GEOMETRY AND TOPOLOGY (pp. 306–314)
edited by G. M. Rassias & G. M. Stratopoulos

HARMONIC POLYNOMIAL MAPS BETWEEN SPHERES AND COMPLEX PROJECTIVE SPACES

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ABSTRACT: We construct compact convex moduli spaces of harmonic maps between spheres and complex projective spaces.

§1. Introduction and generalities.

Let M be a compact oriented (isotropy) irreducible Riemannian homogeneous space. We write $M = G/K$, where G is a transitive Lie group of isometries of M and the isotropy subgroup K acts irreducibly on the tangent space $T_o(M)$ at the origin $o = \{K\}$. Let $\lambda \in Spec(M)$ and consider the associated (finite dimensional) eigenspace $V_\lambda \subset C^\infty(M)$. We endow V_λ with the normalized L_2-scalar product $\langle\, , \rangle$ defined by

$$\langle \mu , \mu' \rangle = \frac{n(\lambda)+1}{\int_M vol(M)} \int_M \mu \cdot \mu' \, vol(M),$$

where $\mu, \mu' \in V_\lambda$, and $n(\lambda)+1 = \dim V_\lambda (=$ multiplicity of $\lambda)$. Precomposing eigenfunctions with isometries on M gives rise to an orthogonal G-module structure on V_λ [3].A map $f : M \to S^n$ into the Euclidean n-sphere is said to be a λ-*eigenmap* if the components $f^i, i = 0, ..., n$, of f with respect to $S^n \subset \mathbf{R}^{n+1}$ belong to V_λ. Such maps are *harmonic* in the sense of J. Eells and J.H. Sampson [7] and, in fact, they can be characterized as harmonic maps of constant energy density $(= \frac{\lambda}{2})$ [8]. A map $f : M \to S^n$ is said to be *full* if the image of f is not contained in a proper linear subspace of $\mathbf{R}^{n+1}$. Two maps $f, f' : M \to S^n$ are said to be *equivalent* if there exists an isometry $U \in O(n+1)$ such that $f' = U \circ f$. One of the fundamental problems in harmonic map theory posed by J. Eells and L. Lemaire [8] and R.T. Smith [15] in 1972 (for spherical domains) is to classify, for fixed $\lambda \in Spec(M)$, the equivalence classes of all full λ-eigenmaps $f : M \to S^n$.
The standard minimal immersion $f_\lambda : M \to S^{n(\lambda)}$ is the prototype of λ-eigenmaps

*Research carried out in part while the author was a member of the Institute for Advanced Study. 1980 Mathematics Subject Classification (1985 Revision). Primary 58E20, Secondary 53C42.

which is, in fact, a full (minimal) homothetic immersion. Its components comprise an orthonormal base of V_λ; different choices of the base give rise to equivalent standard minimal immersions. From here on we fix $f_\lambda : M \to S^{n(\lambda)}$ and thereby an isomorphism $\mathbb{R}^{n(\lambda)+1} \cong V_\lambda$.The G- module structure of V_λ then carries over to a G- module structure $\rho_\lambda : G \to SO(n(\lambda)+1)$ of $\mathbb{R}^{n(\lambda)+1}$. By construction, $f_\lambda : M \to S^{n(\lambda)}$ is equivariant with respect to the homomorphism ρ_λ.
A full λ-eigenmap $f : M \to S^n$ can then be written as $f = A \cdot f_\lambda$, where A is an $(n+1) \times (n(\lambda)+1)$-matrix of maximal rank. The symmetric matrix

$$\langle f \rangle = A^t \cdot A - I_{n(\lambda)+1} \in S^2(\mathbb{R}^{n(\lambda)+1})$$

