CURVATURE AND TORSION OF CONTACT RIEMANNIAN THREE-MANIFOLDS

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Abstract

Let M be a compact and orientable three-manifold with a contact structure (ω, X_0, g) where the metric g is critical with respect to the energy $\int |L_{X_0} g|^2 d\, vol(M, g)$. Then, if its generalized Tanaka-Webster scalar curvature $r^*(g)$ is sufficiently large, g has positive Ricci curvature. As a consequence if the characteristic vector field X_0 is a Killing field and r^* is positive, g may be deformed to a contact metric of positive curvature. If in addition, r^* is a constant, then g may be deformed to a contact metric of constant curvature. It is also shown that the variational problem for the above energy is equivalent with the variational problem for the functional $\int (r^* - r) d\, vol(M, g)$, where r is the scalar curvature of g.

1. Introduction

Y. Carriére [3] has classified Riemannian flows on compact three- manifolds. Th difficulty encountered in the study of Riemannian flows is that they are not au tomatically Killing flows. A compact three-manifold M admitting a nonsingula Killing vector field is a Seifert manifold (see §4 for a proof), so if M is simply con nected it is diffeomorphic to the standard three-sphere S^3. Chern and Hamilto [4] introduced the torsion $|\tau|$ (the length of τ) in their study of compact three manifolds (M, g), where $\tau (= L_{X_0} g)$ is the Lie derivative of the contact metric with respect to the characteristic vector field X_0 of the contact structure, and the conjectured that for fixed *contact form* $\omega = g(X_0, \cdot)$, with X_0 inducing a Seifert fo liation, there exists a complex structure $\phi|B$ on $B = \ker\, \omega$ such that the Dirichle energy

$$\mathcal{E}(g) = \frac{1}{2} \int_M |\tau|^2 d\ \mathrm{vol}(M, g) \tag{1.1}$$

*Supported by the Natural Sciences and Engineering Rresearch Council of Canada.
†Supported by funds 60% of the Ministero Pubblica Istruzione.
1980 MATHEMATICS SUBJECT CLASSIFICATION. Primary 53C15; Secondary 53C20.
KEY WORDS AND PHRASES. Contact Riemannian three-manifolds, critical metric, curvature

is critical over all CR structures. (A CR structure on an orientable th
is a contact structure together with a complex structure on B. Sin
is equivalent to a conformal structure, a Riemannian metric on a co
manifold gives rise to a CR structure.)

Let M be a $(2n+1)$-dimensional contact manifold with a fixed
ω. Denote the space of all associated Riemannian metrics to the co
by $\mathcal{M}(\omega)$. Let g be a point of $\mathcal{M}(\omega)$, and denote by $\{g(t)\}$ a curve i
$g(0) = g$. Tanno [11] showed that g is a critical point of $\mathcal{E}$, if and onl

$$\nabla_{X_0}\tau = 2\tau \cdot \phi. \tag{1.2}$$

Thus, for $n = 1$, $\mathcal{E}(g)$ is critical over all CR structures if and on
satisfied. (This differs from the condition $\nabla_{X_0}\tau = 0$ incorrectly obt
Theorem 5.4.) In the sequel, a critical point of $\mathcal{E}$ will be called a *cri*
Note that g is a critical point of $\mathcal{E}$ if X_0 is a Killing vector field with

Can $\mathcal{E}$ have a critical point g such that $\mathcal{E}(g) \neq 0$? Blair [1] show
answer is no if the contact structure on M is regular, i.e., if every poi
a neighborhood such that any integral curve of the characteristic ve
which passes through the neighborhood does so only once. (A theorem
and Wang says that a compact regular contact manifold is a principal
over a symplectic manifold whose fundamental form has integral perio
showed that the standard contact metric structure on the unit tange
a compact surface of constant negative curvature is not regular [2], a
metric g is a critical point of $\mathcal{E}$ such that $\tau \neq 0$.

