The Problem of Plateau (pp. 327–335)
ed. Th. M. Rassias

ON THE NUMBER OF RIGID MINIMAL IMMERSIONS BETWEEN SPHERES

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Abstract

We show, by a Baire Category argument applied to the parameter space of all minimal immersions between spheres, that linearly rigid minimal immersions abound for sufficiently high degree.

1 Introduction and Preliminaries

It is well known that a (full) homothetic immersion $f : S^m \to S^n$ between Euclidean spheres is minimal iff the components of f are (linearly independent) spherical harmonics on S^m of a fixed order k, or equivalently,

$$\Delta f = \lambda_k f,$$

where $\lambda_k = k(k+m-1)$ is the kth eigenvalue of the Laplace-Beltrami operator Δ on S^m. In this case, k is said to be the (algebraic) degree of f. The metric induced by f on S^m then has (constant) curvature m/λ_k.

The universal example is given by the *standard minimal immersion*

$$f_{\lambda_k} : S^m \to S^{n(\lambda_k)}$$

whose components comprise an orthonormal basis $\{f^i_{\lambda_k}\}_{i=0}^{n(\lambda_k)}$ in the space $\mathcal{H}^k_{S^m}$ of spherical harmonics on S^m of order k endowed with the normalized L^2-scalar product

$$\langle \mu, \mu' \rangle = \frac{n(\lambda_k)+1}{\mathrm{vol}\,(S^m)} \int_{S^m} \mu\mu' \cdot v_{S^m}, \ \mu, \mu' \in \mathcal{H}^k_{S^m},$$

where

$$n(\lambda_k)+1=\dim \mathcal{H}^k_{S^m}=(m+2k-1)\frac{(m+k-2)!}{k!(m-1)!}.$$

Here f_{λ_k} is universal in the sense that, for any full minimal immersion $f: S^m \to S^n$ of degree k, we have

$$f = A \cdot f_{\lambda_k}$$

for some (uniquely determined) $(n+1)\times(n(\lambda_k)+1)$-matrix of maximal rank. The Do Carmo-Wallach parametrization associates to f the symmetric matrix

$$\langle f\rangle_{\lambda_k} = A^{\top}A - I \in S^2(\mathbf{R}^{n(\lambda_k)+1}),$$

where I is the identity. Two full minimal immersions $f': S^m \to S^n$ and $f'': S^m \to S^n$ of degree k are said to be *range* (resp. *domain*) *equivalent* if there exists $U \in O(n+1)$ (resp. $a \in SO(m+1)$) such that $f'' = U \cdot f'$ (resp. $f'' = f' \circ a$). If $f'' = U \cdot f' \circ a$ for some $U \in O(n+1)$ and $a \in SO(m+1)$ then f' and f'' are said to be *equivalent*. Clealy, $\langle f\rangle_{\lambda_k}$ depends only on the range equivalence class of f. As in [2,6], we obtain that the space of range equivalence classes of full minimal immersions $f: S^m \to S^n$, of degree k, (hence $n \leq n(\lambda_k)$), is parametrized by the compact convex body

$$\mathcal{L}_{\lambda_k} = \{C \in \mathcal{F}_{\lambda_k} | C + I \geq 0\}, \tag{1}$$

where the linear subspace $\mathcal{F}_{\lambda_k} \subset S^2(\mathbf{R}^{n(\lambda_k)+1})$ is the orthogonal complement of the set of all projections

$$\operatorname{proj}[(f_{\lambda_k})_*(X)] = (f_{\lambda_k})_*(X)\cdot(f_{\lambda_k})_*(X) \in S^2(\mathbf{R}^{n(\lambda_k)+1}),\ X \in T(S^m). \tag{2}$$

(Here, $(f_{\lambda_k})_*(X)$ is shifted to the origin of $\mathbf{R}^{n(\lambda_k)+1}$ and $\cdot$ stands for the symmetric tensor product.) By birth, $f_{\lambda_k}: S^m \to S^{n(\lambda_k)}$ is equivariant with respect to the homomorphism $\rho_{\lambda_k}: SO(m+1) \to SO(n(\lambda_k)+1)$ that is just the orthogonal $SO(m+1)$-module structure on $\mathcal{H}^k_{S^m} \cong \mathbf{R}^{n(\lambda_k)+1}$, where the isomorphism is given by the orthonormal basis chosen for f_{λ_k}. Taking the symmetric square of the representation given by ρ_{λ_k}, we obtain that $\mathcal{F}_{\lambda_k}$ is an $SO(m+1)$-submodule of $S^2(\mathbf{R}^{n(\lambda_k)+1})$ with $\mathcal{L}_{\lambda_k} \subset \mathcal{F}_{\lambda_k}$ an $SO(m+1)$-invariant subspace. In fact,

$$a\cdot\langle f\rangle_{\lambda_k} = \langle f\circ a^{-1}\rangle_{\lambda_k},\ a \in SO(m+1). \tag{3}$$

