D. BARBASCH*, J. F. GLAZEBROOK AND G. TOTH**

HARMONIC MAPS BETWEEN COMPLEX PROJECTIVE SPACES

ABSTRACT. Using the DoCarmo–Wallach theory, we classify (homogeneous) polynomial harmonic maps of complex projective spaces into spheres and complex projective spaces in terms of finite dimensional moduli spaces. We make use of representation theory of the (special) unitary group to give, for a spherical range, the exact dimension and, for complex projective spaces, a lower bound of the dimension of the moduli spaces.

1. INTRODUCTION AND PRELIMINARY CONSTRUCTIONS

One of the fundamental problems in harmonic map theory as posed by Eells and Lemaire in 1980 [3] is to construct and classify harmonic maps between complex projective spaces. Initiated by the work of Din and Zakrzewski [1] and Glaser and Stora [6] all harmonic maps of $\mathbf{C}P^1$ (or, more generally, all isotropic harmonic maps of a compact Riemann surface) into $\mathbf{C}P^n$ were obtained and explicitly described by Eells and Wood in [4]. These ideas have led to a fairly complete picture of harmonic maps of $\mathbf{C}P^1$ into flag manifolds [5] and all stable harmonic maps of $\mathbf{C}P^1$ into irreducible Hermitian symmetric spaces [10]. Energy minimizing harmonic maps of $\mathbf{C}P^1$ into $\mathbf{C}P^n$ or, more generally, into compact Kähler manifolds of positive biholomorphic sectional curvature were used (and shown to be (anti-)holomorphic) by Siu and Yau in the main step of their solution to the Frankel conjecture [11]. Since holomorphic maps are known and expected to behave more rigidly than harmonic maps in general, it is natural to ask if harmonic maps of $\mathbf{C}P^m$ into $\mathbf{C}P^n$ abound for $m \geqslant 2$.

Our construction, which we proceed to describe, yields parameter spaces of large dimension of harmonic nonholomorphic maps of $\mathbf{C}P^m$, $m \geqslant 2$, into $\mathbf{C}P^n$ for various n.

First, let $p > q \geqslant 0$ and denote by $\mathscr{H}^{p,q} = \mathscr{H}^{p,q}_{m+1}$ the Hermitian vector space of homogeneous harmonic polynomials in the variables $z_0, \bar{z}_0, \ldots, z_m$, $\bar{z}_m \in \mathbf{C}$ of bidegree (p, q). The Hermitian structure on $\mathscr{H}^{p,q}$ is given by the normalized L^2-scalar product

$$\langle \mu, \mu' \rangle = \frac{m!(n(p,q)+1)}{2\pi^{m+1}} \int_{S^{2m+1}} \mu \cdot \bar{\mu}' \operatorname{vol}(S^{2m+1}), \quad \mu, \mu' \in \mathscr{H}^{p,q},$$

where $\operatorname{vol}(S^{2m+1})$ is the volume element of the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$ (with

*Supported in part by NSF Grant DMS 8603172.
**Research done in part while the third named author was on leave at University of California, Berkeley.

Geometriae Dedicata **33**: 37–50, 1990.

total volume $2\pi^{m+1}/m!$) and

$$n(p, q) + 1 = \dim_{\mathbf{C}} \mathscr{H}^{p,q} = \binom{m+p}{p}\binom{m+q}{q} - \binom{m+p-1}{p-1}\binom{m+q-1}{q-1}.$$

Precomposition by unitary transformations (of $\mathbf{C}^{m+1}$) gives rise to a unitary (irreducible) $U(m+1)$-module structure on $\mathscr{H}^{p,q}$ which is given by the homomorphism $\rho_{p,q}\colon U(m+1) \to U(\mathscr{H}^{p,q})$, $\rho_{p,q}(u)\cdot\mu = \mu \circ u^{-1}$, $u \in U(m+1)$, $\mu \in \mathscr{H}^{p,q}$ [8]. The central subgroup $S^1 \subset U(m+1)$ acts, via $\rho_{p,q}$, on $\mathscr{H}^{p,q}$ by the single weight $p - q > 0$.

DEFINITION. We call $f\colon S^{2m+1} \to S^{2n+1}$ a (*homogeneous*) *polynomial harmonic map of bidegree* (p, q) if the components of f in $\mathbf{C}^{m+1} (\supset S^{2m+1})$ belong to $\mathscr{H}^{p,q}$. In this case, f is a harmonic map between spheres since $\mathscr{H}^{p,q}$ consists of (complex valued) spherical harmonics of order $p + q$ [8]. In addition, we assume that f is *full*, i.e. the image of f is not contained in any proper (complex) linear subspace of $\mathbf{C}^{n+1}$.

A (full) polynomial harmonic map $f\colon S^{2m+1} \to S^{2n+1}$ of bidegree (p, q) is equivariant with respect to the homomorphism $\rho_{p-q}\colon S^1 \to S^1$, $\rho(u) = u^{p-q}$, $u \in S^1 \subset \mathbf{C}$, in particular, f factors through the Hopf bundle maps $\pi\colon S^{2m+1} \to \mathbf{C}P^m$ and $\pi\colon S^{2n+1} \to \mathbf{C}P^n$ inducing a map $\tilde{f}\colon \mathbf{C}P^m \to \mathbf{C}P^n$.

