TRANSVERSAL JACOBI FIELDS
FOR HARMONIC FOLIATIONS

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1. Introduction. A foliation $\mathcal{F}$ on a manifold $M$ is given by the exact sequence of vectorbundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0,$$

where $L$ is the tangent bundle and $Q$ the normal bundle of $\mathcal{F}$. If $V(\mathcal{F})$ denotes the Lie algebra of infinitesimal automorphisms of $\mathcal{F}$, we have an exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathcal{F}) \xrightarrow{\pi} Q^L \to 0,$$

where $Q^L$ denotes the invariant sections of $Q$ under the action of $\Gamma L$ by Lie derivatives [4; 9]. We assume throughout that $\mathcal{F}$ is Riemannian, with a bundle-like metric $g_M$ on $M$ inducing the holonomy invariant metric $g_Q$ on $Q \equiv L^\top$ [10]. $\nabla$ denotes the unique metric and torsion-free connection in $Q$ (see, e.g., [3; 9; 10]). Associated to $\nabla$ are transversal curvature data, in particular the (transversal) Ricci operator $\rho_\nabla: Q \to Q$ and the Jacobi operator $J_\nabla = \Delta - \rho_\nabla: \Gamma Q \to \Gamma Q$ [4]. In this paper we study geometric properties of infinitesimal automorphisms $Y \in V(\mathcal{F})$ such that $\tilde{Y} = \pi(Y) \in \Gamma Q^L$ satisfies the Jacobi condition $J_\nabla \tilde{Y} = 0$. In view of the variational meaning of $J_\nabla$ [4], it is then natural to assume $\mathcal{F}$ to be harmonic; that is, all leaves of $\mathcal{F}$ are minimal submanifolds [3].

THEOREM A. Let $\mathcal{F}$ be a transversally orientable harmonic Riemannian foliation on a compact orientable Riemannian manifold $(M, g_M)$, and $Y$ an infinitesimal automorphism of $\mathcal{F}$. Then the following properties are equivalent:

(i) $\tilde{Y}$ is a transversal Killing field, that is, $\theta(Y) g_Q = 0$;
(ii) $\tilde{Y}$ is a transversally divergence-free Jacobi field;
(iii) $\tilde{Y}$ is transversally affine, that is, $\theta(Y) \nabla = 0$.

REMARKS. (1) If $\mathcal{F}$ is given by the fibers of a harmonic submersion $f: M \to N$, the equivalence of (i) and (ii) specializes to the statement that a projectable vector field $v = V \circ f$ ($V \in \Gamma TN$) along $f$ is a divergence-free Jacobi field along $f$ if and only if $V$ is a Killing vector field on $N$ (see [12] for the particular case $N = S^n$). Note, however, that the Jacobi condition for $v$ in the harmonic map theory uses the pull-back of the Riemannian connection of $N$ which, in general, differs from the canonical connection $\nabla$ in $Q$ [3].

(2) For the foliation of $M$ by points, the equivalence of (i) and (ii) is the classical characterization of Killing vector fields given by Lichnerowicz [7] and Yano [14]; the implication (iii) $\Rightarrow$ (i) is due to Yano [14].

Received April 4, 1986.

The work of the first two authors was supported in part by a grant from the National Science Foundation.

THEOREM B. Let \( \mathcal{F} \) and \( Y \) be as in Theorem A with \( \text{codim} \mathcal{F} = 2 \). Then the following properties are equivalent:

(i) \( \bar{Y} \) is a transversal conformal field, that is, \( \theta(Y)g_Q = \sigma \cdot g_Q \);
(ii) \( \bar{Y} \) is a transversal Jacobi field.

REMARK. For the point foliation this result goes back to Lichnerowicz [7]. A sharpening for Jacobi fields along conformal diffeomorphisms was given by Smith [11] (see also [12]).

The key to obtaining Theorems A and B is the transversal divergence theorem given in Section 2 (Theorem C). In Section 3, we generalize the operators \( \delta, \delta^* \) occurring in the Berger–Ebin decomposition [1] to the foliation context. They play a crucial role in deriving a basic identity relating the trace Laplacian and curvature (Theorem D, (i)). The proofs of Theorem A and B are given in Section 4. Finally, in Section 5, we give a few examples.

The terminology for foliations is based on [3–6; 9; 10]. For the related concepts and results in harmonic map theory we refer to Eells–Lemaire [2].