(depends only on and) represents the equivalence class of f uniquely. (From here on, we use [16] as a standard reference, cf. also [19].) The condition $\langle f, f \rangle = 1$ translates into $\langle f \rangle \in E_\lambda$, where E_λ is the orthogonal complement of

$$span\{f_\lambda(x)^2 | x \in M\} \subset S^2(\mathbb{R}^{n(\lambda)+1}),$$

where $f_\lambda(x)^2$ is the symmetric square of the unit vector $f_\lambda(x) \in \mathbb{R}^{n(\lambda)+1}$ which is, in fact, orthogonal projection onto $\mathbb{R} \cdot f_\lambda(x)$. The orthogonal complement is taken with respect to the scalar product $\langle B, B' \rangle = trace\, B^t \cdot B', B, B' \in S^2(\mathbb{R}^{n(\lambda)+1})$. Clearly, E_λ is a G-submodule of $S^2(\mathbb{R}^{n(\lambda)+1})$ with respect to the induced module structure $Ad\rho_\lambda$ on $S^2(\mathbb{R}^{n(\lambda)+1})$.Setting

$$L_\lambda = \{C - I_{n(\lambda)+1} \in E_f | C \geq 0\}$$

(where $\geq$ stands for positive semidefinite) the correspondence $f \longmapsto \langle f \rangle$ gives rise to a parametrization of the equivalence classes of full λ- eigenmaps $f : M \to S^n$ by the compact convex body L_λ of $\mathbf{E}_\lambda$. The G-module structure of E_λ leaves L_λ invariant and, on L_λ, the G-action is induced by precomposing λ-eigenmaps with isometries on M. For fixed $f : M \to S^n$, the isotropy subgroup $G_f = G_{\langle f \rangle} = \{g \in G | \exists U \in O(n+1)$ such that $f \circ g = U \circ f\}$ is nothing but the (maximal) *symmetry group* of f. The classification problem for λ-eigenmaps posed above can then be translated into the problem of understanding the geometry of L_λ (in particular, its boundary ∂L_λ) in E_λ.As a first step we introduce a natural cell structure of L_λ as follows. For a fixed full λ-eigenmap $f : M \to S^n$, define

$$E_f = (span\{f(x)^2) | x \in M\})^\perp \in S^2(\mathbb{R}^{n+1})$$

and

$$L_f = \{C - I_{n+1} \in E_f | C \geq 0\}.$$

Then the affine map $\phi : L_f \to L_\lambda$ defined by $\phi(C - I_{n+1}) = A^t \cdot C \cdot A - I_{n(\lambda)+1}$, injects L_f onto a compact convex set $\bar{I}_f$ containing $\langle f \rangle$. For a full λ-eigenmap

$f' : M \to S^{n'}, \langle f' \rangle \in \bar{I}_f$ iff $f' = A' \cdot f$ for some $(n'+1) \times (n+1)$-matrix A' of maximal rank, or equivalently, iff the components of f' are contained in the linear span of the components of f in V_λ. Denoting by $\mathcal{A}_f$ the affine subspace of E_λ spanned by $\bar{I}_f$, we have $\mathcal{A}_f \cap L_\lambda = \bar{I}_f$ and the interior I_f of $\bar{I}_f$ in $\mathcal{A}_f$ is a convex body containing $\langle f \rangle$. The convex sets I_f, for the various f, give rise to a cell decomposition of L_f. Clearly, $I_{f_\lambda} = int\, L_\lambda$ and when passing to the boundary of a cell the range dimension of the corresponding full λ-eigenmaps decrease. Note also that the range dimension is constant on any I_f, in particular, it is $n(\lambda)$ on I_{f_λ}. For $g \in G$, we have $g \cdot I_f = I_{f \circ g^{-1}}$ so that the G-action on L_λ respects the cell structure. We now subdivide the classification problem introduced above into the following problems:

I. Compute $\dim L_\lambda = \dim E_\lambda$.

II. Decompose $E_\lambda (\otimes_{\mathbb{R}} \mathbb{C})$ into irreducible components and determine the highest weight vectors of the components.

