\frac{2p}{2p+m-1}+\frac{p}{2p+m-1}=\frac{p+m-1}{2p+m-1}$$

and the proposition follows.

REMARK. $f^\pm$ are not full even if f is. As in Section 1 we will also denote by $f^\pm$ the full $\lambda_{p\pm1}$-eigenmap derived from $f^\pm$ by restriction.

We now define

$$\Phi_\pm : \mathcal{L}_{\lambda_p} \longrightarrow \mathcal{L}_{\lambda_{p\pm1}}$$

by

$$\Phi_\pm : (\langle f \rangle_{\lambda_p}) = \langle f^\pm \rangle_{\lambda_{p\pm1}}, \tag{14}$$

where $f: S^m \to S_V$ is a full λ_p-eigenmap. (Note that the definition makes sense since f_1 is equivalent with f_2 implies that $f_1^\pm$ is equivalent with $f_2^\pm$.)

Φ_+ is the equivariant imbedding mentioned in Section 1. To show injectivity of Φ_+ it is, however, more convenient to change the setting and define $\Phi_\pm$ in terms of the moduli space only. This we will do in the next section. In the end of Section 3 we will show that the two definitions agree.

We finish this section with the following proposition that will be a useful computational tool in the sequel.

PROPOSITION 2. *For $h \in \mathcal{H}^{p+1}$ and $h' \in \mathcal{H}^p$, we have*

$$\left\langle \frac{\partial h}{\partial x_i}, h' \right\rangle_p = \mu_p \langle h, H(x_i h') \rangle_{p+1},$$

where

$$\mu_p = (p+1)\frac{2p+m-1}{p+m-1}.$$

PROOF. Homogeneous harmonic polynomials of different degree are L_2-orthogonal. Using this, the harmonic projection formula and (1) we compute:

$$\left\langle \frac{\partial h}{\partial x_i}, h' \right\rangle_p = \frac{n(\lambda_p)+1}{vol(S^m)}\int_{S^m} \frac{\partial h}{\partial x_i} h' v_{S^m}$$

$$= \frac{n(\lambda_p)+1}{vol(S^m)}(2p+m-1)\int_{S^m} \frac{\rho^2}{2p+m-1}\frac{\partial h}{\partial x_i} h' v_{S^m}$$

$$= \frac{n(\lambda_p)+1}{vol(S^m)}(2p+m-1)\int_{S^m}(x_i h - H(x_i h))h' v_{S^m}$$

$$= \frac{n(\lambda_p)+1}{vol(S^m)}(2p+m-1)\int_{S^m} h x_i h' v_{S^m}$$

$$= \frac{n(\lambda_p)+1}{vol(S^m)}(2p+m-1)\int_{S^m} h H(x_i h') v_{S^m}$$

$$= \frac{n(\lambda_p)+1}{n(\lambda_{p+1})+1}(2p+m-1)\langle h, H(x_i h')\rangle_{p+1}.$$

Using (2) the constant becomes μ_p as above.

3. Realization in the tensor product $\mathcal{H}^1 \otimes \mathcal{H}^p$.

A convenient $SO(m+1)$-module in which we encode all the data of both moduli spaces $\mathcal{L}_{\lambda_{p\pm1}}$ (in terms of the degree raising and lowering operators) is $\mathcal{H}^1 \otimes \mathcal{H}^p$. Indeed, $\mathcal{H}^{p\pm1}$ are both submodules of $\mathcal{H}^1 \otimes \mathcal{H}^p$. To see this, let

$$\iota_\pm : \mathcal{H}^{p\pm1} \longrightarrow \mathcal{H}^1 \otimes \mathcal{H}^p$$

be defined by

$$\iota_+(h) = \sum_{i=0}^m y_i \otimes \frac{\partial h}{\partial x_i}, \qquad h \in \mathcal{H}^{p+1}$$

and

$$\iota_-(h') = \sum_{i=0}^m y_i \otimes H(x_i h'), \qquad h' \in \mathcal{H}^{p-1}.$$

Simple computation shows that $\iota_\pm$ are module monomorphisms with respect to the tensor product module structure on $\mathcal{H}^1 \otimes \mathcal{H}^p$.

PROPOSITION 3. *On $\mathcal{H}^{p\pm1}$, we have*

$$\iota_\pm^\top \circ \iota_\pm = d_p^\pm I,$$

where

$$d_p^+ = \left(\frac{\mu_p}{c_p^+}\right)^2 \quad \text{and} \quad d_p^- = \frac{1}{(\mu_{p-1} c_p^-)^2}.$$

PROOF. Let $h \in \mathcal{H}^{p+1}$ and $\sum_{i=0}^m y_i \otimes h_i \in \mathcal{H}^1 \otimes \mathcal{H}^p$. Using Proposition 2, we have

$$\left\langle \iota_+(h), \sum_{i=0}^m y_i \otimes h_i \right\rangle = \sum_{i=0}^m \left\langle \frac{\partial h}{\partial x_i}, h_i \right\rangle_p = \mu_p \sum_{i=0}^m \langle h, H(x_i h_i)\rangle_{p+1}$$

$$= \left\langle h, \mu_p \sum_{i=0}^m H(x_i h_i) \right\rangle_{p+1}$$

so that

$$\iota_+^T\Big(\sum_{i=0}^m y_i\otimes h_i\Big)=\mu_p\sum_{i=0}^m H(x_i h_i). \tag{15}$$

Combining this with ι_+, we obtain

$$\iota_+^T\circ\iota_+(h)=\iota_+^T\Big(\sum_{i=0}^m y_i\otimes\frac{\partial h}{\partial x_i}\Big)=\mu_p\sum_{i=0}^m H\Big(x_i\frac{\partial h}{\partial x_i}\Big)$$

$$=\mu_p(p+1)H(h)=\mu_p(p+1)h\,.$$

Comparing the constants in (12) and in Proposition 2, we get

$$\mu_p(p+1)=(p+1)^2\frac{2p+m-1}{p+m-1}=\Big(\frac{\mu_p}{c_p^+}\Big)^2$$

which completes the proof for ι_+. The verification for ι_- is similar. For future reference we note that

$$\iota_-^T\Big(\sum_{i=0}^m y_i\otimes h_i\Big)=\frac{1}{\mu_{p-1}}\sum_{i=0}^m\frac{\partial h_i}{\partial x_i}. \tag{16}$$