In particular, the $SO(m+1)$-orbit of $\langle f\rangle_{\lambda_k}$ corresponds to those full minimal immersions that are (domain) equivalent to f. A full (minimal) immersion

$f: S^m \to S^n$ of degree k is said to be *linearly rigid* [2,6] if whenever A is an $(n+1)\times(n+1)$-matrix that satisfies $A(f(S^m)) \subset S^n$ and $A\cdot f: S^m \to S^n$ is a homothetic (minimal) immersion (of degree k) then $A \in O(n+1)$. Linear rigidity, when applied to f_{λ_k}, is clearly equivalent to $\mathcal{L}_{\lambda_k} = \mathcal{F}_{\lambda_k} = \{0\}$. For $m = 2$, linear rigidity of f_{λ_k} was proved by Calabi in [1]. Do Carmo and Wallach [2,6] showed that f_{λ_k} is linearly rigid for $k \le 3$. Finally, they also proved that, for $m \ge 3$ and $k \ge 4$, f_{λ_k} is not linearly rigid. More precisely, for $m \ge 3$, we have

$$\mathcal{F}_{\lambda_k} \otimes_{\mathbf{R}} \mathbf{C} \supset S^2(\mathcal{H}^k_{S^m})/\{\sum_{j=0}^{k} \mathcal{H}^{2j}_{S^m} \oplus \sum_{j=1}^{k-1} V^{(2j,2,0,\dots,0)}_{SO(m+1)}\} = \sum_{(a,b)\in\Delta,\, a,b \text{ even}} V^{(a,b,0,\dots,0)}_{SO(m+1)}$$

as $SO(m+1)$-modules, where $\Delta \subset \mathbf{R}^2$ is the closed triangular domain with vertices $(4,4)$, (k,k) and $(2k-4,4)$. In these formulas we use standard terminology in representation theory, namely, $V^{\rho}_{SO(m+1)}$ is the complex irreducible $SO(m+1)$-module with highest weight ρ (whose components are with respect to the standard maximal torus) and, for the moment, the spherical harmonics are complex valued. In particular, by the Weyl dimension formula, we obtain

$$\begin{aligned}\dim \mathcal{L}_{\lambda_k} &= \dim \mathcal{F}_{\lambda_k} \ge \frac{1}{2}(n(\lambda_k)+1)(n(\lambda_k)+2) - \sum_{j=0}^{k}(n(\lambda_{2j})+1)\\ &- \frac{1}{2}(m+1)(m-2)\sum_{j=1}^{k-1}\frac{2j-1}{2j+1}\frac{2j+m}{2j+m-2}(n(\lambda_{2j})+1). \qquad (4)\end{aligned}$$

Remark As shown by Muto [4] the lower estimate is sharp for $m = 3$ and $k = 4$, i.e., in this case, $\dim \mathcal{L}_{\lambda_2} = 18$.

The purpose of this paper is to show that linearly rigid minimal immersions abound.

Theorem 1 *Let $m \ge 3$ and $k \ge 4$ and assume that*

$$\begin{aligned}(n(\lambda_k)+1)(n(\lambda_k)+2) &> (n(\lambda_k)-m)(m(m+1)+2) + 2\sum_{j=0}^{k}(n(\lambda_{2j})+1)\\ &+ (m+1)(m-2)\sum_{j=1}^{k-1}\frac{2j-1}{2j+1}\frac{2j+m}{2j+m-2}(n(\lambda_{2j})+1)(5)\end{aligned}$$

Then there exist $\aleph_1$ mutually inequivalent full linearly rigid minimal immersions $f: S^m \to S^n$ of degree k. Moreover, for each $m \ge 3$, there exists $k(m) \ge 4$ such that (5) holds for $k \ge k(m)$.