III. Describe the cell structure of L_λ modulo the action of G.

Almost nothing is known for rank $M \geq 2$ (cf. [17]). For M rank 1 , E_λ is the sum of those irreducible G- submodules of $S^2(\mathbb{R}^{n(\lambda)+1})$ which are not class 1 with respect to (G, K) (i.e. which when restricted to K do not contain the trivial K-module) [17]. For $M = S^m$, I and II have completely been resolved [16]; III is known for the first nonrigid range ($m = 3$ and $\lambda = 8$) [19] while for $m \geq 5$ odd a cell on ∂L_λ is known corresponding to λ-eigenmaps arising from the Hopf- Whitehead construction applied to orthogonal multiplications [18].For $M = \mathbb{C}P^m$,I has been resolved in [1] using subtle representation theory while,for II,some components of E_λ have been determined previously [11,12,20].Some components of E_λ have also been discovered for $M = \mathbb{H}P^m$ whose dimensions thereby give a lower bound for I.

In §2 we treat the spherical case and select among the legion of examples some classical and recent ones of interest. The results are then applied in §3 to construct specific cells on ∂L_λ which give rise to a massive amount of new examples for *harmonic nonholomorphic maps between complex projective spaces*; an other fundamental question in harmonic map theory (cf. [8]). They include those discovered by A. Din and W. Zakrzewski [5,6] and further classified by J. Eells and J.C. Wood [9].

§2. Harmonic polynomial maps between spheres.

For $M = S^m$ with $(G, K) = (SO(m+1), SO(m))$,we have $\lambda = \lambda_a = a(a+m-1) \in Spec(S^m)$ and the associated eigenspace V_{λ_a} is nothing but the irreducible $SO(m+1)$-module of spherical harmonics of order a on $S^m \subset \mathbb{R}^{m+1}$ [3]. To simplify the notation, from now on, we write $V_{\lambda_a} = V_a$, $E_{\lambda_a} = E_a$, etc. By the rigidity theorem of E. Calabi [4], for $m = 2$, we have $L_a = \{0\}$, i.e. the only full λ_a-eigenmap is $f_a : S^2 \to S^{2a}$, the standard minimal immersion, which is nothing but the classical Veronese surface in S^{2a} [3]. For $a = 1$ it is elementary that $L_a = \{0\}$. However, for $m \geq 3$ and $k \geq 2$, we have $\dim L_a > 0$. More precisely, for $m = 3$,

$$E_a \otimes_{\mathbf{R}} \mathbf{C} \cong \sum_{\substack{(b,c)\in\Delta_a \\ b,c \text{ even}}} \left\{ V_3^{(b,c)} \oplus V_3^{(b,-c)} \right\}$$

and, for $m \geq 4$,

$$E_a \otimes_{\mathbf{R}} \mathbf{C} \cong \sum_{\substack{(b,c)\in\Delta_a \\ b,c \text{ even}}} V_m^{(b,c,0,\ldots 0)},$$

where $\Delta_a \subset \mathbf{R}^2$ is the closed triangle with vertices $(2,2)$,(a,a) and $(2a-2,2)$ and V_m^ρ is the (complex) irreducible $SO(m+1)$ -module with highest weight $\rho = (\rho_1, \rho_2, \ldots, \rho_l) \in (1/2 \cdot \mathbf{Z})^l$, $l = [(m+1)/2]$ (cf. [16]). Applying the Weyl dimension formula, $\dim L_a$ can be computed.

The first nonrigid range $m = 3$ and $a = 2$, i.e. the structure of full quadratic eigenmaps $f : S^3 \to S^n$, is of particular interest [19]. Using coordinates (x,y,u,v) on $S^3 \subset \mathbf{R}^4$, let $f_n : S^3 \to S^n, 2 \leq n \leq 8, n \neq 3$, be defined by $f_n(x,y,u,v) =$

$$\begin{aligned}
&(x^2+y^2-u^2-v^2, 2(xu-yv), 2(xv+yu)), (Hopf\ map), n = 2\\
&(x^2+y^2-u^2-v^2, 2xu, 2xv, 2yu, 2yv), n = 4\\
&(x^2-y^2, u^2-v^2, 2xy, \sqrt{2}(xu+yv), \sqrt{2}(yu-xv), 2uv), n = 5\\
&(1/\sqrt{2}(x^2+y^2-u^2-v^2), 1/\sqrt{2}(x^2-y^2), 1/\sqrt{2}(u^2-v^2),\\
&\sqrt{2}xy, \sqrt{3}(xu+yv), \sqrt{3}(yu-xv), \sqrt{2}uv), n = 6\\
&(x^2-y^2, u^2-v^2, 2xy, \sqrt{2}xu, \sqrt{2}xv, \sqrt{2}yu, \sqrt{2}yv, 2uv), n = 7\\
&f_{\lambda_2}(x,y,u,v), (f_{\lambda_2}\ a\ standard\ minimal\ immersion), n = 8.
\end{aligned}$$