We now turn to the standard minimal immersion $f_{\lambda_p}: S^m\to S_{\mathcal{H}^p}$ and define

$$f_{\lambda_p}^+(x)=c_p^+\sum_{i=0}^m\sum_{j=0}^{n(\lambda_p)}H(x_i f_{\lambda_p}^j)(x)y_i\otimes f_{\lambda_p}^j \tag{17}$$

and

$$f_{\lambda_p}^-(x)=c_p^-\sum_{i=0}^m\sum_{j=0}^{n(\lambda_p)}\frac{\partial f_{\lambda_p}^j}{\partial x_i}(x)y_i\otimes f_{\lambda_p}^j\,. \tag{18}$$

By Proposition 1, $f_{\lambda_p}^\pm: S^m\to S_{\mathcal{H}^1\otimes\mathcal{H}^p}$ are $\lambda_{p\pm1}$-eigenmaps.

PROPOSITION 4. *$f_{\lambda_p}^\pm$ are standard, i.e. equivalent to $f_{\lambda_{p\pm1}}$.*

PROOF. Since $\iota_\pm$ is an isometry up to a constant multiple, it is enough to show that

$$\iota_\pm(f_{\lambda_{p\pm1}}(x))=\sqrt{d_p^\pm}f_{\lambda_p}^\pm(x)\,,\qquad x\in S^m\,. \tag{19}$$

Using again Proposition 2 and (18), for $x\in S^m$, we have

$$\iota_-(f_{\lambda_{p-1}}(x))=\sum_{l=0}^{n(\lambda_{p-1})}f_{\lambda_{p-1}}^l(x)\iota_-(f_{\lambda_{p-1}}^l)$$

$$=\sum_{l=0}^{n(\lambda_{p-1})}f_{\lambda_{p-1}}^l(x)\sum_{i=0}^m y_i\otimes H(x_i f_{\lambda_{p-1}}^l)$$

$$=\sum_{l=0}^{n(\lambda_{p-1})}f_{\lambda_{p-1}}^l(x)\sum_{i=0}^m\sum_{j=0}^{n(\lambda_p)}\langle f_{\lambda_p}^j, H(x_i f_{\lambda_{p-1}}^l)\rangle_p y_i\otimes f_{\lambda_p}^j$$

$$=\frac{1}{\mu_{p-1}}\sum_{l=0}^{n(\lambda_{p-1})} f_{\lambda_{p-1}}^{l}(x)\sum_{i=0}^{m}\sum_{j=0}^{n(\lambda_p)}\left\langle\frac{\partial f_{\lambda_p}^{j}}{\partial x_i}, f_{\lambda_{p-1}}^{l}\right\rangle_{p-1} y_i\otimes f_{\lambda_p}^{j}$$

$$=\frac{1}{\mu_{p-1}}\sum_{i=0}^{m}\sum_{j=0}^{n(\lambda_p)}\frac{\partial f_{\lambda_p}^{j}}{\partial x_i}(x)y_i\otimes f_{\lambda_p}^{j}=\frac{1}{\mu_{p-1}c_p^{-}}f_{\lambda_p}^{-}(x).$$

The computation for ι_+ is similar and hence is omitted.

REMARK. A different (and somewhat less explicit) proof can be given by using the fact that equivariant eigenmaps are standard. (In fact, an eigenmap is equivariant iff the corresponding point in the moduli space is left fixed by $SO(m+1)$. However, $\mathcal{E}_{\lambda_p}$ has no trivial component.) To check equivariance, for $a\in SO(m+1)$, we compute (using (6) and (17)):

$$f_{\lambda_p}^{+}(ax)=c_p^{+}\sum_{i=0}^{m}\sum_{j=0}^{n(\lambda_p)} H(x_i f_{\lambda_p}^{j})(ax)y_i\otimes f_{\lambda_p}^{j}$$

$$=c_p^{+}\sum_{i=0}^{m}\sum_{j=0}^{n(\lambda_p)}\left((ax)_i f_{\lambda_p}^{j}(ax)-\frac{\rho^2}{2p+m-1}\frac{\partial f_{\lambda_p}^{j}}{\partial x_i}(ax)\right)y_i\otimes f_{\lambda_p}^{j}$$

$$=c_p^{+}\sum_{i,k=0}^{m}\sum_{j=0}^{n(\lambda_p)} a_{ik}H(x_k f_{\lambda_p}^{j}\circ a)(x)y_i\otimes f_{\lambda_p}^{j}$$

$$=c_p^{+}\sum_{k=0}^{m}\sum_{j,l=0}^{n(\lambda_p)}\rho_{\lambda_p}(a)_{jl}H(x_k f_{\lambda_p}^{l})(x)(a^{-1}y)_k\otimes f_{\lambda_p}^{j}$$

$$=c_p^{+}\sum_{k=0}^{m}\sum_{l=0}^{n(\lambda_p)} H(x_k f_{\lambda_p}^{l})(x)(a^{-1}y)_k\otimes(f_{\lambda_p}^{l}\circ a^{-1})$$

$$=a\cdot f_{\lambda_p}^{+}(x).$$

We have concluded above that $\mathcal{H}^{p\pm1}$ are components of $\mathcal{H}^1\otimes\mathcal{H}^p$. To complete the picture, we claim that

$$\mathcal{H}^1\otimes\mathcal{H}^p=\mathcal{H}^{p+1}\oplus\mathcal{H}^{p-1}\oplus\mathcal{K}, \tag{20}$$

where

$$\mathcal{K}=\left\{\sum_{i=0}^{m} y_i\otimes\phi_i \,|\, \phi_i\in\mathcal{H}^p \quad\text{and}\quad \phi_i+\sum_{k=0}^{m}A_{ik}\phi_k=0,\ i=0,\cdots,m\right\}$$

is an irreducible submodule. Here

$$A_{ik}=x_k\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial x_k}$$

is the infinitesimal rotation in the (x_i, x_k)-plane. That $\mathcal{K}$ is an $SO(m+1)$-module is a straightforward computation. (In fact, using the commutation relations of the infinitesimal rotations, $so(m+1)$-invariance follows easily.) To show that $\mathcal{K}$ is nontrivial, let $\psi_i\in\mathcal{H}^p$, $i=0,\cdots,m$, be such that $\sum_{i=0}^{m}\partial\psi_i/\partial x_i=0$.

