CURVATURE OF CONTACT RIEMANNIAN THREE-MANIFOLDS WITH CRITICAL METRICS

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1. Introduction. The following odd-dimensional analogue of the Mori and Siu-Yau Theorems (see [11]) which gave rise to the study of compact three-manifolds in [6] forms the background for this paper. A compact normal regular contact Riemannian manifold M of positive curvature is covered by a sphere S^{2n+1}. This was known for three-manifolds as a result of the classification of circle bundles over surfaces. However, a stronger statement was obtained in [5], namely, a compact simply connected normal contact Riemannian three-manifold of nonnegative curvature is homeomorphic with S^3. In dimensions greater than 3, the regularity of the contact structure allows one to employ the Boothby-Wang fibration of M in which the base manifold N is a compact Kaehler manifold of positive curvature. The homotopy sequences of the fiberings $S^1 \to S^{2n+1} \to \mathbb{CP}_n$ and $S^1 \to M \to N$ then show that M is of the same homotopy type as S^{2n+1}, so by Smale's solution of the generalized Poincaré conjecture M is homeomorphic with S^{2n+1} for $n > 1$.

The normality condition for contact manifolds is the analogue of the integrability condition for an almost complex structure. When $n = 1$, to say that a contact Riemannian manifold is normal is equivalent to the statement that the characteristic vector field of the contact structure is a Killing field. Y. Carrière [3] has classified Riemannian flows on compact three-manifolds, but the difficulty encountered is that they are not automatically Killing flows. A compact three-manifold M admitting a nonsingular Killing vector field is a Seifert manifold (see §4 for a proof), so if M is simply connected it is diffeomorphic to the standard three-sphere S^3. Chern and Hamilton [4] introduced the torsion $|\tau|$ (the length of τ) in their study of compact contact three-manifolds (M,g), where $\tau(= L_{X_0}g)$ is the Lie derivative of the contact metric g with respect to the characteristic vector field X_0 of the contact structure, and they conjectured that

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for fixed <u>contact form</u> $\omega = g(X_0,\cdot)$, with X_0 inducing a Seifert foliation, there exists a complex structure $\phi|B$ on $B = \ker \omega$ such that the Dirichlet energy

$$(1.1) \qquad \mathcal{E}(g) = \frac{1}{2}\int_M |\tau|^2 d\,\mathrm{vol}(M,g)$$

is critical over all CR structures. (A CR structure on an orientable three-manifold is a contact structure together with a complex structure on B. Since the latter is equivalent to a conformal structure, a Riemannian metric on a contact three-manifold gives rise to a CR structure.)

Let M be a $(2n+1)$-dimensional contact manifold with a fixed contact form ω. Denote the space of all associated Riemannian metrics to the contact form ω by $\mathcal{M}(\omega)$. Let g be a point of $\mathcal{M}(\omega)$, and denote by $\{g(t)\}$ a curve in $\mathcal{M}(\omega)$ with $g(0) = g$. Tanno [12] showed that g is a critical point of $\mathcal{E}$, if and only if

$$(1.2) \qquad \nabla_{X_0}\tau = 2\tau\cdot\phi.$$

Thus, for $n = 1$, $\mathcal{E}(g)$ is critical over all CR structures if and only if (1.2) is satisfied. (This differs from the condition $\nabla_{X_0}\tau = 0$ incorrectly obtained in [4], Theorem 5.4.) In the sequel, a critical point of $\mathcal{E}$ will be called a <u>critical metric</u>. Note that g is a critical point of $\mathcal{E}$ if X_0 is a Killing vector field with respect to g.

Can $\mathcal{E}$ have a critical point g such that $\mathcal{E}(g) \neq 0$? Blair [1] showed that the answer is no if the contact structure on M is regular, i.e., if every point of M has a neighborhood such that any integral curve of the characteristic vector field X_0 which passes through the neighborhood does so only once. (A theorem of Boothby and Wang says that a compact regular contact manifold is a principal circle bundle over a symplectic manifold whose fundamental form has integral periods.) He also showed that the standard contact metric structure on the unit tangent bundle of a compact surface of constant negative curvature is not regular [2], and that this metric g is a critical point of $\mathcal{E}$ such that $\tau \neq 0$.

