Classification of Quadratic Harmonic Maps of S^3 into Spheres

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§1. **Introduction and statement of the result.** In 1972 R. T. Smith posed the problem of classifying all harmonic maps $f\colon S^m \to S^n$, $m \geq 2$, between Euclidean spheres whose components are homogeneous (harmonic) polynomials of (fixed) degree k (cf. [2, 6, 7] and [3], Problem (4.4), p. 70). Full linear ($k=1$) harmonic maps are isometries of S^m and, by Calabi's rigidity theorem, any full k-homogeneous harmonic map of S^2 is a standard minimal immersion $f_{\lambda_k}\colon S^2 \to S^{2k}$ [1, 8, 9]. The object of this paper is to give a classification of full k-homogeneous harmonic maps in the first (nonrigid) range of m and k not covered by the results above, i.e., when $m=3$ and $k=2$.

Classification Theorem. *Full quadratic ($k=2$) harmonic maps of S^3 into S^n exist only if $2 \leq n \leq 8$ and $n \neq 3$. Moreover, if $f\colon S^3 \to S^n$ is such a map, then there exist $U \in O(4)$, $V \in O(n+1)$ and a symmetric positive definite matrix $B \in S^2(\mathbf{R}^{n+1})$ such that*

$$V \circ f \circ U = B \circ f_n,$$

where $f_n\colon S^3 \to S^n$ defined by

$$f_n(x,y,u,v) = \begin{cases} \left(x^2+y^2-u^2-v^2, 2(xu-yv), 2(xv+yu)\right), \text{ [Hopf map]} & n=2 \\ (x^2+y^2-u^2-v^2, 2xu, 2xv, 2yu, 2yv), & n=4 \\ \left(x^2-y^2, u^2-v^2, 2xy, \sqrt{2}(xu+yv), \sqrt{2}(yu-xv), 2uv\right), & n=5 \\ \left(\frac{1}{\sqrt{2}}(x^2+y^2-u^2-v^2), \frac{1}{\sqrt{2}}(x^2-y^2), \frac{1}{\sqrt{2}}(u^2-v^2), \right. & \\ \qquad \left. \sqrt{2}xy, \sqrt{3}(xu+yv), \sqrt{3}(yu-xv), \sqrt{2}uv\right), & n=6 \\ (x^2-y^2, u^2-v^2, 2xy, \sqrt{2}xu, \sqrt{2}xv, \sqrt{2}yu, \sqrt{2}yv, 2uv), & n=7 \\ f_{\lambda_2}(x,y,u,v), \text{ [}f_{\lambda_2} = \text{a standard minimal immersion]} & n=8 \end{cases}$$

For fixed n, the matrices B (corresponding to the various maps f) form an (open) convex body I_{f_n} lying in a finite-dimensional vector space. Finally, $I_{f_2} =$ point, *$I_{f_4} =$* segment, *$I_{f_5} = 2-$*disk, *$I_{f_6} =$* (finite) *solid cone, $\dim I_{f_7} = 5$ and* $\dim I_{f_8} = 10$.

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Remark. By way of contrast (to the nonexistence of full quadratic harmonic maps $f\colon S^3 \to S^3$), quaternionic square induces a full quadratic map of S^3 onto S^3 which is evidently nonharmonic (cf. also [10]). Note also that the gradient of cubic isoparametric functions give rise to full quadratic harmonic maps $f\colon S^n \to S^n$ in dimensions $n = 4, 7, 13, 25$ [11].

§2. **Generalities on the parameter space.** By the generalized Do Carmo–Wallach classification theorem, for fixed k and m, the equivalence classes of full k-homogeneous harmonic maps $f\colon S^m \to S^n$ can be parametrized by a compact convex body L° lying in a finite-dimensional vector space E (cf. [8], pp. 297–304). The parametrization is given by associating to f the symmetric matrix

$$\langle f\rangle = A^t \cdot A - I_{n(k)+1} \in S^2\big(\mathbf{R}^{n(k)+1}\big),$$

where A is the $(n+1)\times\big(n(k)+1\big)$-matrix defined by $f = A \circ f_{\lambda_k}$ with $f_{\lambda_k}\colon S^m \to S^{n(k)}$ a fixed standard minimal immersion. Then

$$E = (W^\circ)^\perp = \big(\operatorname{span}\{f_{\lambda_k}(x)^2 | x \in S^m\}\big)^\perp \subset S^2\big(\mathbf{R}^{n(k)+1}\big)$$

