Moduli Spaces of Polynomial Minimal Immersions between Complex Projective Spaces

G. Toth

1. Introduction and Preliminaries

In [1; 6] Do Carmo and Wallach showed that the space of full $k$-homogeneous polynomial minimal immersions of the $m$-sphere $S^m$ into any $n$-sphere $S^n$ (for various $n$) can be parametrized by a compact convex body lying in a finite-dimensional vector space. They also gave a lower bound for the dimension in terms of $m$ and $k$. The objective of this note is to construct Do Carmo–Wallach type moduli spaces of (homotopically nontrivial) minimal immersions between complex projective spaces. More precisely, for $m \geq 2$ and $p > q \geq 0$, we consider $\mathcal{C}^{p,q} = \mathcal{C}^{p,q}_{m+1}$, the complex vector space of harmonic polynomials on $\mathbb{C}^{m+1}$ of degree $p$ in $z_0, \ldots, z_m \in \mathbb{C}$ and degree $q$ in $\bar{z}_0, \ldots, \bar{z}_m \in \mathbb{C}$. An element of $\mathcal{C}^{p,q}$ is completely determined by its restriction to the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$. A map $f: S^{2m+1} \rightarrow S^{2n+1}$ between the unit spheres of $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$ is said to be a polynomial map of bidegree $(p, q)$ if the coordinates of $f$ belong to $\mathcal{C}^{p,q}$. In this case, as $\mathcal{C}^{p,q}$ consists of (complex-valued) spherical harmonics, $f$ is a harmonic map in the sense of Eells and Sampson [2; 3]. There are three immediate consequences of homogeneity:

1. $f$ factors through the Hopf bundle maps $\pi: S^{2m+1} \rightarrow \mathbb{C}P^m$ and $\pi: S^{2n+1} \rightarrow \mathbb{C}P^n$ inducing a map $F: \mathbb{C}P^m \rightarrow \mathbb{C}P^n$;

2. $F$ pulls back the canonical line bundle of $\mathbb{C}P^n$ to the $(p-q)$th power of that of $\mathbb{C}P^m$, in particular, $F$ has degree $p-q>0$ (on second cohomology) and, consequently, $m \leq n$;

3. the induced map $F: \mathbb{C}P^m \rightarrow \mathbb{C}P^n$ is harmonic if and only if $f$ is horizontal with respect to the Hopf fibrations (i.e., if the differential of $f$ maps $(\ker \pi_*)^\perp \subset T(S^{2m+1})$ into $(\ker \pi_*)^\perp \subset T(S^{2n+1})$). (This follows from the reduction theorem of Smith [2].)

If, in addition to $f$ being horizontal, $F$ is homothetic then it is minimal [2], and we call $F: \mathbb{C}P^m \rightarrow \mathbb{C}P^n$ the polynomial minimal immersion of bidegree $(p, q)$ induced by $f$. To formulate our main result we recall that a map $F: \mathbb{C}P^m \rightarrow \mathbb{C}P^n$ is said to be full if the image of $F$ is not contained in a

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proper complex linear subspace of $\mathbb{CP}^n$. For a full polynomial minimal immersion $F: \mathbb{CP}^m \to \mathbb{CP}^n$ of bidegree $(p, q)$ we then have $n \leq n(p, q)$, where $n(p, q) + 1 = \dim \mathcal{J}^{p,q}$. Finally, recall that two maps $F, F': \mathbb{CP}^m \to \mathbb{CP}^n$ are said to be equivalent if there exists a unitary transformation $U \in U(n+1)$ such that $F' = U \circ F$.

**Theorem 1.** For fixed $m \geq 2$ and $p > q \geq 0$, the equivalence classes of full polynomial minimal immersions $F: \mathbb{CP}^m \to \mathbb{CP}^n$ (for various $n \leq n(p, q)$) can be parametrized by a compact convex body $\mathcal{L}^{p,q}$ lying in a finite-dimensional vector space $\mathcal{E}^{p,q}$. The interior of $\mathcal{L}^{p,q}$ corresponds to those maps with maximal $n = n(p, q)$. For $m \geq 3$ and $q \geq 2$, we have

$$
\dim \mathcal{L}^{p,q} = \dim \mathcal{E}^{p,q} \geq \left\{ \begin{array}{l}
\left( \begin{array}{c}
m + p \\
p
\end{array} \right) \left( \begin{array}{c}
m + q \\
q
\end{array} \right) - \left( \begin{array}{c}
m + q - 1 \\
p - 1
\end{array} \right) \left( \begin{array}{c}
m + p - 1 \\
q - 1
\end{array} \right)^2 \\
- \sum_{b=0}^{p+q} \min\{b+1, q+1, p+q-b+1\} \left( \begin{array}{c}
m + b - 1 \\
b
\end{array} \right)^2 \frac{m+2b}{m} \\
- 2 \sum_{b=1}^{p+q} \min\{b, q, p+q-b\} \left( \begin{array}{c}
m + b - 1 \\
b + 1
\end{array} \right) \frac{b(m+b+1)(m+2b+1)}{m(m-1)} \\
- \sum_{b=1}^{p+q} \min\{b, q, p+q-b\} \left( \begin{array}{c}
m + b - 2 \\
b + 1
\end{array} \right) \frac{2b^2(m+b)^2(m+2b)}{(m-1)^2m}.
\end{array} \right.
$$

For $m = 3$ and $q \geq 2$, the same estimate holds with the last summation absent.

**Example.** Enumerating, we find that the space of equivalence classes of full quintic minimal immersions $F: \mathbb{CP}^2 \to \mathbb{CP}^n$, $n \leq 42$, of bidegree $(3, 2)$ is of dimension $\geq 887$.