In [6] it is shown that if the scalar curvature $r > -2$ on a compact contact three-manifold (M,g) whose characteristic vector field is a Killing field, then g may be deformed to a contact metric of positive Ricci curvature. It is the main purpose of this paper to show that g may in fact be deformed to a contact metric of positive sectional curvature. This is a consequence of Theorem 1 which also yields the statement that if r is a constant greater than -2 (equivalently, if the generalized Tanaka–Webster scalar curvature is a positive constant), then g may be deformed to a contact metric of (positive) constant curvature. More generally, if the contact metric g is critical, then g is of constant curvature, if and only if the characteristic field is a Killing field and the sectional curvature is 1.

2. Compact three-manifolds. A (2n+1)-dimensional manifold M is said to be a contact manifold if it carries a global 1-form $\omega \neq 0$ with the property that $\omega \wedge (d\omega)^n \neq 0$ everywhere. It has an underlying almost contact structure (ϕ, X_0, ω), where $\omega(X_0) = 1$, $\phi X_0 = 0$ and $\phi^2 = -I + \omega \otimes X_0$, I being the identity field. A metric g, called an associated metric, can then be found such that $\omega = g(X_0, \cdot)$ and $d\omega(X,Y) = g(\phi X, Y)$. (It should be noted that g is not unique.) If the almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, f d/dt) = (\phi X - f X_0, \omega(X) d/dt)$, where f is a real-valued function, is integrable, the contact structure is said to be normal. In this case, X_0 is a Killing vector field with respect to g. Conversely, if $n = 1$ and X_0 is a Killing vector field, the contact structure on M is normal. To facilitate the study of compact three-manifolds, one may apply the following important result due to Lutz and Martinet [8], namely, 'every compact and orientable three-manifold has a contact structure.'

In the sequel, we denote the Ricci tensor by S, and set $\sigma = S(X_0, \cdot)|B$.

THEOREM 1. Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where g is critical. Then, if the scalar curvature r satisfies the inequality

$$r > 2\left(1 - \frac{c^2}{4}\right) + \frac{|\sigma|^2}{1 - \frac{c^2}{4}} + 2c, \quad c = |\tau| < 2, \tag{2.1}$$

g has positive Ricci curvature.

PROOF. As in the proof of the Theorem in [6], to show that the Ricci tensor S is positive definite, we determine at each point $x \in M$, a suitable basis $\{E, \phi E, X_0\}$ of $T_x M$, and verify that the subdeterminants along the main diagonal are positive.

Assume $\sigma_x \neq 0$. Since σ_x is a linear form on B, there exists a vector $X \in B$ such that $\sigma_x = g(X, \cdot)$. Hence, by choosing $E = -\phi(X/|X|)$, we have $|E| = 1$, $\sigma_x(E) = 0$ and $\sigma_x(\phi E) = |\sigma|$.

The sectional curvatures $K(X_0, Y)$ of plane sections containing X_0 satisfy

$$(\nabla_{X_0} \tau)(X,X) = K(X_0, \phi X) - K(X_0, X)$$

for any unit vector $X \in B$ (see [12], Lemma 7.1). Moreover, since the metric g is critical, then by ([6], Proposition 1, formula (ii)) and (1.2), (2.2) $$\nabla_{X_0}\tau = -2\psi,$$

where $\psi(X,Y) = g((L_{X_0}\phi)X,Y)$. Thus,

$$S(E,E) = S(\phi E,\phi E) + 2\psi(E,E).$$

By polarization,

$$S(E,\phi E) = \psi(E,\phi E)$$

since by ([6], Proposition 1), trace $\psi = 0$ and ϕ is symmetric with respect to ψ. It follows that

$$S(E,E) = \frac{r}{2} + \frac{c^2}{4} - 1 + \psi(E,E)$$

and

$$S(\phi E,\phi E) = \frac{r}{2} + \frac{c^2}{4} - 1 - \psi(E,E).$$

Thus,

(2.3) $$S = \begin{bmatrix} \frac{r}{2} + \frac{c^2}{4} - 1 + \psi(E,E) & \psi(E,\phi E) & 0 \\ \psi(\phi E,E) & \frac{r}{2} + \frac{c^2}{4} - 1 - \psi(E,E) & |\sigma| \\ 0 & |\sigma| & 2(1-\frac{c^2}{4}) \end{bmatrix}.$$

