Moduli Spaces of Polynomial Minimal Immersions between Complex Projective Spaces

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1. Introduction and Preliminaries

In [1; 6] Do Carmo and Wallach showed that the space of full k-homogeneous polynomial minimal immersions of the m-sphere S^m into any n-sphere S^n (for various n) can be parametrized by a compact convex body lying in a finite-dimensional vector space. They also gave a lower bound for the dimension in terms of m and k. The objective of this note is to construct Do Carmo–Wallach type moduli spaces of (homotopically nontrivial) minimal immersions between complex projective spaces. More precisely, for $m \geq 2$ and $p > q \geq 0$, we consider $\mathcal{H}^{p,q} = \mathcal{H}^{p,q}_{m+1}$, the complex vector space of harmonic polynomials on $\mathbf{C}^{m+1}$ of degree p in $z_0, \dots, z_m \in \mathbf{C}$ and degree q in $\bar{z}_0, \dots, \bar{z}_m \in \mathbf{C}$. An element of $\mathcal{H}^{p,q}$ is completely determined by its restriction to the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$. A map $f: S^{2m+1} \to S^{2n+1}$ between the unit spheres of $\mathbf{C}^{m+1}$ and $\mathbf{C}^{n+1}$ is said to be a *polynomial map of bidegree* (p,q) if the coordinates of f belong to $\mathcal{H}^{p,q}$. In this case, as $\mathcal{H}^{p,q}$ consists of (complex-valued) spherical harmonics, f is a harmonic map in the sense of Eells and Sampson [2; 3]. There are three immediate consequences of homogeneity:

(1) f factors through the Hopf bundle maps $\pi: S^{2m+1} \to \mathbf{C}P^m$ and $\pi: S^{2n+1} \to \mathbf{C}P^n$ inducing a map $F: \mathbf{C}P^m \to \mathbf{C}P^n$;

(2) F pulls back the canonical line bundle of $\mathbf{C}P^n$ to the $(p-q)$th power of that of $\mathbf{C}P^m$, in particular, F has degree $p-q>0$ (on second cohomology) and, consequently, $m \leq n$;

(3) the induced map $F: \mathbf{C}P^m \to \mathbf{C}P^n$ is harmonic if and only if f is *horizontal* with respect to the Hopf fibrations (i.e., if the differential of f maps $(\ker \pi_*)^\perp \subset T(S^{2m+1})$ into $(\ker \pi_*)^\perp \subset T(S^{2n+1})$). (This follows from the reduction theorem of Smith [2].)

If, in addition to f being horizontal, F is homothetic then it is minimal [2], and we call $F: \mathbf{C}P^m \to \mathbf{C}P^n$ *the polynomial minimal immersion of bidegree* (p,q) induced by f. To formulate our main result we recall that a map $F: \mathbf{C}P^m \to \mathbf{C}P^n$ is said to be *full* if the image of F is not contained in a

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proper complex linear subspace of $\mathbf{C}P^n$. For a full polynomial minimal immersion $F: \mathbf{C}P^m \to \mathbf{C}P^n$ of bidegree (p,q) we then have $n \le n(p,q)$, where $n(p,q)+1 = \dim_{\mathbf{C}} \mathcal{H}^{p,q}$. Finally, recall that two maps $F, F': \mathbf{C}P^m \to \mathbf{C}P^n$ are said to be *equivalent* if there exists a unitary transformation $U \in U(n+1)$ such that $F' = U \circ F$.

THEOREM 1. *For fixed $m \ge 2$ and $p > q \ge 0$, the equivalence classes of full polynomial minimal immersions $F: \mathbf{C}P^m \to \mathbf{C}P^n$ (for various $n \le n(p,q)$) can be parametrized by a compact convex body $\mathcal{L}^{p,q}$ lying in a finite-dimensional vector space $\mathcal{E}^{p,q}$. The interior of $\mathcal{L}^{p,q}$ corresponds to those maps with maximal $n = n(p,q)$. For $m \ge 3$ and $q \ge 2$, we have*

$$\dim \mathcal{L}^{p,q} = \dim \mathcal{E}^{p,q} \ge \left\{ \binom{m+p}{p}\binom{m+q}{q} - \binom{m+p-1}{p-1}\binom{m+q-1}{q-1} \right\}^2$$
$$- \sum_{b=0}^{p+q} \min\{b+1, q+1, p+q-b+1\} \binom{m+b-1}{b}^2 \frac{m+2b}{m}$$
$$-2 \sum_{b=1}^{p+q} \min\{b, q, p+q-b\} \binom{m+b-1}{b+1}^2 \frac{b(m+b+1)(m+2b+1)}{m(m-1)}$$
$$- \sum_{b=1}^{p+q} \min\{b, q, p+q-b\} \binom{m+b-2}{b+1}^2 \frac{b^2(m+b)^2(m+2b)}{(m-1)^2 m}.$$

For $m = 3$ and $q \ge 2$, the same estimate holds with the last summation absent.

EXAMPLE. Enumerating, we find that the space of equivalence classes of full quintic minimal immersions $F: \mathbf{C}P^2 \to \mathbf{C}P^n$, $n \le 42$, of bidegree $(3,2)$ is of dimension ≥ 887.