To prove Theorem 1, in Section 2 we realize $\mathcal{E}^{p,q}$ as a (real) submodule of $\mathcal{J}^{p,q} \otimes \mathbb{R}^{a,p}$. Frobenius reciprocity is then used, in Section 3, to give an estimable lower bound $\bar{\mathcal{E}}^{p,q} \subseteq \mathcal{E}^{p,q} \otimes \mathbb{C}$. Furthermore, in Theorem 2 we give a complete decomposition of $\mathcal{J}^{p,q} \otimes \mathbb{C}^{a,p}$ into irreducible components; this is also a result of independent interest. Theorem 2 is then used to give a decomposition of $\bar{\mathcal{E}}^{p,q}$ and to compute $\dim \mathcal{C} \bar{\mathcal{E}}^{p,q} \leq \dim \mathcal{L}^{p,q}$. Finally, Section 4 is devoted to the proof of Theorem 2.

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## 2. Construction of the Moduli Space

We endow $\mathcal{J}^{p,q} = \mathcal{J}^{p,q}_{m+1}$ $(m \geq 2, p > q \geq 0)$ with a normalized Hermitian $L^2$-scalar product and choose the normalization constant as $m!(n(p, q) + 1)/2\pi^{m+1}$, where $2\pi^{m+1}/m!$ is the total volume of $S^{2m+1} \subseteq \mathbb{C}^{m+1}$ and
\[ n(p, q) + 1 = \dim_{\mathbb{C}} \mathcal{H}^{p, q} = \binom{m + p}{p} \binom{m + q}{q} - \binom{m + p - 1}{p - 1} \binom{m + q - 1}{q - 1}. \]

Precomposition by unitary transformations (of \( \mathbb{C}^{m+1} \)) induces a unitary \( U(m + 1) \)-module structure on \( \mathcal{H}^{p, q} \) given by \( \rho_{p, q}: U(m + 1) \to U(\mathcal{H}^{p, q}) \), \( \rho_{p, q}(u) \mu = \mu \circ u^{-1} \), for \( u \in U(m + 1), \mu \in \mathcal{H}^{p, q} \) [5]. The central subgroup \( S^1 \subset U(m + 1) \) (consisting of diagonal matrices and inducing the fibres of the Hopf bundle map) acts on \( \mathcal{H}^{p, q} \) by the single weight \( p - q > 0 \).

Fixing an orthonormal base \( \{ f_{p, q}^j \}_{j=0}^{n(p, q)} \subset \mathcal{H}^{p, q} \) which, at the same time, identifies \( \mathcal{H}^{p, q} \) with \( \mathbb{C}^{n(p, q) + 1} \), we define

\[ f_{p, q} = (f_{p, q}^0, \ldots, f_{p, q}^{n(p, q)}): S^{2m+1} \to \mathbb{C}^{n(p, q) + 1}. \]

Then \( f_{p, q} \) is equivariant with respect to \( \rho_{p, q} \). As \( U(m + 1) \) acts transitively on \( S^{2m+1} \), by the choice of the normalizing constant above, \( f_{p, q} \) maps into the unit sphere of \( \mathbb{C}^{n(p, q) + 1} \). We obtain that \( f_{p, q}: S^{2m+1} \to S^{2n(p, q) + 1} \) is a full polynomial harmonic map of bidegree \((p, q)\).

**REMARK.** For \( q = 0 \), as an easy calculation shows, \( f_{p, 0} \) is the only full polynomial harmonic map of bidegree \((p, 0)\), and it induces the classical Veronese mapping \( F_{p, 0}: \mathbb{C}P^m \to \mathbb{C}P^{n(p, 0)} \). From here on we may therefore assume that \( q > 0 \).

**PROPOSITION 1.** The map \( F_{p, q}: \mathbb{C}P^m \to \mathbb{C}P^{n(p, q)} \) induced by \( f_{p, q}: S^{2m+1} \to S^{2n(p, q) + 1} \) is homothetic. Moreover, \( f_{p, q} \) is horizontal; in particular, \( F_{p, q} \) is a minimal immersion.

**Proof.** Consider the (complex-valued) Hermitian symmetric 2-tensor \( \omega = \sum_{j=0}^{n(p, q)} df_{p, q}^j \otimes d\bar{f}_{p, q}^j \) on \( T(S^{2m+1}) \). Clearly, \( \omega \) is \( U(m + 1) \)-invariant; in particular, it projects down yielding a Hermitian symmetric 2-tensor \( \Omega \) on \( \mathbb{C}P^m \). Denoting by \( U(m) = [1] \oplus U(m) \subset U(m + 1) \) the isotropy subgroup corresponding to \( \sigma = (1, 0, \ldots, 0) \in \mathbb{C}^{m+1} \), \( U(m) \) acts irreducibly on \( T_0(\mathbb{C}P^m) \), \( \pi(\sigma) = 0 \), so that \( \Omega_{|O} \) is a (real) constant multiple of the standard metric at \( O \). By \( U(m + 1) \)-invariance, this is valid throughout \( \mathbb{C}P^m \) so that \( F_{p, q} \) is homothetic.