2. Transversal divergence theorem. Let \( \mathcal{F} \) be as in Theorem A and let \( \Omega^p_{\mathcal{F}}(\mathcal{F}) \subset \Omega(M) \) be the subcomplex of basic forms (forms killed by \( \iota(X) \), \( \theta(X) \) for \( X \in \Gamma L \), cf. [5; 6; 9; 10]). The transversal orientation and \( g_Q \) give rise to a transversal volume form \( \nu \in \Omega^0_{\mathcal{F}}(\mathcal{F}) \), \( q = \text{codim} \mathcal{F} \). Clearly \( d\nu = 0 \). The characteristic form of \( \mathcal{F} \) (a volume form along the leaves) is given by \( \chi_{\mathcal{F}} = * \nu \in \Omega^p(M) \), \( p + q = n = \dim M \), with respect to the Hodge star operator of \( g_M \) on \( \Omega(M) \). Then \( \mu = \nu \wedge \chi_{\mathcal{F}} \) is the Riemannian volume form of \( (M, g_M) \). Given an infinitesimal automorphism \( Y \) of \( \mathcal{F} \), the transversal divergence \( \text{div}_B \bar{Y} \) is defined as the unique basic scalar satisfying

\[ \theta(Y)\nu = \text{div}_B \bar{Y} \cdot \nu. \]

It depends only on \( \bar{Y} = \pi(Y) \).

THEOREM C. Let \( \mathcal{F} \) and \( Y \) be as in Theorem A. Then

\[ \int_M \text{div}_B \bar{Y} \cdot \mu = 0. \]

Proof. We have

\[ \text{div}_B \bar{Y} \cdot \mu = (\text{div}_B \bar{Y} \cdot \nu) \wedge \chi_{\mathcal{F}} = \theta(Y)\nu \wedge \chi_{\mathcal{F}} = (\iota(Y)\nu) \wedge \chi_{\mathcal{F}} \]

\[ = d(\iota(Y)\nu \wedge \chi_{\mathcal{F}}) + (-1)^q \iota(Y)\nu \wedge d\chi_{\mathcal{F}}. \]

By Stokes’ theorem it suffices to show that the second term is in fact zero. To prove this, we consider the canonical multiplicative filtration [5; 6]

\[ F^q\Omega^m = \{ \omega \in \Omega^m \mid \iota(X_1) \cdots \iota(X_{m-r+1})\omega = 0, X_j \in \Gamma L, j = 1, \ldots, m-r+1 \}, \]

which breaks off above \( q \). We have \( \nu \in F^q \) and \( \iota(Y)\nu \in F^{q-1} \). The harmonicity of \( \mathcal{F} \) is expressed by the \( \mathcal{F} \)-triviality of \( d\chi_{\mathcal{F}} \) or equivalently by \( d\chi_{\mathcal{F}} \in F^2 \) [5; 6]. Thus \( \iota(Y)\nu \wedge d\chi_{\mathcal{F}} \) has filter degree \( (q-1)+2 = q+1 \) and hence vanishes. \( \square \)
3. Operators $\delta, \delta^*$ and fundamental identities. To introduce various differential operators below it is convenient to use the following special (orthonormal) moving frames on $M$. For $x \in M$, let $\{e_\alpha\}_{\alpha=1}^n \subset T_x M$ be an (oriented) orthonormal basis with $\{e_i\}_{i=1}^p \subset L_x$ and $\{e_\alpha\}_{\alpha=p+1}^n \subset Q_x \equiv L_x^\perp$. Let $U$ be a distinguished (flat) neighborhood of $x$ for some local (Riemannian) submersion $f: U \to B$. For $\alpha = p+1, \ldots, n$, let $E_\alpha \in \Gamma(U, Q)$ be the pull-back of the extension of $f_* e_\alpha$ to a vector field on $B$ by parallel transport along geodesic segments emanating from $f(x)$ (use [10, Prop. 4.2]). Then we complete $\{E_\alpha\}_{\alpha=p+1}^n$ by the Gram–Schmidt process to a moving frame $\{E_\alpha\}_{\alpha=1}^n$ by adding $E_i \in \Gamma(U, L)$ with $(E_i)_x = e_i, \ i = 1, \ldots, p$. We have then for $\alpha, \beta = p+1, \ldots, n$:

$$\nabla_{e_\alpha} E_\beta = (\nabla_{E_\alpha} E_\beta)_x = 0;$$

and, as a consequence of torsion-freeness [3, 1.5], $[E_\alpha, E_\beta]_x \in L_x$. Furthermore, as the $E_\alpha$ are infinitesimal automorphisms, we have

$$\nabla_X E_\alpha = \pi[X, E_\alpha] = 0, \ X \in \Gamma(U, L).$$

Generalizing to the foliation context the operators occurring in the Berger–Ebin decomposition [1], we define $\delta: \Gamma S^2 Q^* \to \Gamma Q^*, \ S^2 = \text{symmetric square}$, by the local formula

$$\delta h = -\sum_{\alpha=p+1}^n (\nabla_{E_\alpha} h)(E_\alpha, \cdot), \ h \in \Gamma S^2 Q^*$$

and $\delta^*: \Gamma Q^* \to \Gamma S^2 Q^*$ by

$$(\delta^* \omega)(V, W) = \frac{1}{2} ((\nabla_V \omega)(W) + (\nabla_W \omega)(V)), \ \omega \in \Gamma Q^*, \ V, W \in \Gamma Q.$$ 

Note that $\Omega^2_B(\mathcal{F}) \subset \Gamma Q^*$. Similarly, the basic symmetric 2-forms (killed by $i(X), \theta(X)$ for $X \in \Gamma L$) will be identified with a subspace of $\Gamma S^2 Q^*$.

**PROPOSITION 1.** $\delta$ and $\delta^*$ map basic forms to basic forms.

**Proof.** For $X \in \Gamma L$, a direct calculation yields the commutation relations

$$(\theta(X), \delta)h)(e_\alpha) = -\sum_{\beta=p+1}^n [X, E_\beta]_x \{h(E_\alpha, E_\beta)\}, \ h \in \Gamma S^2 Q^*,$$

$$(2[\theta(X), \delta^*] \omega)(E_\alpha, E_\beta) = [X, E_\alpha]_x \omega(E_\beta) + [X, E_\beta]_x \omega(E_\alpha), \ \omega \in \Gamma Q^*.$$ 

If $h$ is basic, then $\theta(X)h = 0$ and the right-hand side of the first formula vanishes, as $h(E_\alpha, E_\beta) \in \Omega^1_B(\mathcal{F})$ and $[X, E_\beta]_x \in L_x$. Thus $\theta(X)\delta h = 0$ and $\delta h$ is basic. The argument for $\delta^*$ is similar. \qed

**PROPOSITION 2.** For basic $h \in \Gamma S^2 Q^*$ and $\omega \in \Omega^1_B(\mathcal{F})$ we have $\langle \delta h, \omega \rangle = \langle h, \delta^* \omega \rangle$ with respect to the global scalar product on basic forms.

**Proof.** Indeed, by local computation, we find for the pointwise scalar product that

$$(\delta h, \omega)_x = - (\text{div}_B Z)_x + (h, \delta^* \omega)_x,$$

where $Z \in \Gamma Q^L$ is the $g_Q$-dual of the basic 1-form $\lambda$ given locally by
\[ \lambda = \sum_{\beta = p + 1}^{n} h(\cdot, E_{\beta}) \omega(E_{\beta}). \]

The proposition follows now by integration, applying Theorem C. \( \square \)

Let \( d_B \) be the restriction of \( d \) to basic forms. The adjoint is denoted by \( \delta_B : \Omega_B^{\ast}(\mathcal{F}) \rightarrow \Omega_B^{\ast -1}(\mathcal{F}) \). For a harmonic Riemannian \( \mathcal{F} \) it follows from \([5; 6]\) that \( \delta_B \) on \( \omega \in \Omega_B^{1}(\mathcal{F}) \) is given by the formula

\[ \delta_B \omega = - \sum_{\alpha} (\nabla_{E_{\alpha}} \omega)(E_{\alpha}). \]

The range of a summation over a greek index is here, and everywhere below, to extend from \( p + 1 \) to \( n \).