and

$$L^\circ = \{C - I_{n(k)+1} \in E | C \geq 0\}.$$

In a similar vein, for fixed f, define

$$E_f = \big(W_f^\circ\big)^\perp = \big(\operatorname{span}\{f(x)^2 | x \in S^m\}\big)^\perp \subset S^2\big(\mathbf{R}^{n+1}\big)$$

and

$$L_f^\circ = \{C - I_{n+1} \in E_f | C \geq 0\}.$$

Then the affine map $\varphi\colon L_f^\circ \to L^\circ$ defined by $\varphi(C - I_{n+1}) = A^t \cdot C \cdot A - I_{n(k)+1}$, injects L_f° onto a compact convex set $\bar{I}_f$. Denoting by A_f the affine (= flat) subspace of E spanned by $\bar{I}_f$, we have $A_f \cap L^\circ = \bar{I}_f$ and the interior I_f of $\bar{I}_f$ in A_f is a convex body containing $\langle f\rangle$. The convex sets I_f, for the various f, give rise to a cell-decomposition of L°. Clearly, $I_{f_{\lambda_k}} = \operatorname{int} L^\circ$ and the points of $I_{f_{\lambda_k}}$ correspond to maps $f\colon S^m \to S^n$ with maximal $n = n(k)$.

Precomposing harmonic maps with isometries of S^m induces an $SO(m+1)$-action on the set of equivalence classes which, in turn, corresponds to the restriction (to L°) of the $SO(m+1)$-module structure of E given by $\operatorname{Ad}\rho_{\lambda_k}$, where $\rho_{\lambda_k}\colon SO(m+1) \to SO\big(n(k)+1\big)$ is the homomorphism associated with the equivariance of f_{λ_k}. Then, for $a \in SO(m+1)$, we have $a \cdot I_f = I_{f\circ a^{-1}}$, i.e., the $SO(m+1)$-action on L° respects the cell structure of L° introduced above. Note also that $\dim L^\circ = \dim E > 0$ if and only if $m \geq 3$ and $k \geq 2$ [8]. For fixed $f\colon S^m \to S^n$, the isotropy subgroup $SO(m+1)_{\langle f\rangle}$ is the "symmetry group" of f defined by

$$SO(m+1)_f = \{a \in SO(m+1) |\, \exists U \in O(n+1) \text{ such that } f \circ a = U \circ f\}.$$

Its Lie algebra is given by $so(m+1)_f = \{X \in so(m+1) | \exists A \in so(n+1)$ such that $f_*(X) = A \circ f\}$.

Let $K(f)$ denote the (finite-dimensional) vector space of divergence free Jacobi fields along f. Then ([8], pp. 278–290)

$$so(n+1) \circ f + E_f \circ f + f_*\big(so(m+1)\big) \subset K(f). \tag{1}$$

The composition

$$so(m+1) \to f_*\big(so(m+1)\big) \to \frac{f_*\big(so(m+1)\big)}{\big(f_*\big(so(m+1)\big) \cap so(n+1) \circ f\big)}$$

given by f_* and the canonical projection has kernel $so(m+1)_f$ ($\supset \ker f_*$) and so we have

$$T_{\langle f\rangle} = T_{\langle f\rangle}\big(SO(m+1)(\langle f\rangle)\big) \simeq \frac{so(m+1)}{so(m+1)_f}$$

$$\simeq \frac{f_*\big(so(m+1)\big)}{\big(f_*\big(so(m+1)\big) \cap so(n+1) \circ f\big)} \subset \frac{K(f)}{so(n+1) \circ f}. \tag{2}$$

More explicitly, the isomorphism is given as follows: For $X_{\langle f\rangle} \in T_{\langle f\rangle}$, let $\big(a(t)\big) \subset SO(m+1)$ be a 1-parameter subgroup with $\frac{d}{dt}\big(a(t) \cdot \langle f\rangle\big)|_{t=0} = X_{\langle f\rangle}$. Denoting by X the infinitesimal isometry on S^m induced by $\big(a(t)\big)$, to $X_{\langle f\rangle}$, the isomorphism associates $f_*(X) \bmod \big(f_*\big(so(m+1)\big) \cap so(n+1) \circ f\big)$.