Then

$$\sum_{i=0}^{m} y_i \otimes \phi_i - \frac{1}{p+1} \sum_{i=0}^{m} y_i \otimes \frac{\partial}{\partial x_i}\Big(\sum_{k=0}^{m} H(x_k \phi_k)\Big)$$

is a (typical) element of $\mathcal{K}$. (Note that this is nothing but the projection of $\sum_{i=0}^{m} y_i \otimes \phi_i \in \iota_-(\mathcal{H}^{p-1})^{\perp}$ onto $\mathcal{K}$.)

$\mathcal{K}$ is orthogonal to $\mathcal{H}^{p\pm1}$. Indeed, for $\sum_{i=0}^{m} y_i \otimes \phi_i \in \mathcal{K}$, we have

$$\sum_{i=0}^{m} \frac{\partial \phi_i}{\partial x_i} = -\sum_{i,k=0}^{m} \frac{\partial}{\partial x_i}(A_{ik}\phi_k) = (p+m-1)\sum_{i=0}^{m} \frac{\partial \phi_i}{\partial x_i}$$

so that $\sum_{i=0}^{m} \partial \phi_i / \partial x_i = 0$. Similarly:

$$\sum_{i=0}^{m} H(x_i \phi_i) = -\sum_{i,k=0}^{m} H(x_i A_{ki} \phi_k) = -p \sum_{i=0}^{m} H(x_i \phi_i)$$

and hence $\sum_{i=0}^{m} H(x_i \phi_i) = 0$. Now orthogonality is a simple application of Proposition 2.

Finally irreducibility of $\mathcal{K}$ can also be obtained by elementary computation. Note however that using the DoCarmo-Wallach decomposition of the tensor product [**1**], we have

$$V_{m+1}^{(1,0,\cdots,0)} \otimes V_{m+1}^{(p,0\ \cdots,0)} \cong V_{m+1}^{(p+1,0,\cdots,0)} \oplus V_{m+1}^{(p-1,0,\cdots,0)} \oplus V_{m+1}^{(p,1,0,\cdots,0)}.$$

Comparing this with (20) we see that $\mathcal{K} \otimes_{\boldsymbol{R}} \boldsymbol{C} \cong V_{m+1}^{(p,1,0,\cdots,0)}$.

We now define

$$\Phi_{\pm} : S^2(\mathcal{H}^p) \longrightarrow S^2(\mathcal{H}^{p\pm1})$$

by

$$\Phi_{\pm}(C) = \frac{1}{d_p^{\pm}} \iota_{\pm}^{\mathrm{T}} \circ (I \otimes C) \circ \iota_{\pm}\,. \tag{21}$$

Clearly, $\Phi_{\pm}$ are module homomorphisms.

THEOREM 1. *We have*

$$\Phi_{\pm}(\mathcal{E}_{\lambda_p}) \subset \mathcal{E}_{\lambda_{p\pm1}}$$

and

$$\Phi_{\pm}(\mathcal{L}_{\lambda_p}) \subset \mathcal{L}_{\lambda_{p\pm1}}\,.$$

PROOF. Using (17), (19) and (21), we compute

$$\begin{aligned}\langle \Phi_+(C), f_{\lambda_{p+1}}(x) \odot f_{\lambda_{p+1}}(x) \rangle &= \langle \Phi_+(C) f_{\lambda_{p+1}}(x), f_{\lambda_{p+1}}(x) \rangle_{p+1} \\ &= \frac{1}{d_p^+} \langle (I \otimes C)\iota_+(f_{\lambda_{p+1}}(x)), \iota_+(f_{\lambda_{p+1}}(x)) \rangle_p \\ &= \langle (I \otimes C) f_{\lambda_p}^+(x), f_{\lambda_p}^+(x) \rangle_p\end{aligned}$$

$$= (c_p^+)^2 \sum_{i,k=0}^{m} \sum_{j,l=0}^{n(\lambda_p)} H(x_i f_{\lambda_p}^j) H(x_k f_{\lambda_p}^l)$$

$$\times \langle (I\otimes C) y_i \otimes f_{\lambda_p}^j,\ y_k \otimes f_{\lambda_p}^l \rangle$$

$$= (c_p^+)^2 \sum_{i=0}^{m} \sum_{j,l=0}^{n(\lambda_p)} H(x_i f_{\lambda_p}^j) H(x_i f_{\lambda_p}^l) \langle C f_{\lambda_p}^j,\ f_{\lambda_p}^l \rangle$$

$$= (c_p^+)^2 \sum_{i=0}^{m} \sum_{j,l=0}^{n(\lambda_p)} c_{jl} H(x_i f_{\lambda_p}^j) H(x_i f_{\lambda_p}^l)$$

$$= (c_p^+)^2 \sum_{i=0}^{m} \sum_{j,l=0}^{n(\lambda_p)} c_{jl} \left(x_i f_{\lambda_p}^j - \frac{1}{2p+m-1} \frac{\partial f_{\lambda_p}^j}{\partial x_i} \right)$$

$$\times \left(x_i f_{\lambda_p}^l - \frac{1}{2p+m-1} \frac{\partial f_{\lambda_p}^l}{\partial x_i} \right)$$

$$= (c_p^+)^2 \sum_{i=0}^{m} \sum_{j,l=0}^{n(\lambda_p)} \left(c_{jl} f_{\lambda_p}^j f_{\lambda_p}^l - \frac{2}{2p+m-1} c_{jl} f_{\lambda_p}^j x_i \frac{\partial f_{\lambda_p}^l}{\partial x_i} \right.$$

$$\left. + \frac{1}{(2p+m-1)^2} c_{jl} \frac{\partial f_{\lambda_p}^j}{\partial x_i} \frac{\partial f_{\lambda_p}^l}{\partial x_i} \right)$$

$$= (c_p^+)^2 \Big(\langle C f_{\lambda_p}(x),\ f_{\lambda_p}(x) \rangle - \frac{2p}{2p+m-1}$$

$$\times \langle C f_{\lambda_p}(x), f_{\lambda_p}(x) \rangle + \frac{1}{2(2p+m-1)^2} \Delta \langle C f_{\lambda_p}, f_{\lambda_p} \rangle (x) \Big).$$

Assuming $C \in \mathcal{E}_{\lambda_p}$, by (4), this is zero so that $\Phi_+(C) \in \mathcal{E}_{\lambda_{p+1}}$. The computation for Φ_- is simpler and hence is omitted.