The subdeterminants along the main diagonal are positive. For, the inequality (2.1) implies

$$S(E,E) \geq \frac{r}{2} + \frac{c^2}{4} - 1 - c > 0,$$

$$\begin{aligned} S(E,E)S(\phi E,\phi E) - S(E,\phi E)^2 &= (\tfrac{r}{2} + \tfrac{c^2}{4} - 1 - c)(\tfrac{r}{2} + \tfrac{c^2}{4} - 1 + c) \\ &\geq (\tfrac{r}{2} + \tfrac{c^2}{4} - 1 - c)^2 > 0 \end{aligned}$$

since $c^2 = \psi(E,E)^2 + \psi(E,\phi E)^2$, and

$$\begin{aligned} \det S = 2(1 - \tfrac{c^2}{4})\{(\tfrac{r}{2} + \tfrac{c^2}{4} - 1 + \psi(E,E))(\tfrac{r}{2} + \tfrac{c^2}{4} - 1 - \psi(E,E)) \\ - \psi(E,\phi E)^2\} - |\sigma|^2(\tfrac{r}{2} + \tfrac{c^2}{4} - 1 + \psi(E,E)) \end{aligned}$$

$$\geq 2(1 - \frac{c^2}{4})\{(\frac{r}{2} + \frac{c^2}{4} - 1)^2 - c^2\} - |\sigma|^2(\frac{r}{2} + \frac{c^2}{4} - 1 + c)$$

$$= (\frac{r}{2} + \frac{c^2}{4} - 1 + c)\{2(1 - \frac{c^2}{4})(\frac{r}{2} + \frac{c^2}{4} - 1 - c) - |\sigma|^2\} > 0$$

by (2.1).

If $\sigma = 0$, the same argument applies to an arbitrary basis of the form $\{E, \varphi E, X_0\}$.

COROLLARY 1. Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where X_0 is a Killing vector field. Then, if $r > -2$, M admits a contact metric structure $(a\omega, a^{-1}X_0, ag + a(a-1)\omega \otimes \omega)$ of positive sectional curvature for some constant a, $0 < a \leq 1$.

PROOF. Since X_0 is a Killing field, g is a critical metric. In addition, σ and τ both vanish. Thus, the matrix (2.3) reduces to

$$S = \begin{bmatrix} \frac{r}{2} - 1 & 0 & 0 \\ 0 & \frac{r}{2} - 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The components of the Riemann curvature tensor R with respect to the orthonormal basis $\{e_1, e_2, e_3\} = \{E, \phi E, X_0\}$ are

$$(2.4) \qquad R_{ijk\ell} = \delta_{ik}S_{j\ell} + \delta_{j\ell}S_{ik} - \delta_{i\ell}S_{jk} - \delta_{jk}S_{i\ell} - \frac{r}{2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}).$$

Consequently, since $S_{ij} = 0$ for $i \neq j$, the only nonvanishing components of R are those with two indices different.

Let P be a 2-dimensional subspace of the tangent space T_xM at $x \in M$, and let $\{X = \Sigma a_i e_i,\ Y = \Sigma b_j e_j\}$ be an orthonormal basis of P. Then, the sectional curvature $K(X,Y)$ of P is given by

$$g(R(X,Y)X,Y) = (a_1b_2 - a_2b_1)^2R_{1212} + (a_1b_3 - a_3b_1)^2R_{1313} + (a_2b_3 - a_3b_2)^2R_{2323}.$$

Thus, if $r > 4$, $K(X,Y) > 0$.