In [6] it is shown that if the scalar curvature $r > -2$ on a com
three-manifold (M, g) whose characteristic vector field is a Killing
may be deformed to a contact metric of positive Ricci curvature. It
purpose of this paper to show that g may in fact be deformed to a co
of positive sectional curvature. This is a consequence of Theorem
yields the statement that if r is a constant greater than -2 (equiva
generalized Tanaka-Webster scalar curvature is a positive constant),
be deformed to a contact metric of (positive) constant curvature. Mo
if the contact metric g is critical, then g is of constant curvature, if an
characteristic field is a Killing field.

The interpretation of the Chern-Hamilton variational problem given in §4 is the essential difference between this paper and the one presented at the III Internationa Symposium on Differential Geometry, Peniscola, Spain, 1988.

2. Compact three-manifolds

A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it carrie a global 1-form $\omega \neq 0$ with the property that $\omega \wedge (d\omega)^n \neq 0$ everywhere. It ha an underlying almost contact structure (ϕ, X_0, ω), where $\omega(X_0) = 1$, $\phi X_0 = 0$ an $\phi^2 = -I + \omega \otimes X_0$, I being the identity field. A metric g, called an *associate metric*, can then be found such that $\omega = g(X_0, \cdot)$ and $d\omega(X,Y) = g(\phi X, Y)$. (I should be noted that g is not unique.) If the almost complex structure J o $M \times I\!R$ defined by $J(X, f d/dt) = (\phi X - f X_0, \omega(X) d/dt)$, where f is a real-value function, is integrable, the contact structure is said to be *normal*. In this case X_0 is a Killing vector field with respect to g. Conversely, if $n = 1$ and X_0 is Killing vector field, the contact structure on M is normal. To facilitate the stud of compact three-manifolds, one may apply the following important result due t Lutz and Martinet [8], namely, 'every compact and orientable three-manifold ha a contact structure.'

In the sequel, we denote the Ricci tensor by S, and set $\sigma = S(X_0, \cdot)\,|B$.

Theorem 1. *Let M be a compact and orientable three-manifold with contact metr structure (ω, X_0, g), where g is critical. Then, if the scalar curvature r satisfies th inequality*

$$r > 2\left(1 - \frac{c^2}{4}\right) + \frac{|\sigma|^2}{1 - \frac{c^2}{4}} + 2c, \qquad c = |\tau| < 2, \tag{2.1}$$

g has positive Ricci curvature.

Proof. As in the proof of the Theorem in [6], to show that the Ricci tensor S positive definite, we determine at each point $x \in M$, a suitable basis $\{E, \phi E, X_0$ of T_xM, and verify that the subdeterminants along the main diagonal are positiv

Assume $\sigma_x \neq 0$. Since σ_x is a linear form on B, there exists a vector $X \in$ such that $\sigma_x = g(X, \cdot)$. Hence, by choosing $E = -\phi(X/|X|)$, we have $|E|$ 1, $\sigma_x(E) = 0$ and $\sigma_x(\phi E) = |\sigma|$.

The sectional curvatures $K(X_0, Y)$ of plane sections containing X_0 satisfy

$$(\nabla_{X_0}\tau)(X,X) = K(X_0,\phi X) - K(X_0,X)$$

for any unit vector $X \in B$ (see [11] Lemma 7.1). Moreover, since the me
critical by ([6], Proposition 1, formula (ii)) and (1.2),

$$\nabla_{X_0}\tau = -2\psi, \tag{2.2}$$

where $\psi(X,Y) = g((L_{X_0}\phi)X,Y)$. Thus,

$$S(E,E) = S(\phi E,\phi E) + 2\psi(E,E).$$

By polarization,

$$S(E,\phi E) = \psi(E,\phi E)$$

since by ([6], Proposition 1), trace $\psi = 0$ and ϕ is symmetric with respec
follows that

$$S(E,E) = \frac{r}{2} + \frac{c^2}{4} - 1 + \psi(E,E)$$

and

$$S(\phi E,\phi E) = \frac{r}{2} + \frac{c^2}{4} - 1 - \psi(E,E).$$

Thus

$$S = \begin{pmatrix} \frac{r}{2} + \frac{c^2}{4} - 1 + \psi(E,E) & \psi(E,\phi E) & \\ \psi(\phi E,E) & \frac{r}{2} + \frac{c^2}{4} - 1 - \psi(E,E) & | \\ 0 & |\sigma| & 2(1 \end{pmatrix} \tag{2.3}$$