As $E_f \cap so(n+1) = \{0\}$, we will consider $E_f \circ f$ as a linear subspace of $\frac{K(f)}{so(n+1) \circ f}$. It follows that, under the isomorphism above, the linear subspace $A_f \cap T_{\langle f\rangle}$ corresponds to $E_f \circ f \cap \frac{f_*(so(m+1))}{(f_*(so(m+1)) \cap so(n+1) \circ f)}$. Moreover, for $X_{\langle f\rangle} \in A_f \cap T_{\langle f\rangle}$ given by $X_{\langle f\rangle} = \frac{d}{dt}\big(a(t) \cdot \langle f\rangle\big)|_{t=0}$, the orbit $t \mapsto a(t) \cdot \langle f\rangle$, $t \in \mathbf{R}$, is actually contained in I_f.

Factorizing out, we have

$$\frac{E_f \circ f + f_*\big(so(m+1)\big)}{\big(f_*\big(so(m+1)\big) \cap so(n+1) \circ f\big)} \subset \frac{K(f)}{so(n+1) \circ f}. \tag{3}$$

Clearly, for (full) $f\colon S^m \to S^{n(k)}$, equality holds in (1) and (3).

§3. Proof of the Classification Theorem. The entire space of quadratic harmonic polynomials in four variables x, y, u, v is 9-dimensional and is spanned by $x^2 + y^2 - u^2 - v^2$, $x^2 - y^2$, $u^2 - v^2$, xy, xu, xv, yu, yv, uv. Hence, for $2 \le n \le 8$, a full quadratic harmonic map $f\colon S^3 \to S^n$ is given by

$$f(x,y,u,v) = b_1 x^2 + b_2 y^2 + c_1 u^2 + c_2 v^2 + d_1 xy + d_2 xu + d_3 xv + d_4 yu + d_5 yv + d_6 uv, \tag{4}$$

where the vectors b_i, c_i, $d_j \in \mathbf{R}^{n+1}$, $i = 1,2$, $j = 1,\ldots,6$, span $\mathbf{R}^{n+1}$ and $b_1 + b_2 + c_1 + c_2 = 0$. As $\|f(x,y,u,v)\|^2$ is a homogeneous polynomial of degree 4, the condition $\mathrm{Im}(f) \subset S^n$ translates into

$$\|f(x,y,u,v)\|^2 = (x^2+y^2+u^2+v^2)^2 \tag{5}$$

which has to be satisfied for all $(x,y,u,v) \in \mathbf{R}^4$. Substituting (4) into (5) and expanding both sides, we obtain various orthogonality relations between b_i, c_i, $d_j \in \mathbf{R}^{n+1}$. For $n = 2$, a straightforward computation (in the use of the vector cross–product in $\mathbf{R}^3$) gives the general form of a full quadratic harmonic map $f\colon S^3 \to S^2$, namely f is equivalent to

$$\begin{aligned}
&f^{\varepsilon}_{\alpha,\beta}(x,y,u,v)\\
&= \Big(\cos\frac{\alpha}{2}(x^2+y^2-u^2-v^2) + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}(\varepsilon_3 xv + \varepsilon_4 yu)\\
&\qquad -2\sin\frac{\alpha}{2}\cos\frac{\beta}{2}(\varepsilon_2 xu + \varepsilon_5 yv),\\
&\quad \sin\frac{\alpha}{2}\left(x^2 - y^2 - \cos\beta(u^2-v^2)\right)\\
&\qquad +2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}(\varepsilon_2 xu - \varepsilon_5 yv) - 2\cos\frac{\alpha}{2}\sin\frac{\beta}{2}(\varepsilon_3 xv - \varepsilon_4 yu)\\
&\qquad +2\varepsilon_6 \sin\frac{\alpha}{2}\sin\beta uv,\\
&\quad -\sin\frac{\alpha}{2}\sin\beta(u^2-v^2) - 2\sin\frac{\alpha}{2}(\varepsilon_1 xy + \varepsilon_6\cos\beta uv)\\
&\qquad +2\cos\frac{\alpha}{2}\sin\frac{\beta}{2}(\varepsilon_2 xu - \varepsilon_5 yv) + 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}(\varepsilon_3 xv - \varepsilon_4 yu)\Big),
\end{aligned}$$