Assume that there is a point $x_0 \in M$ such that $0 < (r(x_0) + 2)/6 \leq 1$. The minimum k of the function $(r(x) + 2)/6$ then lies in the interval $(0,1]$. Consider the metric $\tilde{g}$ on M defined by

$$\tilde{g} = ag + a(a - 1)\omega \otimes \omega$$

for some constant a, $0 < a < k \leq (r(x) + 2)/6$. If we put $\tilde{\omega} = a\omega$ and $\tilde{X}_0 = a^{-1}X_0$, then $(\tilde{\omega},\tilde{X}_0,\tilde{g})$ is a contact metric structure whose characteristic vector field is a Killing field. By a direct computation (see [5]), the Ricci tensors S and $\tilde{S}$ of the metrics g and $\tilde{g}$, respectively, are related by

$$\tilde{S} = S + 2(1 - a)g - 2(1 - a)(2 + a)\omega \otimes \omega, \tag{2.5}$$

so since $\tilde{g}^{ij} = a^{-1}g^{ij} + (1 - a)a^{-2}X_0^iX_0^j$, $\tilde{r} - 4 = \frac{6}{a}(\frac{r + 2}{6} - a)$. Thus, $\tilde{r} > 4$, from which $\tilde{K}(X,Y) > 0$.

Note that if g is a critical metric, then formula (4) of [6] with $n = 1$ and $b = a^2 - a$ reduces to (2.5) by virtue of (2.2).

3. Constant curvature and critical metrics. Hamilton [7] showed that a metric g of positive Ricci curvature on a compact three-manifold can be deformed to a metric of (positive) constant curvature. If g is a contact metric, we obtain the following

COROLLARY 2. *Let M be a compact and orientable three-manifold with contact metric structure (ω,X_0,g) where X_0 is a Killing vector field. Then, if r is a constant greater than -2, the metric g may be deformed to a contact metric of constant curvature 1.*

PROOF. It is well-known that a Riemannian three-manifold (M,g) is an Einstein manifold if and only if it has constant curvature (cf. (2.4)). The matrix S in the proof of Corollary 1 says that g is an Einstein metric $\Leftrightarrow S = (r/3)g \Leftrightarrow |S|^2 = r^2/3 \Leftrightarrow r = 6$. If r is a constant greater than -2, then the contact metric defined by $\tilde{g} = ag + a(a - 1)\omega \otimes \omega$, $a = (r + 2)/8$, has constant scalar curvature $\tilde{r} = ((r + 2)/a)-2 = 6$.

More generally, if the contact metric is critical, then from (2.3)

$$|S|^2 = 2|\sigma|^2 + 2\psi(E,\phi E)^2 + 4(1 - \frac{c^2}{4})^2 + 2(\frac{r}{2} + \frac{c^2}{4} - 1)^2 + 2\psi(E,E)^2$$

$$= 2|\sigma|^2 + 2c^2 + 2(\frac{r}{2} + \frac{c^2}{4} - 1)^2 + 4(1 - \frac{c^2}{4})^2.$$

Hence, $|S|^2 = r^2/3$, if and only if,

$$[r - 6(1 - \frac{c^2}{4})]^2 + 12(|\sigma|^2 + c^2) = 0.$$

This yields

THEOREM 2. *Let M be a compact and orientable three-manifold with contact metric structure (ω, X_0, g), where g is critical. Then, g is of constant curvature k, if and only if X_0 is a Killing vector field and $k = 1$.*

COROLLARY. *A critical metric on the three-sphere with constant curvature is the standard normal contact metric.*

LEMMA. *Let M be a compact contact $(2n + 1)$-dimensional manifold with contact Riemannian structure (ω, X_0, g) where g is critical. Then, for $a > 0$, $(\tilde{\omega} = a\omega$, $\tilde{X}_0 = a^{-1}X_0$, $\tilde{g} = ag + (a^2 - a)\omega \otimes \omega)$ is also a contact Riemannian structure with critical metric $\tilde{g}$. Moreover, the scalar curvatures of g and $\tilde{g}$ are related by*

$$\tilde{r} = \frac{r + 2n}{a} - 2n + \frac{c^2}{2a^2}(a - 1),$$

and

$$|\tilde{\sigma}| = a^{-3/2}|\sigma|,$$

where $\tilde{\sigma} = S(\tilde{X}_0, \cdot)|B$.