To prove the second statement, we first note that, by (3), $C \in \mathcal{L}_{\lambda_p}$ iff $C \in \mathcal{E}_{\lambda_p}$ and $C + I \geqq 0$. Assuming this and using Proposition 3, for $h \in \mathcal{H}^{p+1}$, we have

$$\langle (\Phi_\pm(C) + I) h,\ h \rangle_{p\pm 1} = \frac{1}{d_p^\pm} \langle ((I\otimes C) + I) \iota_\pm(h),\ \iota_\pm(h) \rangle$$

$$= \frac{1}{d_p^\pm} \langle (I \otimes (C+I)) \iota_\pm(h),\ \iota_\pm(h) \rangle \geqq 0\,.$$

REMARK. Using Proposition 2, an easy computation shows that, for $x \in S^m$, we have

$$\Phi_+(f_{\lambda_p}(x) \odot f_{\lambda_p}(x)) = (c_p^+)^2 \sum_{i=0}^{m} H(x_i f_{\lambda_p})(x) \odot H(x_i f_{\lambda_p})(x)$$

and

$$\Phi_{-}(f_{\lambda_p}(x)\odot f_{\lambda_p}(x))=(c_p^-)^2\sum_{i=0}^{m}\frac{\partial f_{\lambda_p}}{\partial x_i}(x)\odot\frac{\partial f_{\lambda_p}}{\partial x_i}(x)\,.$$

We now show that $\Phi_{\pm}$ are adjoints of each other (up to a constant multiple).

THEOREM 2. *For $C\in S^2(\mathcal{H}^p)$ and $C'\in S^2(\mathcal{H}^{p+1})$, we have*

$$\langle\Phi_{+}(C),\,C'\rangle=\frac{2p+m-1}{p+m-1}\langle C,\,\Phi_{-}(C')\rangle.$$

PROOF. Using (5), Proposition 2 and (16), we compute

$$\begin{aligned}
\langle\Phi_{+}(C),\,C'\rangle &= \frac{1}{d_p^+}\langle\iota_+^{\top}\circ(I\otimes C)\circ\iota_+,\,C'\rangle\\
&= \frac{1}{d_p^+}\sum_{l=0}^{n(\lambda_{p+1})}\langle(\iota_+^{\top}\circ(I\otimes C)\circ\iota_+)f_{\lambda_{p+1}}^l,\,C'f_{\lambda_{p+1}}^l\rangle_{p+1}\\
&= \frac{1}{d_p^+}\sum_{l=0}^{n(\lambda_{p+1})}\sum_{i=0}^{m}\Big\langle(I\otimes C)\Big(y_i\otimes\frac{\partial f_{\lambda_{p+1}}^l}{\partial x_i}\Big),\,\iota_+(C'f_{\lambda_{p+1}}^l)\Big\rangle\\
&= \frac{1}{d_p^+}\sum_{l=0}^{n(\lambda_{p+1})}\sum_{i=0}^{m}\Big\langle C\Big(\frac{\partial f_{\lambda_{p+1}}^l}{\partial x_i}\Big),\,\frac{\partial}{\partial x_i}(C'f_{\lambda_{p+1}}^l)\Big\rangle_p\\
&= \frac{1}{d_p^+}\sum_{l=0}^{n(\lambda_{p+1})}\sum_{j=0}^{n(\lambda_p)}\sum_{i=0}^{m}\Big\langle\frac{\partial f_{\lambda_{p+1}}^l}{\partial x_i},\,f_{\lambda_p}^j\Big\rangle_p\Big\langle Cf_{\lambda_p}^j,\,\frac{\partial}{\partial x_i}(C'f_{\lambda_{p+1}}^l)\Big\rangle_p\\
&= \frac{\mu_p}{d_p^+}\sum_{l=0}^{n(\lambda_{p+1})}\sum_{j=0}^{n(\lambda_p)}\sum_{i=0}^{m}\langle f_{\lambda_{p+1}}^l,\,H(x_i f_{\lambda_p}^j)\rangle_{p+1}\\
&\qquad\times\Big\langle Cf_{\lambda_p}^j,\,\frac{\partial}{\partial x_i}(C'f_{\lambda_{p+1}}^l)\Big\rangle_{p+1}\\
&= \frac{\mu_p}{d_p^+}\sum_{i=0}^{m}\sum_{j=0}^{n(\lambda_p)}\Big\langle Cf_{\lambda_p}^j,\,\frac{\partial}{\partial x_i}(C'H(x_i f_{\lambda_p}^j))\Big\rangle_p\\
&= \frac{\mu_p^2}{d_p^+}\sum_{j=0}^{n(\lambda_p)}\langle Cf_{\lambda_p}^j,\,\iota_-^{\top}\circ(I\otimes C')\circ\iota_-(f_{\lambda_p}^j)\rangle_p\\
&= \frac{\mu_p}{p+1}\langle C,\,\Phi_{-}(C')\rangle
\end{aligned}$$

and the statement follows.