The isotropy representation decomposes as

\[ T_0(S^{2m+1}) = \ker \pi_0 \oplus (\ker \pi_{\sigma})^\perp, \]

where the first term on the right-hand side is the trivial \( U(m) \)-module and the second is isomorphic with \( \mathbb{C}^m \) with \( U(m) \) acting by ordinary matrix multiplication. Schur's lemma implies that \( (f_{p, q})_{\star} \ker \pi_0 \) and \( (f_{p, q})_\star (\ker \pi_{\sigma})^\perp \) are orthogonal. By equivariance of \( f_{p, q} \), we obtain that \( (f_{p, q})_{\star} \ker \pi_0 \) and \( (f_{p, q})_{\star} (\ker \pi_{\sigma})^\perp \) are orthogonal everywhere on \( S^{2m+1} \).

For any full polynomial harmonic map \( f: S^{2m+1} \to S^{2n+1} \) of bidegree \((p, q)\) we have \( f = A \cdot f_{p, q} \), where \( A \) is a complex \((n+1) \times (n(p, q) + 1)\) matrix of maximal rank. We associate to \( f \) the Hermitian symmetric matrix

\[ \langle f \rangle = A^* \cdot A - I \in \text{hom}(\mathcal{H}^{p, q}, \mathcal{H}^{p, q}) = \mathcal{H}^{p, q} \otimes \mathcal{H}^{q, p}, \]

where \( I = \text{identity} \).
To reformulate the condition that the image of $f$ is contained in $S^{2n+1}$ in terms of $\langle f \rangle$, we endow $\mathcal{C}^{p,q} \otimes \mathcal{C}^{q,p}$ with the Hermitian scalar product $\langle C, C' \rangle = \text{trace}(C^* \cdot C)$, $C, C' \in \mathcal{C}^{p,q} \otimes \mathcal{C}^{q,p}$. Then the condition $\text{im}(f) \subset S^{2m+1}$ can be translated into the condition that $\langle f \rangle$ is perpendicular to the projections

$$\text{proj}[f_{p,q}(z)] = \text{proj}_{\mathcal{C}^{p,q}}[f_{p,q}(z)] \in \mathcal{C}^{p,q} \otimes \mathcal{C}^{q,p}, \quad z \in S^{2m+1}.$$  

(For a Hermitian vector space $V$ and a unit vector $v \in V$ we define $\text{proj}[v] = \text{proj}_V[v] \in V \otimes V^*$ by $\text{proj}[v](w) = \langle w, v \rangle v$, $w \in V$.) Indeed, for $z \in S^{2m+1}$, we have

$$|f(z)|^2 - 1 = |A \cdot f_{p,q}(z)|^2 - |f_{p,q}(z)|^2$$

$$= \langle (A^* \cdot A - I)f_{p,q}(z), f_{p,q}(z) \rangle$$

$$= \text{trace}(\text{proj}[f_{p,q}(z)] \cdot \langle f \rangle)$$

$$= \langle \langle f \rangle, \text{proj}[f_{p,q}(z)] \rangle.$$

Assume now that the map $F : CP^m \to CP^n$, induced by $f$, is homothetic with the same homothety constant as that of $f_{p,q} : CP^m \to CP^n$. We claim that this is equivalent to the condition that $\langle f \rangle$ is perpendicular to the projections

$$\text{proj}[(f_{p,q} \cdot X_z)], \quad X_z \in (\ker \pi_z)^{\perp},$$

for all $z \in S^{2m+1}$, where (as usual) the vector $(f_{p,q} \cdot X_z)$ at $f_{p,q}(z)$ is shifted to the origin of $C^{n(p,q)+1}$. Indeed, we have

$$|f_{\cdot}(X_z)|^2 - |(f_{p,q} \cdot X_z)|^2 = |A(f_{p,q} \cdot X_z)|^2 - |(f_{p,q} \cdot X_z)|^2$$

$$= \langle (A^* \cdot A - I)(f_{p,q} \cdot X_z), (f_{p,q} \cdot X_z) \rangle$$

$$= \text{trace}(\text{proj}[(f_{p,q} \cdot X_z)] \cdot \langle f \rangle)$$

$$= \langle \langle f \rangle, \text{proj}[(f_{p,q} \cdot X_z)] \rangle.$$

Finally, we reformulate the condition of horizontality of $f$ in terms of $\langle f \rangle$. Using horizontality of $f_{p,q}$, for $V_z \in \ker \pi_z$ and $X_z \in (\ker \pi_z)^{\perp}$, we compute

$$\langle f_{\cdot}(V_z), f_{\cdot}(X_z) \rangle = \langle (A^* A)(f_{p,q} \cdot V_z), (f_{p,q} \cdot X_z) \rangle$$

$$= \langle (A^* A - I)(f_{p,q} \cdot V_z), (f_{p,q} \cdot X_z) \rangle$$

$$= \langle \langle f \rangle, (f_{p,q} \cdot V_z) \cdot (f_{p,q} \cdot X_z) \rangle,$$

where the dot stands for the Hermitian symmetric product. (In general, given a Hermitian vector space $V$, we denote by $u \cdot v$ the Hermitian symmetric endomorphism defined by $(u \cdot v)(w) = \frac{1}{2}\langle w, u \rangle v + \frac{1}{2}\langle w, v \rangle u$, $w \in V$. Clearly, for $v \in V$, we have $v \cdot v = \text{proj}[v]$.) We observe that $f$ is horizontal if and only if $\langle f \rangle$ is orthogonal to the Hermitian symmetric endomorphism $(f_{p,q} \cdot V_z) \cdot (f_{p,q} \cdot X_z)$ of $\mathcal{C}^{p,q}$ for all $V_z \in \ker \pi_z$ and $X_z \in (\ker \pi_z)^{\perp}$. To put this into a representation-theoretical framework, denote by $S^{p,q}$ the $U(m+1)$-submodule of Hermitian symmetric endomorphisms of $\mathcal{C}^{p,q}$. Clearly, $S^{p,q}$ is a real form of $\mathcal{C}^{p,q} \otimes \mathcal{C}^{q,p}$. Then
\[ \mathcal{W}^{p,q} = \text{span}_R \{ \text{proj}[f_{p,q}(z)] | z \in S^{2m+1} \} \]

