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ON CLASSIFICATION OF ORTHOGONAL MULTIPLICATIONS À LA DO CARMO-WALLACH

ABSTRACT. The space of range-equivalence classes of full orthogonal multiplications $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $n \leqslant p \leqslant n^2$, is shown to be a compact convex body lying in $\mathrm{so}(n) \otimes \mathrm{so}(n)$. Furthermore, the dimension of the space of equivalence classes is determined to be $(n^2(n-1)^2)/4 - n(n-1)$.

1. INTRODUCTION

A fundamental problem is constructive harmonic map theory posed by R. T. Smith [7] is to classify orthogonal multiplications $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ ([3, (4.6) Problem]; cf. also [1], [2], [5], [6], [8]). The connection with harmonic maps is given by the Hopf–Whitehead construction which, when applied to such F, gives rise to a (quadratic) harmonic map

$$f_F: S^{2n-1} \to S^p$$

defined by

$$f_F(x, y) = (\|x\|^2 - \|y\|^2, 2F(x, y)), x, y \in \mathbb{R}^n, \|x\|^2 + \|y\|^2 = 1.$$

Important examples include the various Hopf maps and, for any n, $f_{F_\otimes}: S^{2n-1} \to S^{n^2}$ with $F_\otimes: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n^2}$ the tensor product.

CLASSIFICATION THEOREM. *The space of range-equivalence classes of full orthogonal multiplications* $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $n \leqslant p \leqslant n^2$, *can be parametrized by a compact convex body* L_n *lying in a finite dimensional vector space* E_n. *As an* $\mathrm{SO}(n) \times \mathrm{SO}(n)$ *module* (*induced by precomposition*) E_n *is isomorphic with* $\mathrm{so}(n) \otimes \mathrm{so}(n)$ (*given by* $\mathrm{Ad} \otimes \mathrm{Ad}$). *For* $n \geqslant 3$, *it has finite principal isotropy type. In particular, the space of equivalence classes of orthogonal multiplications* $(= L_n/\mathrm{SO}(n) \times \mathrm{SO}(n))$ *is of dimension* $(n^2(n-1)^2)/4 - n(n-1)$.

REMARKS 1. As shown by Parker [6], for $n = 2$, $L$is a line segment with boundary points corresponding to the Hopf map and its dual. In fact, a complete classification of full quadratic harmonic maps of S^3 into S^n, $2 \leqslant n \leqslant 8$, is given in [10], [11]. Moreover, as in [6], for $n = 3$, $\dim(L_3/\mathrm{SO}(3) \times \mathrm{SO}(3)) = 3$ can be obtained by explicit computation.

2. Besides the Hopf–Whitehead method, one can construct quadratic harmonic maps $f: S^{4n-1} \to S^p$ from orthogonal multiplications $F_i: \mathbb{R}^n \times$

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$\mathbb{R}^n \to \mathbb{R}^{p_i}$, $i = 1, \ldots, 6$, by

$$f(x, y, u, v) = (\|x\|^2 + \|y\|^2 - \|u\|^2 - \|v\|^2,\\ 2F_1(x, u), 2F_2(x, v), 2F_3(y, u), 2F_4(y, v)).$$

and

$$f(x, y, u, v) = (\|x\|^2 - \|y\|^2, \|u\|^2 - \|v\|^2, 2F_1(x, y),\\ \sqrt{2}\, F_2(x, u), \sqrt{2}\, F_3(x, v), \sqrt{2}\, F_4(y, u),\\ \sqrt{2}\, F_5(y, v), 2F_6(u, v)),\ x, y, u, v \in \mathbb{R}^n,\ \|x\|^2 + \|y\|^2 + \|u\|^2 + \|y\|^2 = 1.$$

3. With respect to the cell structure of the parameter space L^0 of equivalence classes of full harmonic maps of spheres with (fixed) constant energy density, the parameter space L given in the theorem above can be shown to compose a cell on ∂L^0 [11].

2. Proof of the Classification Theorem

Recall first that an orthogonal multiplication is a bilinear map $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ satisfying $\|F(x, y)\| = \|x\| \cdot \|y\|$, x, $y \in \mathbb{R}^n$. If F is full, i.e. surjective, then we have $n \leqslant p \leqslant n^2$. Two orthogonal multiplications F, $F': \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ are said to be equivalent if $F(Vx, Wy) = UF'(x, y)$, x, $y \in \mathbb{R}^n$, holds for some V, $W \in \mathrm{SO}(n)$ and $U \in \mathrm{O}(p)$. F and F' are range-equivalent if they are equivalent with $V = W = I_n$ (=identity).