The subdeterminants along the main diagonal are positive. For, the
(2.1) implies

$$S(E,E) \geq \frac{r}{2} + \frac{c^2}{4} - 1 - c > 0,$$

$$\begin{aligned} S(E,E)S(\phi E,\phi E) - S(E,\phi E)^2 &= (\frac{r}{2} + \frac{c^2}{4} - 1 - c)(\frac{r}{2} + \frac{c^2}{4} - \\ &\geq (\frac{r}{2} + \frac{c^2}{4} - 1 - c)^2 > 0 \end{aligned}$$

since $c^2 = \psi(E,E)^2 + \psi(E,\phi E)^2$, and

$$\begin{aligned}
\det S &= 2(1-\frac{c^2}{4})\{(\frac{r}{2}+\frac{c^2}{4}-1+\psi(E,E))(\frac{r}{2}+\frac{c^2}{4}-1-\psi(E,E)) \\
&\quad -\psi(E,\phi E)^2\} - |\sigma|^2(\frac{r}{2}+\frac{c^2}{4}-1+\psi(E,E)) \\
&\geq 2(1-\frac{c^2}{4})\{\frac{r}{2}+\frac{c^2}{4}-1-c^2\} - |\sigma|^2(\frac{r}{2}+\frac{c^2}{4}-1+c) \\
&= (\frac{r}{2}+\frac{c^2}{4}-1+c)\{2(1-\frac{c^2}{4})(\frac{r}{2}+\frac{c^2}{4}-1-c)-|\sigma|^2)\} > 0
\end{aligned}$$

by (2.1).

If $\sigma = 0$, the same argument applies to an arbitrary basis of the form $\{E, \varphi E, X_0$

Corollary 1. *Let M be a compact and orientable three-manifold with contac metric structure (ω, X_0, g), where X_0 is a Killing vector field. Then, if $r > -2$, M admits a contact metric structure $(a\omega,\ a^{-1}X_0,\ ag + a(a-1)\omega\otimes\omega)$ of positiv sectional curvature for some constant a, $0 < a \leq 1$.*

Proof. Since X_0 is a Killing field, g is a critical metric. In addition, σ and both vanish. Thus, the matrix (2.3) reduces to

$$S = \begin{pmatrix} \frac{r}{2}-1 & 0 & 0 \\ 0 & \frac{r}{2}-1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The components of the Riemann curvature tensor R with respect to the or thonormal basis $\{e_1, e_2, e_3\} = \{E, \phi E, X_0\}$ are

$$R_{ijk\ell} = \delta_{ik}S_{j\ell} + \delta_{j\ell}S_{ik} - \delta_{i\ell}S_{jk} - \delta_{jk}S_{i\ell} - \frac{r}{2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}). \tag{2.4}$$

Consequently, since $S_{ij} = 0$ for $i \neq j$, the only nonvanishing components of R ar those with two indices different.

Let P be a 2-dimensional subspace of the tangent space T_xM at $x \in M$, an let $\{X = \sum a_i e_i,\ Y = \sum b_j e_j\}$ be an orthonormal basis of P. Then, the section curvature $K(X,Y)$ of P is given by

$$\begin{aligned}
g(R(X,Y)X,Y) &= (a_1b_2 - a_2b_1)^2 R_{1212} + (a_1b_3 - a_3b_1)^2 R_{1313} \\
&\quad + (a_2b_3 - a_3b_2)^2 R_{2323}
\end{aligned}$$

Thus, if $r > 4$, $K(X,Y) > 0$.