and

\[ \mathcal{Y}^{p,q} = \text{span}_R \{ (f_{p,q})_*(V_z) \cdot (f_{p,q})_*(X_z) | V_z \in \ker \pi_z, X_z \in (\ker \pi_z)^\perp, z \in S^{2m+1} \} \]

and

\[ \mathcal{Z}^{p,q} = \text{span}_R \{ \text{proj}[(f_{p,q})_*(X_z)] | X_z \in (\ker \pi_z)^\perp, z \in S^{2m+1} \} \]

are \( U(m+1) \)-submodules of \( \mathcal{S}^{p,q} \). Finally, let \( \mathcal{E}^{p,q} \) be the orthogonal complement of \( \mathcal{W}^{p,q} + \mathcal{Y}^{p,q} + \mathcal{Z}^{p,q} \) in \( \mathcal{S}^{p,q} \) and set

\[ \mathcal{L}^{p,q} = \{ C \in \mathcal{E}^{p,q} | C + I \geq 0 \}, \]

where \( \geq \) means positive semidefinite. Clearly, \( \mathcal{L}^{p,q} \subset \mathcal{E}^{p,q} \) is a \( U(m+1) \)-invariant convex body containing the origin in its interior. Moreover, as the orthogonal complement of \( \mathcal{W}^{p,q} \) in \( \mathcal{S}^{p,q} \) consists of traceless endomorphisms of \( \mathcal{C}^{p,q} \) (which follows by integrating over \( S^{2m+1} \) the defining equality \( \sum_{i,j=0}^n c_{ij} f_{p,q}(z) f_{p,q}(z) = 0 \), \( (c_{ij})_{i,j=0} = C \in (\mathcal{W}^{p,q})^\perp \)), we obtain that \( \mathcal{L}^{p,q} \) is compact.

Summarizing, we associated to each full polynomial minimal immersion \( F: \mathbb{C}P^m \to \mathbb{C}P^n \) of bidegree \( (p, q) \) an endomorphism \( \langle f \rangle = A^* \cdot A - I \in \mathcal{L}^{p,q} \) via \( f = A \cdot f_{p,q} \), where \( F \) is induced by \( f \). By polar decomposition, the parametrization is injective on the equivalence classes of maps. Furthermore, as the square root of a positive semidefinite endomorphism may be taken, the parametrization is surjective. The first two statements of Theorem 1 follow.

### 3. The Module Structure of \( \mathcal{E}^{p,q} \)

First note that, as \( \text{proj}[f_{p,q}(o)] \), \( o = (1, 0, \ldots, 0) \in \mathbb{C}^{m+1} \), is left fixed by \( U(m) \subset U(m+1) \), every irreducible \( U(m+1) \)-submodule of

\[ \mathcal{W}^{p,q} = \text{span}_R \{ (U(m+1) \cdot \text{proj}[f_{p,q}(o)]) \} \]

is class 1 with respect to \( (U(m+1), U(m)) \); that is, it contains a \( U(m) \)-fixed vector or (equivalently) a copy of \( \mathcal{C}^{0,0}_m \). Denoting by \( \mathcal{W}^{p,q} \) the sum of those complex irreducible \( U(m+1) \)-submodules of \( \mathcal{C}^{p,q} \otimes \mathcal{C}^{q,p} \) which, when restricted to \( U(m) \), contain \( \mathcal{C}^{0,0}_m \), after complexification we obtain that

\[ \mathcal{W}^{p,q} \otimes \mathbb{C} \subset \mathcal{W}^{p,q}. \]

Similarly, we define \( \mathcal{J} = (f_{p,q}_*(\ker \pi_o)^\perp \) (shifted to the origin of \( \mathcal{C}^{p,q} \)). Clearly, \( \mathcal{J} \) is a real irreducible \( U(m) \)-submodule of \( \mathcal{C}^{p,q} \) (by restriction). Moreover,

\[ \dim_R \mathcal{J} = \dim_R (\ker \pi_o)^\perp = 2m \]

as \( F_{p,q} \) is an immersion. Introducing the \( U(m) \)-module

\[ \mathcal{M} = \text{span}_R \{ \text{proj}_{\mathcal{C}^{p,q}}[Y] | Y \in \mathcal{J} \} \subset \mathcal{S}^{p,q} \subset (\mathcal{C}^{p,q} \otimes \mathcal{C}^{q,p}), \]
we have
\[ \mathcal{Z}^{p,q} = \text{span}_R \{ U(m+1) \cdot \mathcal{R} \} . \]

**Proposition 2.** As $U(m)$-modules, we have
\[ \mathcal{R} \otimes \mathcal{C} \cong \mathcal{K}^{0,0}_m \oplus \mathcal{K}^{1,1}_m . \]

**Proof.** Denote by $\mathfrak{J}_c$ the complex closure of $\mathfrak{J}$ in $\mathcal{K}^{p,q}$. Then we can write
\[ \mathcal{R} = \text{span}_R \{ \text{proj}_{\mathfrak{J}_c}[Y] | Y \in \mathfrak{J} \} . \]

As $\mathfrak{J}$ is irreducible, it is either a complex or a totally real $U(m)$-submodule of $\mathcal{K}^{p,q}$. 