To prove Theorem 1, in Section 2 we realize $\mathcal{E}^{p,q}$ as a (real) submodule of $\mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}$. Frobenius reciprocity is then used, in Section 3, to give an estimable lower bound $\tilde{\mathcal{E}}^{p,q} \subset \mathcal{E}^{p,q} \otimes \mathbf{C}$. Furthermore, in Theorem 2 we give a complete decomposition of $\mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}$ into irreducible components; this is also a result of independent interest. Theorem 2 is then used to give a decomposition of $\tilde{\mathcal{E}}^{p,q}$ and to compute $\dim_{\mathbf{C}} \tilde{\mathcal{E}}^{p,q} \le \dim \mathcal{L}^{p,q}$. Finally, Section 4 is devoted to the proof of Theorem 2.

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2. Construction of the Moduli Space

We endow $\mathcal{H}^{p,q} = \mathcal{H}^{p,q}_{m+1}$ $(m \ge 2, p > q \ge 0)$ with a normalized Hermitian L^2-scalar product and choose the normalization constant as $m!(n(p,q)+1)/2\pi^{m+1}$, where $2\pi^{m+1}/m!$ is the total volume of $S^{2m+1} \subset \mathbf{C}^{m+1}$ and

$$n(p,q)+1=\dim_{\mathbf{C}}\mathcal{H}^{p,q}=\binom{m+p}{p}\binom{m+q}{q}-\binom{m+p-1}{p-1}\binom{m+q-1}{q-1}.$$

Precomposition by unitary transformations (of $\mathbf{C}^{m+1}$) induces a unitary $U(m+1)$-module structure on $\mathcal{H}^{p,q}$ given by $\rho_{p,q}\colon U(m+1)\to U(\mathcal{H}^{p,q})$, $\rho_{p,q}(u)\mu=\mu\circ u^{-1}$, for $u\in U(m+1)$, $\mu\in\mathcal{H}^{p,q}$ [5]. The central subgroup $S^1\subset U(m+1)$ (consisting of diagonal matrices and inducing the fibres of the Hopf bundle map) acts on $\mathcal{H}^{p,q}$ by the single weight $p-q>0$.

Fixing an orthonormal base $\{f_{p,q}^j\}_{j=0}^{n(p,q)}\subset\mathcal{H}^{p,q}$ which, at the same time, identifies $\mathcal{H}^{p,q}$ with $\mathbf{C}^{n(p,q)+1}$, we define

$$f_{p,q}=(f_{p,q}^0,\dots,f_{p,q}^{n(p,q)})\colon S^{2m+1}\to\mathbf{C}^{n(p,q)+1}.$$

Then $f_{p,q}$ is equivariant with respect to $\rho_{p,q}$. As $U(m+1)$ acts transitively on S^{2m+1}, by the choice of the normalizing constant above, $f_{p,q}$ maps into the unit sphere of $\mathbf{C}^{n(p,q)+1}$. We obtain that $f_{p,q}\colon S^{2m+1}\to S^{2n(p,q)+1}$ is a full polynomial harmonic map of bidegree (p,q).

REMARK. For $q=0$, as an easy calculation shows, $f_{p,0}$ is the only full polynomial harmonic map of bidegree $(p,0)$, and it induces the classical Veronese mapping $F_{p,0}\colon\mathbf{C}P^m\to\mathbf{C}P^{n(p,0)}$. From here on we may therefore assume that $q>0$.

PROPOSITION 1. *The map $F_{p,q}\colon\mathbf{C}P^m\to\mathbf{C}P^{n(p,q)}$ induced by $f_{p,q}\colon S^{2m+1}\to S^{2n(p,q)+1}$ is homothetic. Moreover, $f_{p,q}$ is horizontal; in particular, $F_{p,q}$ is a minimal immersion.*

Proof. Consider the (complex-valued) Hermitian symmetric 2-tensor $\omega=\sum_{j=0}^{n(p,q)}df_{p,q}^j\otimes d\bar{f}_{p,q}^j$ on $T(S^{2m+1})$. Clearly, ω is $U(m+1)$-invariant; in particular, it projects down yielding a Hermitian symmetric 2-tensor Ω on $\mathbf{C}P^m$. Denoting by $U(m)=[1]\oplus U(m)\subset U(m+1)$ the isotropy subgroup corresponding to $o=(1,0,\dots,0)\in\mathbf{C}^{m+1}$, $U(m)$ acts irreducibly on $T_O(\mathbf{C}P^m)$, $\pi(o)=O$, so that Ω_O is a (real) constant multiple of the standard metric at O. By $U(m+1)$-invariance, this is valid throughout $\mathbf{C}P^m$ so that $F_{p,q}$ is homothetic. □

The isotropy representation decomposes as

$$T_o(S^{2m+1})=\ker\pi_{*o}\oplus(\ker\pi_{*o})^\perp,$$

where the first term on the right-hand side is the trivial $U(m)$-module and the second is isomorphic with $\mathbf{C}^m$ with $U(m)$ acting by ordinary matrix multiplication. Schur's lemma implies that $(f_{p,q})_*\ker\pi_{*o}$ and $(f_{p,q})_*(\ker\pi_{*o})^\perp$ are orthogonal. By equivariance of $f_{p,q}$, we obtain that $(f_{p,q})_*\ker\pi_*$ and $(f_{p,q})_*(\ker\pi_*)^\perp$ are orthogonal everywhere on S^{2m+1}.