THEOREM 3. *Φ_{+} is injective (and hence Φ_{-} is surjective).*

PROOF. By Theorem 2 it is enough to show that Φ_{-} is onto. We begin by considering the differential operator

$$D=\sum_{i=0}^{m}\frac{\partial^2}{\partial x_i\partial y_i}:\mathcal{H}^{p+1}\otimes\mathcal{H}^{p+1}\longrightarrow\mathcal{H}^p\otimes\mathcal{H}^p$$

defined by

$$D(h\otimes h')=\sum_{i=0}^{m}\frac{\partial h}{\partial y_i}\otimes\frac{\partial h'}{\partial x_i}.$$

Write $C\in S^2(\mathcal{H}^{p+1})$ as

$$\sum_{j,l=0}^{n(\lambda_{p+1})} c_{jl}f^j_{\lambda_{p+1}}\otimes f^l_{\lambda_{p+1}}\in\mathcal{H}^{p+1}\otimes\mathcal{H}^{p+1}.$$

Applying D to this we obtain

$$\begin{aligned}
&\sum_{i=0}^{m}\sum_{j,l=0}^{n(\lambda_{p+1})} c_{jl}\frac{\partial f^j_{\lambda_{p+1}}}{\partial x_i}\otimes\frac{\partial f^l_{\lambda_{p+1}}}{\partial x_i}\\
&=\sum_{i=0}^{m}\sum_{j,l=0}^{n(\lambda_{p+1})}\sum_{r,s=0}^{n(\lambda_p)} c_{jl}\left\langle\frac{\partial f^j_{\lambda_{p+1}}}{\partial x_i}, f^r_{\lambda_p}\right\rangle_p\left\langle\frac{\partial f^l_{\lambda_{p+1}}}{\partial x_i}, f^s_{\lambda_p}\right\rangle_p f^r_{\lambda_p}\otimes f^s_{\lambda_p}\\
&=\mu_p^2\sum_{i=0}^{m}\sum_{j,l=0}^{n(\lambda_{p+1})}\sum_{r,s=0}^{n(\lambda_p)} c_{jl}\langle f^j_{\lambda_{p+1}}, H(x_if^r_{\lambda_p})\rangle_{p+1}\langle f^l_{\lambda_{p+1}}, H(x_if^s_{\lambda_p})\rangle_{p+1} f^r_{\lambda_p}\otimes f^s_{\lambda_p}\\
&=\mu_p^2\sum_{i=0}^{m}\sum_{r,s=0}^{n(\lambda_p)}\langle C(H(x_if^r_{\lambda_p})), H(x_if^s_{\lambda_p})\rangle_{p+1} f^r_{\lambda_p}\otimes f^s_{\lambda_p}\\
&=\mu_p\sum_{i=0}^{m}\sum_{r,s=0}^{n(\lambda_p)}\left\langle\frac{\partial}{\partial x_i}C(H(x_if^r_{\lambda_p})), f^s_{\lambda_p}\right\rangle_p f^r_{\lambda_p}\otimes f^s_{\lambda_p}.
\end{aligned}$$

Rewriting this as an element of $S^2(\mathcal{H}^p)$ it follows that

$$D(C)(h)=\mu_p\sum_{i=0}^{m}\frac{\partial}{\partial x_i}C(H(x_ih)),\qquad h\in\mathcal{H}^p.$$

On the other hand, using (16) and (21), for $h\in\mathcal{H}^p$, we have

$$\begin{aligned}
\Phi_-(C)(h)&=\frac{1}{d^+_{p+1}}\iota^T_-(I\otimes C)\iota_-(h)\\
&=\frac{1}{d^-_{p+1}}\iota^T_-(I\otimes C)\sum_{i=0}^{m}y_i\otimes H(x_ih)\\
&=\frac{1}{d^-_{p+1}}\iota^T_-\Big(\sum_{i=0}^{m}y_i\otimes C(H(x_ih))\Big)\\
&=\frac{1}{\mu_pd^-_{p+1}}\sum_{i=0}^{m}\frac{\partial}{\partial x_i}C(H(x_ih)).
\end{aligned}$$

Combining these with Proposition 3, we obtain

$$D(C)=\frac{1}{(c^-_{p+1})^2}\Phi_-(C).$$

According to a result of DoCarmo-Wallach [1], D is onto and this finishes the proof.

Finally, to complete the circle, we show that the definitions of $\Phi_{\pm}$ given in Sections 2 and 3 ((14) and (21)) are the same on $\mathcal{L}_{\lambda_p}$.

THEOREM 4. *Let* $f: S^m \to S_V$ *be a full* λ_p*-eigenmap. Then* $\Phi_{\pm}$ *defined by* (21) *satisfies* (14).

PROOF. Write $f = A \circ f_{\lambda_p}$, where $A: \mathcal{H}^p \to V$ is a linear map. Choosing an orthonormal basis $\{e_i\}_{i=0}^n \subset V$, we have

$$f^j = \sum_{l=0}^{n(\lambda_p)} a_{jl} f^l_{\lambda_p}.$$

Applying the raising and lowering operators $^{\pm}$, we obtain

$$(f^+)^j_i = \sum_{l=0}^{n(\lambda_p)} a_{jl} (f^+_{\lambda_p})^l_i.$$

Viewing $f^{\pm}$ as maps $f^{\pm}: S^m \to S_{\mathcal{H}^1 \otimes V}$ (cf. (17) and (18)) and using (19), this latter equality translates into

$$f^{\pm} = (I \otimes A) f^{\pm}_{\lambda_p} = \frac{1}{\sqrt{d^{\pm}_p}} (I \otimes A) \iota_{\pm} (f_{\lambda_{p \pm 1}}).$$

Finally, by the definition of parametrization of the moduli space, we have

$$\begin{aligned} \langle f^{\pm} \rangle_{p \pm 1} &= \frac{1}{d^{\pm}_p} \iota^{\mathrm{T}}_{\pm} (I \otimes A)^{\mathrm{T}} (I \otimes A) \iota_{\pm} - I \\ &= \frac{1}{d^{\pm}_p} \iota^{\mathrm{T}}_{\pm} (I \otimes (A^{\mathrm{T}} A - I)) \iota_{\pm} \\ &= \Phi_{\pm}(\langle f \rangle_p), \end{aligned}$$

where we used Proposition 3 and (21). The proof is complete.

4. Minimal immersions.

Our main objective in this section is to show that

$$\Phi_{\pm}(\mathcal{M}_{\lambda_p}) \subset \mathcal{M}_{\lambda_{p \pm 1}}. \tag{22}$$

By Theorem 4, we can use the definition of $\Phi_{\pm}$ given in (14).