By homogeneity, we have

$$\tilde{f}^*(\gamma_{\mathbf{C}P^n}) = \otimes^{p-q}\gamma_{\mathbf{C}P^m},$$

where γ stands for the canonical line bundle. Taking first Chern classes, it follows that $\tilde{f}$ has degree $p - q$ (on second cohomology), in particular, $m \leqslant n$. By the Smith Reduction theorem [3], $\tilde{f}\colon \mathbf{C}P^m \to \mathbf{C}P^n$ is harmonic iff $f\colon S^{2m+1} \to S^{2n+1}$ is horizontal, i.e. the differential of f maps the horizontal distribution $(\ker \pi_*)^\perp \subset T(S^{2m+1})$ into the horizontal distribution $(\ker \pi_*)^\perp \subset T(S^{2n+1})$. For a fixed orthonormal base $\{f^j_{p,q}\}_{j=0}^{n(p,q)} \subset \mathscr{H}^{p,q}$ (over $\mathbf{C}$) which, at the same time identifies $\mathscr{H}^{p,q}$ with $\mathbf{C}^{n(p,q)+1}$, define $f_{p,q} = (f^0_{p,q}, \ldots, f^{n(p,q)}_{p,q})\colon S^{2m+1} \to \mathbf{C}^{n(p,q)+1}$. Clearly, $f_{p,q}$ is equivariant with respect to $\rho_{p,q}\colon U(m+1) \to U(n(p,q)+1)$, in particular, by the choice of the normalizing constant above, $f_{p,q}$ maps into the unit sphere of $\mathbf{C}^{n(p,q)+1}$. We obtain that $f_{p,q}\colon S^{2m+1} \to S^{2n(p,q)+1}$ is a full polynomial harmonic map of bidegree (p, q).

PROPOSITION 1. *$f_{p,q}$: $S^{2m+1} \to S^{2n(p,q)+1}$ is horizontal, in particular, the induced map $\tilde{f}_{p,q}$: $\mathbf{C}P^m \to \mathbf{C}P^{n(p,q)}$ is harmonic.*

Proof. Let $o = (1, 0, \ldots, 0) \in \mathbf{C}^{m+1}$ be a base point and $U(m) = [1] \times U(m) \subset U(m+1)$ the corresponding isotropy subgroup. The isotropy representation of $U(m)$ on $T_o(S^{2m+1})$ decomposes as

$$T_o(S^{2m+1}) = \ker \pi_{*_o} \oplus (\ker \pi_{*_o})^{\perp},$$

where $\ker \pi_{*_o} \cong \mathbf{R}$ is trivial and $U(m)$ acts on $(\ker \pi_{*_o})^{\perp} \cong \mathbf{C}^m$ by ordinary matrix multiplication. Equivariance of $f_{p,q}$ with respect to $\rho_{p,q}$ implies that $(f_{p,q})_*(\ker \pi_{*_o})$ and $(f_{p,q})_*(\ker \pi_{*_o})^{\perp}$ (shifted to the origin) are *real* irreducible $U(m)$-submodules of the restriction $\mathscr{H}^{p,q}_{m+1}|_{U(m)}$. They are distinct since $f_{p,q}$ is nonconstant. By Schur's lemma (applied to the orthogonal projection of their linear span onto one of them), they are orthogonal. By equivariance, $(f_{p,q})_*(\ker \pi_*)$ and $(f_{p,q})_*(\ker \pi_*)^{\perp}$ are orthogonal everywhere on S^{2m+1}.

REMARK. Since $\mathbf{C}P^m = U(m+1)/U(1) \times U(m)$ is isotropy irreducible, it is easy to show that $(f_{p,q})_*$ is homothetic on $(\ker \pi_*)^{\perp}$. In particular, $(f_{p,q})_*$ $(\ker \pi_{*_o})^{\perp} \cong \mathbf{C}^m$ as a real $U(m)$-module. Moreover $\tilde{f}_{p,q}$: $\mathbf{C}P^m \to \mathbf{C}P^{n(p,q)}$ is then a homothetic immersion. Being harmonic it is therefore minimal.

Given a full polynomial harmonic map f: $S^{2m+1} \to S^{2n+1}$ of bidegree (p, q) we can write $f = A \cdot f_{p,q}$, where A is a complex $(n+1) \times (n(p,q)+1)$-matrix of maximal rank. We associate to f the Hermitian symmetric matrix $\langle f \rangle = A^* \cdot A - I \in \mathrm{Hom}(\mathscr{H}^{p,q}, \mathscr{H}^{p,q}) = \mathscr{H}^{p,q} \otimes \bar{\mathscr{H}}^{p,q} = \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$. The condition that f maps into S^{2n+1} can then be translated to the condition that $\langle f \rangle$ is perpendicular to the orthogonal projection $\mathrm{proj}[f_{p,q}(z)] \in \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$ (onto $\mathbf{C} \cdot f_{p,q}(z)$) for all $z \in S^{2m+1}$, where $\mathrm{Hom}\,(\mathscr{H}^{p,q}, \mathscr{H}^{p,q}) = \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$ is endowed with the Hermitian scalar product $\langle C, C' \rangle = \mathrm{trace}(C'^* \cdot C)$, $C, C' \in \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$. Indeed, for $z \in S^{2m+1}$, we have

$$\begin{aligned} |f(z)|^2 - 1 &= \langle A f_{p,q}(z), A f_{p,q}(z) \rangle - \langle f_{p,q}(z), f_{p,q}(z) \rangle \\ &= \langle (A^* \cdot A - I) f_{p,q}(z), f_{p,q}(z) \rangle \\ &= \mathrm{trace}(\mathrm{proj}[f_{p,q}(z)] \cdot \langle f \rangle). \end{aligned}$$