For any full polynomial harmonic map $f\colon S^{2m+1}\to S^{2n+1}$ of bidegree (p,q) we have $f=A\cdot f_{p,q}$, where A is a complex $(n+1)\times(n(p,q)+1)$ matrix of maximal rank. We associate to f the Hermitian symmetric matrix $\langle f\rangle=A^*\cdot A-I\in\hom(\mathcal{H}^{p,q},\mathcal{H}^{p,q})=\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$, where $I=$ identity.

To reformulate the condition that the image of f is contained in S^{2n+1} in terms of $\langle f\rangle$, we endow $\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$ with the Hermitian scalar product $\langle C, C'\rangle=\operatorname{trace}(C'^{*}\cdot C)$, $C, C'\in\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$. Then the condition $\operatorname{im}(f)\subset S^{2m+1}$ can be translated into the condition that $\langle f\rangle$ is perpendicular to the projections

$$\operatorname{proj}[f_{p,q}(z)]=\operatorname{proj}_{\mathcal{H}^{p,q}}[f_{p,q}(z)]\in\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p},\quad z\in S^{2m+1}.$$

(For a Hermitian vector space V and a unit vector $v\in V$ we define $\operatorname{proj}[v]=\operatorname{proj}_V[v]\in V\otimes V^*$ by $\operatorname{proj}[v](w)=\langle w, v\rangle v$, $w\in V$.) Indeed, for $z\in S^{2m+1}$, we have

$$\begin{aligned}|f(z)|^2-1&=|A\cdot f_{p,q}(z)|^2-|f_{p,q}(z)|^2\\&=\langle (A^*\cdot A-I)f_{p,q}(z), f_{p,q}(z)\rangle\\&=\operatorname{trace}(\operatorname{proj}[f_{p,q}(z)]\cdot\langle f\rangle)\\&=\langle\langle f\rangle, \operatorname{proj}[f_{p,q}(z)]\rangle.\end{aligned}$$

Assume now that the map $F: \mathbf{C}P^m\to\mathbf{C}P^n$, induced by f, is homothetic with the same homothety constant as that of $F_{p,q}: \mathbf{C}P^m\to\mathbf{C}P^{n(p,q)}$. We claim that this is equivalent to the condition that $\langle f\rangle$ is perpendicular to the projections

$$\operatorname{proj}[(f_{p,q})_*(X_z)],\quad X_z\in(\ker\pi_{*_z})^\perp,$$

for all $z\in S^{2m+1}$, where (as usual) the vector $(f_{p,q})_*(X_z)$ at $f_{p,q}(z)$ is shifted to the origin of $\mathbf{C}^{n(p,q)+1}$. Indeed, we have

$$\begin{aligned}|f_*(X_z)|^2-|(f_{p,q})_*(X_z)|^2&=|A(f_{p,q})_*(X_z)|^2-|(f_{p,q})_*(X_z)|^2\\&=\langle (A^*\cdot A-I)(f_{p,q})_*(X_z), (f_{p,q})_*(X_z)\rangle\\&=\operatorname{trace}(\operatorname{proj}[(f_{p,q})_*(X_z)]\cdot\langle f\rangle)\\&=\langle\langle f\rangle, \operatorname{proj}[(f_{p,q})_*X_z]\rangle.\end{aligned}$$

Finally, we reformulate the condition of horizontality of f in terms of $\langle f\rangle$. Using horizontality of $f_{p,q}$, for $V_z\in\ker\pi_{*_z}$ and $X_z\in(\ker\pi_{*_z})^\perp$, we compute

$$\begin{aligned}\langle f_*(V_z), f_*(X_z)\rangle&=\langle (A^*A)(f_{p,q})_*(V_z), (f_{p,q})_*(X_z)\rangle\\&=\langle (A^*A-I)(f_{p,q})_*(V_z), (f_{p,q})_*(X_z)\rangle\\&=\langle\langle f\rangle, (f_{p,q})_*(V_z)\cdot(f_{p,q})_*(X_z)\rangle,\end{aligned}$$

where the dot stands for the Hermitian symmetric product. (In general, given a Hermitian vector space V, we denote by $u\cdot v$ the Hermitian symmetric endomorphism defined by $(u\cdot v)(w)=\frac{1}{2}\langle w, u\rangle v+\frac{1}{2}\langle w, v\rangle u$, $w\in V$. Clearly, for $v\in V$, we have $v\cdot v=\operatorname{proj}[v]$.) We obtain that f is horizontal if and only if $\langle f\rangle$ is orthogonal to the Hermitian symmetric endomorphism $(f_{p,q})_*(V_z)\cdot(f_{p,q})_*(X_z)$ of $\mathcal{H}^{p,q}$ for all $V_z\in\ker\pi_{*_z}$ and $X_z\in(\ker\pi_{*_z})^\perp$. To put this into a representation-theoretical framework, denote by $\mathcal{S}^{p,q}$ the $U(m+1)$-submodule of Hermitian symmetric endomorphisms of $\mathcal{H}^{p,q}$. Clearly, $\mathcal{S}^{p,q}$ is a real form of $\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$. Then

$$\mathcal{W}^{p,q} = \operatorname{span}_{\mathbf{R}}\{\operatorname{proj}[f_{p,q}(z)] \mid z \in S^{2m+1}\}$$