where $0 \le \alpha,\beta \le \pi$ and $\varepsilon = (\varepsilon_j)_{j=1}^{6} \in \mathbf{Z}_2^6$ obeys the sign relations $\varepsilon_1\varepsilon_2\varepsilon_4 = -\varepsilon_1\varepsilon_3\varepsilon_5 = \varepsilon_2\varepsilon_3\varepsilon_6 = -\varepsilon_4\varepsilon_5\varepsilon_6 = 1$. For fixed ε, all $f^{\varepsilon}_{0,\beta}$, $0 \le \beta \le \pi$, are equivalent. Passing to the equivalence classes, we obtain a homeomorphic embedding of the triangle $\frac{[0,\pi]^2}{\{0\}\times[0,\pi]}$ into ∂L° (induced by $(\alpha,\beta) \to f^{\varepsilon}_{\alpha,\beta}$). By the sign relations, these 8 triangles (corresponding to the various ε) are easily seen to be pasted together along their edges to form two disjoint copies $\mathbf{R}P_1^2$ and $\mathbf{R}P_2^2$ of the real projective plane containing the (equivalence class of the) Hopf map f_2 and its "dual"

$$f_2'(x,y,u,v) = (x^2+y^2-u^2-v^2,\quad 2(xu+yv),\quad 2(xv-yu)),$$

respectively. The symmetry group $SO(4)_{f_2}$, being the isotropy subgroup of $\langle f_2\rangle \in \partial L^\circ$, is then at least 4–dimensional since the respective orbit is contained in $\mathbf{R}P_1^2$. On the other hand, $SO(4)_{f_2} \subset SO(4)$ is a proper subgroup since the

Hopf map is not equivariant. It follows that $\dim SO(4)_{f_2} = 4$ and therefore $SO(4)(\langle f_2 \rangle) = \mathbf{R}P_1^2$. Passing to $O(4)$, we recover the other copy $\mathbf{R}P_2^2$. The statement of the theorem follows for $n = 2$.

For the nonexistence of full quadratic harmonic maps $f\colon S^3 \to S^3$, we have to show that there is no system of vectors b_i, c_i, $d_j \in \mathbf{R}^4$, $i = 1,2$, $j = 1,\ldots,6$, spanning $\mathbf{R}^4$ such that they satisfy the orthogonality relations equivalent to (5). This can be done by tedious but elementary computation separating the cases $\dim \operatorname{span}\{b_i,c_i\}_{i=1}^2 = 1,2$ or 3. (Note that in the last case, it is convenient to use the vector cross–product on span $\{b_i,c_i\}_{i=1}^2$.)

Let V_1 and V_2 denote the affine span of $\mathbf{R}P_1^2$ and $\mathbf{R}P_2^2$, respectively. As $SO(4)$ acts on E without fixed points, V_1 and V_2 are actually linear ($SO(4)$–invariant) subspaces of E. We claim that $E = V_1 \oplus V_2$ is an orthogonal direct sum with V_1 and V_2 irreducible $SO(4)$–submodules. First note that in [8] we determined the irreducible components of the complex $SO(m+1)$–module $E \otimes_{\mathbf{R}} \mathbf{C}$ in general in terms of the (coordinates of the) highest weights which, in our case $m = 3$ and $k = 2$, specializes to $E \otimes_{\mathbf{R}} \mathbf{C} \simeq V_3^{2,2} \oplus V_3^{2,-2}$, in particular, $\dim E = 2 \dim V_3^{2,2} = 10$. (This alone implies the theorem for $n = 8$.) Now, computation shows that $SO(4)_{f_2} \cap SO(4)_{f_2'}$ is 2–dimensional (in fact, the identity component is the standard maximal torus $T \subset SO(4)$), or equivalently, $SO(4)_{f_2}$ acts transitively on $\mathbf{R}P_2^2$ from which the decomposition follows easily. To describe the orbit structure of the irreducible orthogonal $SO(4)$–module V_1 first note that the possible (positive) dimensions of the orbits may only be 2 and 3. A straightforward topological argument shows then that the 3–dimensional orbits actually occur. In fact, if all nontrivial orbits were 2–dimensional, or equivalently, if there were no singular orbits on the unit sphere S_1^4 of V_1, then S_1^4 would split into a product, a contradiction. It follows that the orbits on S_1^4 form a homogeneous (isoparametric) family of hypersurfaces with no exceptional orbit and two singular orbits (= focal varieties) corresponding to $\mathbf{R}P_1^2$ and its antipodal [5]. The same description applies to $\partial L_1 = \partial L^\circ \cap V_1$ as it is equivariantly homeomorphic with S_1^4. Now, consider $\bar{I}_{f_5}$. By straightforward computation, we obtain that $\bar{I}_{f_5}$ is a closed 2–disk with center $\langle f_5 \rangle$ and boundary $\partial I_{f_5} = \{\langle f_2^{\varphi} \rangle | \varphi \in \mathbf{R}\} \subset \mathbf{R}P_1^2$, where, using complex coordinates $z = x + iy$ and $w = u + iv$, $f_2^{\varphi}(z,w) = \left(e^{i\varphi} z^2 + w^2, \operatorname{Im}(2e^{i(\varphi/2)} z \bar{w})\right)$. Moreover, ∂I_{f_5} is an orbit under the action of the circle group