Remarks 1. The last assertion of Theorem 1 is clear. If fact, for fixed $m \geq 3$, all the terms in (5) are polynomials in k with positive coefficients. The left hand side of (5) is of degree $2(m-1)$ and, on the right hand side, the first term is of degree $m-1$ while the second and third are of degree m. In particular, as easy computation shows, $k(3) = 7$.

2. Similar result can be obtained for harmonic λ_k-eigenmaps $f : S^m \to S^n$ (omitting the condition that the maps are homothetic immersions [3]). By contrast, note that, for $m = 3$ and $k = 2$, up to equivalence, the only full linearly rigid λ_2-eigenmap is the Hopf map $f : S^3 \to S^2$ (and its 'dual') [5].

3. For further explicit examples of λ_2-eigenmaps consider the case $m = 4$ and $k = 2$. Let S^4 be the unit sphere in $\mathbf{C}^2 \times \mathbf{R}$ with coordinates $z, w \in \mathbf{C}$ and $t \in \mathbf{R}$. Define $f : S^4 \to S^7$ and $g : S^4 \to S^4$ by

$$f(z,w,t) = (\frac{1}{4}(|z|^2 + |w|^2) - t^2, \frac{\sqrt{15}}{4}(|z|^2 - |w|^2), \frac{\sqrt{15}}{2}zw, \sqrt{\frac{5}{2}}zt, \sqrt{\frac{5}{2}}wt)$$

and

$$g(z,w,t) = (\frac{1}{2}|z|^2 - |w|^2 + t^2, \frac{\sqrt{3}}{2}z^2 - 2wt, \sqrt{3}\bar{z}w + zt).$$

Computation shows that f and g are both linearly rigid. (Note that g is the gradient of a cubic isoparametric function on S^4 [3].)

To prove Theorem 1, in §2, we introduce an $SO(m+1)$-saturation on $\mathcal{L}_{\lambda_k}$ with the property that the one point cells correspond to the linearly rigid immersions. In §3 we then use an inductive argument with respect to the dimension of the cells of the saturation of $\mathcal{L}_{\lambda_k}$ to get an upper bound for $\dim \mathcal{L}_{\lambda_k}$ provided that the cardinality of the saturation modulo $SO(m+1)$ is $\leq \aleph_0$. This, for k large, will contradict to the lower estimate of $\dim \mathcal{L}_{\lambda_k}$ given in (4).

2 The fine structure of the parameter space

Let $f : S^m \to S^n$ and $f' : S^m \to S^{n'}$ be full minimal immersions of degree k and k', respectively. f' is said to be *derived* from f, written as $f' \leftarrow f$, if there exists an $(n'+1) \times (n+1)$-matrix A such that $f' = A \cdot f$. In this case $k = k'$.

Given a full minimal immersion $f : S^m \to S^n$ of degree k, we define

$$\mathcal{L}_f = \{C' \in \mathcal{F}_f | C' + I \geq 0\}, \tag{6}$$

where the linear subspace $\mathcal{F}_f \subset S^2(\mathbf{R}^{n+1})$ is the orthogonal complement of the set of all projections

$$\mathrm{proj}\,[f_*(X)] \in S^2(\mathbf{R}^{n+1}),\ X \in T(S^m).$$

Clearly, we have $\mathcal{L}_{f_{\lambda_k}} = \mathcal{L}_{\lambda_k}$ and $\mathcal{F}_{f_{\lambda_k}} = \mathcal{F}_{\lambda_k}$ and the argument of Do Carmo and Wallach applies yielding that $\mathcal{L}_f \subset \mathcal{F}_f$ is a compact convex body that parametrizes the range equivalence classes of full minimal immersions $f' : S^m \to S^{n'}$ that are derived from f. Let

$$\iota : S^2(\mathbf{R}^{n+1}) \to S^2(\mathbf{R}^{n(\lambda_k)+1})$$

be the affine map defined by

$$\iota(C') = A^{\mathsf{T}} C' A + \langle f \rangle_{\lambda_k} = A^{\mathsf{T}}(C' + I)A - I,\ C' \in S^2(\mathbf{R}^{n+1}), \tag{7}$$

where $f = A \cdot f_{\lambda_k}$.