Assume that there is a point $x_0 \in M$ such that $0 < (r(x_0) + 2)/6 \leq$ minimum k of the function $(r(x) + 2)/6$ then lies in the interval $(0,1]$. C the metric $\tilde{g}$ on M defined by

$$\tilde{g} = ag + a(a-1)\omega \otimes \omega$$

for some constant a, $0 < a < k \leq (r(x) + 2)/6$. If we put $\tilde{\omega} =$ $\tilde{X}_0 = a^{-1}X_0$, then $(\tilde{\omega}, \tilde{X}_0, \tilde{g})$ is a contact metric structure whose chara vector field is a Killing field. By a direct computation (see [5]), the Ricc S and $\tilde{S}$ of the metrics g and $\tilde{g}$, respectively, are related by

$$\tilde{S} = S + 2(1-a)g - 2(1-a)(2+a)\omega \otimes \omega, \tag{2.5}$$

so since $\tilde{g}^{ij} = a^{-1}g^{ij} + (1-a)a^{-2}X_0^iX_0^j$, $\tilde{r} - 4 = \frac{6}{a}(\frac{r+2}{6} - a)$. Thus, $\tilde{r} >$ which $\tilde{K}(X,Y) > 0$.

Note that if g is a critical metric, then formula (4) of [6] with n $b = a^2 - a$ reduces to (2.5) by virtue of (2.2).

3. Constant curvature and critical metrics.

Hamilton [7] showed that a metric g of positive Ricci curvature on a comp manifold can be deformed to a metric of (positive) constant curvature. contact metric, we obtain the following

Corollary 2. *Let M be a compact and orientable three-manifold wi metric structure (ω, X_0, g) where X_0 is a Killing vector field. Then constant greater than -2, the metric g may be deformed to a contact constant curvature 1.*

Proof. It is well-known that a Riemannian three-manifold (M,g) is a manifold if and only if it has constant curvature (cf. (2.4)). The matr proof of Corollary 1 says that g is an Einstein metric $\iff S = (r/3)g$ $r^2/3 \iff r = 6$. If r is a constant greater than -2, then the contact me by $\tilde{g} = ag + a(a-1)\omega \otimes \omega$, $a = (r+2)/8$, has constant scala $\tilde{r} = ((r+2)/a) - 2 = 6$.

More generally, if the contact metric is critical, then from (2.3)

$$\begin{aligned} |S|^2 &= 2|\sigma|^2 + 2\psi(E,\phi E)^2 + 4(1-\frac{c^2}{4})^2 + 2(\frac{r}{2}+\frac{c^2}{4}-1)^2 + 2\psi(E,E)^2 \\ &= 2|\sigma|^2 + 2c^2 + 2(\frac{r}{2}+\frac{c^2}{4}-1)^2 + 4(1-\frac{c^2}{4})^2. \end{aligned}$$

Hence, $|S|^2 = r^2/3$, if and only if,

$$[r - 6(1-\frac{c^2}{4})]^2 + 12(|\sigma|^2 + c^2) = 0.$$

This yields

Theorem 2. *Let M be a compact and orientable three-manifold with contact metr structure (ω, X_0, g), where g is critical. Then, g is of constant curvature k, if ar only if X_0 is a Killing vector field and $k = 1$.*

Corollary. *A critical metric on the three-sphere with constant curvature is t standard normal contact metric.*

Lemma. *Let M be a compact contact $(2n+1)$-dimensional manifold with conta Riemannian structure (ω, X_0, g) where g is critical. Then, for $a > 0$, $(\tilde{\omega}$ $a\omega$, $\tilde{X}_0 = a^{-1}X_0$, $\tilde{g} = ag + (a^2 - a)\omega \otimes \omega)$ is also a contact Riemannian structu with critical metric $\tilde{g}$. Moreover, the scalar curvatures of g and $\tilde{g}$ are related by*

$$\tilde{r} = \frac{r+2n}{a} - 2n + \frac{c^2}{2a^2}(a-1),$$

and

$$|\tilde{\sigma}| = a^{-3/2}|\sigma|,$$

where $\tilde{\sigma} = S(\tilde{X}_0, \cdot)\ |B$.