PROOF. The Ricci tensors S and $\tilde{S}$ are related by [6], formula (4), namely,

$$\tilde{S} = S - 2(a - 1)g + 2(a - 1)(na + n + 1)\omega \otimes \omega \tag{3.1}$$

since g is a critical metric. Thus,

$$\begin{aligned} \tilde{S}(\tilde{X}_0, \tilde{X}_0) &= \frac{1}{a^2}[S(X_0, X_0) + 2n(a^2 - 1)] \\ &= \frac{1}{a^2}[2(n - \frac{c^2}{4}) + 2n(a^2 - 1)] \\ &= 2n - \frac{c^2}{2a^2}. \end{aligned}$$

On the other hand,

$$\tilde{S}(\tilde{X}_0, \tilde{X}_0) = 2(n - \frac{\tilde{c}^2}{4}), \quad \text{where} \quad \tilde{c}^2 = |L_{\tilde{X}_0}\tilde{g}|^2,$$

from which $\tilde{c} = a^{-1}c$. Hence,

$$\mathcal{E}(\tilde{g}) = \frac{1}{2}\int_M \tilde{c}^2 d\ \mathrm{vol}(M,\tilde{g}) = \frac{a^{n-1}}{2}\int_M c^2 d\ \mathrm{vol}(M,g) = a^{n-1}\mathcal{E}(g)$$

since $d\ \mathrm{vol}(M,\tilde{g}) = \frac{1}{2^n n!}\tilde{\omega}\wedge(d\tilde{\omega})^n = a^{n+1}d\ \mathrm{vol}(M,g)$. Since g is critical, it follows that $\tilde{g}$ is also a critical metric. Now, let $\{\tilde{E}_i,\varphi\tilde{E}_i,\tilde{X}_0\}$, $i = 1,\ldots,n$, be a φ-basis with respect to $\tilde{g}$. Then, $\{E_i,\varphi E_i,X_0\}$, $i = 1,\ldots,n$, is a φ-basis with respect to g, where $E_i = a^{1/2}\tilde{E}_i$, and

$$\begin{aligned}\tilde{r} = \mathrm{trace}\ \tilde{S} &= \tilde{S}(\tilde{X}_0,\tilde{X}_0) + \sum_1^n\{\tilde{S}(\tilde{E}_i,\tilde{E}_i) + \tilde{S}(\varphi\tilde{E}_i,\varphi\tilde{E}_i)\}\\ &= 2n - \frac{c^2}{2a^2} + \frac{1}{a}\sum_1^n\{S(E_i,E_i) + S(\varphi E_i,\varphi E_i)\} - \frac{4n}{a}(a-1)\\ &= \frac{r+2n}{a} - 2n + \frac{c^2}{2a^2}(a-1).\end{aligned}$$

Moreover,

$$\begin{aligned}|\tilde{\sigma}|^2 &= \sum_1^n\{\tilde{S}(\tilde{X}_0,\tilde{E}_i)^2 + \tilde{S}(\tilde{X}_0,\varphi\tilde{E}_i)^2\}\\ &= \frac{1}{a^3}\sum_1^n\{S(X_0,E_i)^2 + S(X_0,\varphi E_i)^2\}\\ &= \frac{|\sigma|^2}{a^3}.\end{aligned}$$

The lemma and Theorem 1 give rise to the following theorem (see also [6]).

THEOREM 3. *Let M be a compact and orientable three-manifold with contact metric structure (ω,X_0,g), where g is critical. If there exists a constant a such that $c < 2a$ and*

$$r + \frac{c^2}{2} > \frac{|\sigma|^2}{(a^2 - c^2/4)} + 2(2a + c - 1), \tag{3.2}$$

then M admits a contact critical metric of positive Ricci curvature.

Another application of (3.1) yields the following extension of Theorem 2 of [5].