We wish to reformulate the condition of horizontality of a full polynomial harmonic map f: $S^{2m+1} \to S^{2n+1}$ of bidegree (p, q) in terms of $\langle f \rangle = A^*A - I$, where $f = A \cdot f_{p,q}$. Using horizontality of $f_{p,q}$, for $V_z \in \ker \pi_{*_z}$ and $X_z \in (\ker \pi_{*_z})^{\perp}$, we compute

$$\begin{aligned} \langle f_* V_z, f_* X_z \rangle &= \langle (A^*A)(f_{p,q})_* V_z, (f_{p,q})_* X_z \rangle \\ &= \langle (A^*A - I)(f_{p,q})_* V_z, (f_{p,q})_* X_z \rangle \\ &= \langle \langle f \rangle, (f_{p,q})_* V_z \cdot (f_{p,q})_* X_z \rangle, \end{aligned}$$

where the dot stands for the Hermitian symmetric product. (Given a Hermitian vector space H and $v, w \in H$, we denote by $v \cdot w$ the Hermitian symmetric endomorphism of H given by $(v \cdot w)(u) = \frac{1}{2}\langle u, v\rangle w + \frac{1}{2}\langle u, w\rangle v$, $u \in H$. Now the last equality is a simple computation. Note also that, for $H = \mathscr{H}^{p,q}$, we have $f_{p,q}(z) \cdot f_{p,q}(z) = \mathrm{proj}[f_{p,q}(z)]$.) We obtain that $f: S^{2m+1} \to S^{2n+1}$ is horizontal iff $\langle f \rangle$ is perpendicular to the Hermitian symmetric endomorphism $(f_{p,q})_* V_z \cdot (f_{p,q})_* X_z$ for all $V_z \in \ker \pi_{*_z}$ and $X_z \in (\ker \pi_{*_z})^{\perp}$. To put these into a formal description, denote by $S^{p,q}$ the *real* $U(m+1)$-submodule of Hermitian symmetric endomorphisms of $\mathscr{H}^{p,q}$. Clearly, $S^{p,q}$ is a real form of $\mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$. Then

$$W^{p,q} = \mathrm{span}_{\mathbf{R}}\{\mathrm{proj}\,[f_{p,q}(z)] \mid z \in S^{2m+1}\}$$

and

$$Z^{p,q} = \mathrm{span}_{\mathbf{R}}\{(f_{p,q})_* V_z \cdot (f_{p,q})_* X_z \mid V_z \in \ker \pi_{*_z}, X_z \in (\ker \pi_{*_z})^{\perp}\}$$

are (real) $U(m+1)$-submodules of $S^{p,q}$. Finally, let $E^{p,q}$ be the orthogonal complement of $W^{p,q} + Z^{p,q}$ in $S^{p,q}$ and set

$$L^{p,q} = \{C \in E^{p,q} \mid C + I \geqslant 0\},$$

where '$\geqslant$' means positive semidefinite. Clearly, $L^{p,q} \subset E^{p,q}$ is a $U(m+1)$-invariant convex body containing the origin in its interior.

Moreover, $E^{p,q}$ consists of traceless endomorphisms of $\mathscr{H}^{p,q}$, in particular, $L^{p,q}$ is compact.

To see this, we integrate the defining equality, for $C = (c_{ij})_{i,j=0}^{n(p,q)} \in E^{p,q}$,

$$\sum_{i,j=0}^{n(p,q)} c_{ij} \cdot f^i_{p,q}(z) \bar{f}^j_{p,q}(z) = 0$$

over S^{2m+1} to get $\Sigma_{i=0}^{n(p,q)} c_{ii} = \mathit{trace}\ C = 0$ by orthogonality.

Summarizing, we associated to each full polynomial horizontal harmonic map $f: S^{2m+1} \to S^{2n+1}$ of bidegree (p, q) an endomorphism $\langle f \rangle = A^* \cdot A - I \in L^{p,q}$ via $f = A \cdot f_{p,q}$. As for the $U(m+1)$-action on $L^{p,q}$, we clearly have $u \cdot \langle f \rangle = \langle f \circ u^{-1} \rangle$, $u \in U(m+1)$. By the polar decomposition, the parametrization is injective on the equivalence classes of maps where two full polynomial harmonic maps $f, f': S^{2m+1} \to S^{2n+1}$ are said to be *equivalent* if $f' = U \cdot f$ for some $U \in U(n+1)$. Furthermore, as square root can be taken from positive semidefinite endomorphisms, the parametrization is clearly surjective. Thus we obtain the following:

THEOREM 1. *The equivalence classes of full polynomial horizontal harmonic maps* $f: S^{2m+1} \to S^{2n+1}$ *of bidegree* (p, q), $p > q \geqslant 0$, *can be parametrized by* *a compact convex body $L^{p,q}$ lying in a finite dimensional vector space $E^{p,q}$. The interior of $L^{p,q}$ corresponds to those maps with maximal $n = n(p, q)$. $E^{p,q}$ is a (real) $U(m+1)$-submodule of the $U(m+1)$-module $S^{p,q}$ of Hermitian symmetric endomorphisms of $\mathscr{H}^{p,q}$. The parameter space $L^{p,q}$ is $U(m+1)$-invariant and the action is induced by precomposing harmonic maps by unitary transformations.*

REMARK. The result above is a DoCarmo–Wallach type classification theorem [2].