and

$$\mathcal{Y}^{p,q} = \operatorname{span}_{\mathbf{R}}\{(f_{p,q})_*(V_z)\cdot(f_{p,q})_*(X_z) \mid V_z \in \ker \pi_{*z}, \quad X_z \in (\ker \pi_{*z})^{\perp}, z \in S^{2m+1}\}$$

and

$$\mathcal{Z}^{p,q} = \operatorname{span}_{\mathbf{R}}\{\operatorname{proj}[(f_{p,q})_*(X_z)] \mid X_z \in (\ker \pi_{*z})^{\perp}, z \in S^{2m+1}\}$$

are $U(m+1)$-submodules of $\mathcal{S}^{p,q}$. Finally, let $\mathcal{E}^{p,q}$ be the orthogonal complement of $\mathcal{W}^{p,q} + \mathcal{Y}^{p,q} + \mathcal{Z}^{p,q}$ in $\mathcal{S}^{p,q}$ and set

$$\mathcal{L}^{p,q} = \{C \in \mathcal{E}^{p,q} \mid C + I \geq 0\},$$

where "$\geq$" means positive semidefinite. Clearly, $\mathcal{L}^{p,q} \subset \mathcal{E}^{p,q}$ is a $U(m+1)$-invariant convex body containing the origin in its interior. Moreover, as the orthogonal complement of $\mathcal{W}^{p,q}$ in $\mathcal{S}^{p,q}$ consists of traceless endomorphisms of $\mathcal{H}^{p,q}$ (which follows by integrating over S^{2m+1} the defining equality $\sum_{i,j=0}^{n(p,q)} c_{ij} f_{p,q}^{i}(z) \bar{f}_{p,q}^{j}(z) = 0$, $(c_{ij})_{i,j=0}^{n(p,q)} = C \in (\mathcal{W}^{p,q})^{\perp}$), we obtain that $\mathcal{L}^{p,q}$ is compact.

Summarizing, we associated to each full polynomial minimal immersion $F: \mathbf{C}P^m \to \mathbf{C}P^n$ of bidegree (p,q) an endomorphism $\langle f \rangle = A^* \cdot A - I \in \mathcal{L}^{p,q}$ via $f = A \cdot f_{p,q}$, where F is induced by f. By polar decomposition, the parametrization is injective on the equivalence classes of maps. Furthermore, as the square root of a positive semidefinite endomorphism may be taken, the parametrization is surjective. The first two statements of Theorem 1 follow.

3. The Module Structure of $\mathcal{E}^{p,q}$

First note that, as $\operatorname{proj}[f_{p,q}(o)]$, $o = (1, 0, \ldots, 0) \in \mathbf{C}^{m+1}$, is left fixed by $U(m) \subset U(m+1)$, every irreducible $U(m+1)$-submodule of

$$\mathcal{W}^{p,q} = \operatorname{span}_{\mathbf{R}}\{U(m+1)\cdot \operatorname{proj}[f_{p,q}(o)]\}$$

is class 1 with respect to $(U(m+1), U(m))$; that is, it contains a $U(m)$-fixed vector or (equivalently) a copy of $\mathcal{H}_m^{0,0}$. Denoting by $\tilde{\mathcal{W}}^{p,q}$ the sum of those complex irreducible $U(m+1)$-submodules of $\mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}$ which, when restricted to $U(m)$, contain $\mathcal{H}_m^{0,0}$, after complexification we obtain that

$$\mathcal{W}^{p,q} \otimes \mathbf{C} \subset \tilde{\mathcal{W}}^{p,q}.$$

Similarly, we define $\mathfrak{J} = (f_{p,q})_*(\ker \pi_{*o})^{\perp}$ (shifted to the origin of $\mathcal{H}^{p,q}$). Clearly, $\mathfrak{J}$ is a real irreducible $U(m)$-submodule of $\mathcal{H}_{m+1}^{p,q}$ (by restriction). Moreover,

$$\dim_{\mathbf{R}} \mathfrak{J} = \dim_{\mathbf{R}} (\ker \pi_{*o})^{\perp} = 2m$$

as $F_{p,q}$ is an immersion. Introducing the $U(m)$-module

$$\mathcal{R} = \operatorname{span}_{\mathbf{R}}\{\operatorname{proj}_{\mathcal{H}^{p,q}}[Y] \mid Y \in \mathfrak{J}\} \subset \mathcal{S}^{p,q} \, (\subset \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}),$$

we have

$$\mathcal{Z}^{p,q} = \operatorname{span}_{\mathbf{R}}\{U(m+1)\cdot \mathcal{R}\}.$$

PROPOSITION 2. *As $U(m)$-modules, we have*

$$\mathcal{R}\otimes\mathbf{C} \cong \mathcal{H}_m^{0,0} \oplus \mathcal{H}_m^{1,1}.$$

Proof. Denote by $\mathfrak{J}_c$ the complex closure of $\mathfrak{J}$ in $\mathcal{H}^{p,q}$. Then we can write

$$\mathcal{R} = \operatorname{span}_{\mathbf{R}}\{\operatorname{proj}_{\mathfrak{J}_c}[Y] \mid Y \in \mathfrak{J}\}.$$

As $\mathfrak{J}$ is irreducible, it is either a complex or a totally real $U(m)$-submodule of $\mathcal{H}^{p,q}$.