Proof. The Ricci tensors S and $\tilde{S}$ are related by ([5], formula (4)), viz.,

$$\tilde{S} = S - 2(a-1)g + 2(a-1)(na+n+1)\omega \otimes \omega \tag{2.6}$$

since g is a critical metric. Thus,

$$\tilde{S}(\tilde{X}_0, \tilde{X}_0) = \frac{1}{a^2}[S(X_0, X_0) + 2n(a^2 - 1)]$$

$$= \frac{1}{a^2}[2(n - \frac{c^2}{4}) + 2n(a^2 - 1)]$$

$$= 2n - \frac{c^2}{2a^2}.$$

On the other hand,

$$\tilde{S}(\tilde{X}_0, \tilde{X}_0) = 2(n - \frac{\tilde{c}^2}{4}), \quad \text{where } \tilde{c}^2 = |L_{\tilde{X}_0}\tilde{g}|^2,$$

from which $\tilde{c} = a^{-1}c$. Hence,

$$\mathcal{E}(\tilde{g}) = \frac{1}{2}\int_M \tilde{c}^2 d \text{ vol } (M, \tilde{g}) = \frac{a^{n-1}}{2}\int_M c^2 d \text{ vol } (M, g) = a^{n-1}\mathcal{E}(g$$

since $d \text{ vol } (M, \tilde{g}) = \frac{1}{2^n n!}\tilde{\omega} \wedge (d\tilde{\omega})^n = a^{n+1} d \text{ vol } (M, g)$. Since g is critical, it that $\tilde{g}$ is also a critical metric. Now, let $\{\tilde{E}_i, \varphi\tilde{E}_i, \tilde{X}_0\}$, $i = 1, \ldots, n$, be a with respect to $\tilde{g}$. Then $\{E_i, \varphi E_i, X_0\}$, $i = 1, \ldots, n$, is a φ-basis with re g, where $E_i = a^{1/2}\tilde{E}_i$, and

$$\tilde{r} = \text{trace}\tilde{S} = \tilde{S}(\tilde{X}_0, \tilde{X}_0) + \sum_1^n \{S(\tilde{E}_i, \tilde{E}_i) + \tilde{S}(\varphi\tilde{E}_i, \varphi\tilde{E}_i)\}$$

$$= 2n - \frac{c^2}{2a^2} + \frac{1}{a}\sum_1^n \{S(E_i, E_i) + S(\varphi E_i, \varphi E_i)\} - \frac{4n}{a}(a - 1)$$

$$= \frac{r + 2n}{a} - 2n + \frac{c^2}{2a^2}(a - 1).$$

Moreover,

$$|\tilde{\sigma}|^2 = \sum_1^n \{\tilde{S}(\tilde{X}_0, \tilde{E}_i)^2 + \tilde{S}(\tilde{X}_0, \varphi\tilde{E}_i)^2\}$$

$$= \frac{1}{a^3}\sum_1^n \{S(X_0, E_i)^2 + S(X_0, \varphi E_i)^2\}$$

$$= \frac{|\sigma|^2}{a^3}.$$

The lemma and Theorem 1 give rise to the following theorem (see a

Theorem 3. *Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where g is critical. If there exists a constant a such that $c < 2a$ and*

$$r + \frac{c^2}{2} > \frac{|\sigma|^2}{(a^2 - c^2/4)} + 2(2a + c - 1), \tag{2.7}$$

then M admits a contact critical metric of positive Ricci curvature.

Another application of (2.6) yields the following extension of Theorem 2 of [5].

Theorem 4. *Let M be a compact $(2n+1)$-dimensional manifold with contact metric structure (ω, X_0, g), where g is a critical metric and $\sigma = S(X_0, \cdot)\ |B = 0$. Then, if $S + \lambda g$ is positive definite for some $\lambda < 2 - c/\sqrt{n}$, the first Betti number $b_1(M)$ of M is zero. if, in addition, M is simply connected and $n = 1$, it is diffeomorphic with S^3.*

Proof. Consider the deformation defined by

$$\tilde{g} = ag + (a^2 - a)\omega \otimes \omega$$

for some constant $a = (2-\lambda)/2 > c/2\sqrt{n} > 0$. Then, (2.6) becomes

$$\tilde{S} = S + \lambda g - \lambda(2n - \frac{n\lambda}{2} + 1)\omega \otimes \omega.$$

Since $\tilde{\sigma} = \sigma = 0$, to see that $\tilde{S}$ is positive definite we need only consider $\tilde{S}(X, X)$ with X horizontal and X vertical.