We now take $p = q$. In this case $\mathscr{H}^{p,p}$ has a natural real form $\mathscr{H}_o^{p,p}$ which is the real $U(m+1)$-module of (real valued) homogeneous harmonic polynomials in the variables $z_0, \bar{z}_0, \ldots, z_m, \bar{z}_m \in \mathbf{C}$ of bidegree (p, p). Note that $\mathscr{H}_o^{p,p}$ is nothing but the eigenspace of the Laplacian on $\mathbf{C}P^m$ corresponding to the eigenvalue $4p(p+m)$.

We call $f\colon S^{2m+1} \to S^n$ a polynomial harmonic map of bidegree (p, p) if the components of f in $\mathbf{R}^{n+1}(\supset S^n)$ belong to $\mathscr{H}_o^{p,p}$. In this case f factors through the Hopf bundle map $\pi\colon S^{2m+1} \to \mathbf{C}P^m$ yielding a harmonic map $f\colon \mathbf{C}P^m \to S^n$. We define the fullness of f with respect to proper *real* linear subspaces of $\mathbf{R}^{n+1}$ and equivalence with respect to the orthogonal group $O(n+1)$. Defining $f_{p,p}\colon S^{2m+1} \to S^{n(p,p)}$, $n(p,p) + 1 = \dim_{\mathbf{R}} \mathscr{H}_o^{p,p}$, $W^{p,p}$, $E^{p,p}$ and $L^{p,p}$ analogously (over $\mathbf{R}$) and repeating the construction we arrive at the following:

THEOREM 2. *The equivalence classes of full polynomial harmonic maps $f\colon S^{2m+1} \to S^n$ of bidegree (p, p) can be parametrized by a compact convex body $L^{p,p}$ lying in a finite dimensional vector space $E^{p,p}$. The interior of $L^{p,p}$ corresponds to those maps with maximal $n = n(p, p)$. $E^{p,p}$ is a (real) $U(m+1)$-submodule of the symmetric square $S^2(\mathscr{H}_o^{p,p})$.*

2. Estimates on the dimension of the parameter space

To pin down the $U(m+1)$-module structure of $E^{p,q}$, let $o = (1, 0, \ldots, 0) \in \mathbf{C}^{m+1}$ and denote by $U(m) = [1] \times U(m) \subset U(m+1)$ the corresponding isotropy subgroup.

For $p > q \geqslant 0$ the $U(m+1)$-module $W^{p,q}$ can then be written as

$$W^{p,q} = \operatorname{span}_{\mathbf{R}}\{U(m+1) \cdot \operatorname{proj}[f_{p,q}(o)]\}.$$

As $\operatorname{proj}[f_{p,q}(o)]$ is left fixed by $U(m)$, it follows that every irreducible $U(m+1)$-submodule of $W^{p,q}$ is class 1 with respect to $(U(m+1), U(m))$ (i.e. contains a $U(m)$-fixed vector). After complexification, noting that the representations occurring here are absolutely irreducible, we obtain that

$W^{p,q} \otimes \mathbf{C}$ is contained in the sum of complex class 1 subrepresentations of $\mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$ with respect to $(U(m+1), U(m))$. By Cartan [15], every complex class 1 representation of $(U(m+1), U(m))$ in $\mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$ is of the form $\mathscr{H}^{b,b}$. We obtain the estimate

$$\dim_{\mathbf{R}} W^{p,q} \leqslant \sum_b m[\mathscr{H}^{b,b}: \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}] \cdot \dim_{\mathbf{C}} \mathscr{H}^{b,b}.$$

In a similar vein, we have

$$Z^{p,q} = \operatorname{span}_{\mathbf{R}}\{U(m+1) \cdot [(f_{p,q})_* V_o \cdot (f_{p,q})_* X_o] | V_o \in \ker \pi_{*_o}, X_o \in (\ker \pi_{*_o})^{\perp}\}.$$

PROPOSITION 2. *Given a real $U(m)$-submodule Z_o of $S^{p,q}$, define*

$$Z = \operatorname{span}_{\mathbf{R}}\{U(m+1) \cdot Z_o\} \subset S^{p,q}.$$

Let $\bar{Z}$ be the sum of those complex irreducible $U(m+1)$-submodules of $\mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$ that, when restricted to $U(m)$, contain an irreducible component of $Z_o \otimes_{\mathbf{R}} \mathbf{C}$. Then, we have

$$Z \otimes_{\mathbf{R}} \mathbf{C} \subset \bar{Z}.$$

Proof. This is an application of the Frobenius Reciprocity. The proof follows in exactly the same way as that of Lemma 12.2 in [15].