Turning to the proof of the theorem, let $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a full orthogonal multiplication. By the universal property of the tensor product, there exists a unique $(p \times n^2)$-matrix A of maximal rank such that $F = A \cdot F_{\otimes}$. Now, for x, $y \in \mathbb{R}^n$, we have

$$\|x \otimes y\|^2 = \|x\|^2 \cdot \|y\|^2 = \|F(x, y)\|^2 = \langle A^t A(x \otimes y), x \otimes y \rangle\\ = \langle A^t A, (x \otimes y)^2 \rangle,$$

or equivalently,

$$\langle A^t A - I_{n^2}, (x \otimes y)^2 \rangle = 0,$$

where ^{2} = (symmetric) tensor square and the scalar product is taken in $S^2(\mathbb{R}^{n^2})$. Putting $W_n = \mathrm{span}\{(x \otimes y)^2 | x,\ y \in \mathbb{R}^n\} \subset S^2(\mathbb{R}^{n^2})$, $E_n = W_n^{\perp}$ and $L_n = \{C \in E_n | C + I_{n^2} \geqslant 0\}$ ('$\geqslant$' symmetric positive semidefinite), we obtain a map from the space of range-equivalence classes of orthogonal multiplications into L_n which sends the class of F to $A^t A - I_{n^2} \in L_n$. This correspondence is easily seen to be bijective onto L (in fact, the inverse is given by associating to $C \in L_n$ the orthogonal multiplication $F = \sqrt{C + I_{n^2}} \cdot F_{\otimes}$). With respect to the standard base $\{e_i \otimes e_j\}_{i,j=1}^n \subset \mathbb{R}^n \otimes \mathbb{R}^n$, for a symmetric endomorphism C of $\mathbb{R}^n \otimes \mathbb{R}^n$, we have the polynomial expansion

$$\langle C(x \otimes y), x \otimes y \rangle = \sum_{i,j,k,l=1}^{n} c_{ijkl} x_i y_j x_k y_l,$$

where

$$x = \sum_{i=1}^{n} x_i e_i, \; y = \sum_{j=1}^{n} y_j e_j \quad \text{and} \quad c_{ijkl} = \langle C(e_i \otimes e_j), e_k \otimes e_l \rangle.$$

Comparing coefficients, we obtain that $C \in E_n$ is equivalent to the double skew-symmetry of c_{ijkl} in i, k and j, l. Thus, $E_n \cong \mathrm{so}(n) \otimes \mathrm{so}(n)$ with an isomorphism which, in fact, respects the $\mathrm{SO}(n) \times \mathrm{SO}(n)$ module structures on E_n and $\mathrm{so}(n) \otimes \mathrm{so}(n)$. Assume now that the connected principal isotropy type $(H^0_{E_n})$ is nontrivial. For $n \neq 4$, E_n is irreducible and by a result of Wu-Yi Hsiang ([4, pp. 93–94]), we have

$$(H^0_{E_n}) = (H^0_{E_n|\mathrm{SO}(n) \times \{1\}}) \times (H^0_{E_n|\{1\} \times \mathrm{SO}(n)}).$$

As $\mathrm{SO}(n)$-modules,

$$\begin{aligned} E_n|\mathrm{SO}(n) \times \{1\} &\cong E_n|\{1\} \times \mathrm{SO}(n) \\ &\cong \mathrm{so}(n) \oplus \cdots \oplus \mathrm{so}(n) \left(\frac{n(n-1)}{2} \text{ times}\right) \end{aligned}$$

and since the generic intersection of $n(n-1)/2$ maximal tori is finite, we obtain a contradiction. For $n = 4$, we first split $E_4 \cong \mathrm{so}(4) \otimes \mathrm{so}(4)$ into four irreducible components and then apply the argument above.

REFERENCES

1. Adem, J., 'Construction of some Normed Maps', *Boll. Soc. Mexicana* **2** (1975), 59–75.
2. Baird, P., *Harmonic Maps with Symmetry*, Pitman, 1983.
3. Eells, J. and Lemaire, L., 'Selected Topics in Harmonic Maps', *CBMS Reg. Conf. Ser.* 50, 1980.
4. Hsiang, Wu-Yi., *Cohomology Theory of Topological Transformation Groups*, Springer, 1975.
5. Lam, K. Y., 'Construction of Nonsingular Bilinear Maps', *Topology* **6** (1967), 423–426.
6. Parker, M., 'Orthogonal Multiplications in Small Dimensions', *Bull. London Math. Soc.* **15** (1983), 368–372.
7. Smith, R. T., 'Harmonic Mappings of Spheres', Thesis, Warwick Univ., 1972.
8. . 'Harmonic Mappings of Spheres', *Amer. J. Math.* **97** (1975), 364–385.
9. Toth, G. and D'Ambra, G., 'Parameter Space for Harmonic Maps of Constant Energy Density into Spheres', *Geom Dedicata* **17** (1984), 61–67.
10. Toth, G., 'On Classification of Quadratic Harmonic Maps of S^3' (to appear).
11. . 'Classification of Quadratic Harmonic maps of S^3 into Spheres' (to appear).

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