THEOREM 5. *Let* $f: S^m \to S^n$ *be a full (homothetic) minimal* λ_p*-eigenmap. Then* $f^{\pm}: S^m \to S^{(m+1)(n+1)-1}$ *given by* (11) *and* (12) *are (homothetic) minimal* $\lambda_{p \pm 1}$*-eigenmaps. In particular,* (22) *holds.*

REMARK. For $p=1$, f^- is constant.

PROOF OF THEOREM 5. We first claim that f^+ is homothetic with homothety λ_{p+1}/m iff f_- is homothetic with homothety λ_{p-1}/m. In fact we show that, for any vector field X on S^m, we have

$$(p+m-1)|f_*^+|^2=(m-1)|X|^2+(m+3)|f_*(X)|^2+p|f_*^-(X)|^2. \tag{23}$$

Comparing the homothety constants, the claim will then follow. Turning to the proof of (23), using the harmonic projection formula and $\sum_{i=0}^m x_i \check{X}^i=0$, we compute

$$\begin{aligned}
\frac{p+m-1}{2p+m-1}|f_*^+(X)|^2 &= \sum_{i=0}^m \sum_{j=0}^n \left[\sum_{k=0}^m \frac{\partial H(x_i f^j)}{\partial x_k}\check{X}^k\right]^2\\
&= \sum_{i=0}^m \sum_{j=0}^n \left[\sum_{k=0}^m \frac{\partial}{\partial x_k}\left(x_i f^j-\frac{\rho^2}{2p+m-1}\frac{\partial f^j}{\partial x_i}\right)\Big|_{\rho=1}\check{X}^k\right]^2\\
&= \sum_{i=0}^m \sum_{j=0}^n \Bigg[f^j\check{X}^i+x_i\sum_{k=0}^m\frac{\partial f^j}{\partial x_k}\check{X}^k\\
&\qquad -\frac{1}{2p+m-1}\sum_{k=0}^m\frac{\partial^2 f^j}{\partial x_i\partial x_k}\check{X}^k\Bigg]^2\\
&= |X|^2+|f_*(X)|^2+\frac{p}{2p+m-1}|f_*^-(X)|^2\\
&\qquad -\frac{2}{2p+m-1}\sum_{i,k=0}^m\sum_{j=0}^n f^j\frac{\partial^2 f^j}{\partial x_i\partial x_k}\check{X}^i\check{X}^k\\
&\qquad -\frac{2}{2p+m-1}\sum_{i=0}^m\sum_{j=0}^n\left(\sum_{r=0}^m\frac{\partial f^i}{\partial x_r}\check{X}^r\right)\\
&\qquad \times\left(\sum_{s=0}^m x_i\frac{\partial^2 f^j}{\partial x_i\partial x_s}\check{X}^s\right).
\end{aligned}$$

For the last but one term we have

$$\begin{aligned}
\sum_{i,k=0}^m\sum_{j=0}^n f^j\frac{\partial^2 f^j}{\partial x_i\partial x_k}\check{X}^i\check{X}^k &= \sum_{i,k=0}^m\sum_{j=0}^n\frac{\partial}{\partial x_i}\left(f^j\frac{\partial f^j}{\partial x_k}\right)\check{X}^i\check{X}^k\\
&\qquad -\sum_{i,k=0}^m\sum_{j=0}^n\frac{\partial f^j}{\partial x_i}\frac{\partial f^j}{\partial x_k}\check{X}^i\check{X}^k\\
&= \frac{1}{2}\sum_{i,k=0}^m\frac{\partial^2}{\partial x_i\partial x_k}(\rho^{2p})\check{X}^i\check{X}^k\\
&\qquad -\sum_{j=0}^n\left(\sum_{i=0}^m\frac{\partial f^j}{\partial x_i}\check{X}^i\right)^2\\
&= p|X|^2-|f_*(X)|^2.
\end{aligned}$$

For the last term we use homogeneity to obtain

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\Big(\sum_{r=0}^{m}\frac{\partial f^j}{\partial x_r}\check{X}^r\Big)\Big(\sum_{s=0}^{m}x_i\frac{\partial^2 f^j}{\partial x_i\partial x_s}\check{X}^s\Big)=(p-1)\sum_{j=0}^{n}\Big(\sum_{i=0}^{m}\frac{\partial f^j}{\partial x_i}\check{X}^i\Big)^2$$
$$=(p-1)|f_*(X)|^2.$$

Putting these together (23) follows.

It remains to show that f^- is homothetic. To do this, without loss of generality, we may assume that X is conformal. Using quadratic extension of $\check{X}$ to $\boldsymbol{R}^{m+1}$, this means that

$$\check{X}=\rho^2A-\langle A,\,x\rangle x\,,\qquad A\in\boldsymbol{R}^{m+1}.$$

To simplify the notation, we introduce the gradient vectors (on $\boldsymbol{R}^{m+1}$):

$$F^j=\operatorname{grad} f^j\,,\qquad j=0,\,\cdots,\,n\,.$$

Using this, (11), (12) and homogeneity, we have (on S^m):

$$p(2p+m-1)|f_*^-(X)|^2=\sum_{i=0}^{m}\sum_{j=0}^{n}\Big|\Big(\frac{\partial f^j}{\partial x_i}\Big)_*(X)\Big|^2$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\Big(\Big\langle\frac{\partial F^j}{\partial x_i},\,A\Big\rangle-(p-1)\langle A,\,x\rangle\frac{\partial f^j}{\partial x_i}\Big)^2$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\Big\langle\frac{\partial F^j}{\partial x_i},\,A\Big\rangle^2+(p-1)^2\langle A,\,x\rangle^2\sum_{i=0}^{m}\sum_{j=0}^{n}\Big(\frac{\partial f^j}{\partial x_i}\Big)^2$$
$$-2(p-1)\langle A,\,x\rangle\sum_{i=0}^{m}\sum_{j=0}^{n}\Big\langle\frac{\partial F^j}{\partial x_i},\,A\Big\rangle\frac{\partial f^j}{\partial x_i}.$$