Setting

$$Z_o = \operatorname{span}_{\mathbf{R}}\{(f_{p,q})_* V_o \cdot (f_{p,q})_* X_o \mid V_o \in \ker \pi_{*_o}, X_o \in (\ker \pi_{*_o})^{\perp}\},$$

the $U(m+1)$-module Z in Proposition 2 specializes to $Z^{p,q}$. Moreover, as $U(m)$ acts on $\ker \pi_{*_o}$ trivially, we obtain that $Z_o \cong \mathbf{C}^m$ as real $U(m)$-modules. Complexifying, we get

$$Z_o \otimes_{\mathbf{R}} \mathbf{C} \cong \mathscr{H}_m^{1,0} \oplus \mathscr{H}_m^{0,1}$$

as complex $U(m)$-modules. Now, Proposition 2 implies that each irreducible $U(m+1)$-submodule of $Z^{p,q} \otimes_{\mathbf{R}} \mathbf{C}$ contains $\mathscr{H}_m^{1,0}$ or $\mathscr{H}_m^{0,1}$ (by restriction to $U(m) \subset U(m+1)$). To get a step further, recall that an irreducible complex $U(m+1)$-module $V^{\lambda} = V_{m+1}^{\lambda}$ is uniquely determined by its highest weight λ, which, with respect to the standard (diagonal) maximal torus of $U(m+1)$, is an element of $\mathbf{Z}^{m+1}$ with decreasing entries. In particular, we have

$$\mathscr{H}_{m+1}^{p,q} = V_{m+1}^{(p,0,\ldots,0,-q)}.$$

Moreover, the Branching rule [18] takes the form

$$V_{m+1}^{(\lambda_1,\ldots,\lambda_{m+1})}|_{U(m)} = \sum_{\sigma} V_m^{\sigma},$$

where the summation runs over all $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbf{Z}^m$ for which

$$\lambda_1 \geqslant \sigma_1 \geqslant \cdots \geqslant \lambda_m \geqslant \sigma_m \geqslant \lambda_{m+1}.$$

In particular, an irreducible $U(m+1)$-submodule of $Z^{p,q} \otimes_{\mathbf{R}} \mathbf{C} \subset \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}$ that contains $\mathscr{H}^{1,0}_m$ (resp. $\mathscr{H}^{0,1}_m$) is of the form $V^{(b,1,0,\ldots,0,-b-1)}$ (resp. $V^{(b+1,0,\ldots,0,-1,-b)}$). Summarizing, we obtain the estimate

$$\dim_{\mathbf{R}} Z^{p,q} \leqslant 2 \sum_b m[V^{(b,1,0,\ldots,0,-b-1)}: \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}] \dim_{\mathbf{C}} V^{(b,1,0,\ldots,0,-b-1)}.$$

To determine the multiplicities in the tensor product above we apply the Littlewood–Richardson rule [7] together with Weyl's duality [16] between representations of $\mathrm{GL}(V)$ and the symmetric group S_n on $\otimes^n V$ for a vector space V. We get

$$m[\mathscr{H}^{b,b}: \mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p}] = \min\{b, q, p+q-b\} + 1$$

and

$$m[V^{(b,1,0,\ldots,0,-b-1)}: \mathscr{H}^{p,q} \otimes \mathscr{H}^{p,q}] = \min\{b, q, p+q-b\}.$$

For completeness, we give here the details for the proof of the first formula. The proof of the second formula is analogous. (Note also that, for $m = 2$, these multiplicity formulas can easily be derived independently using Steinberg's formula.) In terms of the highest weights we have to determine the multiplicity

$$m = m[V^{(b,0,\ldots,0,-b)}: V^{(p,0,\ldots,0,-q)} \otimes V^{(q,0,\ldots,0,-p)}].$$

Setting $a = p + q$, m is zero for $b > a$ (e.g. by Steinberg's formula) so that we will assume that $b \leqslant a$. In the first step of the Littlewood–Richardson rule, to each weight vector we add a suitable element of $\mathbf{Z}_+ \cdot (1, \ldots, 1)$ to make the entries $\geqslant 0$. We obtain

$$(b, 0, \ldots, 0, -b) + (a, a, \ldots, a) = (a+b, a, .., a, a-b),$$
$$(p, 0, \ldots, 0, -q) + (q, q, \ldots, q) = (a, q, \ldots, q, 0),$$
$$(q, 0, \ldots, 0, -p) + (p, p, \ldots, p) = (a, p, \ldots, p, 0).$$

Each vector represents a tableau consisting of $m + 1$ rows; the coordinates representing the length of the respective row. We then superimpose the two largest tableaux, i.e. which correspond to the first and third vectors, to obtain the system shown in Figure 1.

In the second step we fill in the complementary boxes with the numbers $1, 2, \ldots, m$ and from each of these we use the amount given by the respective

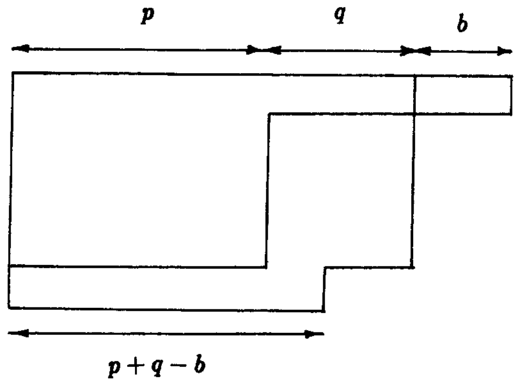

Fig. 1.

coordinate of the second vector, i.e. we use a 1's, q 2's, ..., q m's. The rules for filling in are as follows:

(1) In each row the numbers are nondecreasing,

(2) In each column the numbers are (strictly) increasing,

(3) When reading the sequence of numbers from right to left (a) the 1's are always O.K.; (b) given $i + 1$ in the sequence, the number of previous i's is greater than the number of previous $(i + 1)$'s. Now, the required multiplicity m is the number of possible ways of filling-in.