If X is vertical, that is, if $X = tX_0$, then

$$\begin{aligned}
\tilde{S}(X,X) &= t^2\tilde{S}(X_0, X_0) \\
&= t^2[S(X_0, X_0) + \lambda - \lambda(2n - \frac{n\lambda}{2} + 1)] \\
&= t^2[2n - \frac{c^2}{2} - n\lambda(2 - \frac{\lambda}{2})] \\
&= \frac{t^2}{2}[n(2-\lambda)^2 - c^2] > 0, \quad \lambda < 2 - \frac{c}{\sqrt{n}}.
\end{aligned}$$

If X is horizontal, that is, if $\omega(X) = 0$, then

$$\tilde{S}(X,X) = (S + \lambda g)(X,X) > 0.$$

Thus, $\tilde{S}$ is positive definite, and so $b_1(M) = 0$. The last part is a consequence o Hamilton [7].

4. The Chern-Hamilton energy functional.

The quantity $r + c^2/2$ appearing in (2.1) and (2.7) is equal to $r^* -$ is the generalized Tanaka-Webster scalar curvature defined in [11]. curvature W studied by Chern and Hamilton in [4] is equal to $r^*/8$. T on the scalar curvature r in Corollaries 1 and 2 may therefore be rep assumption that r^* or W be positive.

Since $S(X_0, X_0) = 2 - \operatorname{tr} h^2$ where $h = L_{X_0}\varphi/2$ (see [1]), and $2g(X, h\varphi Y)$, it is seen that g is a critical point of the functional $\int \mathrm{Ric}(X$ on the space $\mathcal{M}(\omega)$ if and only if it is a critical point of $\mathcal{E}(g)$. Applyin $r^* = r - S(X_0, X_0) + 4$ due to Tanno [11, p. 21], we conclude that th problem considered by Chern and Hamilton is the same as that for t $\int_M (r^* - r) d\,\mathrm{vol}(M, g)$. They also studied the functional $\int_M r^* \ d$ v showed that it is critical over all CR structures with a fixed contac only if X_0 is a Killing field. ~~Thus, if r is a constant, $\mathcal{E}(g)$ is critical X_0 is a Killing field.~~

The main result of [4] says that every contact structure on a orientable three-manifold has a contact Riemannian metric for whic a constant ≤ 0 or else $W > 0$. ~~If W is a constant, then g is a critical if and only if g is a critical point of $\int_M r\, d\,\mathrm{vol}(M, g)$ on $\mathcal{M}(\omega)$.~~

R. Schoen pointed out that the classical Yamabe problem should as a part of the variational theory for the Einstein-Hilbert variati namely, the elliptic version of the action principle governing the gravitational field in General Relativity. Yamabe viewed this probl lytic approach to the solution of the classical Poincaré conjecture.

Corollary 2 is not too surprising since a compact simply co manifold M which admits a nonsingular Killing vector field is dif S^3. To see this, let G be a compact Lie group of isometries of by $\mu_t \in G$ the one-parameter group generated by $X(\partial\mu_t/\partial t = X)$ gives rise to an abelian subgroup of G. The closure of μ_t inside G approaches μ_t inside this torus by S^1. Since the action of μ_t is lo can take S^1 sufficiently close to μ_t so that it acts locally free on gives rise to a Seifert manifold structures on M. But a simply co manifold is diffeomorphic to S^3.

The following remark is due to Grant Cairns. G. Monna [9] has constructed a flow on S^3 which is not a Killing flow, so it is neither regular nor normal. In his example, he deforms the Hopf fibration, namely, S^3 is considered as the union of two solid tori in such a way that in each torus the Hopf fibration induces a flow tangent to each of the 2-tori parallel to the boundary. The flows induced on these 2-tori are just linear flows with 'constant' rational gradient. When one deforms the flow as in [5], p. 655, one simply changes the gradient to another constant (possibly irrational) gradient. However, the gradient can be changed in such a way that it is not a constant (i.e., it depends on the 2-torus) and yet the deformed flow is still a contact flow. The resulting flow has infinitely many closed leaves and infinitely many non-closed leaves, so it cannot be a Killing flow.

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