Then, $I_{f_n}, 2 \leq n \leq 8, n \neq 3$, comprise all cells of L_2 modulo the action of $O(4)$. Moreover, I_{f_2} = point, I_{f_4} = segment, I_{f_5} = 2-disk, I_{f_6} =(finite) solid cone, $\dim I_{f_7} = 5$ and $\dim I_{f_8} = 10$. Note also that the $O(4)$-orbit of the point $\langle f \rangle$ corresponding to the Hopf map $f_2 : S^3 \to S^2$ has 2 components which are imbedded in the appropriate 4-spheres of the 5-dimensional components $V_3^{(2,2)}$ and $V_3^{(2,-2)}$ of E_2 as *Veronese surfaces*.

An other discovery of cells in ∂L_2 is offered by the Hopf-Whitehead construction. Recall that a bilinear map $F : \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \to \mathbf{R}^n$ is called an *orthogonal multiplication* if $|F(x,y)| = |x| \cdot |y|, x, y \in \mathbf{R}^{m+1}$. For given F, the map $f_F : S^{2m+1} \to S^n$ defined by

$$f(x,y) = (|x|^2 - |y|^2, 2F(x,y)),$$
$$x, y \in \mathbf{R}^{m+1}, |x|^2 + |y|^2 = 1,$$

is a quadratic eigenmap (which is full iff F is surjective) [2]. Notice that f_2 and f_4 introduced above correspond to complex multiplication and real tensor product on $\mathbf{R}^2$, respectively. In general, we consider the cell $\bar{I}_{\otimes} = \bar{I}_{f_{\otimes}} \subset \partial L_2$, where $f_{\otimes} : S^{2m+1} \to S^{(m+1)^2}$ is associated with the tensor product $\otimes : \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \to \mathbf{R}^{(m+1)^2}$. Since $f_{\otimes}$ is equivariant with respect to $\rho_{\otimes} = \otimes : SO(m+1) \times SO(m+1) \to SO((m+1)^2)$, the point $\langle f_{\otimes} \rangle \in \bar{I}_{\otimes}$ is left fixed by $SO(m+1) \times SO(m+1)$ so that setting it as the origin of the affine span $\mathcal{A}_{\otimes}$ of $\bar{I}_{\otimes}$, we obtain an $SO(m+1) \times SO(m+1)$-module $\mathcal{A}_{\otimes}$. In fact, $\mathcal{A}_{\otimes} \cong so(m+1) \otimes so(m+1)$, where on the right hand side the module structure is given by $Ad \otimes Ad$ [18]. It follows that $\dim I_{\otimes} = (m(m+1)/2)^2$. Note that, for $m = 1, 2, \bar{I}_{\otimes}$ has been determined by M. Parker explicitly [13].

§3. Harmonic polynomial maps between complex projective spaces.

For $p > q \geq 0, p + q = a$, let $\mathcal{H}^{p,q}$ be the (complex) irreducible $U(m+1)$-module of complex harmonic polynomials on $\mathbf{C}^{m+1}$ of bidegree (p, q) [3]. A base $\{f^i_{p,q}\}_{i=0}^{n(p,q)} \subset \mathcal{H}^{p,q}$, $\dim_{\mathbf{C}} \mathcal{H}^{p,q} = n(p,q) + 1$, with respect to a normalized Hermitian L_2-scalar product on $\mathcal{H}^{p,q}$ induces a full λ_a-eigenmap $f_{p,q} : S^{2m+1} \to S^{2n(p,q)+1}$, where the components of $f_{p,q}$ are $\{Re(f^i_{p,q}), Im(f^i_{p,q})\}_{i=0}^{n(p,q)}$. Then, $f_{p,q}$ is equivariant with respect to the homomorphism $\rho_{p,q} : U(m+1) \to SO(2(n(p,q)+1))$. Moreover as the central (diagonal) subgroup $S^1 \subset U(m+1)$ acts on $\mathcal{H}^{p,q}$ via $\rho_{p,q}$ by the single weight $p - q$, the map $f_{p,q}$ projects down to a map $\tilde{f}_{p,q} : \mathbf{C}P^m \to \mathbf{C}P^{n(p,q)}$ such that $\pi \circ f_{p,q} = \tilde{f}_{p,q} \circ \pi$, where π stands for the respective Hopf maps.