Turning to the proof, we first note that, by (3), the empty boxes in the first row are filled up with b 1's. By (2), the empty boxes of the second row can only be 1's and 2's and, by (3), the 1's precede the 2's. Any column which begins at a box in the second row contains consecutive integers. For a column which begins with 1, this is a consequence of (3). For a column which begins with 2, this follows from (2) as the length of the column is m.

Case I. $1 \leqslant b < q$

In this case there are $q - b$ empty boxes in the mth row which overlap with those of the $(m - 1)$th row. Consequently, the empty boxes in the second row have to start with $q - b$ 1's. The remaining b boxes give $b + 1$ possibilities for filling in 1's and 2's. Once the second row is filled up, the rest is determined. Thus, in this case, the multiplicity is $b + 1$.

Case II. $q \leqslant b \leqslant p$

The empty boxes in the $(m - 1)$th and mth row do not overlap. Filling the empty boxes in the second row gives $q + 1$ possibilities. Note that, since

$p + q - b \geqslant q$, there are enough empty boxes for the leftover m's in the last row. The multiplicity is then $q + 1$

Case III. $p < b \leqslant p + q$.

In this case, there are $p + q - b + 1$ possibilities for filling the empty boxes in the last row with 1's and m's. The rest is determined so that the multiplicity is $p + q - b + 1$.

Having exhausted all possibilities, the multiplicity formula follows. Putting everything together, we obtain the following

THEOREM 3. *For the parameter space* $L^{p,q}$, $p > q \geqslant 0$, *we have*

$$\dim L^{p,q} \geqslant \left\{\binom{m+p}{p}\binom{m+q}{q} - \binom{m+p-1}{p-1}\binom{m+q-1}{q-1}\right\}^2$$

$$- \sum_{b=0}^{p+q} \min\{b+1, q+1, p+q-b+1\}\left\{\binom{m+b}{b}^2 - \right.$$

$$\left. - \binom{m+b-1}{b-1}^2\right\}$$

$$-2\sum_{b=1}^{p+q} \min\{b, q, p+q-b\}\binom{m+b-1}{b+1}$$

$$\frac{b(m+b+1)(m+2b+1)}{m(m-1)}.$$

REMARKS. 1. We conjecture that the lower bound for the dimension is sharp so that actually equality holds. Without the horizontality condition, for $m = 2$ and $p + q \leqslant 4$, we will show this to be the case in the forthcoming section.

2. For $q = 0$, easy computation shows that $L^{p,0}$ contains the origin only which corresponds to the classical Veronese mapping.

For $p = q$ the situation is simpler because $\mathscr{H}_0^{p,p}$ is the (real) eigenspace of the Laplacian on $\mathbf{C}P^m$ corresponding to the eigenvalue $4p(p + m)$. Since $\mathbf{C}P^m$ is rank 1, the irreducible components of $E^{p,p} \subset S^2(\mathscr{H}_0^{p,p})$ are *not* class 1 with respect to $(U(m + 1), U(m))$ [15], [12]. Thus, after complexification, we have the equality

$$\dim W^{p,p} = \sum_b m[\mathscr{H}^{b,b}\colon S^2(\mathscr{H}^{p,p})] \dim_{\mathbf{C}} \mathscr{H}^{b,b}.$$

To determine the multiplicity in the symmetric square we apply the combinatorial rules of Chapter 3 in [9] and obtain

$$m[\mathscr{H}^{b,b}\colon S^2(\mathscr{H}^{p,p})] = \left[\left|\frac{\min\{b+1, p+1, 2p-b+1\}}{2}\right|\right] + \varepsilon, \quad b \leqslant 2p,$$

where $[|\cdot|]$ denotes the 'greatest integer $\leqslant$' function and $\varepsilon = 1$ for b even and $\varepsilon = 0$ for b odd.

THEOREM 4. *For the parameter space* $L^{p,p}$, *we have*

$$\begin{aligned}\dim L^{p,p} = \frac{1}{2}\left\{\binom{m+p}{p}^2 - \binom{m+p-1}{p-1}^2\right\}&\left\{\binom{m+p}{p}^2 - \binom{m+p-1}{p-1}^2 - 1\right\} \\ &- \sum_{b=0}^{2p}\left\{\left[\left|\frac{\min\{b+1, p+1, 2p-b+1\}}{2}\right|\right] + \varepsilon\right\}. \\ &\left\{\binom{m+b}{b}^2 - \binom{m+b-1}{b-1}^2\right\}.\end{aligned}$$

REMARK A lower bound for $\dim L^{p,p}$ has been obtained by Urakawa [14].

3. Polynomial harmonic maps of S^5 into S^{2n+1}

From here on we put $m = 2$ and $p \geqslant q \geqslant 0$. Then, by the multiplicity formula above, we have

$$\begin{aligned}\mathscr{H}^{p,q} \otimes \mathscr{H}^{q,p} \cong \sum_{b=0}^{p+q} \min\{b+1, q+1, p+q-b+1\} V^{(b,0,-b)} \oplus \\ \oplus \sum_{c=1}^{q} \sum_{b=0}^{p+q-2c} \min\{b+1, q-c+1, \\ p+q-b-2c+1\} \times \\ \times \left\{V^{(b+c,c,-b-2c)} \oplus V^{(b+2c,-c,-b-c)}\right\}.\end{aligned}$$

