CONTACT THREE-MANIFOLDS WITH POSITIVE GENERALIZED TANAKA-WEBSTER SCALAR CURVATURE

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ABSTRACT. The metric of a compact contact Riemannian three-manifold whose characteristic vector field generates a one-parameter group of isometries may be deformed to a contact metric of positive curvature if the generalized Tanaka-Webster scalar curvature r^* is positive. If, in addition, r^* is a constant then the metric may be deformed to a contact metric of constant curvature.

1. Introduction. Y. Carrière [2] has classified Riemannian flows on compact three-manifolds. The difficulty encountered in the study of Riemannian flows is that they are not automatically Killing flows. A compact three-manifold M admitting a nonsingular Killing vector field is a Seifert manifold, so if M is simply connected it is diffeomorphic to the standard three-sphere S^3. Chern and Hamilton [3] introduced the torsion $|\tau|$ (the length of τ) in their study of compact contact three-manifolds (M,g), where $\tau(= L_{X_0} g)$ is the Lie derivative of the contact metric g with respect to the characteristic vector field X_0 of the contact structure, and they conjectured that for fixed contact form $\omega = g(X_0,\cdot)$, with X_0 inducing a Seifert foliation, there exists a complex structure $\phi|B$ on $B = \ker \omega$ such that the Dirichlet energy

1980 Mathematics Subject Classification. Primary 53C15; Secondary 53C20.
Key words and phrases. Contact Riemannian three-manifolds, critical metric, curvature.

[1]Supported by the Natural Sciences and Engineering Research Council of Canada.
[2]Supported by funds 60% of the Ministero Pubblica Intruzione.

(1.1) $$\mathcal{E}(g) = \frac{1}{2}\int_M |\tau|^2 d\,\mathrm{vol}(M,g)$$

is critical over all CR structures.

Let M be a contact manifold with a fixed contact form ω. Denote the space of all associated Riemannian metrics to the contact form ω by $\mathcal{M}(\omega)$. Let g be a point of $\mathcal{M}(\omega)$, and denote by $\{g(t)\}$ a curve in $\mathcal{M}(\omega)$ with $g(0) = g$. Tanno [8] showed that g is a critical point of $\mathcal{E}$, if and only if

(1.2) $$\nabla_{X_0}\tau = 2\tau \cdot \phi.$$

Thus, $\mathcal{E}(g)$ is critical over all CR structures if and only if (1.2) is satisfied. (This differs from the condition $\nabla_{X_0}\tau = 0$ incorrectly obtained in [3], Theorem 5.4.) In the sequel, a critical point of $\mathcal{E}$ will be called a <u>critical metric</u>. Note that g is a critical point of $\mathcal{E}$ if X_0 is a Killing vector field with respect to g.

In [5] it is shown that if the scalar curvature $r > -2$ on a compact contact three-manifold (M,g) whose characteristic vector field is a Killing field, then g may be deformed to a contact metric of positive Ricci curvature. It is the main purpose of this paper to show that g may in fact be deformed to a contact metric of positive sectional curvature. This is a consequence of Theorem 1 which also yields the statement that if r is a constant greater than -2, then g may be deformed to a contact metric of (positive) constant curvature.

2. <u>Compact three-manifolds</u>. To facilitate the study of compact three-manifolds, one may apply the following important result due to Lutz and Martinet [7], namely, 'every compact and orientable three-manifold has a contact structure.' The reader is referred to [1] for details and other properties of contact manifolds. In the sequel, we denote the Ricci tensor by S, and set $\sigma = S(X_0,\cdot)|B$.

THEOREM 1. *Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where g is critical. Then, if the scalar curvature r satisfies the inequality*

$$r > 2\left(1 - \frac{c^2}{4}\right) + \frac{|\sigma|^2}{1 - \frac{c^2}{4}} + 2c, \quad c = |\tau| < 2, \tag{2.1}$$

g has positive Ricci curvature.

PROOF. As in the proof of the Theorem in [5], to show that the Ricci tensor S is positive definite, we determine at each point $x \in M$, a suitable ϕ-basis $\{E, \phi E, X_0\}$ of T_xM, and verify that the subdeterminants along the main diagonal are positive.

COROLLARY 1. *Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where X_0 is a Killing vector field. Then, if $r > -2$, M admits a contact metric structure $(a\omega, a^{-1}X_0, ag + a(a-1)\omega \otimes \omega)$ of positive sectional curvature for some constant* a, $0 < a \leq 1$.