Proposition 1 *ι is injective and maps $\mathcal{F}_f$ into $\mathcal{F}_{\lambda_k}$. Moreover, we have*

$$\iota(\mathcal{L}_f) = \iota(\mathcal{F}_f) \cap \mathcal{L}_{\lambda_k}.$$

Proof. f is full so that A is of maximal rank. Hence A^{T} has zero kernel and injectivity of ι follows. Given $C' \in \mathcal{F}_f$, for $X \in T(S^m)$, we have

$$\begin{aligned}
\langle \iota(C'), \mathrm{proj}\,[(f_{\lambda_k})_*(X)] \rangle &= \langle \iota(C')(f_{\lambda_k})_*(X), (f_{\lambda_k})_*(X) \rangle \\
&= \langle (C' + I) f_*(X), f_*(X) \rangle - \langle (f_{\lambda_k})_*(X), (f_{\lambda_k})_*(X) \rangle \\
&= \langle C', \mathrm{proj}\,[f_*(X)] \rangle,
\end{aligned}$$

where the last equality is because f and f_{λ_k} have the same homothety constant. The second assertion follows. Finally, the third is obtained by comparing (1) and (6) via (7).

For a full minimal immersion $f : S^m \to S^n$, we define

$$I_f = \iota(\mathcal{L}_f^\circ) \subset \mathcal{L}_{\lambda_k},$$

where the circle stands for the interior. Clearly, I_f is convex and open in $\iota(\mathcal{F}_f)$ and contains $\langle f\rangle_{\lambda_k}$. Then I_f is said to be the *cell* associated to f. Its points correspond to those full minimal immersions $f' : S^m \to S^n$ of degree k that are derived from f, i.e. $f' = A \cdot f$ with A invertible. The cells I_f, for the various f, give rise to a decomposition of $\mathcal{L}_{\lambda_k}$ into mutually disjoint convex sets that comprise the natural saturation $\mathcal{I}_{\lambda_k} = \{I_f | f \leftarrow f_{\lambda_k}\}$ of $\mathcal{L}_{\lambda_k}$. One of the most important property of $\mathcal{I}_{\lambda_k}$ is that when passing to the boundary of a cell the dimension of the range of the corresponding minimal immersions strictly decreases. In particular, $I_{f_{\lambda_k}} = \mathcal{L}^\circ_{\lambda_k}$ so that the natural saturation is of interest only on the boundary $\partial\mathcal{L}_{\lambda_k}$.

The action of $SO(m+1)$ respects $\mathcal{I}_{\lambda_k}$, in fact, by (3), we have

$$a \cdot I_f = I_{f\circ a^{-1}}, \quad a \in SO(m+1).$$

We obtain that $\mathcal{I}_{\lambda_k}$ is an $SO(m+1)$-saturation of $\mathcal{L}_{\lambda_k}$. By construction, a full minimal immersion $f : S^m \to S^n$ of degree k is linearly rigid iff I_f is a one point cell.

Two full minimal immersions $f' : S^m \to S^{n'}$ and $f'' : S^m \to S^{n''}$ of degree k are said to be *geometrically distinct* if, for each $U' \in O(n'+1)$, $U'' \in O(n''+1)$ and $a', a'' \in SO(m+1)$, none of the minimal immersions

$$U' \cdot f' \circ a' \quad \text{and} \quad U'' \cdot f'' \circ a''$$

can be derived from each other. In terms of $\mathcal{I}_{\lambda_k}$, this holds iff none of the orbits

$$SO(m+1) \cdot I_{f'} \quad \text{and} \quad SO(m+1) \cdot I_{f''}$$

is contained in the other. In the special case when f' and f'' are both linearly rigid, f' and f'' are geometrically distinct iff they are inequivalent. Theorem 1 will therefore be proved if we show the following:

Theorem 2 *Assume that*

$$\dim \mathcal{L}_{\lambda_k} > (n(\lambda_k) - m)(\dim SO(m+1) + 1). \tag{8}$$

Then the cardinality of the orbit space $\mathcal{I}_{\lambda_k}/SO(m+1)$ is $\aleph_1$. In particular there exist $\aleph_1$ geometrically distinct full minimal immersions $f : S^m \to S^n$ of degree k. Moreover, there exist $\aleph_1$ inequivalent full linearly rigid minimal immersions $f : S^m \to S^n$ of degree k.