LEMMA. *The map $f_{p,q} : S^{2m+1} \to S^{2n(p,q)+1}$ is horizontal with respect to $\pi : S^{2m+1} \to \mathbf{C}P^m$, i.e. $(f_{p,q})_*(ker\pi_*)$ and $(f_{p,q})_*((ker\pi_*)^{\perp})$ are orthogonal in $T(S^{2n(p,q)+1})$*

PROOF: For $z \in S^{2m+1} \subset \mathbf{C}^{m+1}$, the horizontal subspace $(ker\pi_*)^{\perp}_z$ is the orthogonal complement of $\mathbf{C} \cdot z$ in $\mathbf{C}^{m+1}$ (shifted to z). Given $w \in (ker\pi_*)^{\perp}_z$, we have to show that

$$\langle df_{p,q}(w)\,,\, df_{p,q}(iz) \rangle = 0,$$

where we used Hermitian scalar product in $\mathbf{C}^{n(p,q)+1}$ and iz is considered in $T_z(S^{2m+1})$. By homogeneity, we have

$$\begin{aligned} df_{p,q}(iz) &= \frac{d}{dt} f_{p,q}(e^{it}z)|_{t=0} = \frac{d}{dt} e^{i(p-q)t}|_{t=0} \cdot f_{p,q}(z) \\ &= i(p-q) f_{p,q}(z) = i\frac{p-q}{a} df_{p,q}(z). \end{aligned}$$

On the other hand, differentiating $|f_{p,q}(\cos t \cdot z + \sin t \cdot w)|^2 = 1$ at $t = 0$, we obtain $\langle df_{p,q}(w)\,,\, df_{p,q}(z) \rangle = 0$ and the proof is complete.

Applying the Reduction Theorem of R.T. Smith [15], we obtain that $\tilde{f}_{p,q} : CP^m \to CP^{n(p,q)}$ is a harmonic map.

REMARKS:
1.For $m = 1, n(p,q)+1 = (p+1)(q+1)-pq = p+q+1 = a+1$ we obtain the harmonic maps $\tilde{f}_{p,q} : CP^1 \to CP^a$. They and their conjugates comprise all harmonic maps of CP^1 into CP^a. (For a even we also have to add $\tilde{f}_{a/2,a/2} : CP^1 \to CP^a$ which is induced by $f_{a/2,a/2} : S^3 \to S^{2a+1}$ which is not full.) These are the harmonic maps which were discovered by A. Din and W. Zakrzewski [5,6] and classified by J. Eells and J.C. Wood [9].
2. For $q = 0$, the map $\tilde{f}_{a,0} = \tilde{v}_a : CP^m \to CP^{n(a,0)}, n(a,0) = \binom{m+a}{a}$ is nothing but the (holomorphic) Veronese map [14] induced by $v_a : S^{2m+1} \to S^{2n(a,0)+1}$, where

$$v_a(z_0, ..., z_m) = ((a!/i_0!...i_m!)^{1/2} z_0^{i_0} ... z_m^{i_m})_{\substack{i_0+...+i_m=a \\ i_0,...,i_m \geq 0}}.$$