For $q \geqslant 2$, we have

$$\dim L^{p,q} \geqslant 2 \sum_{c=2}^{q} \sum_{b=0}^{p+q-2c} \min\{b+1, q-c+1, p+q-b-2c+1\}$$
$$(b+1)(b+c+1)(b+3c+1) > 0$$

so that we obtain an abundance of full polynomial harmonic maps $\tilde{f}\colon \mathbf{C}P^2 \to \mathbf{C}P^n$. For example, for quintic polynomial harmonic maps we get $\dim L^{3,2} \geqslant 170$. To simplify the exposition, we now set aside horizontality by putting $p = 2$ and $q = 1$. We have

$$\mathscr{H}^{2,1} \otimes \mathscr{H}^{1,2} \cong \mathscr{H}^{0,0} \oplus 2\mathscr{H}^{1,1} \oplus 2\mathscr{H}^{2,2} \oplus \mathscr{H}^{3,3} \oplus$$
$$\oplus V^{(1,1,-2)} \oplus V^{(2,-1,-1)} \oplus V^{(2,1,-3)} \oplus V^{(3,-1,-2)}.$$

THEOREM 5. *The irreducible components of* $(W^{2,1})^{\perp}$ *are not class* 1 *with respect to* $(U(3), U(2))$, *i.e. we have (as* $U(3)$*modules)*

$$(W^{2,1})^{\perp} \otimes \mathbf{C} \cong V^{(1,1-2)} \oplus V^{(2,-1,-1)} \oplus V^{(2,1,-3)} \oplus V^{(3,-1,-2)}.$$

Proof. Assuming the contrary means the existence of a Hermitian symmetric endomorphism $C \in (W^{2,1})^{\perp}$ of $\mathscr{H}^{2,1}$, which is equivariant with respect to the subgroup $U(2) \subset U(3)$. By the Branching theorem[18], we have

$$\mathscr{H}_3^{2,1}|_{U(2)} \cong \sum_{\substack{0 \leqslant r \leqslant 2 \\ 0 \leqslant s \leqslant 1}} \mathscr{H}_2^{r,s}$$

so that C is (real) diagonal with respect to this decomposition. Now we pick an orthonormal base $\{f_{p,q}^i\}_{i=0}^{14}$ in $\mathscr{H}^{2,1}$ consisting of weight vectors of $U(2)$ in $\mathscr{H}_2^{r,s}$, $0 \leqslant r \leqslant 2$, $0 \leqslant s \leqslant 1$. Namely, using the coordinates $z, w, t \in \mathbf{C}$, we set (in $\mathscr{H}_3^{2,1}$):

$$\mathscr{H}_2^{0,0} = \mathrm{span}\{(|z|^2 - |w|^2 - |t|^2)z\},$$

$$\mathscr{H}_2^{1,0} = \mathrm{span}\left\{\frac{1}{\sqrt{2}}(|w|^2 - 2|z|^2)w, \frac{1}{\sqrt{2}}(|t|^2 - 2|z|^2)t\right\},$$

$$\mathscr{H}_2^{2,0} = \mathrm{span}\{\sqrt{3}\, w^2\bar{z}, \sqrt{3}\, t^2\bar{z}, \sqrt{2}\, \bar{z}wt\},$$
$$\mathscr{H}_2^{0,1} = \mathrm{span}\{\sqrt{3}\, z^2\bar{w}, \sqrt{3}\, z^2\bar{t}\},$$
$$\mathscr{H}_2^{1,1} = \mathrm{span}\{(|w|^2 - |t|^2)z, \sqrt{2}\, z\bar{w}t, \sqrt{2}\, zw\bar{t}\},$$

$$\mathscr{H}_2^{2,1} = \mathrm{span}\left\{\frac{1}{\sqrt{2}}(|w|^2 - 2|t|^2)w, \frac{1}{\sqrt{2}}(|t|^2 - 2|w|^2)t,\right.$$

$$\left.\sqrt{3}\, w^2\bar{t}, \sqrt{3}\, t^2\bar{w}\right\},$$

where the polynomials in the brackets are (by definition) the components of $f_{2,1}: S^5 \to S^{29}$. The condition $C \in (W^{2,1})^\perp$ translates into $\langle C \cdot f_{2,1}, f_{2,1} \rangle = 0$. Expanding, it follows easily that $C = 0$, a contradiction.

REMARKS. Using an explicit base, a similar result can be obtained (for quartics) for $\mathscr{H}_3^{3,1}$.

Using the base of $\mathscr{H}_3^{2,1}$ occurring in the proof above, various cubic harmonic eigenmaps can be constructed between S^5 and S^{2n+1}, $6 \leqslant n \leqslant 14$. For example, let $f_n: S^5 \to S^{2n+1}$ be defined by

$$\begin{aligned} f_6(z, w, t) = {} & ((|z|^2 - \alpha|w|^2 - \beta|t|^2)z, (|w|^2 - \alpha|t|^2 - \beta|z|^2)w, \\ & (|t|^2 - \alpha|z|^2 - \beta|w|^2)t, \gamma\bar{z}t^2, \gamma\bar{w}z^2, \gamma\bar{t}w^2, \delta\bar{z}wt) \\ & (\alpha = -1 + 2\sqrt{2}, \beta = 3 - 2\sqrt{2}, \gamma = 4\sqrt{-1 + 2\sqrt{2}}, \\ & \delta = 2\sqrt{6}\sqrt{3 + 2\sqrt{2}}), \end{aligned}$$