$$\Gamma = \{\operatorname{diag}(e^{i\alpha}, e^{-i\alpha}) | \alpha \in \mathbf{R}\} \subset SO(4).$$

As $\langle f_5 \rangle$ is in the center of $\bar{I}_{f_5}$, the orbit $SO(4)(\langle f_5 \rangle)$ is singular and is then opposite to $\mathbf{R}P_1^2$ in ∂L°. In fact, as computation shows, $\langle f_5 \rangle \in \mathbf{R} \cdot \langle f_2 \rangle$. The orbit structure on ∂L_1 is that of a homogeneous family of hypersurfaces and hence (by considering a radial segment of $\bar{I}_{f_5}$) we obtain that every $SO(4)$–orbit on ∂L_1 intersects $\bar{I}_{f_5}$, or equivalently, $SO(4)(\bar{I}_{f_5}) = \partial L_1$. As $f_2' = f_2 \circ \operatorname{diag}(1,1,-1,1)$, the orbit and cell structures on $\partial L_2 = \partial L^\circ \cap V_2$ are the same as on ∂L_1.

Next we consider $\bar{I}_{f_4}$. A simple computation shows that $\bar{I}_{f_4} \subset \partial L^\circ$ is a closed segment with boundary points $\langle f_2 \rangle \in \mathbf{R}P_1^2$ and $\langle f_2' \rangle \in \mathbf{R}P_2^2$. Since every isotropy subgroup of $\mathbf{R}P_1^2$ acts transitively on $\mathbf{R}P_2^2$, the group $SO(4)$ acts transitively on the set of segments joining $\mathbf{R}P_1^2$ and $\mathbf{R}P_2^2$ whose union is therefore equal to $SO(4)(\bar{I}_{f_4})$.

Next, $\bar{I}_{f_6} \subset \partial L^\circ$ is a solid cone with base 2–disk $\bar{I}_{f_5} \in \partial L_1$ and vertex $\langle f_2' \rangle \in \mathbf{R}P_2^2$. Note also that $\langle f_6 \rangle$ is on the center segment joining $\langle f_5 \rangle$ and $\langle f_2' \rangle$ and the circle group Γ leaves $\langle f_2' \rangle$ fixed and hence rotates $\bar{I}_{f_6}$. As $\langle f_5 \rangle \in \mathbf{R} \cdot \langle f_2 \rangle$, it follows that $SO(4)$ acts transitively on the set of cones with base 2–disk a 2–cell in ∂L_1 and vertex in $\mathbf{R}P_2^2$. The union of these cones then coincides with $SO(4)(\bar{I}_{f_6})$. The same conclusion holds for V_1 and V_2 interchanged (by applying $\mathrm{diag}(1,1,-1,1) \in O(4)$).

Finally, $\bar{I}_{f_7} \subset \partial L^\circ$ is a 5–dimensional compact convex body which is the convex hull of $\bar{I}_{f_5}$ and $\bar{I}_{f_5'}$, where

$$f_5'(x,y,u,v) = \left(x^2 - y^2, u^2 - v^2, 2xy, \sqrt{2}(xu - yv), \sqrt{2}(yu + xv), 2uv\right)$$

with $\partial I_{f_5'} \subset \mathbf{R}P_2^2$ and $\langle f_5' \rangle \in \mathbf{R} \cdot \langle f_2' \rangle$. Moreover, Γ leaves $\bar{I}_{f_5'}$ pointwise fixed. (In fact, in complex coordinates, $\partial I_{f_5'}$ is given by the equivalence classes of $f_2'^{\varphi}(z,w) = \left(e^{i\varphi}\,\bar{z}^2 + w^2, \mathrm{Im}(2e^{i(\varphi/2)}\overline{zw})\right)$.) Similarly, the circle group $\Gamma' = \{\mathrm{diag}(e^{i\alpha}, e^{i\alpha}) | \alpha \in \mathbf{R}\}$ leaves $\bar{I}_{f_5}$ pointwise fixed and rotates $\bar{I}_{f_5'}$. The group $SO(4)$ acts again transitively on the set of convex hulls spanned by any pairs of 2–cells (C_1, C_2) with $C_i \subset \partial L_i$, $i = 1,2$. The union of this set is contained in $SO(4)(\bar{I}_7) \subset \partial L^\circ$.