**Case 1, $\mathfrak{J} = \mathfrak{J}_c$:** The branching rule
\[ \mathcal{K}^{p,q}_{m+1}[U(m)] = \sum_{0 \leq r \leq p, 0 \leq s \leq q} \mathcal{K}^{r,s}_m \]
implies that $\mathfrak{J} \cong \mathcal{K}^{1,0}_m$ or $\mathcal{K}^{0,1}_m$ as complex $U(m)$-modules. Hence $\mathcal{R}$ consists of all Hermitian symmetric endomorphisms of $\mathfrak{J}$. Complexifying, we have
\[ \mathcal{R} \otimes \mathcal{C} \cong \text{hom}(\mathfrak{J}, \mathfrak{J}) \cong \mathcal{K}^{1,0}_m \otimes \mathcal{K}^{0,1}_m \cong \mathcal{K}^{0,0}_m \oplus \mathcal{K}^{1,1}_m . \]

**Case 2, $\mathfrak{J} \neq \mathfrak{J}_c$:** Again by the branching rule, we have
\[ \mathcal{K}^{1,0}_m \cong \mathcal{K}^{0,0}_m \oplus \mathcal{K}^{0,1}_m . \]
(Recall that $q > 0$ so that $\mathcal{K}^{0,1}_m$ is a $U(m)$-component of $\mathcal{K}^{p,q}_{m+1}$.) Composition with the orthogonal projection $P: \mathfrak{J}_c \to \mathcal{K}^{1,0}_m$ induces a $U(m)$-module homomorphism
\[ P^*: \mathcal{R} \to \text{span}_R \{ \text{proj}_{\mathfrak{J}^{1,0}_m}[Y] | Y \in \mathcal{K}^{1,0}_m \} . \]
(Note that $P^*$ is well defined since, for $Y \in \mathfrak{J}$, $\mathbf{C} \cdot Y \cap \mathcal{K}^{0,1}_m = \{0\}$ and so $\mathbf{C} \cdot Y$ projects down to a complex line in $\mathcal{K}^{1,0}_m$. ) As easy computation shows, $P^*$ is injective. It is also surjective as the range is the $U(m)$-module of all Hermitian symmetric endomorphisms of $\mathcal{K}^{1,0}_m$ (which splits into $\mathbf{R} \cdot \mathbf{I}$ and the (irreducible) traceless part). As in Case 1, we obtain that $\mathcal{R} \otimes \mathcal{C} = \mathcal{K}^{0,0}_m \oplus \mathcal{K}^{1,1}_m$. \qed

**Proposition 3.** Let $\tilde{\mathcal{Z}}^{p,q}$ denote the sum of those irreducible complex $U(m+1)$-submodules of $\mathcal{K}^{p,q} \otimes \mathcal{K}^{q,p}$ which, when restricted to $U(m)$, contain either $\mathcal{K}^{0,0}_m$ or $\mathcal{K}^{1,1}_m$. Then we have
\[ \mathcal{Z}^{p,q} \otimes \mathcal{C} \subset \tilde{\mathcal{Z}}^{p,q} . \]

**Proof.** Let $\rho: \mathcal{K}^{p,q} \otimes \mathcal{K}^{q,p} \to \mathcal{R} \otimes \mathcal{C} (= \mathcal{K}^{0,0}_m \oplus \mathcal{K}^{1,1}_m)$ denote the orthogonal projection, and consider the induced representation
\[ \mathcal{J} = \text{Ind}_{U(m)}^{U(m+1)}(\mathcal{R} \otimes \mathcal{C}) \]
\[ = \{ \psi: U(m+1) \to \mathcal{R} \otimes \mathcal{C} | \psi(uv) = u \cdot \psi(v), \ u \in U(m), \]
\[ \psi \text{ continuous} \} , \quad v \in U(m+1) \} . \]


For $\sigma \in \mathcal{X}^{p,q} \otimes \mathcal{X}^{q,p}$, we define the map

$$\Psi(\sigma): U(m+1) \to \mathcal{R} \otimes \mathcal{C}$$

by $\Psi(\sigma)(v) = p(v \cdot \sigma), v \in U(m+1)$. Then $\Psi(\sigma) \in \mathcal{G}$ so that we obtain a map $\Psi: \mathcal{X}^{p,q} \otimes \mathcal{X}^{q,p} \to \mathcal{G}$, which is actually a homomorphism of $U(m+1)$-modules. Clearly, $\ker \Psi = (\mathcal{Z}^{p,q} \otimes \mathcal{C})^\perp$ so that $\im \Psi \equiv \mathcal{Z}^{p,q} \otimes \mathcal{C} \subset \mathcal{G}$ as $U(m+1)$-modules. Using Frobenius reciprocity [6], we have

$$\dim \hom_{U(m+1)}((\mathcal{Z}^{p,q})^\perp, \mathcal{Z}^{p,q} \otimes \mathcal{C}) \leq \dim \hom_{U(m+1)}((\mathcal{Z}^{p,q})^\perp, \mathcal{G}) = \dim \hom_{U(m)}((\mathcal{Z}^{p,q})^\perp, \mathcal{R} \otimes \mathcal{C}) = 0,$$

and the proposition follows. \hfill \Box

Finally, we turn to horizontality and rewrite $\mathcal{Y}^{p,q}$ as

$$\mathcal{Y}^{p,q} = \text{span}_\mathcal{R} \{ U(m+1) \cdot \mathcal{Q} \},$$

where

$$\mathcal{Q} = \text{span}_\mathcal{R} \{ (f_{p,q})_*, V_0 \cdot (f_{p,q})_* X_0 \mid V_0 \in \ker \pi_{*o}, X_0 \in (\ker \pi_*^o)^\perp \}$$

is a $U(m)$-module of $\mathcal{S}^{p,q}$. Since $(f_{p,q})_*$ is homothetic on $(\ker \pi_*^o)^\perp$, we have $\mathcal{Q} = \mathcal{J}$ as real $U(m)$-modules so that

$$\mathcal{Q} \otimes \mathcal{R} \mathcal{C} \equiv \mathcal{X}^{0,1}_{m} \oplus \mathcal{X}^{1,0}_{m}.$$ 