Using (13) the last sum here rewrites as

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\Big\langle\frac{\partial F^j}{\partial x_i},\,A\Big\rangle\frac{\partial f^j}{\partial x_i}=\sum_{i,k=0}^{m}\sum_{j=0}^{n}\frac{\partial^2 f^j}{\partial x_i\partial x_k}\frac{\partial f^j}{\partial x_i}A^k$$
$$=\frac{1}{2}\sum_{k=0}^{m}\frac{\partial}{\partial x_k}\Big[\sum_{i=0}^{m}\sum_{j=0}^{n}\Big(\frac{\partial f^j}{\partial x_i}\Big)^2\Big]A^k$$
$$=\frac{p(2p+m-1)}{2}\sum_{k=0}^{m}\frac{\partial\rho^{2(p-1)}}{\partial x_k}\Big|_{\rho=1}A^k$$
$$=p(p-1)(2p+m-1)\langle A,\,x\rangle\,.$$

So that we obtain

$$p(2p+m-1)|f_*^-(X)|^2=\sum_{i=0}^{m}\sum_{j=0}^{n}\Big\langle\frac{\partial F^j}{\partial x_i},\,A\Big\rangle^2-p(p-1)^2(2p+m-1)\langle A,\,x\rangle^2. \tag{24}$$

It remains to evaluate the first term on the R.H.S. of (24). To do this we write the homogeneous extension of (8) in coordinates as

$$\sum_{j=0}^{m}\langle F^j,\,\check{X}\rangle=\frac{\lambda_p}{m}\rho^{2(p-1)}|\check{X}|^2. \tag{25}$$

For the L. H. S. of this equation we have

$$\sum_{j=0}^{n}\langle F^j, \check{X}\rangle^2=\sum_{j=0}^{n}[\rho^2\langle F^j, A\rangle-p\langle A, x\rangle f^j]^2$$

$$=\rho^4\sum_{j=0}^{n}\langle F^j, A\rangle^2+p^2\rho^{2p}\langle A, x\rangle^2$$

$$-2p\rho^2\langle A, x\rangle\sum_{j=0}^{n}\langle F^j, A\rangle f^j.$$

For the last term here we have

$$\sum_{j=0}^{n}\langle F^j, A\rangle f^j=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\partial f^j}{\partial x_i}f^jA^i$$

$$=\frac{1}{2}\sum_{i=0}^{m}\frac{\partial\rho^{2p}}{\partial x_i}A^i$$

$$=p\rho^{2(p-1)}\langle A, x\rangle.$$

Combining these with (25) we obtain

$$\sum_{j=0}^{n}\langle F^j, \check{X}\rangle^2=\rho^4\sum_{j=0}^{n}\langle F^j, A\rangle^2-p^2\rho^{2p}\langle A, x\rangle^2$$

$$=\frac{p(p+m-1)}{m}\rho^{2p}(\rho^2|A|^2-\langle A, x\rangle^2).$$

Rearranging we finally have

$$\sum_{j=0}^{n}\langle F^j, A\rangle^2=\frac{p(p+m-1)}{m}\rho^{2(p-1)}|A|^2+\frac{p(p-1)(m-1)}{m}\rho^{2(p-2)}\langle A, x\rangle^2.$$

We now take the Laplacian of both sides. Using the formula

$$\Delta(\rho^{2(p-2)}\langle A, x\rangle^2)|_{\rho=1}=2|A|^2+2(p-2)(2p+m-1)\langle A, x\rangle^2,$$

we obtain

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\left\langle\frac{\partial F^j}{\partial x_i}, A\right\rangle^2=\frac{p(2p+m-1)(p-1)(p+m-2)}{m}|A|^2$$

$$+\frac{p(p-1)(p-2)(m-1)(2p+m-1)}{m}\langle A, x\rangle^2.$$

Combining this with (24), we get

$$p(2p+m-1)|f_*^-(X)|^2=p(2p+m-1)\frac{\lambda_{p-1}}{m}(|A|^2-\langle A, x\rangle^2).$$

which completes the proof.

REMARK. As $\mathcal{M}_{\lambda_p}$ spans $\mathcal{F}_{\lambda_p}$, it follows that

$$\Phi_{\pm}(\mathcal{F}_{\lambda_p}) \subset \mathcal{F}_{\lambda_{p\pm1}}.$$

COROLLARY 1. *Assume that for fixed $m \geqq 3$ and $p_0 \geqq 4$, equality holds in* (9). *Then, for $p \geqq p_0$, the modules $V_{m+1}^{(2l, 2, 0, \cdots, 0)} \subset S^2(\mathcal{H}^p)$, $l=1, \cdots, p_0-1$, are not components of $\mathcal{F}_{\lambda_{p_0}} \otimes_{\boldsymbol{R}} \boldsymbol{C}$.*

PROOF. By Theorem 3, the homomorphism

$$(\Phi_-)^{p-p_0} : \mathcal{E}_{\lambda_p} \longrightarrow \mathcal{E}_{\lambda_{p_0}}$$

is surjective and hence the kernel of its complexification consists of the modules $V_{m+1}^{(a, b, 0, \cdots, 0)}$ with $(a, b) \in \Delta^p \setminus \Delta^{p_0}$, a, b even. If, for some $l=1, \cdots, p_0-1$, the module $V_{m+1}^{(2l, 2, 0, \cdots, 0)}$ were a component of $\mathcal{F}_{\lambda_p} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ then, since $(2l, 2, 0, \cdots, 0) \in \Delta^{p_0}$, $(\Phi_-)^{p-p_0} V_{m+1}^{(2l, 2, 0, \cdots, 0)} \cong V_{m+1}^{(2l, 2, 0, \cdots, 0)}$ and so it would also be a component of $\mathcal{F}_{\lambda_{p_0}} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ which is a contradiction.

Muto has shown that, for $m=3$ and $p_0=4$, equality holds in (9) [**3**]. Hence we obtain the following

COROLLARY 2. *For $m=3$ and for any $p \geqq 4$, the modules*

$$V_4^{(2, \pm 2)} \quad V_4^{(4, \pm 2)} \quad \textit{and} \quad V_4^{(6, \pm 2)}$$

are not components of $\mathcal{F}_{\lambda_p} \otimes_{\boldsymbol{R}} \boldsymbol{C}$.