THEOREM D. Let \( \mathcal{F} \) and \( Y \) be as in Theorem A, and \( \omega \in \Omega_B^{1}(\mathcal{F}) \) the \( g_Q \)-dual of \( \bar{\nabla} = \pi(Y) \). Then we have the following identities:

1. \[ 2\delta \omega = - \text{trace} \nabla^2 \omega - \rho \bar{\nabla}(\omega) + d_B^2 \delta_B \omega; \]
2. \[ \text{div}_B \bar{\nabla} = - \delta_B \omega = (\delta \omega, g_Q); \]
3. \[ |\delta \omega - 1/q \cdot \text{div}_B \bar{\nabla} \cdot g_Q|^2 = |\delta \omega|^2 - (1/q)(\text{div}_B \bar{\nabla})^2 \text{ (pointwise norms)}. \]

Proof. At \( x \in M \), we have

\[ 2(\delta \omega)(e_{\beta}) = - 2 \sum_{\alpha} (\nabla_{e_{\alpha}}(\delta \omega))(e_{\alpha}, e_{\beta}) = - 2 \sum_{\alpha} e_{\alpha}((\delta \omega)(E_{\alpha}, E_{\beta})) \]

\[ = - \sum_{\alpha} e_{\alpha}((\nabla_{E_{\alpha}} \omega)(E_{\beta}) + (\nabla_{E_{\beta}} \omega)(E_{\alpha})) \]

\[ = -(\text{trace} \nabla^2 \omega)(e_{\beta}) - \sum_{\alpha} e_{\alpha} E_{\beta} \{ \omega(E_{\alpha}) \} + \sum_{\alpha} e_{\alpha} \{ \omega(\nabla_{E_{\beta}} E_{\alpha}) \} \]

\[ = -(\text{trace} \nabla^2 \omega)(e_{\beta}) - \sum_{\alpha} e_{\beta} E_{\alpha} \{ \omega(E_{\alpha}) \} - \sum_{\alpha} \omega(\nabla_{e_{\alpha}} \nabla_{E_{\beta}} E_{\alpha}), \]

where in the last equality we used the fact that \( [E_{\alpha}, E_{\beta}]_X \{ \omega(E_{\alpha}) \} = 0 \), as \( \omega(E_{\alpha}) \) is basic and \( [E_{\alpha}, E_{\beta}]_X \in L_X \). On the other hand,

\[ (d_B \delta_B \omega)(e_{\beta}) = - e_{\beta} \left\{ \sum_{\alpha} (\nabla_{E_{\alpha}} \omega)(E_{\alpha}) \right\} \]

\[ = - \sum_{\alpha} e_{\beta} E_{\alpha} \{ \omega(E_{\alpha}) \} + \sum_{\alpha} \omega(\nabla_{e_{\beta}} \nabla_{E_{\alpha}} E_{\alpha}). \]

Noting that \( \nabla_{[E_{\alpha}, E_{\beta}]_X} E_{\alpha} \) does not contribute to the curvature \( R_{\bar{\nabla}}(e_{\alpha}, e_{\beta}) \), as \( [E_{\alpha}, E_{\beta}]_X \in L_X \), we obtain (1). As for (2), we have

\[ (\Theta(Y) \nu)(E_{p + 1}, \ldots, E_n) = Y(\nu(E_{p + 1}, \ldots, E_n)) - \sum_{\alpha} \nu(E_{p + 1}, \ldots, \pi[Y, E_{\alpha}], \ldots, E_n) \]

\[ = - \sum_{\alpha} g_Q(\pi[Y, E_{\alpha}], E_{\alpha}) \]

\[ = \sum_{\alpha} g_Q(\nabla_{E_{\alpha}} \bar{\nabla}, E_{\alpha}) \]

\[ = \sum_{\alpha} (\nabla_{E_{\alpha}} \omega)(E_{\alpha}) \]

\[ = - \delta_B \omega, \]
while the second equality is immediate. Using (2), the left-hand side of (3) can be rewritten as

$$|\delta^* \omega|^2 - \frac{2}{q} \text{div}_B \tilde{\nabla} \cdot (\delta^* \omega, g_Q) + \frac{1}{q} (\text{div}_B \tilde{\nabla})^2 = |\delta^* \omega|^2 - \frac{1}{q} (\text{div}_B \tilde{\nabla})^2,$$

and (3) follows.

\[\square\]