PROOF. Since X_0 is a Killing field, g is a critical metric. In addition, σ and τ both vanish. The matrix for S with respect to the ϕ-basis $\{E, \phi E, X_0\}$ is given by

$$S = \begin{bmatrix} \frac{r}{2} - 1 & 0 & 0 \\ 0 & \frac{r}{2} - 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The components of the Riemann curvature tensor R with respect to the orthonormal basis $\{e_1, e_2, e_3\} = \{E, \phi E, X_0\}$ are

$$(2.2)\quad R_{ijk\ell} = \delta_{ik}S_{j\ell} + \delta_{j\ell}S_{ik} - \delta_{i\ell}S_{jk} - \delta_{jk}S_{i\ell} - \frac{r}{2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}).$$

Consequently, since $S_{ij} = 0$ for $i \neq j$, the only nonvanishing components of R are those with two indices different.

Let P be a 2-dimensional subspace of the tangent space T_xM at $x \in M$, and let $\{X = \Sigma a_i e_i,\ Y = \Sigma b_j e_j\}$ be an orthonormal basis of P. Then, the sectional curvature $K(X,Y)$ of P is given by

$$g(R(X,Y)X,Y) = (a_1b_2 - a_2b_1)^2R_{1212} + (a_1b_3 - a_3b_1)^2R_{1313} + (a_2b_3 - a_3b_2)^2R_{2323}\ .$$

Thus, if $r > 4$, $K(X,Y) > 0$.

Assume that there is a point $x_0 \in M$ such that $0 < (r(x_0) + 2)/6 \leq 1$. The minimum k of the function $(r(x) + 2)/6$ then lies in the interval $(0,1]$. Consider the metric $\tilde{g}$ on M defined by

$$\tilde{g} = ag + a(a-1)\omega \otimes \omega$$

for some constant a, $0 < a < k \leq (r(x) + 2)/6$. If we put $\tilde{\omega} = a\omega$ and $X_0 = a^{-1}X_0$, then $(\tilde{\omega}, \tilde{X}_0, \tilde{g})$ is a contact metric structure whose characteristic vector field is a Killing field. By a direct computation (see [4]), the Ricci tensors S and $\tilde{S}$ of the metrics g and $\tilde{g}$, respectively, are related by

$$\tilde{S} = S + 2(1-a)g - 2(1-a)(2+a)\omega \otimes \omega,$$

so since $\tilde{g}^{ij} = a^{-1}g^{ij} + (1-a)a^{-2}X_0^iX_0^j$, $\tilde{r} - 4 = \frac{6}{a}\left(\frac{r+2}{6} - a\right)$. Thus, $\tilde{r} > 4$, from which $\tilde{K}(X,Y) > 0$.

3. Constant curvature. Hamilton [6] showed that a metric g of positive Ricci curvature on a compact three-manifold can be deformed to a metric of (positive) constant curvature. If g is a contact metric, we obtain the following

COROLLARY 2. *Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g) where X_0 is a Killing vector field. Then, if r is a constant greater than -2, the metric g may be deformed to a contact metric of constant curvature 1.*

PROOF. It is well-known that a Riemannian three-manifold (M,g) is an Einstein manifold if and only if it has constant curvature (cf. (2.2)). The matrix S in the proof of Corollary 1 says that g is an Einstein metric $\Leftrightarrow S = (r/3)g \Leftrightarrow |S|^2 = r^2/3 \Leftrightarrow r = 6$. If r is a constant greater than -2, then the contact metric defined by $\tilde{g} = ag + a(a-1)\omega \otimes \omega$, $a = (r+2)/8$, has constant scalar curvature $\tilde{r} = ((r+2)/a) - 2 = 6$.

More generally, if the contact metric is critical, we obtain

THEOREM 2. *Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where g is critical. Then, g is of constant curvature k, if and only if X_0 is a Killing vector field and $k = 1$.*

COROLLARY. *A critical metric on the three-sphere with constant curvature is the standard normal contact metric.*

REMARK: The quantity $r + c^2/2$ appearing in (2.1) is equal to $r^* - 2$, where r^* is the generalized Tanaka-Webster scalar curvature defined in [8].

The condition on the scalar curvature r in Corollaries 1 and 2 may therefore be replaced by the assumption that r^* be positive. The Webster curvature studied by Chern and Hamilton in [3] is equal to $r^*/8$. Their main result says that every contact structure on a compact and orientable three-manifold has a contact Riemannian metric whose Webster curvature is either a constant ≤ 0 or it is strictly positive everywhere.

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Received May 16, 1988

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