5. Range dimensions.

Returning to the general situation we now study the distribution of values of the range dimension n of full λ_p-eigenmaps $f : S^m \to S^n$. To emphasize the dependence of n on f we put $n = n(f) = n(\langle f \rangle_{\lambda_p})$. Since $rank\, A = rank\,(A^{\mathsf{T}} A)$, for any matrix A, we have

$$n(\langle f \rangle_{\lambda_p}) + 1 = rank\,(\langle f \rangle_{\lambda_p} + I).$$

PROPOSITION 5. *The map $\Phi_+ : \mathcal{L}_{\lambda_p} \to \mathcal{L}_{\lambda_{p+1}}$ does not decrease the range dimension, or equivalently, for any full λ_p-eigenmap $f : S^m \to S^n$, we have*

$$n(f) \leqq n(f^+).$$

(Here f^+ is considered to be full).

PROOF. Since f is full its components $f^0, \cdots, f^n$ are linearly independent. Thus it is enough to show that, for fixed $k=0, \cdots, m$, the polynomials

$$H(x_k f^0), \cdots, H(x_k f^n)$$

are also linearly independent. Assume that

$$\sum_{j=0}^{n} c_j H(x_k f^j) = 0, \qquad c_j \in \boldsymbol{R}, \quad j=0, \cdots, n.$$

Setting $h=\sum_{j=0}^{n}c_j f^j$, we obtain

$$H(x_k h)=x_k h-\frac{\rho^2}{2p+m-1}\frac{\partial h}{\partial x_k}=0\,. \tag{26}$$

We write

$$h(x_0,\cdots,x_m)=x_k^q h'(x_0,\cdots,\hat{x}_k,\cdots,x_m)+h''(x_0,\cdots,x_m),$$

where the monomials in h'' contain x_k with degree $<q$. Substituting this into (26) and comparing the coefficients of x_k^{q+1}, we obtain that $h'=0$. This means that h does not depend on x_k and so, again by (26), it must vanish. Hence $c_j=0$, $j=0,\cdots,n$, and the proof is complete.

The interior of the moduli space $\mathcal{L}_{\lambda_p}$ corresponds to those full λ_p-eigenmaps $f:S^m\to S^n$ for which $n(f)=n(f_{\lambda_p})=n(\lambda_p)$. Hence the boundary of $\mathcal{L}_{\lambda_p}$ can be written as

$$\partial\mathcal{L}_{\lambda_p}=\{C\in\mathcal{E}_{\lambda_p}\mid rank\,(C+I)<n(\lambda_p)+1\}.$$

We now study the restriction of Φ_+ to $\partial\mathcal{L}_{\lambda_p}$. Clearly, if $f:S^m\to S^n$ is a full λ_p-eigenmap with $(m+1)(n+1)<n(\lambda_{p+1})+1$ then Φ_+ maps $\langle f\rangle_{\lambda_p}\in\partial\mathcal{L}_{\lambda_p}$ to a boundary point in $\partial\mathcal{L}_{\lambda_{p+1}}$. (As an example, take the Hopf map $f:S^3\to S^2$ which is quadratic, i.e. $p=2$.) For higher range dimensions however Φ_+ can well map boundary points of $\mathcal{L}_{\lambda_p}$ into the interior of $\mathcal{L}_{\lambda_{p+1}}$ as the following result shows.

THEOREM 6. *For any full λ_p-eigenmap*

$$f:S^m\longrightarrow S^{n(\lambda_p)-1},\qquad m\geqq 3,$$

we have

$$\Phi_+(\langle f\rangle_{\lambda_p})\in int\,\mathcal{L}_{\lambda_{p+1}}\,.$$

PROOF. If the statement were false then there would exist a nonzero $h\in\mathcal{H}^{p+1}$ such that

$$\langle h,\,H(x_i f^j)\rangle_{p+1}=0$$

for all $i=0,\cdots,m$ and $j=0,\cdots,n$. By Proposition 2, this is equivalent to $\partial h/\partial x_i$, $i=0,\cdots,m$ being orthogonal to $span\,\{f^j\mid j=0,\cdots,n\}$. By assumption the latter is of codimension one in $\mathcal{H}^p$ so that we have

$$a_0\frac{\partial h}{\partial x_0}=\cdots=a_m\frac{\partial h}{\partial x_m} \tag{27}$$

for some nonzero constants $a_i\in\boldsymbol{R}$, $i=0,\cdots,m$. Taking partial derivatives, we obtain

$$a_i^2\frac{\partial^2 h}{\partial x_i^2}=a_k^2\frac{\partial^2 h}{\partial x_k^2},\qquad i,\,k=0,\cdots,m\,.$$

By harmonicity of h, we then get

$$\frac{\partial^2 h}{\partial x_i^2}=0$$

for all $i=0, \cdots, m$. Equivalently, for fixed $k=0, \cdots, m$, we can write

$$h(x_0, \cdots, x_m)=x_k h'(x_0, \cdots, \hat{x}_k, \cdots, x_m)+h''(x_0, \cdots, \hat{x}_k, \cdots, x_m).$$

Substituting this into the Euler equation (using (27)):

$$\sum_{i=0}^{m} x_i \frac{\partial h}{\partial x_i}=a_k \frac{\partial h}{\partial x_k} \sum_{i=0}^{m} \frac{x_i}{a_i}=(p+1)h$$

and comparing the coefficients of x_k we obtain $h=0$.

REMARK. According to a result in [6], for $m \geqq 5$ and $p \geqq 2$, full λ_p-eigenmaps $f: S^m \to S^{n(\lambda_p)-1}$ exist.

REMARK. The intersection of $\mathcal{L}_{\lambda_{p+1}}$ with the cokernel of Φ_+ corresponds to λ_{p+1}-eigenmaps $f: S^m \to S^n$ with higher range dimension n. In fact, by Theorem 2, for such maps f, we have $\langle f \rangle_{\lambda_{p+1}} \in ker\, \Phi_-$ and so

$$\Phi_-(\langle f \rangle_{\lambda_{p+1}})=\langle f^- \rangle_{\lambda_p}=\langle f_{\lambda_p} \rangle_{\lambda_p}=0.$$

In particular, $\partial f^j / \partial x_i$, $i=0, \cdots, m$, $j=0, \cdots, n$ span $\mathcal{H}^p$. A necessary condition for this is

$$n(f)+1 \geqq \frac{n(\lambda_p)+1}{m+1}.$$

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