Case 1, $\mathfrak{J} = \mathfrak{J}_c$: The branching rule

$$\mathcal{H}_{m+1}^{p,q} \mid U(m) = \sum_{\substack{0\le r\le p\\ 0\le s\le q}} \mathcal{H}_m^{r,s}$$

implies that $\mathfrak{J} \cong \mathcal{H}_m^{1,0}$ or $\mathcal{H}_m^{0,1}$ as complex $U(m)$-modules. Hence $\mathcal{R}$ consists of *all* Hermitian symmetric endomorphisms of $\mathfrak{J}$. Complexifying, we have

$$\mathcal{R}\otimes\mathbf{C} \cong \hom(\mathfrak{J},\mathfrak{J}) \cong \mathcal{H}_m^{1,0}\otimes\mathcal{H}_m^{0,1} \cong \mathcal{H}_m^{0,0}\oplus\mathcal{H}_m^{1,1}.$$

Case 2, $\mathfrak{J} \neq \mathfrak{J}_c$: Again by the branching rule, we have

$$\mathfrak{J}_c \cong \mathcal{H}_m^{1,0} \oplus \mathcal{H}_m^{0,1}.$$

(Recall that $q>0$ so that $\mathcal{H}_m^{0,1}$ is a $U(m)$-component of $\mathcal{H}_{m+1}^{p,q}$.) Composition with the orthogonal projection $P\colon \mathfrak{J}_c \to \mathcal{H}_m^{1,0}$ induces a $U(m)$-module homomorphism

$$P^*\colon \mathcal{R} \to \operatorname{span}_{\mathbf{R}}\{\operatorname{proj}_{\mathcal{H}_m^{1,0}}[Y] \mid Y \in \mathcal{H}_m^{1,0}\}.$$

(Note that P^* is well defined since, for $Y\in\mathfrak{J}$, $\mathbf{C}\cdot Y \cap \mathcal{H}_m^{0,1} = \{0\}$ and so $\mathbf{C}\cdot Y$ projects down to a complex line in $\mathcal{H}_m^{1,0}$.) As easy computation shows, P^* is injective. It is also surjective as the range is the $U(m)$-module of all Hermitian symmetric endomorphisms of $\mathcal{H}_m^{1,0}$ (which splits into $\mathbf{R}\cdot I$ and the (irreducible) traceless part). As in Case 1, we obtain that $\mathcal{R}\otimes\mathbf{C} = \mathcal{H}_m^{0,0}\oplus\mathcal{H}_m^{1,1}$. $\square$

PROPOSITION 3. *Let $\tilde{\mathcal{Z}}^{p,q}$ denote the sum of those irreducible complex $U(m+1)$-submodules of $\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$ which, when restricted to $U(m)$, contain either $\mathcal{H}_m^{0,0}$ or $\mathcal{H}_m^{1,1}$. Then we have*

$$\mathcal{Z}^{p,q}\otimes\mathbf{C} \subset \tilde{\mathcal{Z}}^{p,q}.$$

Proof. Let $p\colon \mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p} \to \mathcal{R}\otimes\mathbf{C}$ $(=\mathcal{H}_m^{0,0}\oplus\mathcal{H}_m^{1,1})$ denote the orthogonal projection, and consider the induced representation

$$\begin{aligned}\mathcal{I} &= \operatorname{Ind}_{U(m)}^{U(m+1)}(\mathcal{R}\otimes\mathbf{C})\\ &= \{\psi\colon U(m+1)\to\mathcal{R}\otimes\mathbf{C} \mid \psi(uv) = u\cdot\psi(v),\ u\in U(m),\\ &\qquad\qquad v\in U(m+1),\ \psi \text{ continuous}\}.\end{aligned}$$

For $\sigma \in \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}$, we define the map

$$\Psi(\sigma): U(m+1) \to \mathcal{R} \otimes \mathbf{C}$$

by $\Psi(\sigma)(v) = p(v \cdot \sigma)$, $v \in U(m+1)$. Then $\Psi(\sigma) \in \mathcal{G}$ so that we obtain a map $\Psi: \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p} \to \mathcal{G}$, which is actually a homomorphism of $U(m+1)$-modules. Clearly, $\ker \Psi = (\mathcal{Z}^{p,q} \otimes \mathbf{C})^{\perp}$ so that $\operatorname{im} \Psi \cong \mathcal{Z}^{p,q} \otimes \mathbf{C} \subset \mathcal{G}$ as $U(m+1)$-modules. Using Frobenius reciprocity [6], we have

$$\begin{aligned} \dim \hom_{U(m+1)}((\tilde{\mathcal{Z}}^{p,q})^{\perp}, \mathcal{Z}^{p,q} \otimes \mathbf{C}) &\le \dim \hom_{U(m+1)}((\tilde{\mathcal{Z}}^{p,q})^{\perp}, \mathcal{G}) \\ &= \dim \hom_{U(m)}((\tilde{\mathcal{Z}}^{p,q})^{\perp}, \mathcal{R} \otimes \mathbf{C}) = 0, \end{aligned}$$

and the proposition follows. □

Finally, we turn to horizontality and rewrite $\mathcal{Y}^{p,q}$ as

$$\mathcal{Y}^{p,q} = \operatorname{span}_{\mathbf{R}}\{U(m+1) \cdot \mathcal{Q}\},$$

where

$$\mathcal{Q} = \operatorname{span}_{\mathbf{R}}\{(f_{p,q})_* V_o \cdot (f_{p,q})_* X_o \mid V_o \in \ker \pi_{*o}, X_o \in (\ker \pi_{*o})^{\perp}\}$$

is a $U(m)$-module of $\mathcal{S}^{p,q}$. Since $(f_{p,q})_*$ is homothetic on $(\ker \pi_*)^{\perp}$, we have $\mathcal{Q} = \mathfrak{I}$ as real $U(m)$-modules so that

$$\mathcal{Q} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathcal{H}_m^{1,0} \oplus \mathcal{H}_m^{0,1}.$$