Now let $\tilde{\mathcal{Y}}^{p,q}$ denote the sum of those irreducible complex $U(m+1)$-modules of $\mathcal{X}^{p,q} \otimes \mathcal{X}^{q,p}$ which, when restricted to $U(m)$, contain either $\mathcal{X}^{0,1}_{m}$ or $\mathcal{X}^{1,0}_{m}$. By the same argument as given in the proof of Proposition 3, we obtain that

$$\mathcal{Y}^{p,q} \otimes \mathcal{C} \subset \tilde{\mathcal{Y}}^{p,q}.$$ 

We now define $\tilde{\mathcal{E}}^{p,q} = (\tilde{\mathcal{Z}}^{p,q})^\perp \subset \mathcal{X}^{p,q} \otimes \mathcal{X}^{q,p}$ as the sum of those irreducible complex $U(m+1)$-submodules which do not contain $\mathcal{X}^{0,1}_{m}$ and $\mathcal{X}^{1,1}_{m}$. By the above (and since $\mathcal{W}^{p,q}, \mathcal{Y}^{p,q} \subset \tilde{\mathcal{Z}}^{p,q}$ (cf. the branching rule below)), we have

$$\tilde{\mathcal{E}}^{p,q} \subset \mathcal{E}^{p,q} \otimes \mathcal{C}.$$ 

To obtain the lower estimate on $\dim_{\mathcal{R}} \mathcal{E}^{p,q} = \dim_{\mathcal{R}} \mathcal{E}^{p,q} \geq \dim_{\mathcal{C}} \tilde{\mathcal{E}}^{p,q}$ of Theorem 1, we decompose $\tilde{\mathcal{E}}^{p,q}$ into irreducible complex $U(m+1)$-submodules. To do this, recall that an irreducible complex $U(m+1)$-module $V^\lambda = V^\lambda_{m+1}$ is uniquely determined by its highest weight $\lambda$ which, with respect to the standard (diagonal) maximal torus of $U(m+1)$, is an element of $\mathcal{Z}^{m+1}$ (with nonincreasing entries). In particular, we have

$$\mathcal{X}^{p,q} = V^{(p,0,\ldots,0,-q)}.$$ 

The general branching rule states that

$$V^{(\lambda_1,\ldots,\lambda_{m+1})}_{m+1} \mid_{U(m)} = \sum_{\sigma} V^{\sigma}_m,$$

where the summation runs through those $\sigma = (\sigma_1,\ldots,\sigma_m)$ for which

$$\lambda_1 \geq \sigma_1 \geq \cdots \geq \lambda_m \geq \sigma_m \geq \lambda_{m+1}.$$
THEOREM 2. Let $p > q > 0$ and $a = p + q$. Then, for $m \geq 3$, we have

$$
\mathcal{J}^{p,q} \otimes \mathcal{J}^{q,p} \cong \sum_{b=0}^{a} \sum_{c=0}^{\min\{b,q,a-b\}} \sum_{d=0}^{\min\{b-c, b-d, q-c, q-d, b+c-2d, a-b-c\}} [\min\{b-c, b-d, q-c, q-d, b+c-2d, a-b-c\} + 1] V^{(b,c,0,\ldots,0,-d,d-b-c)},
$$

where $e$ denotes the greatest integer $\leq (b+c)/2$. For $m = 2$, we have

$$
\mathcal{J}^{p,q} \otimes \mathcal{J}^{q,p} \cong \sum_{b=0}^{a} [\min\{b, q, a-b\} + 1] \mathcal{J}^{b,b} \oplus \sum_{c=1}^{q} \sum_{b=0}^{a-2c} [\min\{b-c, q-c, a-b-2c\} + 1] \times \{ V^{(b+c,c,-b-2c)} \oplus V^{(b+2c,-c,-b-c)} \}.
$$

We prove Theorem 2 in the next section. We now show how to compute $\dim_{C} \mathcal{E}^{p,q}$ from this decomposition to get the lower bound for $\dim \mathcal{L}^{p,q}$ of Theorem 1. By the branching rule, $V^{(b,c,0,\ldots,0,-d,d-b-c)}$ does not contain $\mathcal{J}^{0,0}_{m}$ or $\mathcal{J}^{1,1}_{m}$ (and consequently does not contain $\mathcal{J}^{1,0}_{m}$ or $\mathcal{J}^{0,1}_{m}$) if and only if $c \geq 2$ or $d \geq 2$. Thus we have

$$
(\mathcal{E}^{p,q})^{\perp} \cong \sum_{b=0}^{p+q} [\min\{b, q, p+q-b\} + 1] \mathcal{J}^{b,b} \oplus \sum_{b=1}^{p+q} \min\{b, q, p+q-b\} \{ V^{(b,1,0,\ldots,0,-b-1)} \oplus V^{(b+1,0,\ldots,-1,-b)} \} \oplus \sum_{b=1}^{p+q} \min\{b, q, p+q-b\} \{ V^{(b,1,0,\ldots,0,-1,-b)} \}.
$$

Computing the dimensions by the Weyl degree formula [5], we arrive at the lower estimate given in Theorem 1.