Summarizing, we obtain that $O(4)\left(\bar{I}_{f_6} \cup \bar{I}_{f_7}\right) \subset \partial L^\circ$ contains all segments which join ∂L_1 and ∂L_2 and hence it coincides with the whole ∂L°. Since

$$O(4)\left(\bar{I}_{f_6} \cup \bar{I}_{f_7}\right) = O(4)\Big(\bigcup_{\substack{n=2 \\ n\neq 3}}^{7} I_{f_n}\Big),$$

the proof is finished.

§4. **Applications.** In his thesis [6], R. T. Smith also posed the following problem:

> Given a Jacobi field v along a harmonic map $f\colon M \to N$ between Riemannian manifolds M and N, does there exist a 1–parameter family $f_t\colon M \to N$, $|t| < \varepsilon$, of harmonic maps such that $f_\circ = f$ and $\frac{\partial f_t}{\partial t}\big|_{t=0} = v$?

By constructing a 1–parameter group of nonharmonic diffeomorphisms of $\mathbf{R}^3$ with induced Jacobi field, he solved the problem negatively [6], pp. 105–107. Here we show, however, that the answer to this question is affirmative for a large number of (full) k–homogeneous harmonic maps $f\colon S^m \to S^n$. We begin with the following:

Lemma. *Let f: $S^m \to S^n$ be a full k-homogeneous harmonic map and assume that equality holds in (1). Then $v \in K(f)$ if and only if there exists a 1-parameter family f_t: $S^m \to S^n$ of full k-homogeneous harmonic maps such that $f_\circ = f$ and $\frac{\partial f_t}{\partial t}\big|_{t=0} = v$.*

Proof. Given $v \in K(f)$, the induced vector-function $\check{v}$: $S^m \to \mathbf{R}^{n+1}$ decomposes as $\check{v} = A \cdot f + B \cdot f + X(f)$, where $A \in so(n+1)$, $B \in E_f$ and $X \in so(m+1)$. Choose $\varepsilon > 0$ such that $2tB + I_{n+1}$ is positive definite for $|t| < \varepsilon$ and define $f_t = \varphi_t \circ \sqrt{2tB + I_{n+1}} \circ f \circ \psi_t$, where $(\varphi_t) = (\exp(tA)) \subset SO(n+1)$ and $(\psi_t) = (\exp(tX)) \subset SO(m+1)$. Then $\Delta^{S^m} f_t = \lambda_k \cdot f_t$, $\lambda_k = k(k+m-1) \in \mathrm{Spec}(S^m)$, i.e., f_t: $S^m \to S^n$ is a (full) k-homogeneous harmonic map for $|t| < \varepsilon$. Clearly, $f_\circ = f$ and $\frac{\partial f_t}{\partial t}\big|_{t=0} = v$. The converse follows from the fact that $v \in K(f)$ if and only if $\Delta^{S^m} \check{v} = \lambda_k \cdot \check{v}$ ([8], p. 280).

As noted at the end of §2, the condition of the lemma is satisfied for any f with equivalence class in $\mathrm{int}\, L^\circ$. Using the (proof of the) Classification Theorem, for $n = 3$ and $k = 2$, we can prove more, namely:

Theorem 1. *For any full quadratic harmonic map f: $S^3 \to S^n$, $2 \le n \le 8$ and $n \ne 3$, equality holds in (1). In particular, every $v \in K(f)$ can be generated by a 1-parameter family of quadratic harmonic maps.*

Proof. Using the notations in the proof of the Classification Theorem, clearly, it is enough to show that equality holds in (1) on the cells I_{f_n}, $2 \le n \le 7$, $n \ne 3$. To argue by dimension comparison, we first note that $\dim(\frac{K(f)}{so(n+1)\circ f})$ is constant on any open cell. Using an appropriate base in the space of quadratic spherical harmonics on S^3, we note that a tedious but straightforward computation yields

$$\dim\left(\frac{K(f_n)}{so(n+1)\circ f_n}\right) = \begin{cases} 2, & n = 2, \\ 5, & n = 4, \\ 4, & n = 5, \\ 7, & n = 6, \\ 9, & n = 7. \end{cases} \tag{6}$$