Now let $\tilde{\mathcal{Y}}^{p,q}$ denote the sum of those irreducible complex $U(m+1)$-modules of $\mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}$ which, when restricted to $U(m)$, contain either $\mathcal{H}_m^{1,0}$ or $\mathcal{H}_m^{0,1}$. By the same argument as given in the proof of Proposition 3, we obtain that

$$\mathcal{Y}^{p,q} \otimes \mathbf{C} \subset \tilde{\mathcal{Y}}^{p,q}.$$

We now define $\tilde{\mathcal{E}}^{p,q} = (\tilde{\mathcal{Z}}^{p,q})^{\perp} \subset \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}$ as the sum of those irreducible complex $U(m+1)$-submodules which do not contain $\mathcal{H}_m^{0,0}$ and $\mathcal{H}_m^{1,1}$. By the above (and since $\tilde{\mathcal{W}}^{p,q}, \tilde{\mathcal{Y}}^{p,q} \subset \tilde{\mathcal{Z}}^{p,q}$ (cf. the branching rule below)), we have

$$\tilde{\mathcal{E}}^{p,q} \subset \mathcal{E}^{p,q} \otimes \mathbf{C}.$$

To obtain the lower estimate on $\dim_{\mathbf{R}} \mathcal{L}^{p,q} = \dim_{\mathbf{R}} \mathcal{E}^{p,q} \ge \dim_{\mathbf{C}} \tilde{\mathcal{E}}^{p,q}$ of Theorem 1, we decompose $\tilde{\mathcal{E}}^{p,q}$ into irreducible complex $U(m+1)$-submodules. To do this, recall that an irreducible complex $U(m+1)$-module $V^{\lambda} = V_{m+1}^{\lambda}$ is uniquely determined by its highest weight λ which, with respect to the standard (diagonal) maximal torus of $U(m+1)$, is an element of $\mathbf{Z}^{m+1}$ (with nonincreasing entries). In particular, we have

$$\mathcal{H}^{p,q} = V^{(p,0,\dots,0,-q)}.$$

The general branching rule states that

$$V_{m+1}^{(\lambda_1,\dots,\lambda_{m+1})}|_{U(m)} = \sum_{\sigma} V_m^{\sigma},$$

where the summation runs through those $\sigma = (\sigma_1, \dots, \sigma_m)$ for which

$$\lambda_1 \ge \sigma_1 \ge \cdots \ge \lambda_m \ge \sigma_m \ge \lambda_{m+1}.$$

THEOREM 2. *Let $p>q>0$ and $a=p+q$. Then, for $m\geq 3$, we have*

$$\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p} \cong \sum_{b=0}^{a}\sum_{c=0}^{\min\{b,q,a-b\}}\sum_{d=0}^{\min\{b,q,e\}}[\min\{b-c,b-d,q-c,q-d,b+c-2d,a-b-c\}+1]V^{(b,c,0,\ldots,0,-d,d-b-c)},$$

where e denotes the greatest integer $\leq (b+c)/2$. For $m=2$, we have

$$\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}\cong\sum_{b=0}^{a}[\min\{b,q,a-b\}+1]\mathcal{H}^{b,b} \oplus\sum_{c=1}^{q}\sum_{b=0}^{a-2c}[\min\{b-c,q-c,a-b-2c\}+1] \times\{V^{(b+c,c,-b-2c)}\oplus V^{(b+2c,-c,-b-c)}\}.$$

We prove Theorem 2 in the next section. We now show how to compute $\dim_{\mathbf{C}}\tilde{\mathcal{E}}^{p,q}$ from this decomposition to get the lower bound for $\dim\mathcal{L}^{p,q}$ of Theorem 1. By the branching rule, $V_{m+1}^{(b,c,0,\ldots,0,-d,d-b-c)}$ does not contain $\mathcal{H}_m^{0,0}$ or $\mathcal{H}_m^{1,1}$ (and consequently does not contain $\mathcal{H}_m^{1,0}$ or $\mathcal{H}_m^{0,1}$) if and only if $c\geq 2$ or $d\geq 2$. Thus we have

$$(\tilde{\mathcal{E}}^{p,q})^{\perp}\cong\sum_{b=0}^{p+q}[\min\{b,q,p+q-b\}+1]\mathcal{H}^{b,b} \oplus\sum_{b=1}^{p+q}\min\{b,q,p+q-b\}\{V^{(b,1,0,\ldots,0,-b-1)}\oplus V^{(b+1,0,\ldots,-1,-b)}\} \oplus\sum_{b=1}^{p+q}\min\{b,q,p+q-b\}V^{(b,1,0,\ldots,0,-1,-b)}.$$

Computing the dimensions by the Weyl degree formula [5], we arrive at the lower estimate given in Theorem 1.