4. Decomposition of $\mathcal{J}^{p,q} \otimes \mathcal{J}^{q,p}$

We have the following.

THEOREM 3. Let $p > q > 0$ and $m \geq 3$. Then, for $b \geq c \geq d > 0$, the multiplicity

$$
m[V^{(b,c,d,\ldots)}; \mathcal{J}^{p,q} \otimes \mathcal{J}^{q,p}] = 0.
$$

Moreover, for $b_{j} \geq c_{j} \geq 0$, $j = 1, 2$, $b_{1} + c_{1} = b_{2} + c_{2}$, we have

$$
m[V^{(b_{1},c_{1},0,\ldots,0,-c_{2},-b_{2})}; \mathcal{J}^{p,q} \otimes \mathcal{J}^{q,p}] = \min\{(b_{1}-c_{1})^{+}, (q-c_{1})^{+}, (b_{2}-c_{2})^{+}, (q-c_{2})^{+}, (b_{2}-c_{1})^{+}, (b_{1}-c_{2})^{+}, (p+q-b_{1}-c_{1})^{+}\} + 1,
$$

where $^{+}$ denotes the positive part.
Once these multiplicity formulas are proved, Theorem 2 follows easily. For \( m = 2 \), the computation is elementary and the decomposition follows from Steinberg's formula. For \( m \geq 3 \), the first multiplicity formula combined with self-duality of \( \mathcal{K}_{P,q} \otimes \mathcal{K}_{q,P} \) imply that the only irreducible components are of the form \( V^{(b_1,c_1,0,\ldots,0,-c_2,-b_2)} \). Also, the center \( S^1 \subset U(m+1) \) acts on \( \mathcal{K}_{P,q} \otimes \mathcal{K}_{q,P} \) trivially, so that \( b_1 + c_1 = b_2 + c_2 \). The rest is a simple computation.

To determine the multiplicities we apply the Littlewood–Richardson rule [4] together with Weyl's duality [7] between representations of \( GL(V) \) and the symmetric group \( S_n \) on \( \bigotimes^n V \) for a vector space \( V \).

To prove the second multiplicity formula first, in the initial step of the Littlewood–Richardson rule, we add suitable elements of \( Z_4 \cdot (1, \ldots, 1) \) to the highest weight vectors of \( \mathcal{K}_{P,q} \) and \( \mathcal{K}_{q,P} \) to make the components descend to zero. We also add the sum of these elements to the highest weight of the representation whose multiplicity is to be computed. We obtain

\[
(b_1, c_1, 0, \ldots, 0, -c_2, -b_2) + (p+q, \ldots, p+q) = (p+q+b_1, p+q+c_1, p+q, \ldots, p+q, p+q-c_2, p+q-b_2),
\]

\[
(p, 0, \ldots, 0, -q) + (q, \ldots, q) = (p+q, q, \ldots, q, 0),
\]

\[
(q, 0, \ldots, -p) + (p, \ldots, p) = (p+q, p, \ldots, p, 0).
\]

Each vector represents a tableau consisting of \( m+1 \) rows; the coordinates representing the length of the respective row. We superimpose the two largest tableaux, which in this case correspond to the first and third vectors, to obtain Figure 1.

![Figure 1](image)

In the second step, we fill in the complementary space with the numbers 1, 2, ..., \( m \) and from each of these use the amount given by the respective coordinate of the second vector; that is, we use \( p+q \) 1's, \( q \) 2's, ..., \( q \) \( m \)'s.
The rules for filling are as follows.

(1) In each row the numbers are nondecreasing.
(2) In each column the numbers are (strictly) increasing.
(3) When reading the sequence of numbers from right to left:
   (a) the 1's are always O.K.; and
   (b) given \( i + 1 \) in the sequence, the number of previous \( i \)'s is greater than the number of previous \( (i+1) \)'s.

The required multiplicity is the number of possible ways of filling in.

We denote by \( R_i \) (\( i = 1, \ldots, m+1 \)) the \( i \)th row of the complementary tableau. By (3) there can only be 1's in \( R_1 \). In particular, \( b_1 \leq p + q \). By duality, we also have \( b_2 \leq p + q \). By (1)-(3), the last \( c_1 \) entries of \( R_2 \) are filled with 2's. 

\textbf{(Proof:)} There cannot be 1's there by (2). If there were an entry \( i \geq 3 \) then the same would be true for the last entry of \( R_2 \), by (1). Now, apply (3) to that entry to get a contradiction.) There are at most \( q \) 2's, so that we obtain \( c_1 \leq q \) (since otherwise the multiplicity is zero). By duality, we also have \( c_2 \leq q \); in particular, the smaller tableau is entirely contained in the larger tableau. By (1), there can only be 1's and 2's in \( R_2 \), the former preceding the latter.

From \( R_3 \) to \( R_{m-1} \) there are \( m-1 \) column entries. Thus, by (2), in \( R_3 \) there can only be 2's, 3's and 4's. However 4 cannot occur in \( R_3 \) since in that case the last entry of \( R_3 \) would be 4 and applying (3) would yield a contradiction. Thus, in \( R_3 \) there are only 2's and 3's. Below the 2's of \( R_2 \) there must be 3's, by (2).

We are interested in how many 3's are in \( R_3 \) under the 1's in \( R_2 \). We call these "jumps" since the respective column in \( R_2 \) and \( R_3 \) will be \( (\frac{1}{3}) \). \textbf{We claim that there are exactly} \( c_1 \) jumps.