THEOREM 4. *Let* M *be a compact* $(2n + 1)$*-dimensional manifold with contact metric structure* (ω, X_0, g), *where* g *is a critical metric and* $\sigma = S(X_0, \cdot)|B = 0$. *Then, if* $S + \lambda g$ *is positive definite for some* $\lambda < 2 - c/\sqrt{n}$, *the first Betti number* $b_1(M)$ *of* M *is zero. If, in addition,* M *is simply connected and* $n = 1$, *it is diffeomorphic with* S^3.

PROOF. Consider the deformation defined by

$$\tilde{g} = ag + (a^2 - a)\omega \otimes \omega$$

for some constant $a = (2 - \lambda)/2 > c/2\sqrt{n} > 0$. Then, (3.1) becomes

$$\tilde{S} = S + \lambda g - \lambda(2n - \frac{n\lambda}{2} + 1)\omega \otimes \omega.$$

Since $\tilde{\sigma} = \sigma = 0$, to see that $\tilde{S}$ is positive definite we need only consider $\tilde{S}(X,X)$ with X horizontal and X vertical.

If X is vertical, that is, if $X = tX_0$, then

$$\begin{aligned}\tilde{S}(X,X) &= t^2\tilde{S}(X_0,X_0)\\ &= t^2[S(X_0,X_0) + \lambda - \lambda(2n - \frac{n\lambda}{2} + 1)]\\ &= t^2[2n - \frac{c^2}{2} - n\lambda(2 - \frac{\lambda}{2})]\\ &= \frac{t^2}{2}[n(2 - \lambda)^2 - c^2] > 0, \quad \lambda < 2 - \frac{c}{\sqrt{n}}.\end{aligned}$$

If X is horizontal, that is, if $\omega(X) = 0$, then

$$\tilde{S}(X,X) = (S + \lambda g)(X,X) > 0.$$

Thus, $\tilde{S}$ is positive definite, and so $b_1(M) = 0$. The last part is a consequence of Hamilton [7].

4. Remarks: (a) The quantity $r + c^2/2$ appearing in (2.1) and (3.2) is equal to $r^* - 2$, where r^* is the generalized Tanaka-Webster scalar curvature defined in [12]. The Webster curvature W studied by Chern and Hamilton in [4] is equal to $r^*/8$. The condition on the scalar curvature r in Corollaries 1 and 2 may therefore be replaced by the assumption that r^* or W be positive. The main result of [4] says that every contact structure on a compact orientable three-manifold has a contact Riemannian metric whose Webster curvature W is either a constant ≤ 0 or $W > 0$.

(b) Corollary 2 is not too surprising since a compact simply connected three-manifold M which admits a nonsingular Killing vector field is diffeomorphic to S^3. To see this, let G be a compact Lie group of isometries of M and denote by $\mu_t \in G$ the one-paramater group generated by $X(\partial\mu_t/\partial t = X)$. The flow μ_t gives rise to an abelian subgroup of G. The closure of μ_t inside G is a torus and approaches μ_t inside this torus by S^1. Since the action of μ_t is locally free, one can take S^1 sufficiently close to μ_t so that it acts locally free on M. This action gives rise to a Seifert manifold structure on M. But a simply connected Seifert manifold is diffeomorphic to S^3.

(c) The following remark is due to Grant Cairns. G. Monna [9] has constructed a flow on S^3 which is not a Killing flow, so it is neither regular nor normal. In his example, he deforms the Hopf fibration, namely, S^3 is considered as the union of two solid tori in such a way that in each torus the Hopf fibration induces a flow tangent to each of the 2-tori parallel to the boundary. The flows induced on these 2-tori are just linear flows with 'constant' rational gradient. When one deforms the flow as in [5], p. 655, one simply changes the gradient to another constant (possibly irrational) gradient. However, the gradient can be changed in such a way that it is not a constant (i.e., it depends on the 2-torus) and yet the deformed flow is still a contact flow. The resulting flow has infinitely many closed leaves and infinitely many non-closed leaves, so it cannot be a Killing flow.

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