$$\begin{aligned} f_9(z, w, t) = {} & (\sqrt{7/8}(|z|^2 - 2|w|^2)z, 1/\sqrt{8}(|w|^2 - 2|z|^2)w, \\ & \sqrt{7/8}(|w|^2 - 2|t|^2)w, 1/\sqrt{8}(|t|^2 - 2|w|^2)t, \\ & \sqrt{7/8}(|t|^2 - 2|z|^2)t, 1/\sqrt{8}(|z|^2 - 2|t|^2)z, \\ & \sqrt{6}z^2\bar{w}, \sqrt{6}w^2\bar{t}, \sqrt{6}t^2\bar{z}, \sqrt{6}\bar{z}wt), \end{aligned}$$

$$\begin{aligned} f_{12}(z, w, t) = {} & ((|z|^2 - |w|^2 - |t|^2)z, 1/\sqrt{2}(|w|^2 - 2|z|^2)w, 1/\sqrt{2}(|t|^2 - 2|z|^2)t, \\ & \sqrt{3}w^2\bar{z}, \sqrt{3}t^2\bar{z}, \sqrt{3}z^2\bar{w}, \sqrt{3}z^2\bar{t}, (|w|^2 - |t|^2)z, \\ & \sqrt{2}z\bar{w}t, 1/\sqrt{2}(|w|^2 - 2|t|^2)w, 1/\sqrt{2}(|t|^2 - 2|w|^2)t, \sqrt{3}w^2\bar{t}, \sqrt{6}\bar{z}wt) \end{aligned}$$

and the remainder can be obtained by replacing $\sqrt{6}\,\bar{z}wt$ by $(\sqrt{3}\,\bar{z}wt, \sqrt{3}\,z\bar{w}t)$ and by $(\sqrt{2}\,\bar{z}wt, \sqrt{2}\,z\bar{w}t, \sqrt{2}\,zw\bar{t})$.

REMARK. We conjecture that there are no cubic harmonic maps $f: S^5 \to S^{2n+1}$ of bidegree (2, 1) for $n \leqslant 5$. This is certainly the case (as computation shows) if we assume that the components belong to the span of that of f_6. (In the terminology of [13], $\langle f_6 \rangle$ is a vertex of $L^{2,1}$.)

Acknowledgement

The authors wish to thank the referee for pointing out an error in the first version of the manuscript.

REFERENCES

1. Din, A. and Zakrzewski, W., 'General classical solutions in the CP^{n-1} model', *Nucl. Phys.* **B174**(1980), 397–406.
2. DoCarmo, M. and Wallach, N., 'Minimal immersions of spheres into spheres', *Ann. Math.* **93**(1971), 43–62.
3. Eells, J. and Lemaire, L., *Selected Topics in Harmonic Maps*, Reg. Conf. Ser. in Math. No. 50, AMS, 1982.
4. Eells, J. and Wood, J. C., 'Harmonic maps from surfaces to complex projective spaces', *Adv. Math.* **49**, No. 3 (1983), 217–263.
5. Guest, M. A., 'Geometry of maps between generalized flag manifolds', *J. Diff. Geom.* **25**(1987), 223–247.
6. Glaser, V. and Stora, R., 'Regular solutions of the CP^n models and further generalizations', CERN, Preprint, 1980.
7. James, G. D., 'The representation theory of the symmetric group', *Lecture Notes in Math.* **682**, Springer, New York, 1978.
8. Knapp, A. W., *Representation Theory of Semisimple Groups.* Princeton University Press, 1986.
9. Robinson, G. de B., *Representation of the Symmetric Group*, University of Toronto Press, 1961.
10. Siu, Y. T., 'Curvature characterization of hyperquadrics', *Duke Math. J* **47**, No. 3 (1980), 641–654.
11. Siu, Y. T. and Yau, S. T., 'Compact Kähler manifolds of positive bisectional curvature', *Invent. Math.* **59** (1980), 189–204.
12. Toth, G., 'On classification of quadratic harmonic maps of S^3, *Proc. Amer. Math. Soc.* **102**, No.1 (1988), 174–176.
13. Toth, G., 'Classification of quadratic harmonic maps of S^3 into spheres', *Indiana Univ. Math J.* **36**, No. 2 (1987), 231–239.
14. Urakawa, H., 'Minimal immersions of projective spaces into spheres', *Tsukuba J. Math.* **9**, No. 2(1985), 321–347.
15. Wallach, N. R., *Minimal immersions of symmetric spaces into spheres*', in *Symmetric Spaces*, Marcel Dekker, New York, 1972, pp. 1–40.
16. Weyl, H., *The Classical Groups*, Princeton Univ. Press, 1946.
17. Wood, J. C., 'Twistor constructions for harmonic maps', in *Lecture Notes in Math.* **1255**, Springer-Verlag, New York, 1987, pp. 130–159.
18. Zhelobenko, D. P., *Compact Lie Groups and their Representations*, Amer. Math. Soc., Providence, Rhode Island, 1973.

Authors' addresses:

D. Barbasch,
Department of Mathematics,
Hill Center,
Rutgers University,
New Brunswick, NJ 08903,
U.S.A.

J. F. Glazebrook,
Department of Mathematics,
Eastern Illinois University,
Charleston, IL 61920,
U.S.A.

G. Toth,
Department of Mathematics,
Rutgers University,
Camden, NJ 08102,
U.S.A.

(Received, January 24, 1989; revised version, June 27, 1989)