For $n = 2$, as $T_{\langle f_2 \rangle} = T_{\langle f_2 \rangle}(\mathbf{R}P_1^2)$ is 2-dimensional, equality holds already in (2). For fixed f: $S^3 \to S^4$, $\langle f \rangle \in I_{f_4}$, we first claim that $E_f \circ f \cap f_*(so(4)) = \{0\}$. In fact, as noted in §2, if $X_{\langle f \rangle} = \frac{d}{dt}\left(a(t) \cdot \langle f \rangle\right)\big|_{t=0} \in A_f \cap T_{\langle f \rangle}$, then the orbit $t \mapsto a(t) \cdot \langle f \rangle$, $t \in \mathbf{R}$, is contained in I_{f_4} which, by orthogonality. is possible only if $X_{\langle f \rangle} = 0$. The action of $SO(4)$ on ∂L° respects the cell structure and so $SO(4)_f = SO(4)_{f_2} \cap SO(4)_{f_2'}$ with identity component $T \subset SO(4)$ the standard maximal torus. Hence $\dim(SO(4)(\langle f \rangle)) = 4$. This and $\dim E_f \circ f = 1$ make up 5 in (6). For $n = 5$, $\bar{I}_{f_5}$ is a 2-disk with center $\langle f_5 \rangle \in \mathbf{R} \cdot \langle f_2 \rangle$, in particular, $SO(4)_{f_5} = SO(4)_{f_2}$ and the orbit $SO(4)(\langle f_5 \rangle)$ is 2-dimensional. As the circle group Γ rotates $\bar{I}_{f_5}$, it follows that $SO(4)(\langle f_5\rangle)\cap \bar{I}_{f_5} = \langle f_5\rangle$ and so $A_{f_5}\cap T_{\langle f_5\rangle} = \{0\}$, or equivalently, $E_{f_5}\circ f_5 \cap (f_5)_*\big(so(4)\big) = \{0\}$. Thus, by (6), equality holds in (1). For fixed $f\colon S^3 \to S^5$ with $\langle f\rangle \in I_{f_5}\setminus\{\langle f_5\rangle\}$, we have $SO(4)_f \subset SO(4)_{f_5}$, which is a proper inclusion of the identity components as $\Gamma \not\subset SO(4)_f$ but $\Gamma \subset SO(4)_{f_5}$. In particular, $\dim SO(4)(\langle f\rangle) \geq 3$. On the other hand, $SO(4)(\langle f\rangle)\cap I_{f_5}$ is a circle (with center $\langle f_5\rangle$) and so $\dim\big(A_f\cap T_{\langle f\rangle}\big) = 1$, or equivalently, $E_f\circ f\cap f_*\big(so(4)\big)$ is 1–dimensional. Thus, equality holds in (1). For $n = 6$, let σ denote the center segment of the cone $\bar{I}_{f_6}$ joining $\langle f_5\rangle$ and the vertex $\langle f_2'\rangle$. For $f\colon S^3 \to S^6$ with $\langle f\rangle \in \sigma$, the orbit $SO(4)(\langle f\rangle)$ is 4–dimensional (as $SO(4)_f = SO(4)_{f_5}\cap SO(4)_{f_2'} = SO(4)_{f_2}\cap SO(4)_{f_2'}$) and, as Γ fixes $\langle f_2'\rangle$, it intersects I_{f_6} only at $\langle f\rangle$. Thus, $E_f\circ f\cap f_*\big(so(4)\big) = \{0\}$ and since $\dim E_f\circ f = 3$, equality holds in (1). For $\langle f\rangle \in I_{f_6}\setminus\sigma$, the orbit $SO(4)(\langle f\rangle)$ is 5–dimensional (as it can easily be seen by projecting $\langle f\rangle$ down to I_{f_5} from $\langle f_2'\rangle$ and it intersects I_{f_6} in the circle $\Gamma(\langle f\rangle)$. In particular, $E_f\circ f\cap f_*\big(so(4)\big)$ is 1–dimensional and we are done. Finally, for $n = 7$, let σ denote the segment in $\bar{I}_{f_7}$ joining $\langle f_5\rangle$ and $\langle f_{5'}\rangle$ and consider first the case $f\colon S^3 \to S^7$ with $\langle f\rangle \in \sigma$. If $\big(a(t)\big) \subset SO(4)$ is a 1–parameter subgroup such that $a(t)\cdot\langle f\rangle \in I_{f_7}$, then $a(t)\big(\bar{I}_{f_7}\big) = \bar{I}_{f_7}$. Hence $a(t)\cdot\big(\bar{I}_{f_5}\big) = \bar{I}_{f_5}$, in particular, $a(t)$ leaves $\langle f_5\rangle$ fixed. Similarly, $a(t)\cdot\langle f_5'\rangle = \langle f_5'\rangle$ and it follows that $a(t)$ leaves $\langle f\rangle$ fixed. Thus, $E_f\circ f\cap f_*\big(so(4)\big) = \{0\}$ and, as $\dim SO(4)(\langle f\rangle) = 4$ and $\dim E_f\circ f = 5$, equality holds in (1). Secondly, assume that $\langle f\rangle \in \sigma'$, where σ' is a segment joining $\langle f_5'\rangle$ and a point $\langle f'\rangle \in I_{f_5}\setminus\{\langle f_5\rangle\}$. Taking a 1–parameter subgroup $\big(a(t)\big) \subset SO(4)$ with $a(t)\cdot\langle f\rangle \in I_{f_7}$, we obtain that either $a(t)$ leaves $\langle f\rangle$ fixed or rotates $\langle f\rangle$ around the circle $\Gamma(\langle f\rangle)$. Thus $\dim\big(E_f\circ f\cap f_*\big(so(4)\big)\big) = 1$, $\dim SO(4)(\langle f\rangle) = 5$ (as $SO(4)_f = SO(4)_{f'}\cap SO(4)_{f_5'} = SO(4)_{f'}\cap SO(4)_{f_2'}$ $(\subset SO(4)_{f_2}\cap SO(4)_{f_2'})$ is 1–dimensional) and we are done. Lastly, assume that $\langle f\rangle \in \sigma''$, where σ'' is a segment joining $\langle f'\rangle \in I_{f_5}\setminus\{\langle f_5\rangle\}$ and $\langle f''\rangle \in I_{f_5'}\setminus\{\langle f_5'\rangle\}$. Using the circle groups Γ and Γ' it follows that $E_f\circ f\cap f_*\big(so(4)\big) = 2$, $\dim SO(4)(\langle f\rangle) = 6$ and the proof is finished.