4. Proof of Theorems A and B. We first observe that \(\tilde{\nabla}\) is transversally Killing if and only if \(\delta^* \omega = 0\), \(\tilde{\nabla}\) is transversally Jacobi if and only if trace \(\nabla^2 \omega + \rho_{\nabla}(\omega) = 0\) (by duality), and \(\tilde{\nabla}\) is transversally divergence-free if and only if \(\delta_B \omega = 0\). The equivalence of (i) and (ii) in Theorem A follows readily from Theorem D. As shown in [4], (i) \(\Rightarrow\) (iii). It therefore suffices to prove (iii) \(\Rightarrow\) (ii). We use the characterization of transversally affine infinitesimal automorphisms by the identity

$$\nabla_{\mathcal{F}} A_{\mathcal{F}}(Y) = R_{\mathcal{F}}(\tilde{\nabla}, V), \quad V \in \Gamma Q,$$

where \(A_{\mathcal{F}}(Y): Q \to Q\) is given by the difference \(\theta(Y) - \nabla_Y\) (and depends only on \(\tilde{\nabla}\), see [4]). Evaluating this identity for \(V = E_\alpha\) and summing over \(\alpha\), we obtain

$$-\text{trace} \nabla^2 \tilde{\nabla} = \rho_{\mathcal{F}}(\tilde{\nabla}),$$

which is precisely the Jacobi condition. It remains to show \(\text{div}_B \tilde{\nabla} = 0\). By Theorem C, it suffices to show that \(\text{div}_B \tilde{\nabla}\) is a constant function. Since \(\text{div}_B \tilde{\nabla} \in \Omega^1_B(\mathbb{F})\), it remains to verify that \(e_\beta \text{div}_B \tilde{\nabla} = 0, \beta = p + 1, ..., n\). Indeed, we have

$$e_\beta \text{div}_B \tilde{\nabla} = e_\beta \left\{ \sum_\alpha g_Q(\nabla_{E_\alpha}, E_\alpha) \right\}$$

$$= \sum_\alpha g_Q(\nabla_{e_\beta}, E_\alpha, E_\alpha)$$

$$= -\sum_\alpha g_Q((\nabla_{e_\beta}, A_{\mathcal{F}})(E_\alpha), E_\alpha)$$

$$= -\sum_\alpha g_Q(R_{\mathcal{F}}(\tilde{\nabla}, e_\beta), e_\alpha)$$

$$= 0,$$

which completes the proof of Theorem A.

To prove Theorem B, we first note that the transversal conformality condition translates to \(2\delta^* \omega = \sigma \cdot g_Q\). By (2) and \((g_Q, g_Q) = q\), this identity is further equivalent to \(\delta^* \omega = 1/q \text{div}_B \tilde{\nabla} \cdot g_Q\). Applying \(\delta\) to both sides, and observing that the holonomy invariance of \(g_Q\) implies \(\delta g_Q = 0\), we have

$$\delta \delta^* \omega = \frac{1}{q} \delta (\text{div}_B \tilde{\nabla} \cdot g_Q) = -\frac{1}{q} d_B(\text{div}_B \tilde{\nabla}) = \frac{1}{q} d_B \delta_B \omega.$$

For \(q = 2\), this reduces (1) to the Jacobi condition for \(\tilde{\nabla}\). Assuming conversely the Jacobi condition for \(\tilde{\nabla}\), we have, again by (1), \(2\delta^* \omega = d_B \delta_B \omega\). Taking the global scalar product with \(\omega\), we obtain

$$2|\delta^* \omega|^2 - \|\text{div}_B \tilde{\nabla}|^2 = 0.$$

For \(q = 2\), identity (3) implies the transversal conformality of \(\tilde{\nabla}\).

\[\square\]
5. Examples. (1) Given a compact oriented Riemannian manifold $M$ with positive semidefinite Ricci tensor, the Albanese (Jacobi) map $J: M \to A(M)$ is the totally geodesic projection of a fibre bundle over the flat Albanese torus $A(M)$, of dimension equal to the first Betti number of $M$ [8; 12]. From the geometric properties of this bundle it follows that the linear space of parallel vector fields on $M$ is isomorphic (via $J_*$) with the linear space of parallel vector fields on $M$. By Theorem A, this space is further isomorphic with the linear space of transversal divergence-free Jacobi automorphisms of the corresponding harmonic Riemannian foliation.

(2) By Theorem B, the linear space of transversal Jacobi automorphisms of the harmonic Hopf fibration $f: S^3 \to S^2$ is isomorphic with the linear space of infinitesimally conformal fields on $S^2$, in particular, it is 6-dimensional. Note further that the nullity of $f$ as a harmonic map equals 8 [13].

REFERENCES