We now take $\bar{I}_{p,q} = \bar{I}_{f_{p,q}}$ and intersect it with the linear subspace $Fix_{Ad\rho_a}(S^1, E_a)$ to obtain the convex set $Fix_{Ad\rho_a}(S^1, \bar{I}_{p,q})$. Given a full λ_a-eigenmap $f : S^{2m+1} \to S^N$ with $\langle f \rangle$ in the intersection, we have $f = A \cdot f_{p,q}$ for some $(N+1) \times (2(n(p,q)+1))$-matrix A of maximal rank. Moreover, as $\langle f \rangle \in Fix_{Ad\rho_a}(S^1, E_a)$, the map f is equivariant with respect to a homomorphism $\rho : S^1 \to SO(N+1)$ and $A : \mathbf{R}^{2(n(p,q)+1)} \to \mathbf{R}^{N+1}$ is intertwining between ρ_a and ρ. As ρ acts on $\mathbf{R}^{2(n(p,q)+1)}$ with the single weight $p - q (> 0)$ the same holds for ρ, in particular, $N = 2n+1$ is odd and $A : \mathbf{C}^{n(p,q)+1} \to \mathbf{C}^{n+1}$ is complex linear. By equivariance, f projects down to a map $\tilde{f} : CP^m \to CP^n$ such that $\pi \circ f = \tilde{f} \circ \pi$. Repeating the proof of the previous lemma, it follows that f is horizontal with respect to $\pi : S^{2m+1} \to CP^m$ so that $\tilde{f} : CP^m \to CP^n$ is harmonic. It follows that the convex set $Fix_{Ad\rho_a}(S^1, \bar{I}_{p,q})$ parametrizes the harmonic maps $\tilde{f} : CP^m \to CP^n$ obtained in the above manner.

REMARK: Examples are easy to construct. For instance, for $m = 2$ and $p = 2, q = 1$,

$$\begin{aligned} f(z,w,t) = (&\sqrt{7/8}(|z|^2 - 2|w|^2)z, \sqrt{1/8}(|w|^2 - 2|z|^2)w, \sqrt{7/8}(|w|^2 - 2|t|^2)w, \\ &\sqrt{7/8}(|z|^2 - 2|w|^2)z... \end{aligned}$$

gives rise to a harmonic map $\tilde{f} : CP^2 \to CP^9$.

Returning to the general situation, we first note that $f_{p,q} : S^{2m+1} \to S^{2n(p,q)+1}$ is equivariant with respect to the homomorphism $\rho_{p,q} : U(m+1) \to SO(2(n(p,q)+1))$ corresponding to the $U(m+1)$-module structure on $\mathcal{H}^{p,q} \cong \mathbf{C}^{n(p,q)+1}$. Setting $\langle f_{p,q} \rangle$ as the origin of the affine span $\mathcal{E}_{p,q}$ of $Fix_{Ad\rho_a}(S^1, \bar{I}_{p,q})$ we obtain that $\mathcal{E}_{p,q}$ is a $U(m+1)$-module and

$$\mathcal{E}_{p,q} \cong Fix_{Ad\rho_a}(S^1, E_{p,q})$$

as $U(m+1)$-modules, where $E_{p,q} = E_{f_{p,q}}$ (cf.§1).
Let $\bar{E}_{p,q}$ be the sum of those irreducible $U(m+1)$- submodules of $S^2(\mathcal{H}^{p,q}_{\mathbf{R}})$ ($\mathcal{H}^{p,q}_{\mathbf{R}} =$ the realification of $\mathcal{H}^{p,q}$) which do not contain $U(m)$-fixed vectors.

PROPOSITION. $\bar{E}_{p,q} \subset E_{p,q}$.

PROOF: Let $p : S^2(\mathcal{H}^{p,q}_{\mathbf{R}}) \to W_0$ denote the orthogonal projection, where $W_0 = \mathbf{R} \cdot f_{p,q}(o)^2$ and $o = (1,0,...,0) \in S^{2m+1}$ is the base point left fixed by $U(m)$. Consider the induced representation

$$I = Ind_{U(m)}^{U(m+1)}(W_0) = \{\psi : U(m+1) \to W_0 | \psi(u \cdot v) = u \cdot \psi(v), u \in U(m), v \in U(m+1), \psi \, continuous\}.$$