Example. The condition for equality in (1) fails to be satisfied if $m > 3$. In fact, for the full harmonic map $f\colon S^5 \to S^9$ given by the Hopf–Whitehead construction applied to the tensor product $\otimes\colon \mathbf{R}^3\times\mathbf{R}^3 \to \mathbf{R}^9$, i.e.,

$$f(x,y) = (\|x\|^2 - \|y\|^2, 2x\otimes y), \qquad x,y \in \mathbf{R}^3,\ \|x\|^2 + \|y\|^2 = 1,$$

we have

$$so(10)\circ f + E_f\circ f + f_*\big(so(6)\big) \neq K(f).$$

In fact, as computation shows, $\dim K(f) = 81$, $\dim so(10)\circ f = 45$, $\dim E_f\circ f = 9$ and $\dim f_*\big(so(6)\big) \leq 15$.

Theorem 2. *For any $m \geq 3$ and $k \geq 2$ (nonrigid range), the principal isotropy type of the parameter space L° with respect to the $SO(m+1)$–action is finite. Equivalently, the symmetry group of a full k–homogeneous harmonic map $f\colon S^m \to S^n$ is generically (i.e., on an open dense subset of L°) is finite.*

Proof. We have just seen from the last step of the proof above that, for $m = 3$ and $k = 2$, there exists finite isotropy and so the principal isotropy type should also be finite. Indicate, for a moment, the dependence of E on m and k by E_m^k. Then, for $k \geq 2$, $E_3^2 \otimes_{\mathbf{R}} \mathbf{C} \subset E_3^k \otimes_{\mathbf{R}} \mathbf{C}$ [8] and the result follows in this case. For $m \geq 4$, $SO(m+1)$ is simple and the (complex) irreducible $SO(m+1)$–modules with nonfinite principal isotropy type were classified by Wu–Yi Hsiang ([4], pp. 83–85). A comparison of his list with the irreducible components of $E_m^k \otimes_{\mathbf{R}} \mathbf{C}$ gives the result.

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