4. Decomposition of $\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$

We have the following.

THEOREM 3. *Let $p>q>0$ and $m\geq 3$. Then, for $b\geq c\geq d>0$, the multiplicity*

$$m[V^{(b,c,d,\ldots)}:\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}]=0.$$

Moreover, for $b_j\geq c_j\geq 0$, $j=1,2$, $b_1+c_1=b_2+c_2$, we have

$$m[V^{(b_1,c_1,0,\ldots,0,-c_2,-b_2)}:\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}] =\min\{(b_1-c_1)^+,(q-c_1)^+,(b_2-c_2)^+,(q-c_2)^+, (b_2-c_1)^+,(b_1-c_2)^+,(p+q-b_1-c_1)^+\}+1,$$

where $^+$ denotes the positive part.

Once these multiplicity formulas are proved, Theorem 2 follows easily. For $m=2$, the computation is elementary and the decomposition follows from Steinberg's formula. For $m\geq 3$, the first multiplicity formula combined with self-duality of $\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$ imply that the only irreducible components are of the form $V^{(b_1,c_1,0,\dots,0,-c_2,-b_2)}$. Also, the center $S^1\subset U(m+1)$ acts on $\mathcal{H}^{p,q}\otimes\mathcal{H}^{q,p}$ trivially, so that $b_1+c_1=b_2+c_2$. The rest is a simple computation.

To determine the multiplicities we apply the Littlewood–Richardson rule [4] together with Weyl's duality [7] between representations of $GL(V)$ and the symmetric group S_n on $\bigotimes^n V$ for a vector space V.

To prove the second multiplicity formula first, in the initial step of the Littlewood–Richardson rule, we add suitable elements of $\mathbf{Z}_+\cdot(1,\dots,1)$ to the highest weight vectors of $\mathcal{H}^{p,q}$ and $\mathcal{H}^{q,p}$ to make the components descend to zero. We also add the sum of these elements to the highest weight of the representation whose multiplicity is to be computed. We obtain

$$\begin{gathered}(b_1,c_1,0,\dots,0,-c_2,-b_2)+(p+q,\dots,p+q)\\ =(p+q+b_1,p+q+c_1,p+q,\dots,p+q,p+q-c_2,p+q-b_2),\\ (p,0,\dots,0,-q)+(q,\dots,q)=(p+q,q,\dots,q,0),\\ (q,0,\dots,-p)+(p,\dots,p)=(p+q,p,\dots,p,0).\end{gathered}$$

Each vector represents a tableau consisting of $m+1$ rows; the coordinates representing the length of the respective row. We superimpose the two largest tableaux, which in this case correspond to the first and third vectors, to obtain Figure 1.

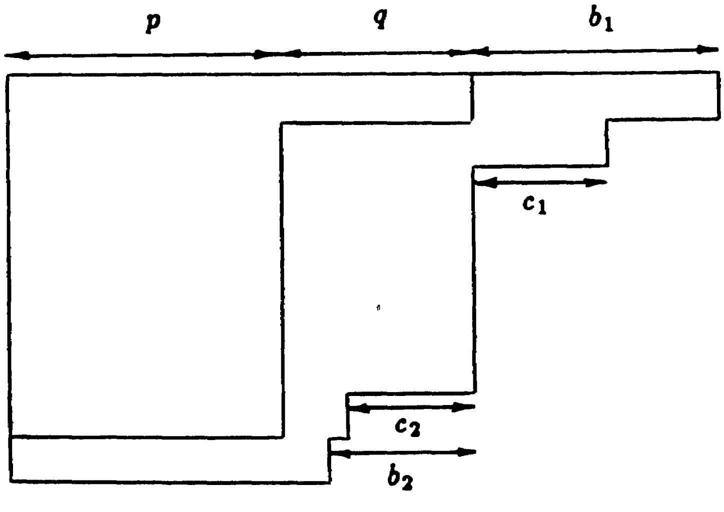

Figure 1

In the second step, we fill in the complementary space with the numbers $1,2,\dots,m$ and from each of these use the amount given by the respective coordinate of the second vector; that is, we use $p+q$ 1's, q 2's, $\dots$, q m's.

The rules for filling are as follows.

(1) In each row the numbers are nondecreasing.
(2) In each column the numbers are (strictly) increasing.
(3) When reading the sequence of numbers from right to left:
  (a) the 1's are always O.K.; and
  (b) given $i+1$ in the sequence, the number of previous i's is greater than the number of previous $(i+1)$'s.

The required multiplicity is the number of possible ways of filling in.

We denote by R_i ($i=1,\dots,m+1$) the ith row of the complementary tableau. By (3) there can only be 1's in R_1. In particular, $b_1 \le p+q$. By duality, we also have $b_2 \le p+q$. By (1)-(3), the last c_1 entries of R_2 are filled with 2's. (*Proof:* There cannot be 1's there by (2). If there were an entry $i \ge 3$ then the same would be true for the last entry of R_2, by (1). Now, apply (3) to that entry to get a contradiction.) There are at most q 2's, so that we obtain $c_1 \le q$ (since otherwise the multiplicity is zero). By duality, we also have $c_2 \le q$; in particular, the smaller tableau is entirely contained in the larger tableau. By (1), there can only be 1's and 2's in R_2, the former preceding the latter.