For $\sigma \in S^2(\mathcal{H}^{p,q}_{\mathbf{R}})$ we define the map $\Psi(\sigma) : U(m+1) \to W_0$ by $\Psi(\sigma)(v) = p(v \cdot \sigma), v \in U(m+1)$. Then $\Psi(\sigma) \in I$ so that we obtain a homomorphism $\Psi : S^2(\mathcal{H}^{p,q}_{\mathbf{R}}) \to I$ of $U(m+1)$-modules. We have $\ker \Psi = (span\{U(m+1) \cdot W_0\})^\perp = (span\{f(x)^2 | x \in S^{2m+1}\})^\perp = E_{p,q}$ so that $im\Psi = E^\perp_{p,q} \subset I$ as $U(m+1)$-modules. By Frobenius reciprocity [21], we have $\dim \hom_{U(m+1)}(\bar{E}_{p,q}, E^\perp_{p,q}) \leq \dim \hom_{U(m+1)}(\bar{E}_{p,q}, I) = \dim \hom_{U(m)}(\bar{E}_{p,q}, W_0) = 0$ and the claim follows.

By the proposition above a lower estimate on $\dim Fix_{Ad\rho_a}(S^1, \bar{I}_{p,q}) = \dim \mathcal{E}_{p,q}$ is provided by

$$L(p,q) = \dim Fix_{Ad\rho_a}(S^1, \bar{E}_{p,q}).$$

To enumerate $L(p,q)$, we complexify

$$\begin{aligned} &Fix_{Ad\rho_a}(S^1, S^2(\mathcal{H}^{p,q}_{\mathbf{R}})) \otimes_{\mathbf{R}} \mathbf{C} \\ &= Fix_{Ad\rho_a}(S^1, S^2(\mathcal{H}^{p,q} \oplus \mathcal{H}^{q,p})) \\ &= \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p} \oplus \mathcal{H}^{q,p} \otimes \mathcal{H}^{p,q} \end{aligned}$$

and obtain that

$$Fix_{Ad\rho_a}(S^1, \bar{E}_{p,q} \otimes_{\mathbf{R}} \mathbf{C}) \subset \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p} \oplus \mathcal{H}^{q,p} \otimes \mathcal{H}^{p,q}$$

is the sum of those irreducible $U(m+1)$-submodules which do not contain $U(m)$-fixed vectors. This is just the condition for spherical harmonics $\mathcal{H}^{b,b}$ so that we obtain.

$$\begin{aligned} L(p,q) = &\, 2 \dim_{\mathbf{C}} \mathcal{H}^{p,q} \otimes \mathcal{H}^{p,q} \\ &- 2 \sum_b m \left[\mathcal{H}^{b,b} : \mathcal{H}^{p,q} \otimes \mathcal{H}^{p,q}\right] \cdot \dim_{\mathbf{C}} \mathcal{H}^{b,b}. \end{aligned}$$

For $m, a \geq 4$, the Littlewood-Richardson rule [10,22] gives the multiplicity

$$m\left[\mathcal{H}^{b,b} : \mathcal{H}^{p,q} \otimes \mathcal{H}^{q,p}\right] = \min\{b+1, q+1, a-b+1\}$$

(cf. also [1]).
Since

$$\dim_{\mathbf{C}} \mathcal{H}^{p,q} = \binom{m+p}{p}\binom{m+q}{q} - \binom{m+p-1}{p-1}\binom{m+q-1}{q-1}$$

we finally get, for $m, a \geq 4$,

$$\dim Fix_{Ad\rho_a}(S^1, \bar{I}_{p,q}) \geq L(p,q) = 2\left[\binom{p+m}{p}\binom{m+q}{q} - \binom{m+p-1}{p-1}\binom{m+q-1}{q-1}\right]^2$$
$$-2\sum_{b=0}^{a}\min\{b+1, q+1, a-b+1\}\left[\binom{m+b}{b}^2 - \binom{m+b-1}{b-1}^2\right].$$

REMARKS:
1. Enumerating, for $m = a = 4$, we find

$$L(3,1) = 36{,}600.$$

2. For $q = 0$, $L(a,0) = 0$. In fact, as can be easily shown, $Fix_{Ad\rho_a}(S^1, \bar{I}_{a,0}) = \{\langle v_a \rangle\}$.

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