3 Proof of Theorem 2

Assume that $\mathcal{I}_{\lambda_k}/SO(m+1)$ is countable. Since the interior of $\mathcal{L}_{\lambda_k}$ is a cell the set of $SO(m+1)$-orbits of cells on $\partial\mathcal{L}_{\lambda_k}$ is also countable. By the Baire Category Theorem, at least one $SO(m+1)$-orbit of a cell has nonempty interior. Let $\langle f_1\rangle_{\lambda_k}$ be an interior point. Then $f_1 : S^m \to S^{n_1}$ is a full minimal immersion of degree k with $n_1 \le n(\lambda_k) - 1$. Moreover, we have

$$\dim \mathcal{L}_{\lambda_k} = \dim(SO(m+1)\cdot I_{f_1}) + 1 \le \dim SO(m+1) + 1 + \dim I_{f_1}. \quad (9)$$

We now take the set of $SO(m+1)$-orbits of cells which lie on the boundary $\partial I_{f_1} = \bar{I}_{f_1} \setminus I_{f_1}$. The intersections of these $SO(m+1)$-orbits with ∂I_{f_1} give a partition of ∂I_{f_1} into countably many subsets. Again by the Baire Category Theorem at least one $SO(m+1)$-orbit of a cell intersects ∂I_{f_1} in a set with nonempty interior in ∂I_{f_1}. Let $\langle f_2\rangle_{\lambda_k}$ be an interior point. Then $f_2 : S^m \to S^{n_2}$ is a full minimal immersion of degree k with $n_2 \le n_1 - 1 \le n(\lambda_k) - 2$. Moreover, I_{f_2} is contained in this intersection so that we have

$$\begin{aligned} \dim I_{f_1} &= \dim(SO(m+1)\cdot I_{f_2} \cap I_{f_1}) + 1 \\ &\le \dim(SO(m+1)\cdot I_{f_2}) + 1 \\ &\le \dim SO(m+1) + 1 + \dim I_{f_2}. \end{aligned}$$

Combining this with (8), we obtain

$$\dim \mathcal{L}_{\lambda_k} \le 2(\dim SO(m+1) + 1) + \dim I_{f_2}$$

and

$$n_2 \le n(\lambda_k) - 2.$$

Repeating this, in the ℓth step we obtain

$$\dim \mathcal{L}_{\lambda_k} \le \ell(\dim SO(m+1) + 1) + \dim I_{f_\ell}$$

and

$$n_\ell \le n(\lambda_k) - \ell.$$

The procedure clearly stops in $n(\lambda_k) - m$ steps yielding

$$\dim \mathcal{L}_{\lambda_k} \le (n(\lambda_k) - m)(\dim SO(m+1) + 1)$$

which contradicts to (8).

To prove the second statement, choose geometrically distinct full minimal immersions $f_\sigma : S^m \to S^{n_\sigma}$, $\sigma \in \Sigma$, where Σ is of cardinality $\aleph_1$. For $\sigma \in \Sigma$, choose a finite set $\Lambda_\sigma \subset \bar{I}_{f_\sigma} (\subset \mathcal{L}_{\lambda_k})$ consisting of points that correspond to linearly rigid full minimal immersions such that the affine span of Λ_k is equal to that of $\bar{I}_{f_\sigma}$. The existence of Λ_σ follows easily by induction with respect to the dimension of the cells comprising $\bar{I}_{f_\sigma}$.

We now claim that, for $\sigma, \sigma' \in \Sigma$, $\sigma \neq \sigma'$, we have $\Lambda_\sigma \neq \Lambda_{\sigma'}$. In fact, $\Lambda_\sigma = \Lambda_{\sigma'}$ iff $I_{f_\sigma} = I_{f_{\sigma'}}$ so that f_σ and $f_{\sigma'}$ are not geometrically distinct; a contradiction. We obtain that the set $\{\Lambda_\sigma | \sigma \in \Sigma\}$ has cardinality $\aleph_1$. Since the set of all finite subsets of a countable set is countable, $\cup_{\sigma \in \Sigma} \Lambda_\sigma$ has cardinality $\aleph_1$. The proof of Theorem 2 is complete.

Remark. The action of $SO(m+1)$ on $\mathcal{L}_{\lambda_k}$ has further interesting properties that are related to the ones used in the proof above. It can be shown, e.g. that the principal isotropy type is finite provided that $\mathcal{L}_{\lambda_k}$ is nontrivial. Furthermore, it can be proved that the orbit of the center of mass of any cell I_f is always transversal (in the weak sense) to I_f. In particular, equality holds in (9), provided that the orbit of the center of mass is principal. For $m = 3$ this is actually the case for the top (5-)dimensional cell on the boundary of the parameter space for λ_2-eigenmaps [5].

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