First of all, if there were fewer than \( c_1 \) jumps then we would run out of the 2's which are \( q \) in number. (We used up \( c_1 \) 2's in the last \( c_1 \) entries of \( R_2 \) and, by (2), under the 2's in \( R_2 \) there must be 3's.) Secondly, if there were more than \( c_1 \) jumps then we would apply (3) to the 3 occurring in the very first jump. The number of 3's then would exceed the number of 2's; this is a contradiction.

As a byproduct we also obtained that we used up all 2's. We call the space occupied by the first \( c_1 \) 3's in \( R_3 \) the \textit{critical box}. We show below that any (allowed) location of the critical box determines the rest of the filling-ins.

In \( R_4 \) there can only be 3's and 4's, by repeating the argument above. Below the 3's of \( R_3 \) there must be 4's and below the 2's of \( R_3 \) there must be 3's, because otherwise the 4's in \( R_4 \) would exceed the 3's in \( R_3 \), violating (3). This argument can be carried out until \( R_{m-1} \) is filled up. It also follows that we would use up all numbers between 2 and \( m-2 \).

We now consider \( R_{m+1} \) which, at this point, can only be filled up by 1's, \((m-1)\)'s, and \( m \)'s. \textbf{We claim that there can only be} 1's and \( m \)'s in \( R_{m+1} \).

Assume the contrary. If there are no \( m \)'s in \( R_{m+1} \) then \( R_{m+1} \) is filled up by 1's and \((m-1)\)’s. We then take the first \( m \) in \( R_m \) (which certainly exists) and apply (3) to get a contradiction. If there is an \( m \) in \( R_{m+1} \) we take the first one
and again apply (3) to get a contradiction. As a byproduct we also obtained
that the \((m - 1)\)'s must be used up in \(R_{m-1}\) and \(R_m\). Thus, below the \((m - 2)\)'s
in \(R_{m-1}\) there must be \((m - 1)\)'s of \(R_m\), and the filling-in is unique.

Summarizing, we showed that: \textit{Once the location of the critical box is fixed, there is only one way to fill in.}

To obtain the exact constraints on the location of the critical box, denote
by \(d\) the distance of the critical box from the right wall (i.e., there are exactly
\(d + c_1\) 2's in \(R_2\)). The critical box occupies \(R_3\), whose length is \(q\), so that \(d \leq q - c_1\). There are at least \((q - b_2)^+ (m - 1)\)'s in \(R_m\) (by (2)) so that there are at least
\((q - b_2)^+ 2\)'s in \(R_3\). This means that \(d \leq q - c_1 - (q - b_2)^+\). Taking into account the previous estimate, we can replace this by \(d \leq b_2 - c_1\). In particular, \(c_1 \leq b_2\) is a general constraint on the tableaux (i.e., otherwise the multiplicity is zero). By duality, we also have \(c_2 \leq b_1\). Summarizing, we obtain

\[d \leq \min\{b_1 - c_1, q - c_1, b_2 - c_1\}.
\]

For the lower bound, first note that the critical box cannot occupy the last
\(c_2 - 1\) entries of \(R_3\) only because in that case the \(q (m - 1)\)'s cannot be filled
in \(R_{m-1}\) and \(R_m\). Thus, \(d \geq c_2 - c_1\). Moreover, in \(R_{m+1}\) there are \(p + q - b_2\)
places for the \(m\)'s, so \(R_m\) ends with at least \((q - (p + q - b_2))^+ = (b_2 - p)^+\)
\(m\)'s. Thus, \(d \geq c_2 - c_1 + (b_2 - p)^+\). By the previous estimate, we can replace
this by \(d \geq b_1 - p\). Summarizing, we obtain

\[d \geq \max\{c_2 - c_1, b_1 - p\}.
\]

Comparing the upper and lower bounds, we obtain that \(b_1 + c_1 \leq p + q\) is
a constraint on the tableaux. Finally we obtain that the multiplicity \(-1\) is the
minimum of the following numbers:

\[b_1 - c_1, \ q - c_1, \ b_2 - c_1\]

and

\[b_1 - c_1 - (c_2 - c_1) = b_1 - c_2, \quad q - c_1 - (b_1 - p) = p + q - b_1 - c_1,\]
\[b_1 - c_1 - (b_1 - p) = p - c_1, \quad b_2 - c_1 - (c_2 - c_1) = b_2 - c_2,\]
\[q - c_1 - (c_2 - c_1) = q - c_2, \quad b_2 - c_1 - (b_1 - p) = p - c_2.
\]

Two of the numbers do not contribute in the minimum as \(p > q\). The second
multiplicity formula of Theorem 3 follows.

To prove the first we apply a similar argument. In the tableaux corre-
responding to

\[m[V^{(b, c, \ldots)}; \mathcal{J}^p, q \otimes \mathcal{J}^q, p],\]

\(R_1\) is filled up by 1's as before. The last \(c\) entries of \(R_2\) are again filled up by
2's, and there can only be 1's and 2's in \(R_2\). Similarly, by (3) the last \(d\) entries
of \(R_3\) are filled up by 3's, and there can only be 2's and 3's in \(R_3\). Denote by
\(J\) the number of jumps \((\frac{1}{3})\) in \(R_2\) and \(R_3\). Then, by (3) we have \(J \leq c - d\). On
the other hand, \(J \geq c\) since otherwise we would run out of the \(q 2\)'s. This is a
contradiction, so the multiplicity is zero.
References


Department of Mathematical Sciences
Rutgers University
Camden, NJ 08102