From R_3 to R_{m-1} there are $m-1$ column entries. Thus, by (2), in R_3 there can only be 2's, 3's and 4's. However 4 cannot occur in R_3 since in that case the last entry of R_3 would be 4 and applying (3) would yield a contradiction. Thus, in R_3 there are only 2's and 3's. Below the 2's of R_2 there must be 3's, by (2).

We are interested in how many 3's are in R_3 under the 1's in R_2. We call these "jumps" since the respective column in R_2 and R_3 will be $\binom{1}{3}$. *We claim that there are exactly c_1 jumps.*

First of all, if there were fewer than c_1 jumps then we would run out of the 2's which are q in number. (We used up c_1 2's in the last c_1 entries of R_2 and, by (2), under the 2's in R_2 there must be 3's.) Secondly, if there were more than c_1 jumps then we would apply (3) to the 3 occurring in the very first jump. The number of 3's then would exceed the number of 2's; this is a contradiction.

As a byproduct we also obtained that we used up all 2's. We call the space occupied by the first c_1 3's in R_3 the *critical box*. We show below that any (allowed) location of the critical box determines the rest of the filling-ins.

In R_4 there can only be 3's and 4's, by repeating the argument above. Below the 3's of R_3 there must be 4's and below the 2's of R_3 there must be 3's, because otherwise the 4's in R_4 would exceed the 3's in R_3, violating (3). This argument can be carried out until R_{m-1} is filled up. It also follows that we would use up all numbers between 2 and $m-2$.

We now consider R_{m+1} which, at this point, can only be filled up by 1's, $(m-1)$'s, and m's. *We claim that there can only be* 1*'s and m's in R_{m+1}.*

Assume the contrary. If there are no m's in R_{m+1} then R_{m+1} is filled up by 1's and $(m-1)$'s. We then take the first m in R_m (which certainly exists) and apply (3) to get a contradiction. If there is an m in R_{m+1} we take the first one and again apply (3) to get a contradiction. As a byproduct we also obtained that the $(m-1)$'s must be used up in R_{m-1} and R_m. Thus, below the $(m-2)$'s in R_{m-1} there must be $(m-1)$'s of R_m, and the filling-in is unique.

Summarizing, we showed that: *Once the location of the critical box is fixed, there is only one way to fill in.*

To obtain the exact constraints on the location of the critical box, denote by d the distance of the critical box from the right wall (i.e., there are exactly $d+c_1$ 2's in R_2). The critical box occupies R_3, whose length is q, so that $d \le q-c_1$. There are at least $(q-b_2)^+$ $(m-1)$'s in R_m (by (2)) so that there are at least $(q-b_2)^+$ 2's in R_3. This means that $d \le q-c_1-(q-b_2)^+$. Taking into account the previous estimate, we can replace this by $d \le b_2-c_1$. In particular, $c_1 \le b_2$ is a general constraint on the tableaux (i.e., otherwise the multiplicity is zero). By duality, we also have $c_2 \le b_1$. Summarizing, we obtain

$$d \le \min\{b_1-c_1, q-c_1, b_2-c_1\}.$$

For the lower bound, first note that the critical box cannot occupy the last c_2-1 entries of R_3 only because in that case the q $(m-1)$'s cannot be filled in R_{m-1} and R_m. Thus, $d \ge c_2-c_1$. Moreover, in R_{m+1} there are $p+q-b_2$ places for the m's, so R_m ends with at least $(q-(p+q-b_2))^+ = (b_2-p)^+$ m's. Thus, $d \ge c_2-c_1+(b_2-p)^+$. By the previous estimate, we can replace this by $d \ge b_1-p$. Summarizing, we obtain

$$d \ge \max\{c_2-c_1, b_1-p\}.$$

Comparing the upper and lower bounds, we obtain that $b_1+c_1 \le p+q$ is a constraint on the tableaux. Finally we obtain that the multiplicity -1 is the minimum of the following numbers:

$$b_1-c_1, \;\; q-c_1, \;\; b_2-c_1$$

and

$$\begin{aligned} b_1-c_1-(c_2-c_1) &= b_1-c_2, & q-c_1-(b_1-p) &= p+q-b_1-c_1,\\ b_1-c_1-(b_1-p) &= p-c_1, & b_2-c_1-(c_2-c_1) &= b_2-c_2,\\ q-c_1-(c_2-c_1) &= q-c_2, & b_2-c_1-(b_1-p) &= p-c_2. \end{aligned}$$

Two of the numbers do not contribute in the minimum as $p>q$. The second multiplicity formula of Theorem 3 follows.

To prove the first we apply a similar argument. In the tableaux corresponding to

$$m[V^{(b,c,d,\ldots)}: \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}],$$

R_1 is filled up by 1's as before. The last c entries of R_2 are again filled up by 2's, and there can only be 1's and 2's in R_2. Similarly, by (3) the last d entries of R_3 are filled up by 3's, and there can only be 2's and 3's in R_3. Denote by J the number of jumps $\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right)$ in R_2 and R_3. Then, by (3) we have $J \le c-d$. On the other hand, $J \ge c$ since otherwise we would run out of the q 2's. This is a contradiction, so the multiplicity is zero.

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